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THE PERTURBATION OF LOVE WAVE SPECTRA

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ABSTRACT

The equations governing the variational principles for Love wave spectra are investigated. It is shown that assumptions used by earlier authors are not necessary to the validity of the variational techniques.

Moreover it is demonstrated that except for a homogeneous plate, these assumptions are false for plane multilayered media and lead to incorrect expressions for group-velocity perturbations. The correct expressions are determined and examples of their use are given.

INTRODUCTION

A potentially important method for the calculation of surface wave dispersion curves and the inversion of observed dispersion data is the use of partial derivatives obtained by techniques based on variational principles. The error in phase velocities calculated in this manner is frequently less than 10 per cent of the difference between the trial and actual values.

On the other hand, group-velocity perturbations using similar techniques yielded velocities which were often closer to the starting or trial values. Motivated by what was felt to be unaccountably large errors for the group velocity, theoretical derivatives were obtained from the dispersion equations of simple models.

While calculating analytic expressions for Love wave energy integrals and their derivatives in a layer over a half-space, it was discovered that unnecessary relations had been used by previous investigators to derive equations for phase-velocity perturbations due to changes in elastic constants. By numerical differentiation, it was verified that Love and Rayleigh wave energy integrals neglected as second order functions of frequency, wave number, or elastic constants in the derivation of group velocity, media response and phase velocity derivatives were actually first order functions of those parameters in a multilayered half-space or plate. The resulting expressions are, fortunately, still correct since the requirement of second order dependence was only a sufficient condition. Unfortunately, the use of these relations give erroneous expressions for the partial derivatives of group velocity and media response for all Love wave models except a homogeneous plate.

In this paper, the relations which govern the variational principles, such as Rayleigh's principle are derived in detail. The previously mentioned relations or assumptions are shown to be unnecessary. Expressions for the plate and the layer over a half-space are derived and values are calculated to show the effect of using the incorrect formulas for group velocity perturbations.

The correct expressions for the perturbations of group velocity and amplitude response are then extended to a multilayered half-space.

Theory

Previous work on the perturbation of Love wave spectra has been based upon variational properties of the Lagrangian

$$\mathfrak{L} \equiv \omega^2 I_0 - k^2 I_1 - I_2 \tag{1}$$

where

$$I_0 = \int_0^\infty \rho v^2 \, dz \tag{2}$$

$$I_1 = \int_0^\infty \mu v^2 \, dz \tag{3}$$

$$I_2 = \int_0^\infty \mu \left(\frac{dv}{dz}\right)^2 dz; \tag{4}$$

and where v is the horizontal transverse displacement, μ the rigidity, ρ the density, k the wave number and ω the frequency. [Meisner, 1926; Jeffreys, 1934, 1961; Takeuchi *et al*, 1962, 1964; Anderson, 1964; Andrianova *et al*, 1965; and Vilkovitch *et al*, 1966]. Before discussing these properties, we will evaluate $\mathfrak{L}$ and obtain the associated relations which govern the perturbation equations.

The equation of motion for horizontally polarized shear waves in a media in which the physical parameters ρ and μ are functions of z only is given by

$$\frac{d}{dz}\left(\mu \frac{dv}{dz}\right) - k^2 \mu v + \omega^2 \rho v = 0. \tag{5}$$

Multiplying equation (5) by v and integrating from $z = 0$ to ∞ yields the desired form

$$\mathfrak{L} = -\left(v\mu \frac{dv}{dz}\right)\bigg|_0^\infty. \tag{6}$$

In obtaining equation (6), the integration of

$$\int_0^\infty v \frac{d}{dz}\left(\mu \frac{dv}{dz}\right) dz$$

was done by parts and by making use of the continuity of displacement, v, and stress, $\mu \, dv/dz$, at discontinuities of the elastic parameters.

For Love waves, we impose the boundary conditions of a free surface at $z = 0$, i.e., $\mu(0) \, dv^{(0)}/dz = 0$ and that the motion vanish at infinity. Therefore for Love waves in a vertically heterogeneous half-space, the Lagrangian vanishes.

$$\mathfrak{L}(\omega, k, \mu, \rho) = 0. \tag{7}$$

For a heterogeneous plate, bounded at say $z = 0$ and $z = H$ by either a free or rigid surface, i.e., $\mu\, dv/dz = 0$ or $v = 0$ respectively, we obtain the same result.

$$\omega^2 I_0 = k^2 I_1 + I_2 . \tag{8}$$

Similarly, if we multiply equation (5) by $(\partial v/\partial p)$ and integrate from $z = 0$ to $z = \infty$ where p is one of the variables ω, k, μ or ρ while keeping the other three constant, we obtain

$$\omega^2 \int_0^\infty \rho \frac{\partial}{\partial p} (v^2)\, dz = k^2 \int_0^\infty \mu \frac{\partial}{\partial p} (v^2)\, dz + \int_0^\infty \mu \frac{\partial}{\partial p} \left[\left(\frac{dv}{dz}\right)^2\right] dz - 2 \left(\frac{\partial v}{\partial p}\, \mu \frac{dv}{dz}\right)\Bigg|_0^\infty \tag{9}$$

where we have used the continuity of $\partial v/\partial p$ across interfaces. Any algorithm which makes use of the continuity of displacement in numerical calculations of the eigenfunctions will ensure this. Again the boundary conditions at $z = 0$ and infinity require that the last term in equation (9) vanish. Thus equation (9) becomes

$$\omega^2 \int_0^\infty \rho \frac{\partial}{\partial p} (v^2)\, dz = k^2 \int_0^\infty \mu \frac{\partial}{\partial p} (v^2)\, dz + \int_0^\infty \mu \frac{\partial}{\partial p} \left[\left(\frac{dv}{dz}\right)^2\right] dz. \tag{10}$$

As we shall see, these four equations (10) govern the perturbation equations for Love wave spectra in vertically heterogeneous media. First let us consider the perturbation due to a small change in wave number, sk. The resulting perturbation of the integrands can be expressed as

$$\delta \int A\, dz = \int \delta A\, dz = \left[\int \left(\frac{\partial A}{\partial k}\right)_\omega dz + U \int \left(\frac{\partial A}{\partial \omega}\right)_k dz\right] \delta k + O(\delta k)^2 \tag{11}$$

where the group velocity U is

$$U = \frac{d\omega}{dk}.$$

Combining this with equations (10) for $p = \omega$ and k, yields

$$\omega^2 \int \rho\delta(v^2)\, dz = k^2 \int \mu\delta(v^2)\, dz + \int \mu\delta \left[\left(\frac{dv}{dz}\right)^2\right] dz + O(\delta k)^2 \tag{12}$$

or

$$\omega^2 \delta I_0 = k^2 \delta I_1 + \delta I_2 + O(\delta k)^2.$$

The energy equation (8) for the new eigensolutions is

$$(\omega + \delta\omega)^2 (I_0 + \delta I_0) = (k + \delta k)^2 (I_1 + \delta I_1) + I_2 + \delta I_2 . \tag{13}$$

Expanding gives

$$(\omega + \delta\omega)^2 I_0 + \omega^2 \delta I_0 = (k + \delta k)^2 I_1 + k^2 \delta I_1 + I_2 + \delta I_2 + O(\delta k)^2$$

and using equation (12) we obtain the expression of Rayleigh's principle found in Jeffreys (1961)

$$(\omega + \delta\omega)^2 I_0 = (k + \delta k)^2 I_1 + I_2 + O(\delta k)^2 \tag{14}$$

which in turn yields

$$U = \frac{d\omega}{dk} = \frac{I_1}{cI_0}.$$

For the perturbation due to a small change in rigidity $\delta\mu$, keeping ω and ρ fixed, we can write the integrand perturbations as

$$\int \delta A_\mu \, dz = \left[\int \left(\frac{\partial A}{\partial \mu}\right)_{\omega,k} dz + \left(\frac{\partial k}{\partial \mu}\right)_\omega \int \left(\frac{\partial A}{\partial k}\right)_\mu dz \right] \delta\mu + O(\delta\mu)^2. \tag{15}$$

Using equation (15) with equation (10) where $p = \mu$, we obtain

$$\omega^2 \int \rho\delta(v^2)_\mu \, dz = k^2 \int \mu\delta(v^2)_\mu \, dz + \int \mu\delta\left[\left(\frac{dv}{dz}\right)^2\right]_\mu dz + O(\delta\mu)^2. \tag{16}$$

The energy equation (8) becomes

$$\omega^2[I_0 + \delta(I_0)_\mu] = (k + \delta k_\mu)^2[I_1 + \delta(I_1)_\mu] + I_2 + \delta(I_2)_\mu . \tag{17}$$

Neglecting terms higher than first order in $\delta\mu$ and δk equation (17) reduces to

$$\omega^2 I_0 + \omega^2\delta(I_0)_\mu = k^2 I_1 + k^2\delta(I_1)_\mu + 2k\delta k_\mu I_1 + I_2 + \delta(I_2)_\mu \tag{18}$$

and by equations (8) and (16) to

$$0 = k^2 \int \delta\mu \cdot v^2 \, dz + 2k\delta k_\mu I_1 + \int \delta\mu \cdot \left(\frac{dv}{dz}\right)^2 dz$$

or

$$\delta c_\mu = -\frac{c}{k}\,\delta k_\mu = \frac{c}{2k^2 I_1}\left[k^2 \int \delta\mu \cdot v^2 \, dz + \int \delta\mu \cdot \left(\frac{dv}{dz}\right)^2 dz\right] \tag{19}$$

where we have used the relations

$$\delta(I_0)_\mu = \int \rho\delta(v^2)_\mu \, dz$$

$$\delta(I_1)_\mu = \int \delta\mu \cdot v^2 \, dz + \int \mu \cdot \delta(v^2)_\mu \, dz$$

$$\delta(I_2)_\mu = \int \delta\mu \left(\frac{dv}{dz}\right)^2 dz + \int \mu \cdot \delta \left[\left(\frac{dv}{dz}\right)^2\right]_\mu dz. \tag{20}$$

Similarly for a small perturbation in density $\delta\rho$ with ω and μ fixed we obtain

$$\delta c_\rho = -\frac{c^2}{2I_1} \int \delta\rho \cdot v^2 \, dz. \tag{21}$$

These are the same expressions obtained in Anderson (1964), Andrianova *et al* (1965) and Vilkovitch *et al* (1966) by assuming that the relations

$$\int \rho\delta(v^2)_p \, dz, \qquad \int \mu\delta(v^2)_p \, dz \quad \text{and} \quad \int \mu\delta \left[\left(\frac{dv}{dz}\right)^2\right]_p dz$$

are second order in δp where p can be k, ω, ρ or μ, i.e.,

$$\int \rho \frac{\partial}{\partial p} (v^2) \, dz = \int \mu \frac{\partial}{\partial p} (v^2) \, dz = \int \mu \frac{\partial}{\partial p} \left[\left(\frac{dv}{dz}\right)^2\right] dz = 0. \tag{22}$$

By using equations (10) to obtain the same result, we see that these assumptions are not necessary for the Rayleigh principle to be valid. In fact, we will demonstrate in the next section that except for a single layer plate these conditions are false for plane multilayered media. Although of no consequence in deriving formulas for U, $\delta c\mu$ and $\delta c\rho$ these invalid assumptions yield incorrect expressions for the group-velocity perturbations in Vilkovitch *et al* (1965). As of yet the spherical case has not been investigated but it would be very fortuitous if the group velocity partial derivatives for a sphere in Vilkovitch *et al* (1966) and Andrianova *et al* (1965) were correct while the corresponding expressions for the half-space and heterogeneous plate were not. Backus and Gilbert (1967) state that the integrals of the variations of the spherical eigenfunctions sum out to zero and that the individual integrals are not necessarily zero for similar relations in the spherical problem.

In a multilayered structure of homogeneous layers, the energy integrals take the form

$$I_0 = \sum_{j=1}^{n} \rho_j \int_{z_{j-1}}^{z_j} v^2 \, dz \equiv \sum_{j=1}^{n} \rho_j D_j$$

$$I_1 = \sum_{j=1}^{n} \mu_j \int_{z_{j-1}}^{z_j} v^2 \, dz \equiv \sum_{j=1}^{n} \mu_j D_j$$

$$I_2 = \sum_{j=1}^{n} \mu_j \int_{z_{j-1}}^{z_j} \left(\frac{dv}{dz}\right)^2 dz \equiv \sum_{j=1}^{n} \mu_j S_j \tag{23}$$

and equations (10) can be written

$$\omega^2 \sum_{j=1}^{n} \rho_j \left(\frac{\partial D_j}{\partial k}\right)_{\omega,\mu,\rho} = k^2 \sum_{j=1}^{n} \mu_j \left(\frac{\partial D_j}{\partial k}\right)_{\omega,\mu,\rho} + \sum_{j=1}^{n} \mu_j \left(\frac{\partial S_j}{\partial k}\right)_{\omega,\mu,\rho}$$

$$\omega^2 \sum_{j=1}^{n} \rho_j \left(\frac{\partial D_j}{\partial \omega}\right)_{k,\mu,\rho} = k^2 \sum_{j=1}^{n} \mu_j \left(\frac{\partial D_j}{\partial \omega}\right)_{k,\mu,\rho} + \sum_{j=1}^{n} \mu_j \left(\frac{\partial S_j}{\partial \omega}\right)_{k,\mu,\rho}$$

$$\omega^2 \sum_{j=1}^{n} \rho_j \left(\frac{\partial D_j}{\partial \rho_m}\right)_{\omega,k,\mu} = k^2 \sum_{j=1}^{n} \mu_j \left(\frac{\partial D_j}{\partial \rho_m}\right)_{\omega,k,\mu} + \sum_{j=1}^{n} \mu_j \left(\frac{\partial S_j}{\partial \rho_m}\right)_{\omega,k,\mu}$$

$$\omega^2 \sum_{j=1}^{n} \rho_j \left(\frac{\partial D_j}{\partial \mu_m}\right)_{\omega,k,\rho} = k^2 \sum_{j=1}^{n} \mu_j \left(\frac{\partial D_j}{\partial \mu_m}\right)_{\omega,k,\rho} + \sum_{j=1}^{n} \mu_j \left(\frac{\partial S_j}{\partial \mu_m}\right)_{\omega,k,\rho} \tag{24}$$

where it is understood that in taking the derivatives with respect to a particular layer ρ_m and μ_m that the other layer ρ's and μ's are held constant.

From equations (24) we can obtain the partial derivatives of the Lagrangian

$$\left(\frac{\partial \mathfrak{L}}{\partial k}\right)_{\omega,\rho,\mu} = -2kI_1$$

$$\left(\frac{\partial \mathfrak{L}}{\partial \omega}\right)_{k,\rho,\mu} = 2\omega I_0$$

$$\left(\frac{\partial \mathfrak{L}}{\partial \rho_m}\right)_{\omega,k,\mu} = \omega^2 D_m$$

$$\left(\frac{\partial \mathfrak{L}}{\partial \mu_m}\right)_{\omega,k,\rho} = -(k^2 D_m + S_m). \tag{25}$$

Making use of the implicit relation

$$\mathfrak{L}(\omega, k, \mu, \rho) = 0$$

it is easy to show that

$$U = -\left(\frac{\partial \mathfrak{L}}{\partial k}\right)_{\omega,\rho,\mu} \Big/ \left(\frac{\partial \mathfrak{L}}{\partial \omega}\right)_{k,\rho,\mu} = \frac{I_1}{cI_0}$$

$$\left(\frac{\partial c}{\partial \rho_m}\right)_{\omega,\mu} = -\frac{c^3 D_m}{2I_1} = \frac{c}{U}\left(\frac{\partial c}{\partial \rho_m}\right)_{k,\mu}$$

$$\left(\frac{\partial c}{\partial \mu_m}\right)_{\omega,\rho} = \frac{c}{k^2}\frac{(k^2 D_m + S_m)}{2I_1} = \frac{c}{U}\left(\frac{\partial c}{\partial \mu_m}\right)_{k,\rho} \tag{26}$$

(Anderson and Harkrider, 1968). These expressions are equivalent to those given in Anderson (1964) and Vilkovitch *et al* (1966).

The Lagrangian for each homogeneous layer, $\mathfrak{L}_j$, can be evaluated in the same manner as equation (6) to give

$$\mathfrak{L}_j = -(v_j\tau_j - v_{j-1}\tau_{j-1}) \tag{27}$$

where v_j is the horizontal displacement and τ_j the tangential stress at the bottom of the jth layer. For the half-space, i.e., layer n, the stress is related to the displacement by

$$\tau_n(z) = \mu_n k r_{\beta n} v_n(z) \tag{28}$$

where

$$r_{\beta n}^* = -\left(1 - \frac{c^2}{\beta_n^2}\right)^{1/2}$$

and the shear velocity

$$\beta_n^2 = \frac{\mu_n}{\rho_n}.$$

The Lagrangian of the half-space is then

$$\mathfrak{L}_n = \tau_{n-1}v_{n-1} = \mu_n k r_{\beta n}^* v_{n-1}^2. \tag{29}$$

In terms of the Thomson-Haskell matrix formulation equations (27) and (29) become

$$\mathfrak{L}_j = k[(A_j)_{11}(A_j^*)_{21} - (A_{j-1})_{11}(A_{j-1}^*)_{21}] \tag{30}$$

and

$$\mathfrak{L}_n = k\mu_n r_{\beta n}^*(A_{n-1})_{11}^2 \tag{31}$$

where $(A_j)_{11}$ and $(A_j^*)_{21}$ are elements of the Thomson-Haskell multilayer matrix for the first j layers. The Lgrangian of the entire system for Love waves is

$$\mathfrak{L} = \sum_{j=1}^{n} \mathfrak{L}_j = k(A_{n-1})_{11}[(A_{n-1}^*)_{21} + \mu_n r_{\beta n}^*(A_{n-1})_{11}] \tag{32}$$

since $(A_0^*)_{21}$ is initialized to zero in the Thomson-Haskell algorithm.

From Harkrider (1964) the spectral amplitude response for a surface source and receiver at a distance of one wavelength for Love waves is given by

$$A = \frac{1}{(A_{n-1})_{11}\left(\frac{\partial F_L}{\partial k}\right)_\omega} \tag{33}$$

where

$$F_L = -[(A_{n-1}^*)_{21} + \mu_n r_{\beta n}^*(A_{n-1})_{11}]. \tag{34}$$

Comparing equations (34) and (32), we see that

$$\mathfrak{L} = -k(A_{n-1})_{11}F_L \tag{35}$$

and

$$\begin{aligned}\left(\frac{\partial \mathfrak{L}}{\partial k}\right)_\omega &= -\left\{(A_{n-1})_{11} + k\left[\frac{\partial (A_{n-1})_{11}}{\partial k}\right]_\omega\right\} F_L - k(A_{n-1})_{11}\left(\frac{\partial F_L}{\partial k}\right)_\omega \\ &= -k(A_{n-1})_{11}\left(\frac{\partial F_L}{\partial k}\right)_\omega \end{aligned} \tag{36}$$

where the period equation $F_L(\omega, k) = 0$ determines the relationship between ω and k for Love waves. Combining equations (36), (33) and (25) yields an expression for the amplitude response in terms of an energy integral, i.e.,

$$A = \frac{1}{2I_1} \tag{37}$$

(Harkrider and Anderson, 1966). This relationship is also valid for the free or rigid multilayered plate. Equation (37) can be obtained in a more elegant and straightforward manner from the inhomogeneous form of the differential equation (5). (Neigauz in Keilis-Borok and Yanovskaya, 1962; Andrianova *et al*, 1965; Vlaar, 1966; and Saito, 1967).

Similar relations to equation (10) are also valid for Rayleigh waves, but will not be discussed in this paper.

Analytical and Numerical Results

For a homogeneous plate of thickness d_1 the energy integrals are given by

$$\begin{aligned} I_0 &= \rho_1 \frac{d_1}{2}\left[1 + \cos Q_1 \frac{\sin Q_1}{Q_1}\right] \\ I_1 &= \mu_1 \frac{d_1}{2}\left[1 + \cos Q_1 \frac{\sin Q_1}{Q_1}\right] \\ I_2 &= \mu_1\left(\frac{\omega^2}{\beta_1^2} - k^2\right)\frac{d_1}{2}\left[1 - \cos Q_1 \frac{\sin Q_1}{Q_1}\right]. \end{aligned} \tag{38}$$

If the plate is bounded on both sides by a free surface the period equation is

$$\sin Q_1 = 0 \tag{39}$$

where

$$Q_1 = kr_{\beta 1}\, d_1$$

and

$$r_{\beta 1} = \left(\frac{c^2}{\beta_1^2} - 1\right)^{1/2}.$$

Substitution of equation (39) in equations (38) yields

$$I_0 = \rho_1 \frac{d_1}{2}$$

$$I_1 = \mu_1 \frac{d_1}{2}$$

$$I_2 = \mu_1 \left(\frac{\omega^2}{\beta_1^2} - k^2\right) \frac{d_1}{2} \tag{40}$$

and from equations (26) and (37) we obtain

$$U = \frac{\beta_1^2}{c}$$

and

$$A = \frac{1}{\mu_1 d_1}. \tag{41}$$

It is obvious that

$$\frac{dI_0}{dk} = \frac{dI_0}{d\omega} = \frac{dI_0}{d\mu_1} = 0 \tag{42}$$

and

$$\frac{dI_1}{dk} = \frac{dI_1}{d\omega} = \frac{dI_1}{d\rho_1} = 0 \tag{43}$$

which agree with the assumptions of Anderson (1964), Andrianova *et al* (1965), and Vilkovitch *et al* (1966).

For the case of a layer over a half-space the energy integrals are

$$I_0 = \rho_1 \frac{d_1}{2}\left\{1 + \cos Q_1 \frac{\sin Q_1}{Q_1}\right\} - \frac{\rho_2}{2} \frac{\cos^2 Q_1}{k r_{\beta 2}^*}$$

$$I_1 = \mu_1 \frac{d_1}{2}\left\{1 + \cos Q_1 \frac{\sin Q_1}{Q_1}\right\} - \frac{\mu_2}{2} \frac{\cos^2 Q_1}{k r_{\beta 2}^*}$$

$$I_2 = \mu_1 \frac{Q_1^2}{2d_1}\left\{1 - \cos Q_1 \frac{\sin Q_1}{Q_1}\right\} - \frac{\mu_2}{2} k r_{\beta 2}^* \cos^2 Q_1. \tag{44}$$

The classical Love wave period equation for this case is

$$\tan Q_1 = -\frac{\mu_2 r_{\beta 2}^*}{\mu_1 r_{\beta 1}}. \tag{45}$$

Since I_2 is not involved in calculating perturbations of U and A, we will restrict the following to integrals I_0 and I_1. Evaluating I_0 and I_1 for values of ω and k given by

TABLE 1

LAYER THICKNESS (D), SHEAR VELOCITY (BETA), DENSITY (RHO) AND RIGIDITY (MU) FOR THE STANDARD AND PERTURBED ONE LAYER MODELS

D (km)	Beta (km/sec)	Rho (g/cm³)	Mu (10^{10} dy/cm²)
	Standard		
40.	3.6	2.8	36.288
	4.5	3.3	66.825
	Perturbed		
40.	3.8	3.0	43.320
	4.5	3.3	66.825

TABLE 2

PHASE VELOCITIES (c), GROUP VELOCITIES (U), AND AMPLITUDE RESPONSES (A) FOR THE STANDARD ONE LAYER MODEL

T (sec)	c (km/sec)	U (km/sec)	A (10^{-15} μ/dy)
120.13	4.4550	4.3677	0.2591
113.90	4.4500	4.3534	0.2896
103.83	4.4400	4.3250	0.3514
95.99	4.4300	4.2970	0.4145
89.65	4.4200	4.2694	0.4788
79.90	4.4000	4.2154	0.6108
69.64	4.3700	4.1377	0.8167
64.51	4.3500	4.0883	0.9587
55.06	4.3000	3.9733	1.3276
48.35	4.2500	3.8713	1.7119
43.16	4.2000	3.7825	2.1059
38.90	4.1500	3.7068	2.5043
35.24	4.1000	3.6440	2.9024
28.99	4.0000	3.5544	3.6820
23.45	3.9000	3.5089	4.4228
18.01	3.8000	3.5027	5.1204
11.96	3.7000	3.5324	5.8013
6.70	3.6350	3.5711	6.3047
3.45	3.6100	3.5909	6.5887

equation (45) yields

$$I_0 = \frac{d_1}{2}\left[\rho_1 - \frac{\cos^2 Q_1}{Q_1}\left(\rho_1 \frac{\mu_2 r_{\beta 2}^*}{\mu_1 r_{\beta 1}} + \rho_2 \frac{r_{\beta 1}}{r_{\beta 2}^*}\right)\right]$$

$$I_1 = \frac{d_1}{2}\left[\mu_1 - \frac{\cos^2 Q_1}{Q_1}\,\mu_2\left(\frac{r_{\beta 2}^*}{r_{\beta 1}} + \frac{r_{\beta 1}}{r_{\beta 2}^*}\right)\right]. \qquad (46)$$

In order to calculate the derivatives of equations (46), the following formulas were used

$$\frac{dI_l}{dk} = \left(\frac{\partial I_l}{\partial k}\right)_\omega + U\left(\frac{\partial I_l}{\partial \omega}\right)_k$$

$$\left(\frac{\partial I_l}{\partial \mu_1}\right)_\omega = \left(\frac{\partial I_l}{\partial \mu_1}\right)_{\omega,k} + \left(\frac{\partial k}{\partial \mu_1}\right)_\omega \left(\frac{\partial I_l}{\partial k}\right)_{\omega,\mu}$$

and

$$\left(\frac{\partial I_l}{\partial \rho_1}\right)_\omega = \left(\frac{\partial I_l}{\partial \rho_1}\right)_{\omega,k} + \left(\frac{\partial k}{\partial \rho_1}\right)_\omega \left(\frac{\partial I_l}{\partial k}\right)_{\omega,\rho} \tag{47}$$

where by equations (26)

$$U = \frac{\left[\mu_1 - \dfrac{\cos^2 Q_1}{Q_1}\mu_2\left(\dfrac{r^*_{\beta 2}}{r_{\beta 1}} + \dfrac{r_{\beta 1}}{r^*_{\beta 2}}\right)\right]}{c\left[\rho_1 - \dfrac{\cos^2 Q_1}{Q_1}\left(\rho_1 \dfrac{\mu_2}{\mu_1}\dfrac{r^*_{\beta 2}}{r_{\beta 1}} + \rho_2 \dfrac{r_{\beta 1}}{r^*_{\beta 2}}\right)\right]}$$

$$\left(\frac{\partial k}{\partial \mu_1}\right)_\omega = -\frac{k}{c}\left(\frac{\partial c}{\partial \mu_1}\right)_\omega$$

$$= -\frac{k\left[\left(1 - \dfrac{\cos^2 Q_1}{Q_1}\dfrac{\mu_2}{\mu_1}\dfrac{r^*_{\beta 2}}{r_{\beta 1}}\right) + r^2_{\beta 1}\left(1 + \dfrac{\cos^2 Q_1}{Q_1}\dfrac{\mu_1}{\mu_2}\dfrac{r^*_{\beta 2}}{r_{\beta 1}}\right)\right]}{2\left[\mu_1 - \dfrac{\cos^2 Q_1}{Q_1}\mu_2\left(\dfrac{r^*_{\beta 2}}{r_{\beta 1}} + \dfrac{r_{\beta 1}}{r^*_{\beta 2}}\right)\right]}$$

$$\left(\frac{\partial k}{\partial \rho_1}\right)_\omega = -\frac{k}{c}\left(\frac{\partial c}{\partial \rho_1}\right)_\omega$$

$$= \frac{\omega^2\left(1 - \dfrac{\cos^2 Q_1}{Q_1}\dfrac{\mu_2}{\mu_1}\dfrac{r^*_{\beta 2}}{r_{\beta 1}}\right)}{2k\left[\mu_1 - \dfrac{\cos^2 Q_1}{Q_1}\mu_2\left(\dfrac{r^*_{\beta 2}}{r_{\beta 1}} + \dfrac{r_{\beta 1}}{r^*_{\beta 2}}\right)\right]} \tag{48}$$

and

$$l = 0 \quad \text{or} \quad 1.$$

Using these equations, partial derivatives of the integrals I_0 and I_1 were calculated for a standard layer over half-space model. For comparison purposes, this model is the same as used in Vilkovitch *et al* (1966). The physical parameters are listed in Table 1, and the spectral values for various periods are given in Table 2.

The numerical results showed that

$$\frac{dI_0}{dk} \neq 0, \qquad \frac{dI_1}{dk} \neq 0, \qquad \frac{dI_2}{dk} \neq 0$$

$$\frac{dI_0}{d\omega} \neq 0, \qquad \frac{dI_1}{d\omega} \neq 0, \qquad \frac{dI_2}{d\omega} \neq 0$$

$$\left(\frac{\partial I_0}{\partial \mu_1}\right)_\omega \neq 0, \qquad \left(\frac{\partial I_0}{\partial \rho_1}\right)_\omega \neq D_1, \qquad \left(\frac{\partial I_1}{\partial \mu_1}\right)_\omega \neq D_1$$

$$\left(\frac{\partial I_1}{\partial \rho_1}\right)_\omega \neq 0, \qquad \left(\frac{\partial I_2}{\partial \rho_1}\right)_\omega \neq 0 \quad \text{and} \quad \left(\frac{\partial I_2}{\partial \mu_1}\right)_\omega \neq S_1$$

which are in direct conflict with the assumptions of Anderson (1964) and Vilkovitch *et al* (1966).

These results require no changes in the formulas for phase-velocity perturbations given in Anderson (1964) and Vilkovitch *et al* (1966). However, additional terms must be added to the expressions for group-velocity perturbations in Vilkovitch *et al* (1966).

For a multilayered media, the correct expressions are

$$\left(\frac{\partial c}{\partial \mu_m}\right)_{\omega,\rho} = \frac{c}{k^2}\frac{(k^2 D_m + S_m)}{2I_1}$$

$$\left(\frac{\partial c}{\partial \rho_m}\right)_{\omega,\mu} = -\frac{c^3 D_m}{2I_1}$$

$$\left(\frac{\partial c}{\partial \beta_m}\right)_{\omega,\rho} = \frac{c}{2I_1}\left[2\beta_m \rho_m D_m + 2\,\frac{\beta_m \rho_m}{k^2} S_m\right]$$

$$\left(\frac{\partial c}{\partial \rho_m}\right)_{\omega,\mu} = \frac{c}{2I_1}\left[\beta_m^{\,2} D_m + \frac{\beta_m^{\,2}}{k^2} S_m - c^2 D_m\right] \tag{49}$$

$$\left(\frac{\partial U}{\partial \mu_m}\right)_{\omega,\rho} = \frac{1}{cI_0}\left[D_m + \sum_{j=1}^{n} \mu_j \left(\frac{\partial D_j}{\partial \mu_m}\right)_{\omega,\rho}\right] - \frac{U}{I_0}\sum_{j=1}^{n} \rho_j \left(\frac{\partial D_j}{\partial \mu_m}\right)_{\omega,\rho} - \frac{U}{c}\left(\frac{\partial c}{\partial \mu_m}\right)_{\omega,\rho}$$

$$\left(\frac{\partial U}{\partial \rho_m}\right)_{\omega,\mu} = \frac{1}{cI_0}\sum_{j=1}^{n} \mu_j \left(\frac{\partial D_j}{\partial \rho_m}\right)_{\omega,\mu} - \frac{U}{I_0}\left[D_m + \sum_{j=1}^{n} \rho_j \left(\frac{\partial D_j}{\partial \rho_m}\right)_{\omega,\mu}\right] - \frac{U}{c}\left(\frac{\partial c}{\partial \rho_m}\right)_{\omega,\mu}$$

$$\left(\frac{\partial U}{\partial \beta_m}\right) = \frac{2\beta_m \rho_m}{cI_0}\left[D_m + \sum_{j=1}^{n} \mu_j \left(\frac{\partial D_j}{\partial \mu_m}\right)_{\omega,\rho}\right] - \frac{2\beta_m \rho_m U}{I_0}\sum_{j=1}^{n} \rho_j \left(\frac{\partial D_j}{\partial \mu_m}\right)_{\omega,\rho} - \frac{U}{c}\left(\frac{\partial c}{\partial \beta_m}\right)_{\omega,\rho}$$

$$\left(\frac{\partial U}{\partial \rho_m}\right)_{\omega,\beta} = \frac{1}{cI_0}\left\{\beta_m^{\,2} D_m + \sum_{j=1}^{n} \mu_j\left[\left(\frac{\partial D_j}{\partial \rho_m}\right)_{\omega,\mu} + \beta_m^{\,2}\left(\frac{\partial D_j}{\partial \mu_m}\right)_{\omega,\rho}\right]\right\}$$
$$- \frac{U}{I_0}\left\{D_m + \sum_{j=1}^{n} \rho_j\left[\left(\frac{\partial D_j}{\partial \rho_m}\right)_{\omega,\mu} + \beta_m^{\,2}\left(\frac{\partial D_j}{\partial \mu_m}\right)_{\omega,\rho}\right]\right\} - \frac{U}{c}\left(\frac{\partial c}{\partial \rho_m}\right)_{\omega,\beta} \qquad (50)$$

and

$$\left(\frac{\partial A}{\partial \mu_m}\right)_{\omega,\rho} = -\frac{A}{I_0}\left[D_m + \sum_{j=1}^{n} \mu_j\left(\frac{\partial D_j}{\partial \mu_m}\right)_{\omega,\rho}\right]$$

$$\left(\frac{\partial A}{\partial \rho_m}\right)_{\omega,\mu} = -\frac{A}{I_1}\sum_{j=1}^{n} \mu_j\left(\frac{\partial D_j}{\partial \rho_m}\right)_{\omega,\mu}$$

$$\left(\frac{\partial A}{\partial \beta_m}\right)_{\omega,\rho} = -\frac{2\beta_m \rho_m A}{I_1}\left[D_m + \sum_{j=1}^{n} \mu_j\left(\frac{\partial D_j}{\partial \mu_m}\right)_{\omega,\rho}\right]$$

$$\left(\frac{\partial A}{\partial \rho_m}\right)_{\omega,\beta} = -\frac{A}{I_1}\left\{\beta_m^{\,2}\left[D_m + \sum_{j=1}^{n} \mu_j\left(\frac{\partial D_j}{\partial \mu_m}\right)_{\omega,\rho}\right] + \sum_{j=1}^{n} \mu_j\left(\frac{\partial D_j}{\partial \rho_m}\right)_{\omega,\mu}\right\}. \qquad (51)$$

Setting the derivatives of D_j to zero in equation (50), yields the Vilkovitch *et al* (1966) formulas for group-velocity perturbations.

As a check on the integral formulation program, the group velocities and the perturbations of phase and group velocity were calculated by the following alternate

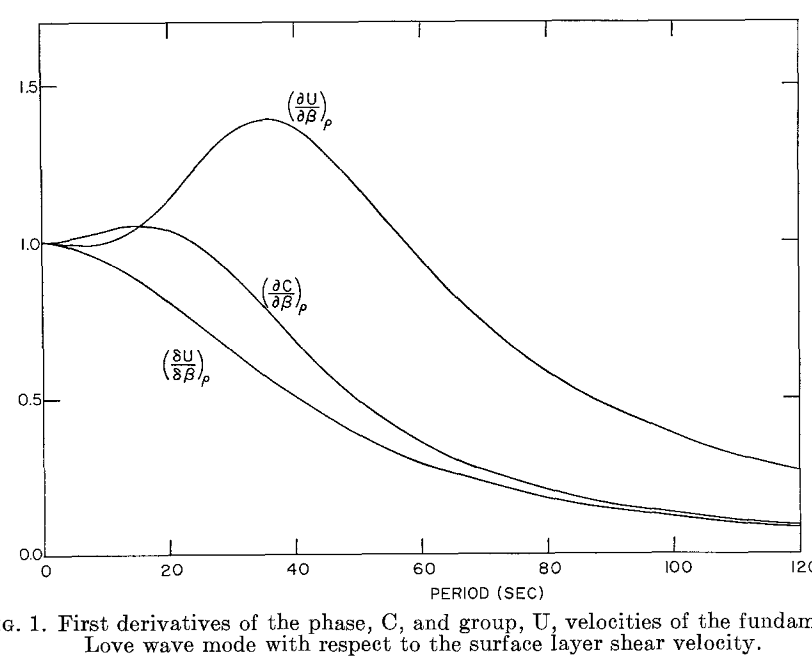

FIG. 1. First derivatives of the phase, C, and group, U, velocities of the fundamental Love wave mode with respect to the surface layer shear velocity.

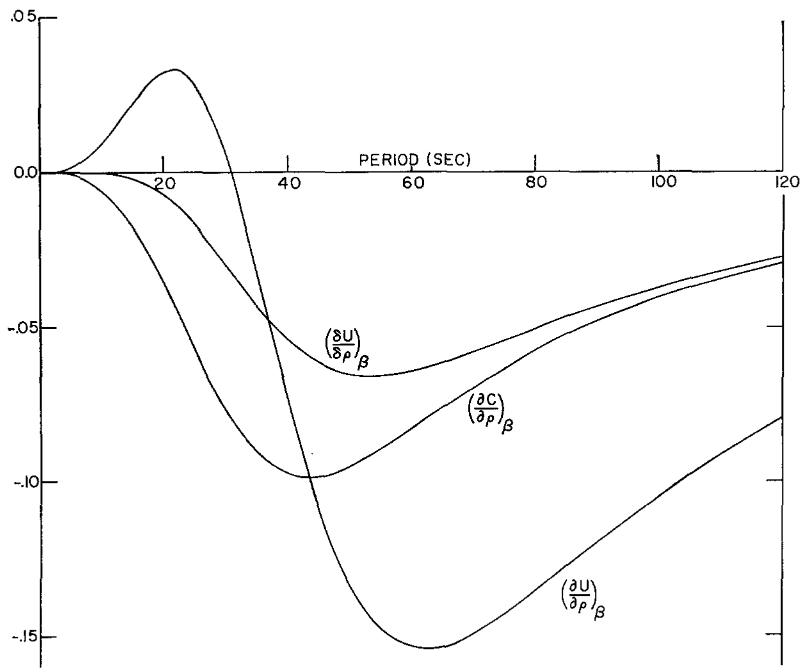

FIG. 2. First derivatives of the phase, C, and group, U, velocities of the fundamental Love wave mode with respect to surface layer density.

technique. The period equation (45) for the layer over a half-space can be rewritten as

$$k = \frac{n\pi + \mathfrak{D}}{r_{\beta 1} d_1} \tag{58}$$

where

$$\mathfrak{D} = \arctan B$$

and

$$B = \frac{\mu_2}{\mu_1} \left[\frac{1 - \left(\dfrac{c}{\beta_2}\right)^2}{\left(\dfrac{c}{\beta_1}\right)^2 - 1} \right]^{1/2} . \tag{59}$$

Restricting our formulas to the fundamental mode, i.e., $n = 0$, we have

$$k = \frac{\mathfrak{D}}{r_{\beta 1} d_1} \tag{60}$$

and

$$U = \frac{d\omega}{dk} = c + k \frac{dc}{dk} = c + \frac{k}{G}$$

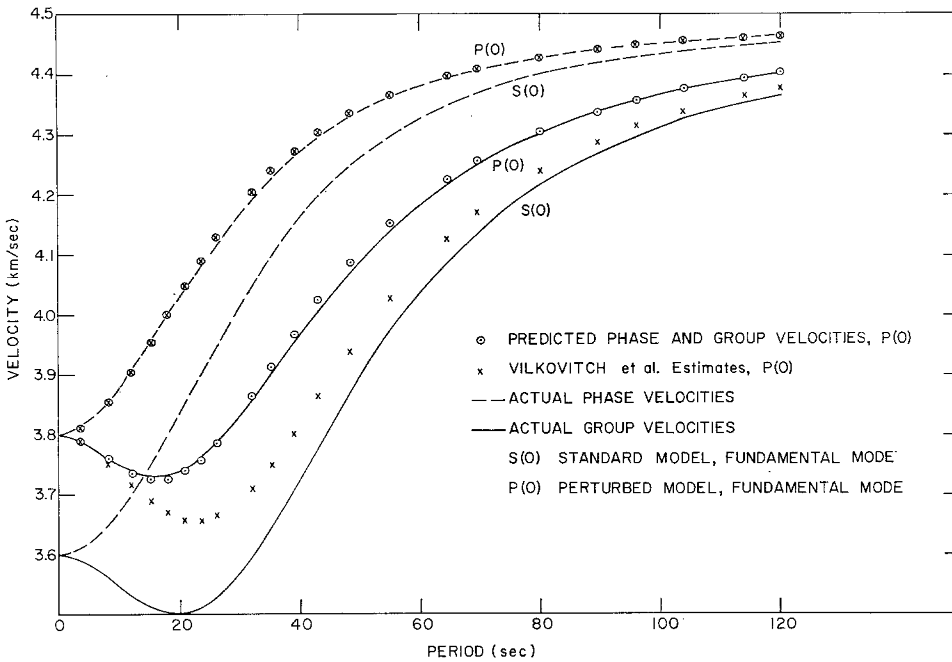

FIG. 3. Standard and perturbed model phase and group velocities with predicted values based on first derivatives.

where

$$G \equiv \frac{dk}{dc}.$$

The explicit formula for k, equation (60), can be differentiated with respect to c to give

$$G = \frac{1}{r_{\beta 1} d_1}\left[\frac{E}{1 + B^2} - \frac{c\mathfrak{D}}{(\beta_1 r_{\beta 1})^2}\right] \tag{61}$$

where

$$E \equiv -cB\left\{\frac{1}{(\beta_2 r_{\beta 2}^*)^2} + \frac{1}{(\beta_1 r_{\beta 1})^2}\right\}. \tag{62}$$

With these expressions we have k and U as explicit functions of c. We can now use the following formulas to calculate the phase-velocity perturbations

$$\left(\frac{\partial k}{\partial a}\right)_\omega = \frac{k}{U}\frac{dc}{dk}\left(\frac{\partial k}{\partial a}\right)_c = \frac{k}{UG}\left(\frac{\partial k}{\partial a}\right)_c \tag{63}$$

and

$$\left(\frac{\partial c}{\partial a}\right)_\omega = -\frac{c}{UG}\left(\frac{\partial k}{\partial a}\right)_c$$

where a can be either ρ, μ, or β. From the explicit relations, we obtain

$$\left(\frac{\partial k}{\partial \mu_1}\right)_c = -\frac{1}{r_{\beta 1} d_1}\left\{\frac{\mathfrak{D}_c^2}{2\mu_1 \beta_1^2 r_{\beta 1}} - \frac{1}{1 + B^2}\frac{B}{2\mu_1}\left[2 - \frac{c^2}{(\beta_1 r_{\beta 1})^2}\right]\right\}$$

TABLE 3

PHASE VELOCITIES (c) AND AMPLITUDE RESPONSES (A) FOR THE PERTURBED MODEL WITH THEIR ESTIMATES (c* AND A*) USING DERIVATIVES

T (sec)	c (km/sec)	c* (km/sec)	A (10^{-15} μ/dy)	A* (10^{-15} μ/dy)
120.13	4.4678	4.4674	0.2115	0.2127
113.90	4.4643	4.4639	0.2353	0.2362
103.83	4.4573	4.4570	0.2830	0.2828
95.99	4.4504	4.4501	0.3309	0.3293
89.65	4.4435	4.4434	0.3791	0.3755
79.90	4.4298	4.4300	0.4759	0.4672
69.64	4.4095	4.4105	0.6226	0.6032
64.51	4.3961	4.3978	0.7214	0.6930
55.06	4.3629	4.3665	0.9718	0.9159
48.35	4.3299	4.3356	1.2275	1.1399
43.16	4.2967	4.3045	1.4889	1.3700
38.90	4.2632	4.2728	1.7566	1.6105
35.24	4.2290	4.2399	2.0311	1.8647
28.99	4.1577	4.1693	2.6018	2.4204
23.45	4.0810	4.0908	3.2041	3.0355
18.01	3.9973	4.0035	3.8422	3.6971
11.96	3.9046	3.9069	4.5395	4.4107
6.70	3.8385	3.8389	5.0962	4.9676
3.45	3.8114	3.8144	5.4205	5.2880

TABLE 4

GROUP VELOCITIES (U) FOR THE PERTURBED MODEL WITH THEIR ESTIMATES, U*, USING DERIVATIVES AND, V, USING VILKOVITCH'S FORMULLA

T (sec)	U (km/sec)	U* (km/sec)	V (km/sec)
120.13	4.4060	4.4056	4.3793
113.90	4.3961	4.3959	4.3663
103.83	4.3766	4.3768	4.3406
95.99	4.3575	4.3583	4.3153
89.65	4.3388	4.3403	4.2904
79.90	4.3024	4.3056	4.2419
69.64	4.2506	4.2568	4.1725
64.51	4.2177	4.2260	4.1286
55.06	4.1410	4.1542	4.0275
48.35	4.0715	4.0880	3.9392
43.16	4.0087	4.0263	3.8638
38.90	3.9521	3.9686	3.8010
35.24	3.9015	3.9151	3.7503
28.99	3.8184	3.8234	3.6833
23.45	3.7608	3.7583	3.6578
18.01	3.7326	3.7272	3.6696
11.96	3.7404	3.7369	3.7161
6.70	3.7707	3.7698	3.7654
3.45	3.7900	3.7899	3.7892

$$\left(\frac{\partial k}{\partial \rho_1}\right)_c = \frac{1}{r_{\beta 1} d_1}\left[\frac{\mathfrak{D}c^2}{2\mu_1 r_{\beta 1}} - \frac{1}{1 + B^2}\frac{B}{2\rho_1}\frac{c^2}{(\beta_1 r_{\beta 1})^2}\right]. \tag{64}$$

For calculating the perturbations of U we used

$$\left(\frac{\partial U}{\partial a}\right)_\omega = \left(\frac{\partial U}{\partial a}\right)_c + \left(\frac{\partial U}{2c}\right)_a \left(\frac{\partial c}{\partial a}\right)_\omega \tag{65}$$

and the explicit function of U in terms of c and the physical constants a.

The phase- and group-velocity perturbations for the standard model are shown

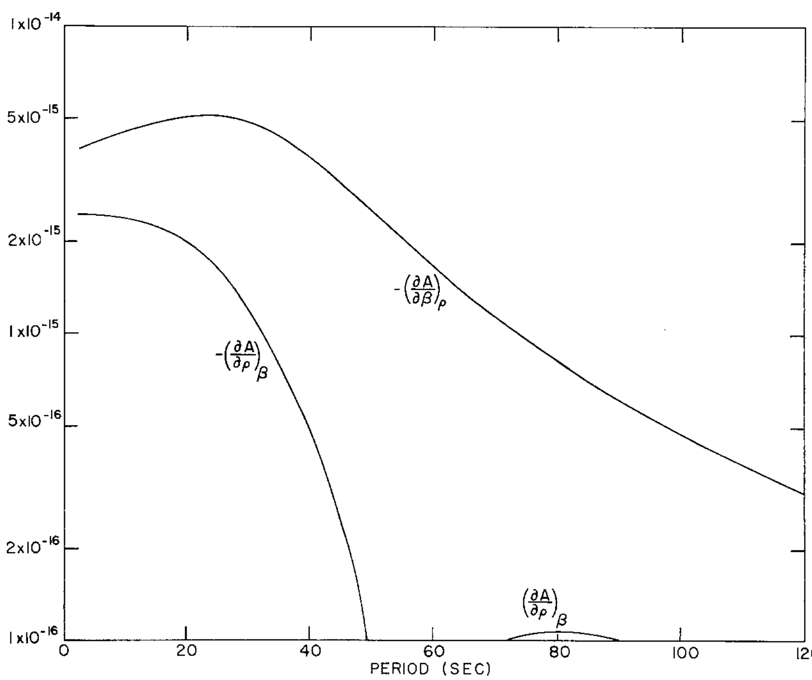

FIG. 4. First derivatives of the amplitude response, A, of the fundamental Love wave mode with respect to surface layer shear velocity and density.

in Figures 1 and 2. The actual group velocity perturbations are seen to differ considerably from the Vilkovitch *et al* (1966) values $(\delta U/\delta\beta_1)_\rho$ and $(\delta U/\delta\rho_1)_\beta$. These values were calculated using the formulas in Vilkovitch *et al* (1966) and agree with their plotted values. The group-velocity derivatives with respect to the surface-layer shear velocity are actually greater than the corresponding phase-velocity derivatives.

An interesting result not predicted from the incorrect group-velocity derivatives is that for high frequencies the group-velocity perturbations differ in sign with the phase-velocity perturbations due to a change in surface density. Increasing the density, increases the group velocity at periods less than 30 seconds while decreasing it at greater periods.

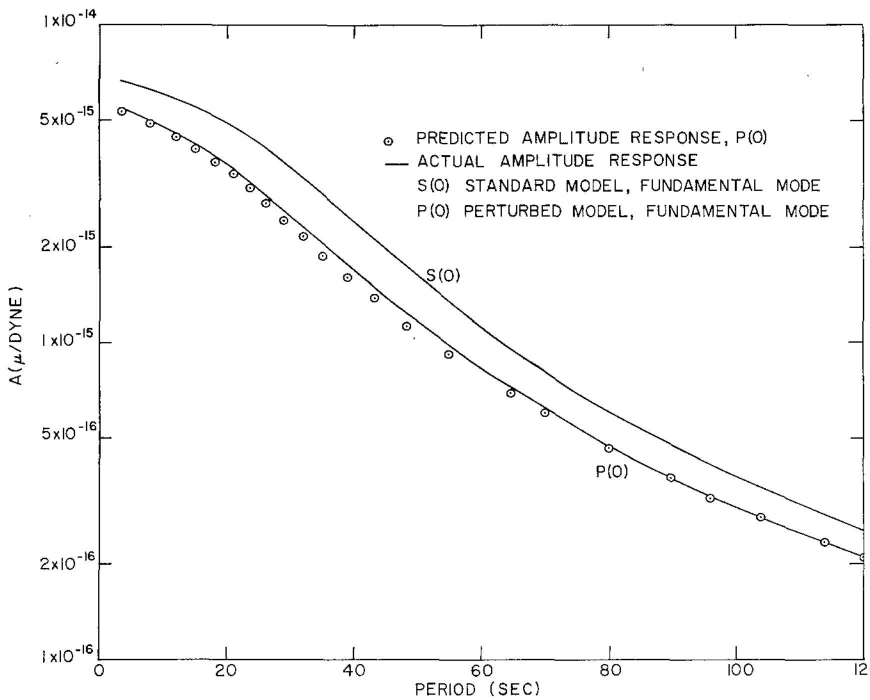

FIG. 5. Standard and perturbed model amplitude responses with predicted valves based on first derivatives.

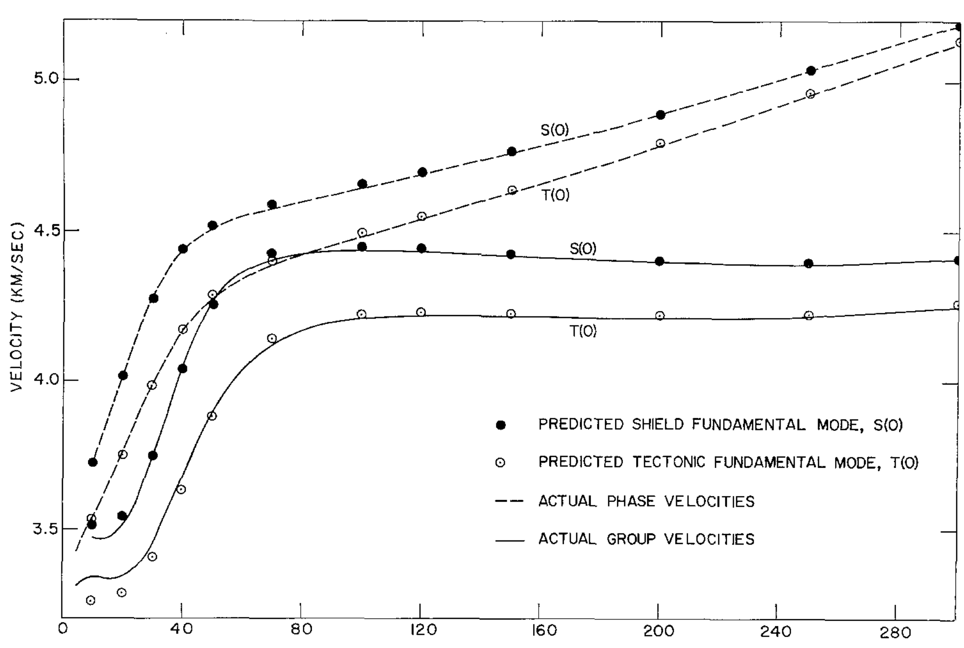

FIG. 6. Shield and Tectonic model phase and group velocities with predicted values based on first derivatives.

In Figure 3 the phase and group velocity are shown for the standard model. In order to demonstrate the effectiveness of using the partial derivatives to estimate the dispersion of similar models, we have calculated the exact and estimated spectra for the perturbed model in Table 1. The estimated and exact values of the phase and group velocity for this model are given in Tables 3 and 4 and are shown in Figure 3. Group velocity estimates based on the Vilkovitch values are given in Table 4 and Figure 3.

From Figure 3, we see that estimates based on first derivatives are very good. Values from Vilkovitch (1966) give poor estimates of the new group velocity. As an example of the technique they calculated the phase and group velocities for a perturbation of the standard model to a model with constant velocity and density

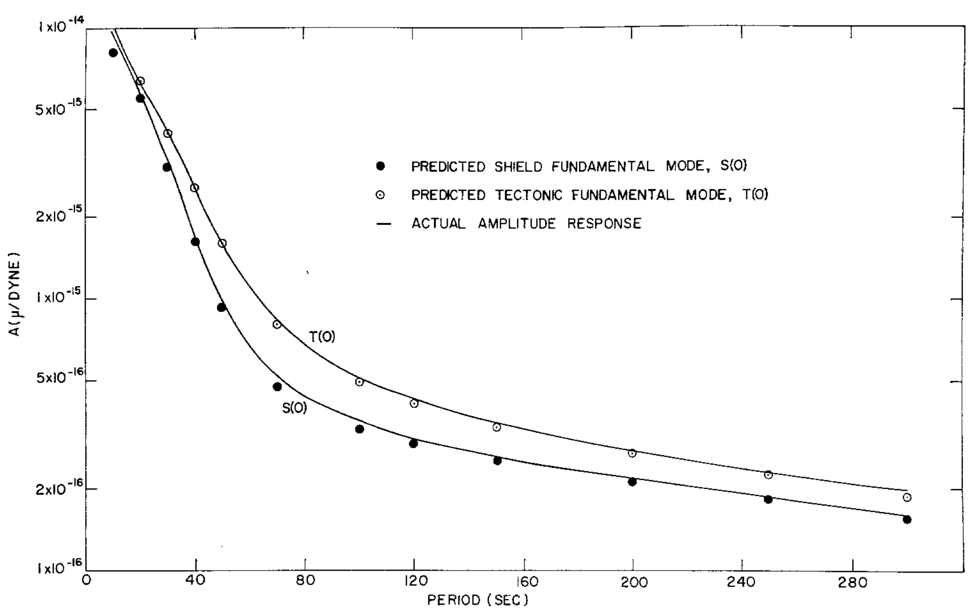

FIG. 7. Shield and Tectonic model amplitude responses with predicted values based on first derivatives.

gradients in the layer and half-space. Even though they made a correction to the values for constant velocity and density perturbations, it is surprising that their group velocity estimates were as good as shown in their figure.

In Figures 4 and 5, the amplitude response perturbations are given along with the resulting estimates for the perturbed model.

For the multilayered half-space and plate, the derivatives were calculated approximately by a difference scheme and analytically by using a tedious but straightforward method described in Anderson and Harkrider (1968). The technique is essentially the n layer extension of equations (44) through (48). Again the assumptions of Anderson (1964), Andrianova *et al* (1965) and Vilkovitch *et al* (1966) were found to be inapplicable.

Two multilayered models along with their mutual spectral estimates are given in Figures 6 and 7. The models and the partial derivative tables used to calculate their predicted spectra can be found in Anderson and Harkrider (1968). The agreement between predicted and actual spectral values for the fundamental modes shown in these figures is extremely good.

Acknowledgments

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