Supplemental Material for: Anomalous Quasiparticle Symmetries and Non-Abelian Defects on Symmetrically Gapped Surfaces of Weak Topological Insulators

DETAILS OF GAP-OPENING INTERACTIONS

This section reviews some aspects of K-matrix formalism [1] that are useful for understanding the gapping interactions used to generate surface topological order. In the main text, we introduced the following K-matrix K, charge vector Q, and time reversal vector X for each interface:

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & -4 & 0 \end{pmatrix}, Q = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \\ 0 \\ 2 \end{pmatrix}, X = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{1}$$

The K-matrix encodes commutation relations of the interface modes

$$\vec{\Phi}^T = (\varphi_R, -\varphi_L, \phi_{a,+}, \phi_{d,+}, \phi_{a,-}, \phi_{d,-}) \tag{2}$$

through

$$[\partial_x \Phi_i(x), \Phi_j(x')] = 2\pi i K^{-1}_{ij} \delta(x - x'). \tag{3}$$

(The sign in front of φ_L follows from defining electrons as $\psi_{R/L} \sim e^{i\varphi_{R/L}}$. An alternative convention where $\psi_{R/L} \sim e^{\pm i\varphi_{R/L}}$ is sometimes used in the literature.) Under time reversal the fields transform according to [2]

$$\mathcal{T}\Phi_i\mathcal{T}^{-1} = T_{ij}\Phi_j + \pi K^{-1}_{ij} X_j \tag{4}$$

where the matrix T is given by

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

In the present case this gives

$$\mathcal{T}\vec{\Phi}^T\mathcal{T}^{-1} = (-\varphi_L, \varphi_R + \pi, -\phi_{a,+}, \phi_{d,+}, -\phi_{a,-}, \phi_{d,-}). \tag{6}$$

A general quasiparticle $\Psi_{QP} \sim e^{i\vec{n}\cdot\vec{\Phi}}$, where $\vec{n}$ is an integer vector, has charge

$$q_{\vec{n}} = n_i K^{-1}_{ij} Q_j. \tag{7}$$

It follows that $a_\pm$ quasiparticles carry charge $e/2$ while $d_\pm$ are neutral.

Local operators correspond to $\vec{n}^T = \vec{l}^T K$ where $\vec{l}$ is an integer vector, i.e.,

$$\Psi_{\text{local}}(\vec{l}) \sim \exp\left\{ i\vec{l}^T K \vec{\Phi} \right\} \tag{8}$$

For example, a right moving electron is expressed as

$$\psi_L = \Psi_{\text{local}}\left(\vec{l}^T = (0,1,0,0,0,0)\right) \sim e^{i\varphi_L} \tag{9}$$

while a local charge-$2e$ boson on the '+' plate is given by

$$a^4_+ = \Psi_{\text{local}}\left(\vec{l}^T = (0,0,0,1,0,0)\right) \sim e^{4i\phi_{a,+}}. \tag{10}$$

To express gap-opening interactions in a convenient form, the main text employed variables $\theta_{c,s,n}$ defined as follows,

$$4\theta_c = \vec{l}_c K \vec{\Phi}, \qquad \vec{l}^T_c = (2,2,0,-1,0,-1) \tag{11}$$

$$4\theta_s = \vec{l}_s K \vec{\Phi}, \qquad \vec{l}^T_s = (-2,2,-1,0,1,0) \tag{12}$$

$$4\theta_n = \vec{l}_n K \vec{\Phi}, \qquad \vec{l}^T_n = (0,0,0,-1,0,1). \tag{13}$$

These satify

$$\vec{l}_c \cdot \vec{Q} = \vec{l}_s \cdot \vec{Q} = \vec{l}_n \cdot \vec{Q} = 0, \tag{14}$$

as well as

$$\vec{l}^T_i K \vec{l}_j = 0 \quad i,j = c,s,n \tag{15}$$

The first condition ensures that charge is conserved while the second is required to simultaneously pin all of the θ_i's. In addition, $e^{4i\theta_{c,s,n}}$ are invariant under time reversal as desired to maintain a fully symmetric gapped surface.

RELATIONSHIP BETWEEN SYMMETRICALLY GAPPED STRONG AND WEAK TOPOLOGICAL INSULATOR SURFACES

Here we comment further on the connections between the symmetric Abelian topological orders for gapped WTI surfaces and the known non-Abelian topological orders for gapped STI surfaces. For the latter, two phases unrelated by simple condensation transitions are possible [3–6]: the T-Pfaffian and the somewhat more elaborate Pfaffian-antisemion phase. Four possibilities for gapping the WTI surface thus immediately arise—each of the two Dirac cone can be gapped independently into either of these phases. As we argue below, we expect that both of the minimal Abelian topological orders that can arise on the WTI surface—corresponding to plates with $K = 4\sigma^x$ and $4\sigma^z$—may be accessed from these non-Abelian states via condensation transitions. Which Abelian states arise depends on which of the four possible parent non-Abelian phases one considers. In particular, for the simplest

non-Abelian 'double T-Pfaffian' state, obtaining topological order corresponding to $4\sigma^z$ can be ruled out leaving $4\sigma^x$ as the only natural possibility.

To relate the possible STI and WTI states, suppose first that the WTI surface can be fully gapped by purely *local* interactions between 2D topologically ordered plates and QSH edges (i.e., acting solely within a given interface). This is precisely the situation we encountered (by construction) in the main text. In this case the WTI surface inherits only the quasiparticles native to the 2D plates, with no additional topologically distinct excitations generated. Continuing to assume the minimal two-dimensional K-matrix for the plates, the quasiparticle content of the symmetric topologically ordered WTI surface thus corresponds to either $K = 4\sigma^x$ or $4\sigma^z$ as discussed below Eq. (1).

Suppose next that the symmetrically gapped WTI surface instead descended from a topologically ordered parent state—e.g., the 'double T-Pfaffian'—via a simple condensation transition. From this viewpoint the quasiparticles and their properties are inherited from the parent phase, and simply correspond to the anyons that remain deconfined after the transition. Again, no additional anyons appear in the process.

We can thus rule out potential parent states by identifying quasiparticles that appear in the descendant state, but not in the putative parent. Symmetric WTI surface topological orders corresponding to $4\sigma^x$ and $4\sigma^z$ both admit quasiparticles with charge $e/2$. In the $4\sigma^x$ case these excitations carry topological spins $e^{in\frac{\pi}{2}}$ for integer n, while for $4\sigma^z$ their topological spins are instead given by $e^{i(2n+1)\frac{\pi}{4}}$. Regarding the parent states, both the T-Pfaffian and Pfaffian-antisemion phases host Abelian $e/2$ excitations with topological spin $\pm i$; the Pfaffian-antisemion (which admits more anyon types) hosts additional $e/2$ quasiparticles with spins ± 1. Both phases also harbor non-Abelian $e/4$ excitations, which carry spin 1 in the T-Pfaffian but spin $e^{i\pi/4}$ in the Pfaffian-antisemion [5]. Thus in either the 'double T-Pfaffian' or 'double Pfaffian-antisemion', all quasiparticles with half-integer charge have topological spins of the form $e^{in\frac{\pi}{2}}$, consistent with $4\sigma^x$ but not with $4\sigma^z$. In contrast, a phase where the two Dirac cones are gapped differently—i.e., one in the T-Pfaffian and one in the Pfaffian-antisemion—exhibits $e/2$ excitations with topological spins $e^{in\pi/4}$. Hence this phase could potentially give way to $4\sigma^z$ upon condensing appropriate combinations of quasiparticles.

We note that the analysis presented in main part of the paper may be interpreted as explicitly carrying out a condensation starting form a symmetric parent state that is closely related to the 'double T-Pfaffian'. In this modified parent state, the two copies of T-Pfaffian topological order are not independently symmetric under time reversal and translations, but instead related by these symmetries; see Fig. 1 and Ref. [7]. This approach has the technical advantage that it permits a controlled analytic treatment in terms of quasi-1D Hamiltonians and bosonization. Apart from this, we do not expect any significant difference between either method.

RELATIONSHIP TO STRONG TOPOLOGICAL INSULATORS WITH MODIFIED TIME REVERSAL SYMMETRY

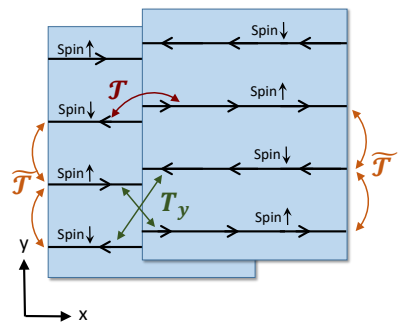

FIG. 1. (color online) Weak topological insulator surface viewed as a bilayer.

We expound on the relation to the STI by equivalently describing the WTI surface as a bilayer system, partitioning the right- and left-movers from each QSH edge as in Fig. 1. Here time-reversal $\mathcal{T}$ and translations T_y interchange the two layers while $\tilde{\mathcal{T}} = \mathcal{T}T_y$ does not. Each layer represents the surface of an antiferromagnetic topological insulator [8, 9]—which also supports a single Dirac cone—and maps to the setup considered in Ref. [7] for studying the correlated STI surface. Interactions can drive each layer into an electrically insulating, symmetry-preserving 'composite Dirac liquid' that hosts a single Dirac cone built from emergent *neutral* fermions that carry a fictitious 'pseudocharge' $\tilde{e}$ (which is not microscopically conserved). Intra-cone Cooper pairing of neutral fermions in the bilayer generates 'double T-Pfaffian' non-Abelian topological order. One can conveniently view the resulting anyons as defects in the paired condensates; in particular, an $hc/2\tilde{e}$ vortex carries *physical* charge $e/4$ and binds a Majorana zero mode. The so-far decoupled layers host independent $hc/2\tilde{e}$ vortices. Our analysis shows how to obtain an Abelian state for which individual $h/2\tilde{e}$ vortices are confined, yielding $e/2$ as the minimal charge. The Abelian state further includes a neutral quasiparticle—an interlayer $\pm e/4$ dipole—that acquires a $\pi/2$ phase when encircling an $e/2$ excitation.

DERIVATION OF DEFECT HAMILTONIAN

We now derive the Hamiltonian in Eq. (11) that describes the dislocation depicted in Fig. 3 (b) of the main text. Continuing to denote the defect position by x_0, the gap-opening interactions for $x < x_0$ are given by

$$\begin{aligned} \hat{\mathcal{O}}_c &\equiv (\psi_R\psi_L)^2 (a_- a_+)^4 + H.c. \\ \hat{\mathcal{O}}_s &\equiv \left(\psi_R^\dagger \psi_L\right)^2 \left(d_-^\dagger d_+\right)^4 + H.c. \qquad (x < x_0) \qquad (16) \\ \hat{\mathcal{O}}_a &\equiv \left(a_-^\dagger a_+\right)^4 + H.c., \end{aligned}$$

which are precisely the terms invoked in the main text to generate symmetric topological order. At $x > x_0$ the upper and lower plates fuse together *without* the aid of an intervening QSH edge mode—which detours into the bulk at x_0 as the figure illustrates. As in the main text we use $x > x_0$ to denote both the QSH edge modes that enter the bulk and plate fields at the surface. The following perturbations describe the fusion of the plates in this region,

$$\hat{\mathcal{O}}_d \equiv \left(d_-^\dagger d_+\right)^4 + H.c.$$
$$\hat{\mathcal{O}}_a \equiv \left(a_-^\dagger a_+\right)^4 + H.c., \quad (x > x_0). \tag{17}$$

When relevant the above terms catalyze condensation of $\langle d_-^\dagger d_+\rangle$ and $\langle a_-^\dagger a_+\rangle$, allowing the anyons to seamlessly pass between plates as desired. It remains to specify the fate of the QSH edge fields $\psi_{R/L}$ that bleed into the bulk. We will add a perturbation (assumed relevant)

$$\hat{\mathcal{O}}_\psi \equiv \left(\psi_R^\dagger \psi_L\right)^2 + H.c., \quad (x > x_0) \tag{18}$$

for those modes; this catalyzes a magnetization $\langle \psi_R^\dagger \psi_L\rangle \neq 0$ without changing the symmetries of the problem since the defect spontaneously breaks time-reversal symmetry as discussed earlier.

Notice that $\hat{\mathcal{O}}_a$ is present on both sides of the defect. We can therefore set $a \equiv a_- \sim a_+$ in all remaining terms. Furthermore, $\hat{\mathcal{O}}_s$ at $x < x_0$ is simply a product of terms present in $\hat{\mathcal{O}}_d$ and $\hat{\mathcal{O}}_\psi$ for $x > x_0$. Minimizing the energy therefore requires fixing $\langle \psi_R^\dagger \psi_L (d_-^\dagger d_+)^2\rangle$ to a uniform constant everywhere. (Otherwise there will be an energy cost when the expectation value 'twists' at x_0.) It follows that $d_-^\dagger d_+$ is slaved to the magnetization $\psi_R^\dagger \psi_L$ in the region $x > x_0$.

All the interesting defect physics has now been distilled into the perturbations $\hat{\mathcal{O}}_c$ and $\hat{\mathcal{O}}_\psi$. In terms of neutral fermions $\tilde{\psi}_{R/L} \equiv \psi_{R/L} a^2$ we have

$$\hat{\mathcal{O}}_c \sim \left(\tilde{\psi}_R \tilde{\psi}_L\right)^2 + H.c. \tag{19}$$

and

$$\hat{\mathcal{O}}_\psi \sim \left(\tilde{\psi}_R^\dagger \tilde{\psi}_L\right)^2 + H.c. \tag{20}$$

Focusing on these crucial pieces, the defect Hamiltonian density becomes

$$\mathcal{H}' = \tilde{\Delta}\Theta(x_0 - x)(\tilde{\psi}_R\tilde{\psi}_L)^2 + \tilde{u}\Theta(x - x_0)(\tilde{\psi}_R^\dagger\tilde{\psi}_L)^2 + H.c.$$

The first and last terms respectively favor condensates with $\langle \tilde{\psi}_R \tilde{\psi}_L\rangle \neq 0$ and $\langle \tilde{\psi}_R^\dagger \tilde{\psi}_L\rangle \neq 0$. The sign of the latter is arbitrary but, importantly, the former is not: The paired neutral-fermion condensate couples to the remaining 2D surface via Josephson coupling and hence its phase is chosen spontaneously but *globally*. For the 1D subsystem describing the zero modes it thus acts like an external superconductor with a fixed phase, and one should replace $(\tilde{\psi}_R\tilde{\psi}_L)^2 \to \tilde{\psi}_R\tilde{\psi}_L$. We then arrive precisely at the Hamiltonian density quoted in Eq. (11).

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