THE

PHYSICS OF FLUIDS

VOLUME 4, NUMBER 5 MAY, 1961

On Cylindrical Magnetohydrodynamic Shock Waves*

CARL GREIFINGER
The Rand Corporation, Santa Monica, California

AND

JULIAN D. COLE
California Institute of Technology, Pasadena, California and The Rand Corporation, Santa Monica, California
(Received December 16, 1960)

If an axial rod is surrounded by an ionized gas, an expanding cylindrical shock wave can be produced by passing through the gas a current which returns along the rod. The azimuthal magnetic field of the current acts like a piston, pushing the plasma away from the rod and leaving behind a cylindrical vacuum region. The case considered is that where a uniform magnetic field parallel to the axis is initially present in the gas; in this case a transverse magnetohydrodynamic shock wave results from the current discharge. The flow is analyzed under the assumptions that the plasma is a nonviscous, nonheat-conducting, ideal gas of infinite electrical conductivity, and that the discharge current increases linearly with time. The analysis is made first on the basis of the "snowplow" theory of Rosenbluth, and then from a similarity solution of the full magnetohydrodynamic equations. The results of the two solutions are compared for the case $\gamma = 7/5$. It is found that the speed predicted by the snowplow theory is in very good agreement with the speed of the contact front obtained from the solution of the full equations over the entire range of shock strength, but that the snowplow speed is a good approximation to the shock speed only in the limit of strong shocks. The effect on the flow of varying the axial field is discussed.

INTRODUCTION

IT is known that if an axial rod is surrounded by an ionized gas, an inverse pinch effect[1] can be produced by passing through the gas a current which returns along the rod. Such a current produces an azimuthal magnetic field which acts like a piston, pushing the plasma away from the rod and leaving behind a cylindrical vacuum region. The plasma boundary is preceded by a shock wave running into and compressing the ionized gas (see Fig. 1).

The usefulness of the inverse pinch effect in the study of plasma dynamics has been discussed by Anderson.[1] Another application of the effect is in the production and study of magnetohydrodynamic shock waves. This can be accomplished with the same experimental arrangement by establishing a magnetic field in the plasma prior to the initiation of the discharge. For example, an axial magnetic field created by an external solenoid can be trapped in the plasma region. The magnetic piston action of the rising pinch current will then result in the propagation into the plasma of a transverse magnetohydrodynamic shock wave. By studying the dependence of these waves on the external field, one might hope to distinguish between different magnetohydrodynamic models of the flow.

A model of the motion which has been proposed for the inverse pinch effect is the so-called "snow-

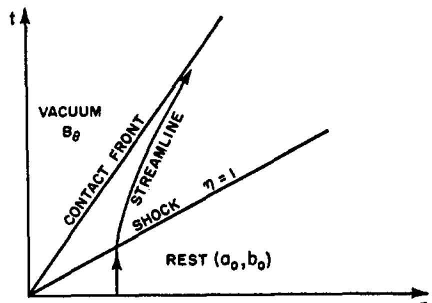

FIG. 1. Expanding shock and contact fronts driven by the azimuthal magnetic field B_θ of the rising discharge current.

* A preliminary report of this work was presented at the Division of Fluid Dynamics of The American Physical Society meeting, Baltimore, Maryland, November 21–23, 1960.

[1] O. A. Anderson, H. P. Furth, J. M. Stone, and R. E. Wright, Phys. Fluids 1, 489 (1958).

plow" model.[1] It is assumed, in this theory, that all the mass swept up by the shock is compressed into an infinitely thin layer immediately behind the shock, so that the contact front and the shock are the same interface. The motion of the interface is determined from the principle that the time rate of change of momentum of the accumulated mass is equal to the force on the interface. The snowplow approximation is valid in the limit of strong shocks in gases for which the specific heat ratio is close to unity. Under such conditions, the compression behind the shock is very large, as required by the theory. In experiments[1,2] where these conditions are fairly well realized, there is reasonable agreement with the theory.

The snowplow theory can be extended without any difficulty to the case where a uniform axial magnetic field is initially present in the plasma. The theory has the advantage of allowing a rather simple calculation of the gross dynamics of the gas enveloped by the shock. However, in its simple form the theory can not give any information about the distribution of the flow quantities in the shocked gas. The detail of the flow between shock and contact front can be obtained only from a solution of the full magnetohydrodynamic equations. In the general case, where the electrical conductivity of the gas is some assumed function of temperature and the current along the axial rod is a known function of time, this entails rather extensive numerical calculations. If, however, it is assumed that the electrical conductivity is infinite and that the current along the rod varies linearly with time, a similarity solution of the magnetodydrodynamic equations can be obtained, valid for shocks of arbitrary strength, from which the details of the flow can be extracted with relatively little numerical effort. The similarity solution thus obtained becomes, in the limit of zero axial field, a solution of the hydrodynamic equations for the ordinary inverse pinch effect.

Another way to obtain approximate information about the distribution of flow quantities between the shock and interface is to construct the snowplow theory as a proper limit (shock strength $\rightarrow \infty$, $\gamma \rightarrow 1$) of the magnetohydrodynamic equations. The analogous calculation in the pure gas-dynamic case is called Newtonian theory.[3] Details of this theory are not presented here.

The assumption of infinite conductivity, which is made also in the snowplow theory, is a reasonable approximation under typical experimental conditions. For example, for a Mach 20 shock (10-ev temperature) progressing through cold deuterium, the characteristic diffusion distance of magnetic field is of the order of a few millimeters for times during which the shock traverses distances of the order of a few centimeters. Under such conditions, dynamic effects will clearly be more important than diffusive effects in determining the motion of the plasma. The requirement of a linear variation of current with time can also be satisfied in practice. Although under typical experimental conditions the current is usually sinusoidal, the ringing time of the circuit can be made sufficiently long that deviation from linearity during traversal of the shock is unimportant. If necessary, it is always possible to estimate the correction for nonlinearity from the snowplow theory, which is not subject to the same limitation as the similarity solution. In the next section, the snowplow theory will be extended to the case of interest.

SNOWPLOW THEORY OF THE MOTION

For the purposes of comparison with the similarity solution, the snowplow theory will be developed for an idealized geometry where the axial rod is assumed to be a line. This idealization, while necessary for the existence of a similarity solution, need not be imposed on the snowplow theory which can be formulated just as easily for rods of finite radius. In practice the similarity solution should become a good approximation to the actual flow at distances from the rod of several times its radius. The snowplow solution, on the other hand, remains valid arbitrarily close to the rod provided, of course, that the shock remains sufficiently strong.

As mentioned in the Introduction, the snowplow model assumes that the magnetic piston action of the driving pinch current pushes the plasma radially outward from the rod. The advancing shock wave sweeps the ionized gas into a thin sheath, the motion of which is determined by equating the time rate of change of momentum of the accumulated mass to the force on the sheath. Assume that the plasma is initially at a uniform pressure p_0 and density ρ_0, and in a uniform magnetic field of strength B_0 directed parallel to the axis. If at a time t after the initiation of the discharge the advancing shock is at a radius r, the accumulated mass per unit length of the sheath is

$$M = \pi \rho_0 r^2, \tag{1}$$

[2] O. A. Anderson, W. R. Baker, S. A. Colgate, J. Ise, Jr., and R. V. Pyle, Phys. Rev. **110**, 1375 (1958).
[3] J. D. Cole, J. Aeronaut. Sci. **24**, 448 (1957).

while the net outward pressure on the sheath is

$$p = \frac{B_\theta^2}{2\mu} - \frac{B_0^2}{2\mu} - p_0 = \frac{\mu I^2}{8\pi^2 r^2} - \frac{B_0^2}{2\mu} - p_0. \qquad (2)$$

(Mks units will be used throughout.) The first term on the right-hand side of Eq. (2) is the magnetic pressure of the azimuthal field B_θ produced by the pinch current I, while the second term is the magnetic pressure of the external field.

The equation of motion is

$$2\pi r p = \frac{d}{dt}\left(M\frac{dr}{dt}\right). \qquad (3)$$

If one assumes a linear pinch current, $I = I_0\omega t$, and introduces as parameters the sound speed $a_0 = (\gamma p_0/\rho_0)^{\frac{1}{2}}$, where γ is the usual ratio of specific heats and the Alfvén speed $b_0 = (B_0^2/\mu\rho_0)^{\frac{1}{2}}$, Eq. (3) becomes

$$\frac{d}{dt}\left(r^2\frac{dr}{dt}\right) = c_0^4\frac{t^2}{r} - r\left(b_0^2 + \frac{2}{\gamma}a_0^2\right), \qquad (4)$$

where

$$c_0 = (\mu I_0^2\omega^2/4\pi^2\rho_0)^{\frac{1}{4}} \qquad (5)$$

is a quantity with the dimensions of a speed. The only physically admissible solution of Eq. (4) which passes through $r = 0$, $t = 0$ is

$$r = kt, \qquad k^2 = \tfrac{1}{4}\left\{-\left(b_0^2 + \frac{2}{\gamma}a_0^2\right) + \left[\left(b_0^2 + \frac{2}{\gamma}a_0^2\right)^2 + 8c_0^4\right]^{\frac{1}{2}}\right\}. \qquad (6)$$

In the limit of strong shocks, where the snowplow theory should give reasonably accurate results, $a_0^2/c_0^2 \ll 1$ and $b_0^2/c_0^2 \ll 1$; Eq. (6) becomes

$$\frac{r}{t} \cong 2^{-\frac{1}{4}}c_0\left[1 - \frac{2^{-11/4}}{c_0^2}\left(b_0^2 + \frac{2}{\gamma}a_0^2\right)\right]. \qquad (7)$$

The second factor in the bracket of Eq. (7) represents the lowest-order correction to the zero field, zero ambient pressure snowplow solution.

SIMILARITY SOLUTION

As in the preceding section, the problem will be idealized by assuming that the return for the pinch current is an axis of infinite extent. The plasma is assumed to be an ideal gas with infinite electrical conductivity; viscosity and heat conduction are neglected. A magnetic field directed parallel to the axis is assumed to exist initially in the plasma. As a consequence of the assumption of infinite conductivity, the magnetic field in the plasma will remain parallel to the axis; the azimuthal magnetic field of the pinch current cannot penetrate into the plasma. Under the assumptions made, the motion of the gas is governed by the following equations:

$$r\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial r}(r\rho u) = 0 \qquad \text{(continuity)}, \qquad (8a)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{1}{\rho}\frac{\partial}{\partial r}\left(\frac{B^2}{2\mu}\right) = 0 \qquad \text{(momentum)}, \qquad (8b)$$

$$r\frac{\partial B}{\partial t} + \frac{\partial}{\partial r}(rBu) = 0 \qquad \text{(induction)}, \qquad (8c)$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)\left(\frac{p}{\rho^\gamma}\right) = 0 \qquad \text{(entropy)}. \qquad (8d)$$

In these equations, p is the pressure of the gas, ρ the density, u the radial velocity, and B the axial magnetic field. With the assumed geometry, all variables are functions only of the radial distance r and the time t. The specific heat ratio γ is assumed constant and the magnetic permeability μ taken to be that of free space. The gas is assumed to be initially in a uniform state p_0, ρ_0 and B_0. For these initial conditions, a solution of Eqs. (8) is sought which represents the flow behind a cylindrical shock wave driven by a magnetic piston.

It is evident that no characteristic length or time enters the initial or boundary conditions of the problem so that from general dimensional considerations[4] the solution must have a conical similarity. The form of the solution is thus

$$\rho = \rho_0\mathcal{R}(\eta), \qquad u = (r/t)\mathcal{V}(\eta),$$
$$p = \rho_0(r/t)^2\mathcal{P}(\eta), \qquad B = (\mu\rho_0)^{\frac{1}{2}}(r/t)\mathcal{B}(\eta), \qquad (9)$$

where the quantities $\mathcal{R}$, $\mathcal{V}$, $\mathcal{P}$, and $\mathcal{B}$ are nondimensional functions of the nondimensional variable

$$\eta = r/c^*t. \qquad (10)$$

c^* is some characteristic parameter with the dimensions of a velocity.

Lines of constant η are straight lines in the (r, t) plane passing through the origin. In particular, the shock wave and contact front are two such lines; these fronts therefore expand with constant radial velocity.

It is convenient to introduce as independent variables, instead of η and t, the stream function ψ defined by

$$\partial\psi/\partial r = (\rho/\rho_0)r, \qquad \partial\psi/\partial t = -(\rho/\rho_0)ur, \qquad (11)$$

[4] L. I. Sedov, *Similarity and Dimensional Methods in Mechanics* (Academic Press, Inc., New York, 1960).

and the variable ϕ, defined as

$$\phi = 2\psi/r^2. \tag{12}$$

Physically, the value of ϕ corresponding to any (r, t) is the ratio of the mass which is at a time t between the axis and the radius r to the mass initially in the same volume. The value of ϕ at the shock, therefore, is $\phi_s = 1$, while at the contact front ϕ takes the value $\phi_c = 0$. Evidently ϕ is constant on a similarity curve $\eta =$ const. In terms of the variables (ψ, ϕ) Eq. (9) takes the form

$$\begin{aligned} \rho &= \rho_0\sigma(\phi), & u &= c^*U(\phi), \\ p &= \rho_0 c^{*2}P(\phi), & B &= (\mu\rho_0)^{\frac{1}{2}}c^*\beta(\phi), \end{aligned} \tag{13}$$

where the quantities σ, U, P, and β are nondimensional functions of the single (nondimensional) variable ϕ. The parameter c^* in Eq. (13) is taken to be the velocity of the shock. The dependence of c^* on the other parameters of this problem is one of the results to be obtained from the integration of the equations of motion.

The Jacobian of the transformation defined by Eqs. (11) and (12) is readily found to be

$$J \equiv \partial(\psi, \phi)/\partial(r, t) = -2\sigma u\phi. \tag{14}$$

Since the Jacobian is not zero (except at certain points), the transformation defines a one-to-one mapping between (r, t) and (ψ, ϕ). One may, in fact, obtain an integral expression for the stream function ψ. The differential equation

$$\frac{\partial t}{\partial \psi} = -\frac{1}{J}\frac{\partial \phi}{\partial r} = \frac{1}{2^{\frac{1}{2}}c^*}\frac{\sigma - \phi}{\sigma\phi^{\frac{1}{2}}U}\frac{1}{\psi^{\frac{1}{2}}} \tag{15}$$

is easily integrated to yield

$$\psi = \tfrac{1}{2}c^{*2}t^2[\sigma^2\phi U^2/(\sigma - \phi)^2]. \tag{16}$$

Finally, it is possible to express the variable η as a function (implicit) of ϕ; by combining Eqs. (10), (12), and (16), one obtains

$$\eta = \sigma U/(\sigma - \phi). \tag{17}$$

If one now substitutes Eq. (13) into Eqs. (8), the equations of motion become

$$\sigma U + 2(\sigma - \phi)\sigma\frac{dU}{d\phi} - 2\phi U\frac{d\sigma}{d\phi} = 0 \qquad \text{(continuity)}, \tag{18a}$$

$$\phi U\frac{dU}{d\phi} + \frac{\phi - \sigma}{\sigma}\left[\frac{dP}{d\phi} + \frac{d}{d\phi}\left(\frac{\beta^2}{2}\right)\right] = 0 \qquad \text{(momentum)}, \tag{18b}$$

$$\beta U + 2(\sigma - \phi)\beta\frac{dU}{d\phi} - 2\phi U\frac{d\beta}{d\phi} = 0 \qquad \text{(induction)}, \tag{18c}$$

$$d/d\phi(P/\sigma^\gamma) = 0 \qquad \text{(entropy)}. \tag{18d}$$

The entropy equation, (18d), has the immediate integral

$$P/\sigma^\gamma = a \quad \text{(const)}, \tag{19}$$

from which it follows that the entropy is constant along a streamline. In fact, since Eq. (13) does not contain ψ explicitly, $p/\rho^\gamma =$ constant; that is, the flow behind the shock is isentropic. This is a consequence of the fact that the shock is one of constant strength. Another integral may be obtained by combining Eqs. (18a) and (18c); namely,

$$\beta/\sigma = b \quad \text{(const)}. \tag{20}$$

This integral expresses the fact that the magnetic field is proportional to the density along a streamline. [It again follows from Eq. (13) that $B/\rho =$ const everywhere behind the shock.] This is a well-known consequence of the infinite conductivity assumption—namely, that the magnetic field is "frozen" into the fluid. The constants a and b appearing in these integrals are not arbitrary, but are determined by the boundary conditions at the shock.

The two integrals (19) and (20) may now be used to eliminate the functions P and β from Eqs. (18). The system (18) is thereby reduced to two coupled ordinary differential equations, for which no further analytic integration appears possible. The remaining integrations must therefore be carried out numerically.

The starting point for the numerical integrations is determined from the boundary conditions at the shock. In terms of physical variables, the shock jump conditions may be written

$$\frac{\rho_0}{\rho_s} = 1 - \frac{u_s}{c^*} \qquad \text{(continuity)}, \tag{21a}$$

$$p_s - p_0 + \frac{1}{2}\mu(B_s^2 - B_0^2) = \rho_0 u_s c^* \qquad \text{(momentum)}, \tag{21b}$$

$$\frac{B_0}{B_s} = 1 - \frac{u_s}{c} \qquad \text{(induction)}, \tag{21c}$$

$$\begin{aligned}[c^{*2} - \tfrac{1}{2}(\gamma + 1)c^*u_s - a_0^2](c^* - u_s)^2 \\ = b_0^2[c^{*2} - \tfrac{1}{2}(\gamma + 1)c^*u_s + \tfrac{1}{2}\gamma u_s^2]\end{aligned} \qquad \text{(wave speed)}, \tag{21d}$$

where the subscript s denotes quantities immediately behind the shock. The quantity c^* is again the shock speed, while a_0 and b_0 are the sound and Alfvén speeds, respectively, in the gas at rest. The wave speed equation, (21d), is derived by combining the usual energy equation with the continuity, momentum, and induction equations.[5]

In terms of the nondimensional variables defined in Eq. (13), the shock relations become

$$1/\sigma_s = 1 - U_s, \qquad (22a)$$

$$P_s - (\lambda_0^2/\gamma) + \tfrac{1}{2}\beta_0^2(\sigma_s^2 - 1) = U_s, \qquad (22b)$$

$$\beta_0/\beta_s = 1 - U_s, \qquad (22c)$$

$$[1 - \lambda_0^2 - \tfrac{1}{2}(\gamma + 1)U_s](1 - U_s) = \beta_0^2(1 - \tfrac{1}{2}\gamma U_s), \qquad (22d)$$

where

$$\lambda_0 = a_0/c^* \qquad (23)$$

is the ratio of the sound speed in the gas at rest to the shock speed, and

$$\beta_0 = B_0/(\mu\rho_0)^{\frac{1}{2}}c^* = b_0/c^* \qquad (24)$$

is the corresponding ratio of Alfvén speed to shock speed.

If the quantities λ_0 and β_0 are considered as parameters, Eqs. (22) define a two-parameter family of shock curves. For a given pair of values of λ_0 and β_0, U_s is determined simply as a root of the quadratic Eq. (22d). One then obtains σ_s and β_s directly from Eqs. (22a) and (22c), respectively, and then P_s from Eq. (22b). The constants a and b are determined by evaluating Eqs. (19) and (20) at the shock; this yields

$$a = P_s/\sigma_s^\gamma, \qquad b = \beta_0. \qquad (25)$$

Finally, from the definition of the variable ϕ,

$$\phi_s = 1. \qquad (26)$$

The location of the shock and the values of the constants a and b are thus completely determined once the values of λ_0 and β_0 are specified. The equations of motion may then be integrated from the shock point, given by Eqs. (22) and (26), to the contact front, which is reached when the variable ϕ takes the value $\phi_c = 0$.

The remaining parameter of the problem, namely the rate of rise of the pinch current, enters through the boundary condition at the contact front. The appropriate boundary condition at the contact front is that the magnetic pressure of the azimuthal field B_θ of the pinch current balances the total pressure in the gas, i.e.,

$$(B_\theta^2/2\mu)_c = p_c + (B_c^2/2\mu), \qquad (27)$$

where the subscript c denotes quantities at the contact front. If the boundary condition is to be compatible with the assumed form of the solution, the pinch current must vary linearly with the time. For, if the current is of the form $I = I_0\omega t$, the azimuthal field becomes a function only of the nondimensional variable η; in particular,

$$B_\theta = \mu I/2\pi r = (\mu I_0\omega/2\pi c^*)(1/\eta). \qquad (28)$$

The boundary condition (28) may now be written in terms of nondimensional variables; the resulting equation is most conveniently expressed in the form

$$c^* = c_0(2\eta_c^2\pi_c)^{-\frac{1}{4}}, \qquad (29)$$

where c_0 is the characteristic speed defined by Eq. (5) and

$$\pi_c \equiv P_c + \tfrac{1}{2}\beta_c^2 = a\sigma_c^\gamma + \tfrac{1}{2}b^2\sigma_c^2 \qquad (30)$$

is the nondimensional total pressure at the contact front. The quantities η_c and π_c are functions of the parameters λ_0 and β_0 which can be determined only from the numerical integration which remains to be performed. The shock speed c^* is evidently determined by the three parameters λ_0, β_0, and c_0.

RESULTS AND CONCLUSIONS

The two coupled ordinary differential equations to which the system (18) reduces have been integrated numerically by the Kutta-Runge method for a specific heat ratio of $\gamma = 7/5$. The results are shown graphically in Figs. 2 through 8. Of particular interest is Fig. 3, in which the speed of the contact front is plotted against the sound speed in the ambient gas for various values of the ratio of Alfvén speed to sound speed in the ambient gas. On the same graph is plotted the speed predicted by the snowplow theory, given by Eq. (6). The agreement between the snowplow theory and the exact solution over the entire range of shock strength is rather remarkable. [The shock Mach number, $M = c^*/(a_0^2 + b_0^2)^{\frac{1}{2}}$, decreases along each curve from $M = \infty$ at $a_0 = 0$ to $M = 1$.] Even in the limit of weak shocks, where the snowplow model is a poor approximation to the actual flow, the two speeds disagree by no more than 10%. It should be noted that this is the case even for a value of γ which is not particularly favorable for the snow-

[5] J. D. Cole, in *The Magnetohydrodynamics of Conducting Fluids*, edited by D. Bershader (Stanford University Press, Stanford, California, 1959), p. 17.

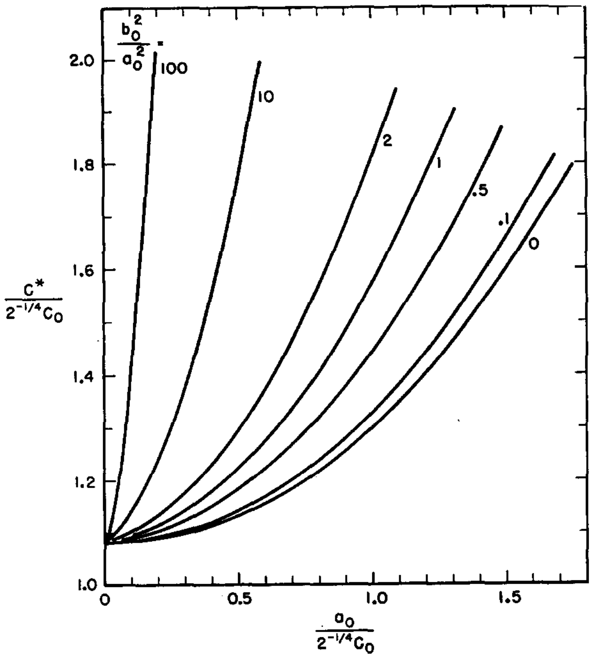

FIG. 2. Dependence of shock speed on ambient gas pressure for different values of the square of the ratio of Alfvén speed to sound speed in the ambient gas. ($\gamma = 7/5$.)

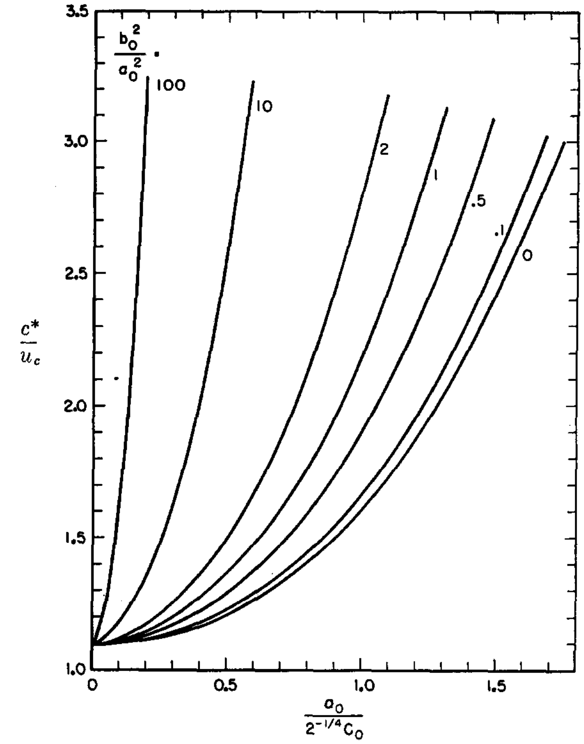

FIG. 4. Dependence of the ratio of shock speed to speed of the contact front on ambient gas pressure for different values of the square of the ratio of Alfvén speed to sound speed in the ambient gas. ($\gamma = 7/5$.)

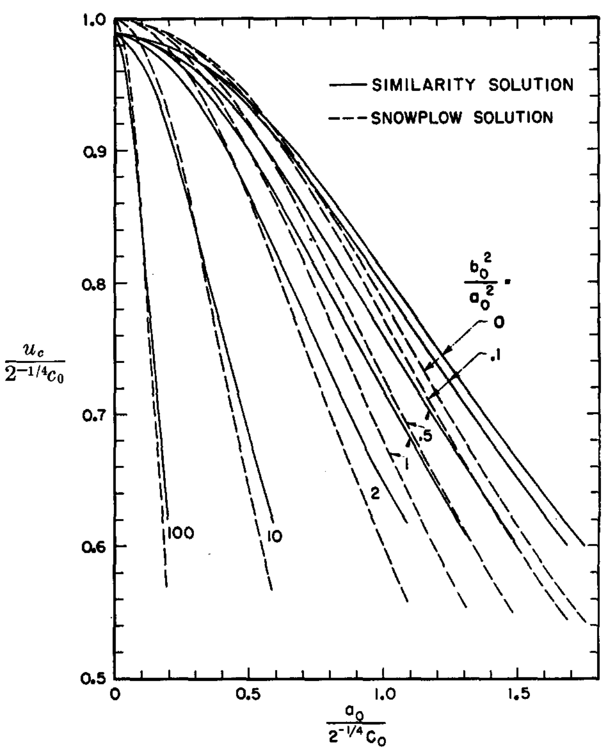

FIG. 3. Dependence of speed of contact front on ambient gas pressure for different values of the square of the ratio of Alfvén speed to sound speed in the ambient gas. ($\gamma = 7/5$.) Dashed line is snowplow speed given by Eq. (6).

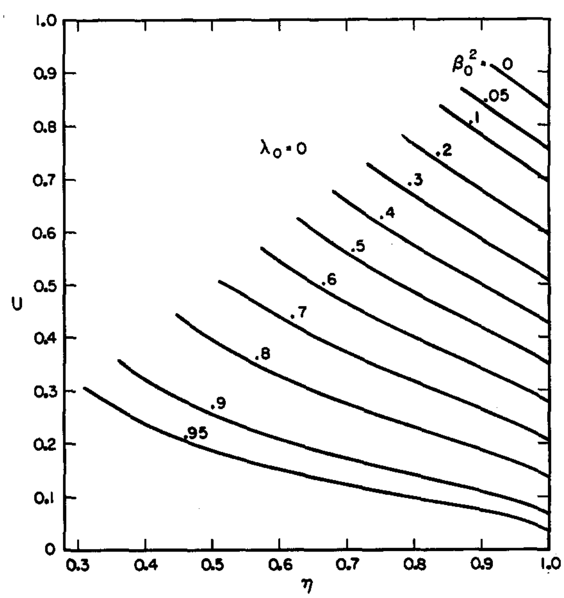

FIG. 5. Flow velocity, in units of c^*, plotted against nondimensional variable $\eta = r/c^*t$ for different values of $\beta_0^2 = (b_0/c^*)^2$ for the case of zero ambient gas pressure. Location of the shock is at $\eta = 1$. ($\gamma = 7/5$.)

plow theory. The reason for this uniformly good agreement is that, independent of shock strength and specific heat ratio, most of the momentum in the flow is carried by the fluid very close to the contact front where the density and velocity are greatest. The snowplow theory, being simply a statement of momentum balance, predicts rather accurately the motion of this portion of the fluid. The theory is therefore very useful for estimating corrections to the speed of the front for nonlinearity of driving current or finite radius of the axial rod. However, except in the limit of very strong shocks, the snowplow speed provides only a poor approximation to the speed of the shock (see Fig. 4).

As can be seen from Fig. 2, the effect of the axial magnetic field on the shock speed is greater for weaker shocks (smaller values of a_0/c_0). For a given ambient pressure, then, the effect of the field will be more pronounced for slow discharges than for fast ones. In the fast discharge experiments of Anderson[1] with deuterium at 1000-μ initial pressure, typical currents of 120–135 ka were generated in rise times of 0.5 μsec. These conditions correspond to a value of $a_0/2^{-\frac{1}{2}}c_0$ of about 0.025. Under these conditions, an axial field of 1000 gauss, corresponding

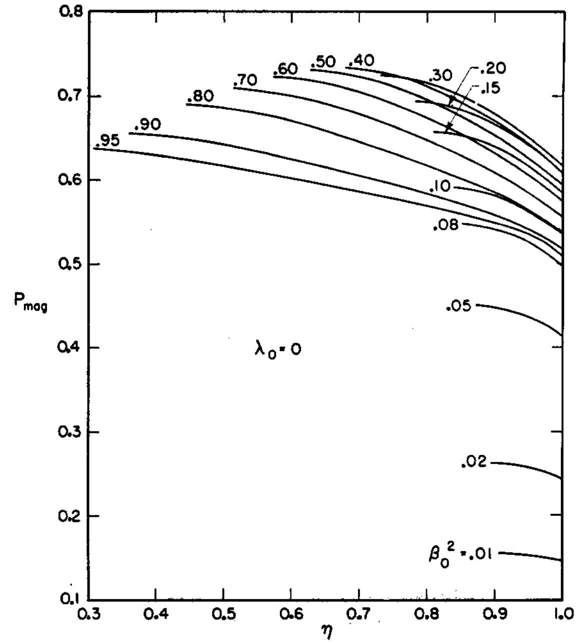

FIG. 7. Magnetic pressure, in units of $\rho_0 c^{*2}$, plotted against nondimensional variable $\eta = r/c^*t$ for different values of $\beta_0^2 = (b_0/c^*)^2$ for the case of zero ambient gas pressure. Location of the shock is at $\eta = 1$. ($\gamma = 7/5$.)

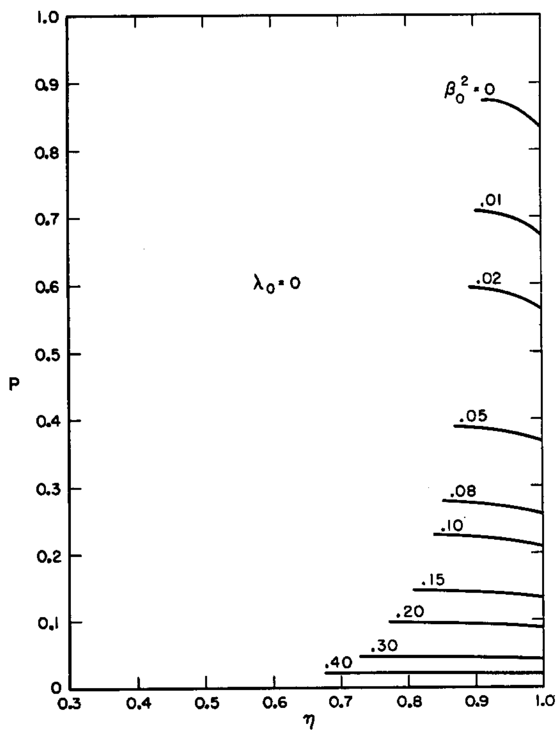

FIG. 6. Gas pressure, in units of $\rho_0 c^{*2}$, plotted against nondimensional variable $\eta = r/c^*t$ for different values of $\beta_0^2 = (b_0/c^*)^2$ for the case of zero ambient gas pressure. Location of the shock is at $\eta = 1$. ($\gamma = 7/5$.)

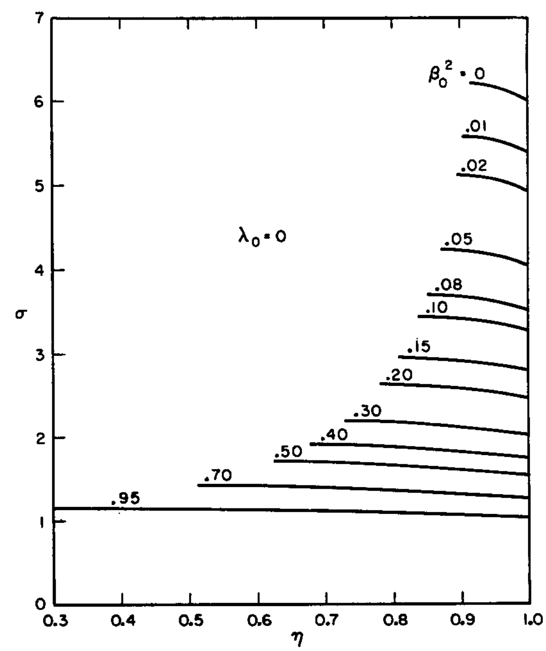

FIG. 8. Density, in units of ρ_0, plotted against nondimensional variable $\eta = r/c^*t$ for different values of $\beta_0^2 = (b_0/c^*)^2$ for the case of zero ambient gas pressure. Location of the shock is at $\eta = 1$. ($\gamma = 7/5$.)

to a value of $b_0/a_0 = 6$, would produce only about a 3% change in the shock speed. On the other hand, if the rate of current rise were reduced by a factor of 100, the same axial field would produce a measurable change in the shock speed of about 50%.

As mentioned in the Introduction, the similarity solution provides also the details of the flow between shock and contact front. In Figs. 5–7, the distribution of flow velocity, gas pressure, and magnetic pressure is plotted against the nondimensional variable η for the case of zero gas pressure in the ambient gas. The curves for different values of β_0^2 correspond, in this case, to different values of the axial magnetic field, all other parameters (ambient density and rate of current rise) remaining constant. The effect of the magnetic field on the flow is contained, therefore, in the variation from one value of β_0^2 to another, with the larger values of β_0^2 corresponding to stronger fields and weaker shocks. It is seen that, for weak fields, the flow velocity is quite linear between the two fronts, with increasing deviation from linearity as field strength increases. The gas pressure behind the shock decreases monotonically with increasing field strength and is relatively constant (as is also the density) between shock and contact front. The magnetic pressure at the shock, on the other hand, has a maximum at a value of β_0^2 between 0.3 and 0.4. In other words, as the strength of the initial field is increased, the magnetic field at the shock first rises and then falls. The reason for the existence of a maximum is that the magnetic field at the shock is proportional both to the initial field and to the density ratio across the shock; as the field increases the shock is weakened so that the density ratio decreases and a maximum is produced.

There is a well-known exact analogy between hypersonic flow past slender bodies and the unsteady gas-dynamic flow produced in a plane as the body passes a fixed observer. According to this analogy the steady hypersonic flow past a cone is equivalent to the unsteady flow produced by a uniformly expanding cylindrical piston. Van Dyke[6] has calculated the flow past the cone and a comparison of his numerical results with the calculations of this paper for the case of zero initial magnetic field shows exact agreement, and is a check on the numerical calculations. Of course, in the comparison the contact front is identified with the cone surface.

It has recently come to the authors' attention that a similarity solution has been independently obtained by Korobeinikov and Riazanov[7] for the equivalent problem of a cylindrical piston advancing with constant speed into a uniform plasma of infinite electrical conductivity in a uniform axial magnetic field.

---

[6] M. D. Van Dyke, NACA Rept. 1194 (1954).

[7] V. P. Korobeinikov and E. V. Riazanov, J. Appl. Math. and Mech. **24**, 144 (1960).