difference being that the definition of ψ^1_{mn} is now (19) instead of (10), and that c is changed to ic in the functions Se.

[1] Stratton, *Proc. Nat. Acad. Sci.*, **1**, 51–56 (1935).

[2] The functions $Se^1_{-1/2,m}(c, \cos u)$ are of course proportional to the Mathieu functions $ce_m(u)$, and the So's to the functions $se_m(u)$.

[3] Whittaker and Watson, *Modern Analysis*, page 396.

[4] Watson, *Theory of Bessel Functions*, page 180.

[5] Whittaker and Watson, *Modern Analysis*, page 398.

[6] Relations can also be obtained between spheroidal waves and cylindrical waves.

ON THE BENDING OF ELECTROMAGNETIC MICRO-WAVES BELOW THE HORIZON

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1. An interesting phase in the development of the modern radio technique are the experiments conducted during the last few years with very short wave-lengths. Marchese Marconi[1] reported about an extensive series of successful radio connections over distances up to 260 km., in which waves of from 50 cm. to 60 cm. were used, concentrated with the help of a parabolic reflector. Clavier and Gallant[2] went even to still shorter waves of only 17.4 cm. which they sent over a distance of 61 km. also concentrating them with a reflector of 3.8 m. in diameter. The most remarkable feature of Marchese Marconi's results is that the distances covered by him exceed several times the range of rectilinear visibility from the sending station.

The memory is still fresh of the great surprise which was caused among physicists by the unusually long range of long wave radio-reception. The explanation of these puzzling facts about long waves was traced, in the meantime,[3] to the influence of the Kennelly-Heaviside layer of the upper atmosphere, and the question, naturally, arises to what extent atmospheric influences are responsible for the phenomena observed by Marchese Marconi with micro-waves. The first step in answering this question must be an investigation of how much bending is to be expected from the point of view of the wave theory *completely neglecting the atmosphere.* Such an investigation is the subject of this paper.

The simple method which we propose is based on Huyghens' principle and treats the surface of the earth as a perfectly absorbing screen. As far as we know, it was not used heretofore and there are good reasons for this: In the case of long waves, the properties of the soil play an important part both in their production and their propagation. The height of the receiving station is only a fraction of the wave-length, so that only the so-called "surface wave" is of practical interest. On the other hand, the micro-waves are produced away from the soil and independently from it and, after they strike the earth, the surface wave is so thin as to be entirely unimportant. The transmission is, in this case, a matter of space propagation on which the physical properties of the earth surface have no material influence. It is therefore, perfectly permissible to replace it by a perfectly absorbing screen.

The results of our calculations and their comparison with Marchese Marconi's observations are summarized in the last section.

2. Suppose that the sending station of the micro-waves is at the point A of figure 1 and the receiver at the point P_2 below the horizon. According to Huyghens' principle every point of the space exposed to waves

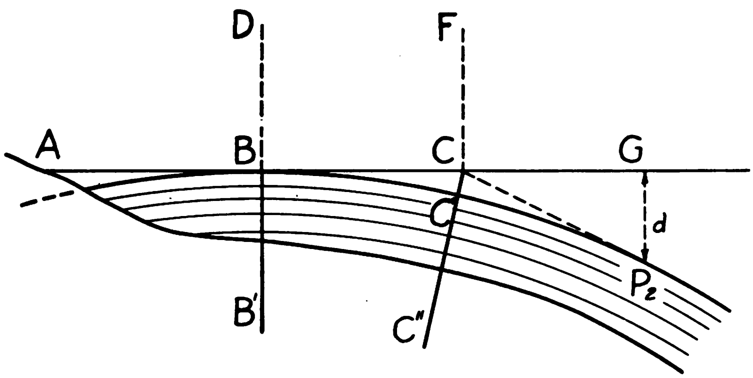

FIGURE 1

can be itself regarded as the origin of a spherical wave. We shall make use of it in the following way: The reflector in the source A throws a diffraction pattern onto the plane DB' at the horizon of which the part BB' is screened off by the earth. The remaining part BD we regard now as the source of spherical waves which produce another diffraction pattern in the plane CF. We repeat the operation taking now every point of the half plane CF as the source of a spherical wave and, finally, compute the intensity which these waves produce in the point of reception P_2. In short, the case is treated as if between the origin (A) with its reflector and the receiver P_2 there were interposed two perfectly absorbing screens, the one being BB', the other CC''.

It is true that the part CC' of the half plane CC'' is not entirely optically empty since it receives some intensity by diffraction below the horizon B. We shall see, however, that the intensity which it can pass onto the point P_2 (by another diffraction below the horizon) is weak compared with that coming from CF. Upon solving this threefold diffraction problem it was found that the screen in BB' has very little influence: Its omission does not appreciably affect the intensity in P_2 when this point is either near the horizon B or far from it, and only produces a slight change in the intermediate case of a medium distance. In order not to encumber our theory unnecessarily, we simplify the problem by considering only one perfectly absorbing screen in CC'', as this gives an altogether sufficient approximation.

3. In the case of small angle diffraction Kirchhoff's formula for Huyghens' principle can be written in the simplified form

$$u_P = \frac{1}{4\pi L}\int ds[\partial u_M/c\partial t - \partial u_M/\partial N]_{t-\frac{l}{c}}, \qquad (1)$$

t represents the time and c the velocity of light while the other notations are sufficiently clear from figure 2.

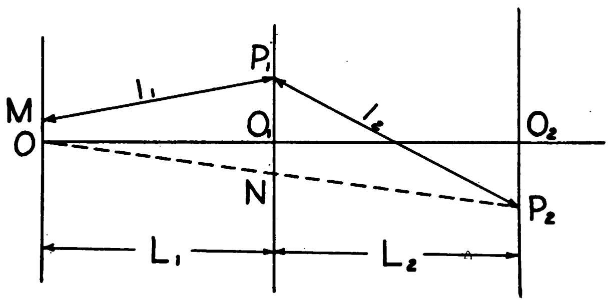

FIGURE 2

Marchese Marconi's source of waves was, virtually, a rectangle oscillating in phase, because his oscillating elements were strung out along the focal line of a reflector in the shape of a parabolic cylinder. We suppose that this rectangle lies in the plane OM with its center in the origin O and we denote its sides by $2a$ and $2b$. Let, moreover, the coördinates of the three planes OM, O_1P_1, O_2P_2 be, respectively, ξ, η; ξ_1, η_1; ξ_2, η_2; the third direction (along the axis OO_2) being denoted by z. We have to take for u_M (wave potential at the rectangular source, $z = 0$) $u_M = A \times \exp ik(ct - z)$ and for the distance l_1 the approximation

$$l_1 = L_1 - (\alpha_1\xi + \beta_1\eta) + \tfrac{1}{2} L_1(\alpha_1^2 + \beta_1^2), \qquad (2)$$

using the abbreviations $\alpha_1 = \xi_1/L_1$, $\beta_1 = \eta_1/L_1$. The integration with respect to $ds = d\xi d\eta$ goes in ξ from $-a$ to $+a$ and in η from $-b$ to $+b$ and gives for the diffraction pattern in the plane O_1P_1 the well-known expression

$$u_1 = \frac{2ikabA}{\pi L_1}\frac{\sin ka\alpha_1}{ka\alpha_1}\frac{\sin kb\beta_1}{kb\beta_1} \times \exp ik[ct - L_1 - \tfrac{1}{2} L_1(\alpha_1^2 + \beta_1^2)]. \quad (3)$$

In order to find the potential in the point P_2, we apply the formula (1) a second time substituting for u_M the expression (3). For the distance l_2 we have to use an approximation taking in terms of the second order

$$l_2 = L_2 + \tfrac{1}{2}\,[(L_1\alpha_1 - L\alpha_2)^2 + (L_1\beta_1 - L\beta_2)^2]/L_2, \quad (4)$$

where $\alpha_2 = \xi_2/L$, $\beta_2 = \eta_2/L$, $L = L_1 + L_2$. The integration is to be taken with respect to $ds = L_1^2 d\alpha_1 d\beta_1$ and gives the result

$$u_2 = -\frac{k^2abAL_1}{\pi^2 L_2}\exp ik[ct - L - \tfrac{1}{2}L(\alpha_2^2 + \beta_2^2)]$$
$$\times \int \frac{\sin ka\alpha_1}{ka\alpha_1}\exp \tfrac{1}{2}\,ikg(\alpha_1 - \alpha_2)^2 d\alpha_1 \int \frac{\sin kb\beta_1}{kb\beta_1}\exp \tfrac{1}{2}\,ikg(\beta_1 - \beta_2)^2 d\beta_1 \quad (5)$$

with

$$g = LL_1/L_2.$$

When taken from $-\infty$ to $+\infty$, either of the integrals has the value

$$(2\pi/kgb^2)^{1/2}\int_0^b \exp(ikb^2/2g)\cos kb\beta_2 db.$$

The ratio of the arguments of the exponential and of the cosine, $b/2g\beta_2$ is in all applications a very small number. Without impairing the approximation, we can substitute 1 for the exponential, obtaining the expression

$$\left(\frac{2\pi}{kg}\right)^{1/2}\frac{\sin kb\beta_2}{kb\beta_2}\exp(-i\pi/4). \quad (6)$$

Turning back to the integrals in eq. (5), we notice that in them also the coefficients which occur in the arguments of the exponential and the sine, namely, $kg/2$ and ka stand in the ratio $g/2a$, i.e., roughly the distance between the sender and the receiver to the opening of the mirror. This ratio is, of course, very large and, unless $\alpha_1 - \alpha_2$ (or resp. $\beta_1 - \beta_2$) is small of the order $2a/g$, the exponential oscillates so rapidly as to give no appreciable contribution to the integral. Therefore, if we integrate from $\alpha_2 - \epsilon$ to $\alpha_2 + \epsilon$ (resp. from $\beta_2 - \epsilon$ to $\beta_2 + \epsilon$) where ϵ is only a few times as large as $2a/g$, we must get practically the same result as integrating from $-\infty$ to $+\infty$, that is to say, the formula (6). It follows from this that when the point $N(\alpha_1 = \alpha_2, \beta_1 = \beta_2)$ lying in the straight line OP_2 with its vicinity is unobstracted by the screen, the intensity in P_2 is practically the same as if there were no screen at all: The property of rectilinear propagation which Fresnal proved for the spherical wave holds also for our diffraction pattern. In fact, if we substitute (6) for the two integrals in the eq. (5) we obtain a result exactly of the same structure as the expression (3) defining this pattern. This justifies our neglecting the influence of the surface CC' (Fig. 1) upon the intensity in P_2 and also explains why omitting the additional screen in BB' makes so little difference.

4. As stated in section 2, we are interested in the case when α_2 is negative (P_2 below the horizon) and the integration in eq. (5) is extended over the half plane $\alpha_1 > 0$. It is true that our expressions are rigorous only for small angles α_1 and β_1, but owing to the structure of the integrands, those parts of the plane for which α_1 and β_1 are not small give no appreciable contribution. We may, therefore, without changing the result, extend the limits of integration, with respect to α_1, from 0 to $+\infty$, and with respect to β_1, from $-\infty$ to $+\infty$. The second integral is, therefore, given by the expression (6), while we use for the first the following notations

$$-\alpha_2 = d/L,$$

$$\Phi(d) = \int_0^\infty \frac{\sin ka\alpha_1}{ka\alpha_1} \exp \tfrac{1}{2}\, ikg(\alpha_1 - \alpha_2)^2 d\alpha_1 \tag{7}$$

d being the distance GP_2 (Fig. 1) below the horizon.

The wave potential u_2 becomes now

$$u_2 = 4abA\lambda^{-3/2}\,(L_1/L_2L)^{1/2}\Phi(d)\,\frac{\sin kb\beta_2}{kb\beta_2}$$
$$\times \exp ik\left[ct - L - \tfrac{1}{2}\,L(\alpha_2^2 + \beta_2^2) - \frac{\pi}{4}\right],$$

If we denote the area of the reflector $S = 4ab$ and the total intensity of the sending station by $I = SA^2$, the intensity in the great circle going through the axis of the projected beam ($\beta_2 = 0$) is

$$E = |\,u\,|^2 = IS\lambda^{-3}\cdot|\,\Phi(d)\,|^2\cdot L_1/L_2L. \tag{8}$$

When d is large (more accurately, $\sqrt{kg}\,d >> L$), the integral (7) can be approximately evaluated by parts

$$\Phi(d) = L/ikgd = L_2\lambda/2\pi i L_1 d,$$

giving the expression for the intensity at large distances

$$E = ISL_2/4\pi^2\lambda d^2 L_1 L. \tag{9}$$

Let the distance from the sender to the horizon be l, then $L_1 = (L + l)/2$ and $L_2 = (L - l)/2$. Denoting the radius of the earth by R, we have for a not too large d the expression

$$d = (L - l)^2/2R, \quad (10)$$

so that the formula (9) takes the form

$$E = ISR^2/\pi^2\lambda(L - l)^3(L + l)L. \quad (11)$$

When l happens to be small compared with L this gives a decline of the intensity inversely proportional to the fifth power of the distance L.

It is worth mentioning that our formulas represent the intensities also in the case of a reflector with circular opening. The theory is then a little more cumbersome but the expression for large distances is strictly the same as (11) and that for shorter ones is practically identical with (8).

5. In Marchese Marconi's experiments the sending station of Rocca di Papa had an elevation of 750 m. corresponding to a distance of the horizon $l = 98$ km. The opening of the concentrating reflector was equal to three wave-lengths ($2a = 2b = 3\lambda$). Under these circumstances the formula (11) is valid from $L = 180$ km. on. For shorter distances must be used the expression (8). In general, the function $\Phi(d)$ is not easy to evaluate, but under the conditions just mentioned ka is so small that $(\sin ka\alpha_1)/ka\alpha_1$ is appreciably equal to 1 for the whole interval in question and $\Phi(d)$ becomes identical with Fresnel's integral. The numerical values of the intensity following from the formulas (8) and (11) are given in relative units in the following table:

L:	40 km.	60	80	98	100	102	105	110	115	120
E:	12.6	5.6	3.2	1.00	.96	.88	.76	.56	.38	.22
φ:	77	70	65	55	54	53	52	49.5	46	41

L:	125 km.	130	135	140	145	150	155	160
E:	.16	.11	7.9×10^{-2}	5.7	4.3	3.2	2.5	2.0
φ:	38.5	35.5	32.5	30	27	25	22	20.5

L:	170 km.	180	190	200	210	220	230	240	250
E:	1.6	1.2	7.4×10^{-3}	5.2	3.6	2.6	1.9	1.4	1.1
φ:	19	16	12	8.8	5.6	2.8	0.0	−2.7	−4.5

Since the reception was acoustic, we have listed under φ the theoretical "sensation level" (the subjective auditory intensity) of the signals computed according to the formula $\varphi = 20 \log_{10} (E/E_0)$, where E_0 is the threshold intensity of audibility.[4] Marchese Marconi mentions that the signals were heard until the maximum distance of 230 km. Therefore, their theoretical intensity at 230 km. was taken as E_0. Under favorable conditions, the reception was clear until 205 km., then it became erratic and only occasionally audible. Comparing this with our table, we must conclude that *the experimental range of the waves did not materially exceed the expectations of a theory disregarding all atmospheric influences. Qualitatively the observations agree with the predictions of our formulas.* This applies also to the fact that communication with the sending station could be again established at Cape Figari at an elevation of 340 m. and distance $L = 270$ km. Our theory gives for this point $E = 3 \times 10^{-3}$, $\varphi = 4$.

Atmospheric agencies had an unquestionable effect inasmuch as they could spoil the reception causing "slow and deep fading" of the signals which, at times, reduced them to complete inaudibility. More quantitative observations are required to answer the question whether they could occasionally also help the reception. It is easy to estimate the influence of the Kennelly-Heaviside layer if one makes definite assumptions as to its height and reflective power for micro waves. Supposing its height to be 80 km. and the reflection complete, the reflected intensity should be from $L = 150$ km. on of the same order of magnitude as the E of our table. These assumptions are, however, very doubtful and, moreover, Marchese Marconi ascertained by turning the reflector of the receiving stations that the signals came from the horizontal direction. It is also not difficult to take into account the influence of atmospheric refraction which would slightly increase the intensities of our table, but the experimental data are not accurate enough to make this correction worth our while.

Note added in proof: Working with wave-lengths from 17 m. to 3.5 m. Schelleng, Burrows and Ferrell (*Inst. Radio Engineers*, **21**, 427 (1933)) found that, even within the range of rectilinear visibility, the transmitted intensities dropped considerably below the inverse square law, being better represented by the inverse fourth power of the distance. The authors attribute these remarkable results to reflection from the surface of the earth. If this explanation be correct, the effect must be weaker for the shorter wave-lengths with which we deal, and at large distances less important than the effect of the curvature of the earth. The sensation levels given in our table should, therefore, continue to be a sufficiently good representation of the actual conditions.

[1] Marchese Marconi, *Proc. Roy. Inst. of G. B.*, **27**, 509 (1933). According to newspaper reports M. M. has lately slightly increased the range of his reception. His more recent publications were, however, inaccessible to the author.

[2] A. Clavier, *L'Onde electrique*, **13**, 101 (1934); A. Clavier and L. C. Gallant, *Radio Engineering*, 14, Feb.-March (1934).

[3] See especially: G. N. Watson, *Proc. Roy. Soc.*, **A95**, 83, 546 (1919).

[4] Compare, e.g., H. Fletcher, *Int. Crit. Tables*, **6**, 450.