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INVERSION OF SURFACE WAVE DISPERSION DATA

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The principal goal of the study to be outlined here was to obtain a systematic and practical means of determining structural models of the earth from the observed dispersion of surface waves or from the free oscillation spectrum. This was accomplished through a perturbation scheme which has its basis in Rayleigh's Principle. That is, an initial test structure is assumed and then perturbed until its theoretical dispersion agress with the observed dispersion to within some preselected degree of accuracy. The problem is therefore to determine the necessary perturbations for rapid convergence of the theoretical spectrum to the observed spectrum.

The essential concepts of the perturbation method were stated and utilized by Rayleigh himself and recently restated by Jeffreys in a context similar to that of the present study. However, up to the present time the method has been employed to predict the effect on the vibrational frequencies due to a specified change in the parameters of the system, such as density and the elastic constants. The present problem is just the inverse of this, that is, to determine changes in the system parameters which give a specified change in the characteristic vibrational frequencies. The application of Rayleigh's Principle to this inverse problem was first suggested to us by Dr. Freeman Gilbert.

We have generalized the method to a form applicable to this inverse problem in a radially inhomogeneous, spherical earth and have programed the procedure for automatic computation of the required structure. In addition, we have formulated the method in a manner which shows, very clearly, the nature of the dependence of the dispersive properties on small changes in the structure parameters. From this formulation we find that this perturbation scheme, which is exact to first order and, therefore, representative of any first order theory, is not by itself sufficient to predict the required first order changes in a test structure. However, if mild supplementary conditions of constraint are introduced, then the perturbation scheme does yield corrections which in practice have given rapid convergence. Thus, constraints, based on knowledge of the structure derived from other sources, have been shown to be a necessary as well as desirable part of the method.

Once having obtained a structure giving the same vibrational spectrum as that observed, the question of uniqueness arises. In practice, only limited portions of the spectrum are observed and the spectral data will also have an associated error. In addition, the surface wave data represents an averaged dispersion over a laterally inhomogeneous earth. Under these conditions there is no question but that the derived structure is *not* unique. That is, there are other structures which will satisfy the data, which is necessarily limited in extent and accuracy, as well or very nearly as well. Further, depending on the conditions of the experiment, the model or models obtained can only represent an averaged structure for the earth, especially for the crust. Therefore, in view of the experimental limitations alone, it is only realistic to consider the structure obtained in the light of other independent evidence.

Thus, for example, travel time curves are being computed from structures obtained from dispersion data for comparison with the arrival times of body phases. In addition, as more and better data becomes available, especially with regard to the higher mode surface waves, the practical uniqueness of the method will improve considerably.

Fig. 1 shows of the essential features of the theory in a condensed and symbolic form. Throughout the next few figures H and H_s

ABSTRACT FORMULATION

Defn. $(u,v) \equiv \int_{D_H} \cdots \int u\,\bar{v}\,dx_1 \cdots dx_k$

Variational Principle

$$\lambda_\ell = \frac{(Hu_\ell, u_\ell)}{(u_\ell, u_\ell)} = MIN \frac{(Hu,u)}{(u,u)}$$

$(u, u_m) = 0, \quad m = 1, 2, \cdots, \ell-1$

Associated Euler-Lagrange Equation

$$Hu_\ell = \lambda_\ell u_\ell$$

(Plus Boundary Conditions)

Approximate Variational Method

$$\lambda_\ell^N = \frac{\sum_{s=1}^{N} \{H_s u_\ell^s, u_\ell^s\}}{\sum_{s=1}^{N} \{u_\ell^s, u_\ell^s\}} = MIN \frac{\sum_{s=1}^{N} \{H_s u^s, u^s\}}{\sum_{s=1}^{N} \{u^s, u^s\}}$$

$\{u^s, u_m^s\} = 0 \;;\; m = 1, 2, \cdots, \ell-1$

$$\lambda_\ell = \lim_{N\to\infty} \lambda_\ell^N$$

Associated Euler-Lagrange Eqs.

$$H_s u_\ell^s = \lambda_\ell u_\ell^s, \quad s = 1, 2, \cdots, N$$

(Plus Boundary Conditions on u_ℓ^s)

Defn. $\{u,v\} = \int_{D_{H_s}} \cdots \int u\,\bar{v}\,dx_1 \cdots dx_k$

Fig. 1

denote differential operators, the $U's$ and $\lambda's$ are the eigenfunctions and eigenvalues belonging to the operators H. The Euler-Lagrange equations are, in the present context, just the equations of motion for Love of Rayleigh waves or equivalently for the torsional or spheroidal oscillations of the earth.

The inner product of two functions is the indicated integral over the region in which the operator H is meaningful (i.e. the domain of H). In the present application this is the interior of our spherical earth model.

The variational principle indicated is fundamental to this theory; the first equality is an identity while the second equality is essentially a statement of Rayleigh's principle. Thus the eigenvalue is obtained by minimizing the indicated functional by varying U subject to the orthogonality condition indicated.

The approximate variational method is in fact the approach used in the present application and corresponds to the usual layered approximation of a radially inhomogeneous earth. Formally it may be obtained from the exact variational formulation by breaking the integrals up into a sum of integrals over short intervals in the radial direction. For sufficiently small intervals the integrand in the individual integrals can be approximated so that the coefficients of the differential operator are constants and the eigenfunctions are solutions of the indicated set of differential equations each appropriate for a layer. The boundary conditions connect the solutions U'_i in each layer so as to maintain continuity and satisfy the original boundary conditions for the exact variational method.

In the limit as the number of integration intervals (or layers) becomes large, the eigenvalue for this approximate theory will approach that of the exact theory.

The perturbation theory shown in Fig. 2 actually follows from the statement of Rayleigh's principle. However, it may be formulated and derived in greater generality in the manner indicated in these equations. Here we indicate how a perturbation affects one of the layer equations. The layer index is momentarily suppressed for clarity.

The unperturbed system corresponds to our test structure while the perturbed system corresponds to a structure giving the observed dispersion. The difference in the structures corresponds to a perturbation of the coefficients of the differential operator. Taking the perturbations in the $\lambda's$ and $P_i's$ to be small so that second order variations may be neglected, it is easy to show that (4) follows from these relationships.

Perturbation Theory

(1) $H u_\ell = \lambda_\ell u_\ell$ (Unperturbed), $H \equiv \frac{d}{dx}\left({}^{(o)}P \frac{d}{dx}\right) + {}^{(o)}P_2$

(2) $\mathcal{H} U_\ell = \Lambda_\ell U_\ell$ (Perturbed), $\mathcal{H} \equiv \frac{d}{dx}\left(P \frac{d}{dx}\right) + P_2$

where: $P_i - {}^{(o)}P_i = \delta P_i$, $i = 1, 2$

$$(3)\quad \begin{cases} \Lambda_\ell - \lambda_\ell = \delta\lambda_\ell \\ \mathcal{H} = H + \sum_i H_i \delta P_i \\ U_\ell = u_\ell + \sum_i \left(\frac{\partial U_\ell}{\partial P_i}\right) \delta P_i + \left(\frac{\partial U_\ell}{\partial \Lambda}\right)_{\lambda_\ell} \delta\lambda_\ell + O(\delta P_i^2, \delta\lambda_\ell^2) \end{cases}$$

Thus $(\lambda_\ell + \delta\lambda_\ell)(U_\ell, U_\ell) = (\mathcal{H} U_\ell, U_\ell)$, and from (3) using (1);

$$(4)\quad \boxed{\sum_i (H_i u_\ell, u_\ell)\, \delta P_i = \delta\lambda_\ell (u_\ell, u_\ell) + O(\delta P_i^2, \delta\lambda_\ell^2)}$$

Partial Variations:

$$\left(\frac{\delta\lambda_\ell}{\delta P_1}\right)_{P_2} = \frac{(H_1 u_\ell, u_\ell)}{(u_\ell, u_\ell)} \quad ; \quad \left(\frac{\delta\lambda_\ell}{\delta P_2}\right)_{P_1} = \frac{(H_2 u_\ell, u_\ell)}{(u_\ell, u_\ell)}$$

Fig. 2

It is convenient to express this result in a form analogous to partial derivatives. Thus, we consider the ratio of the variation in the eigenvalue to the variation or perturbation of one of the elastic parameters with all the other parameters held fixed.

For the layer or step approximation expressed in its full form we have indicated in figure 3 the two relations giving the unperturbed and perturbed eigenvalues. We now can show how the difference between these two eigenvalues depends on the differences between the elastic parameters in each layer. Thus proceeding in the same manner as in the previous figure we get the first order relationship indicated.

Again the result may be expressed in terms of the partial variations, where all but one of the layer parameters are held constant for each partial. These partials are expressed in terms of the known unperturbed eigenfunctions and the known perturbation operators, and may therefore be computed from the test structure.

An application of these ideas to the toroidal oscillations of the earth yields the relation show in figure 4. We shall use this mode of oscillation as an example of an application of the method.

The conditions on the rigidity and density functions indicated denote the layer approximation and a representation involving a continuous variation of the elastic properties which also may be

Fig. 3

Perturbation Theory (Step Approximation)

$$\lambda_\ell^N = \frac{\sum_{s=1}^{N}\{H_S U_\ell^s, U_\ell^s\}}{\sum_{s=1}^{N}\{U_\ell^s, U_\ell^s\}} \quad \text{(unperturbed)}$$

$$\Lambda_\ell^N = \frac{\sum_{s=1}^{N}\{\mathcal{H}_S U_\ell^s, U_\ell^s\}}{\sum_{s=1}^{N}\{U_\ell^s, U_\ell^s\}}, \qquad \mathcal{H}_S = H_S + \sum_i H_i^s \delta P_i^s \quad \text{(Perturbed)}$$

$$\{\mathcal{H}_S U_\ell^s, U_\ell^s\} = \Lambda_\ell^N \{U_\ell^s, U_\ell^s\}$$

$$\{H_S U_\ell^s, U_\ell^s\} = \lambda_\ell^N \{U_\ell^s, U_\ell^s\}$$

From the result (4):

$$\Lambda_\ell^N = \lambda_\ell^N + \delta\lambda_\ell^N = \sum_{s=1}^{N}\left[\frac{\{H_S U_\ell^s, U_\ell^s\}}{\sum_{s=1}^{N}\{U_\ell^s, U_\ell^s\}} + \frac{\sum_i \{H_i^s U_\ell^s, U_\ell^s\}\delta P_i^s}{\sum_{s=1}^{N}\{U_\ell^s, U_\ell^s\}}\right]$$

$$\therefore \boxed{\delta\lambda_\ell^N = \sum_{s=1}^{N}\left[\frac{\sum_i \{H_i^s U_\ell^s, U_\ell^s\}\delta P_i^s}{\sum_{s=1}^{N}\{U_\ell^s, U_\ell^s\}}\right]}$$

Partial Variations:

$$\boxed{\left(\frac{\delta\lambda_\ell^N}{\delta P_i^s}\right)_{P_j^r} = \frac{\{H_i^s U_\ell^s, U_\ell^s\}}{\sum_{s=1}^{N}\{U_\ell^s, U_\ell^s\}} \quad ; \quad P_j^r \text{ fixed for } r = 1, 2, \cdots s-1, s+1, \cdots N}$$

Toroidal Oscillations, Love waves

Rayleighs Principle

$$n\omega_\ell^2 \sum_{s=1}^{N}\int_{r_s}^{r_{s+1}} \rho(r)\left[{}_nW_\ell^{(s)}(r)\right]^2 r^2 dr =$$

$$\sum_{s=1}^{N}\int_{r_s}^{r_{s+1}} \mu(r)\left[\left(\frac{d\,{}_nW_\ell^{(s)}}{dr} - \frac{{}_nW_\ell^{(s)}}{r}\right)^2 + (\ell-1)(\ell+2)\left(\frac{{}_nW_\ell}{r}\right)^2\right] r^2 dr$$

$$\left.\begin{matrix}\mu(r) = \mu_s \\ \rho(r) = \rho_s\end{matrix}\right\} \; r_s \le r \le r_{s+1} \quad \text{or} \quad \mu(r) = \sum_{j=0}^{K_1} \alpha_j r^j, \quad \rho(r) = \sum_{j=0}^{K_2} \beta_j r^j$$

Associated Euler-Lagrange Equations

$$\mu_s \frac{d^2\,{}_nW_\ell^{(s)}}{dr^2} + \frac{2\mu_s}{r}\frac{d\,{}_nW_\ell^{(s)}}{dr} + \mu_s\left[\frac{{}_n\omega_\ell^2}{\beta_n^2} - \frac{\ell(\ell+1)}{r^2}\right]{}_nW_\ell^{(s)} = 0, \quad s = 1, 2, \cdots\cdots, N$$

(Plus Bondary Conditions)

$${}_nW_\ell^{(s)} = {}_nA_\ell^{(s)} j_\ell({}_nk_\ell^{(s)} r) + {}_nB_\ell^{(s)} n_\ell({}_nk_\ell^{(s)} r) \; ; \; {}_nk_\ell^s = \frac{n\omega_\ell}{\beta_s}, \quad \beta_s = [\mu_s/\rho_s]^{1/2}$$

Fig. 4

used. Use of the latter structure representation involves some further complications which we shall not discuss, but is included here to indicate an alternate approach which is used and which is generally superior to the layer approximation.

The associated equations of motion in the individual layers are as indicated and yield solutions in terms of Bessel functions.

This formulation is actually somewhat awkward for numerical calculation. Figure 5 shows an alternate approximate form of the equation of motion which is simple and highly accurate.

Alternate Form (Toroidal Oscillations and Love Waves)

$$\mu_s \frac{d^2 W_s}{dr^2} + \frac{2\mu_s}{r}\frac{dW_s}{dr} + \mu_s\left[\frac{\omega^2}{\beta_s^2} - \frac{(ka)^2}{r^2}\right] W_s = 0 \; ; \; W_s \equiv {}_nW_l^{(s)}, \; (ka)^2 \equiv l(l-1)$$

(Anderson's flat earth approximation)

$$N_s \frac{d^2 V_s}{dz^2} + L_s\left[\frac{\omega^2}{\beta_s'^2} - k^2\right] V_s = 0 \; ; \; V_s = (a-z) W_s$$

where: $N_s \equiv \left(\frac{a-z_s'}{a}\right)\mu_s$, $L_s \equiv \mu_s$ (Anisotropic)

$$\beta_s' = \left(\frac{a}{a-z_s'}\right)\beta_s$$

valid when: $h_s = (z_{s+1} - z_s) << a$

$$\left(\frac{a-z}{a}\right)_s^2 \simeq \left(\frac{a-z_s'}{a}\right)^2, \quad z_s' = z_s + h_s/2$$

solution:

$$V_s = A_s \sin k\gamma_s z + B_s \cos k\gamma_s z \; , \quad \gamma_s = \left(\frac{a}{a-z_s'}\right)\left[\left(\frac{c}{\beta_s'}\right)^2 - 1\right]^{1/2}$$

Fig. 5

By introducing the parameters and new variables indicated we can transform the equation for the toroidal oscillations into an equation wich has the form of an equation for Love waves in a flat layered anisotropic half-space. Thus we can actually consider simultaneously the toroidal oscillations of a sphere or Love wave propagation in a half-space. The approximation is valid when the layer thicknesses are small compared to the earth's radius (a). The solutions are seen to have a very simple form.

Using the previous approximate equation of motion we find that Rayleigh's principle takes the simple form indicated (Fig. 6). Under (a) we have indicated the relationship for the layered approximation and under (b) that for a polynomial approximation for the elastic parameters. In both cases the integrals may be evaluated analyti-

Alternate Rayleigh Principle (Toroidal Oscillations)

$$\omega^2 \sum_{s=1}^{N} \int_{z_s}^{z_{s+1}} \rho(z)\, V_s^2(z)\, dz = \sum_{s=1}^{N} \int_{z_s}^{z_{s+1}} \mu(z) \left[\left(\frac{N_s}{L_s}\right) \left(\frac{dV_s}{dz}\right)^2 + k^2 v_s^2 \right] dz$$

(a) $$\omega^2 \sum_{s=1}^{N} \rho_s I_o^s = k^2 \sum_{s=1}^{N} \mu_s I_o^s + \sum_{s=1}^{N} \mu_s I_1^s$$

$$I_0^s = \int_{z_s}^{z_{s+1}} V_s^2(z)\, dz \quad ; \quad I_1^s = \left(\frac{N_s}{L_s}\right) \int_{z_s}^{z_{s+1}} \left(\frac{dV_s}{dz}\right)^2 dz$$

(Structure Approximated by Step Functions)

(b) $$\omega^2 \sum_{j=1}^{k_1} \alpha_j J_o^j = k^2 \sum_{j=1}^{k_2} \beta_j J_o^j + \sum_{j=1}^{k_2} \beta_j J_1^j$$

$$J_0^j = \sum_{s=1}^{N} \int_{z_s}^{z_{s+1}} z^j V_s^2(z)\, dz \quad ; \quad J_1^j = \sum_{s=1}^{N} \left(\frac{N_s}{L_s}\right) \int_{z_s}^{z_{s+1}} z^j \left(\frac{dV_s}{dz}\right)^2 dz$$

(Structure Approximated by Polynomials)

Fig. 6

cally due to the simplicity of the eigenfunctions V, shown on the previous figure.

The results of the perturbation theory can be applied and we easily obtain the variational partials of interest. Further, the phase and group velocity are defined as indicated (Fig. 7) and can be

Variational Partials

u = Group Velocity = $\frac{d\omega}{dk}$ $\quad V_r$ = P.E. of r^{th} layer $\quad \tau$ = Total K.E.

c = Phase Velocity = ω/k $\quad T_r$ = K.E. of r^{th} layer $\quad E$ = Total Energy = 2τ

$$u = \left(\frac{\delta\omega}{\delta k}\right)_{\mu,\rho} = \frac{1}{c} \left(\frac{\sum_{s=1}^{N} \mu_s I_o^s}{\sum_{s=1}^{N} \rho_s I_o^s} \right)$$

(a) Layered Structure Approximation ($k = k_o = \text{constant}$)

$$\frac{\rho_r}{c} \left(\frac{\delta c}{\delta \rho_r}\right)_{\mu} = \frac{\rho_r}{\omega} \left(\frac{\delta\omega}{\delta \rho_r}\right)_{\mu} = -\frac{1}{2} \left[\frac{\rho_r I_o^r}{\sum_{s=1}^{N} \rho_s I_o^s} \right] = -\left(\frac{T_r}{E}\right)$$

$$\frac{\mu_r}{c} \left(\frac{\delta c}{\delta \mu_r}\right)_{\rho} = \frac{\mu_r}{\omega} \left(\frac{\delta\omega}{\delta \mu_r}\right)_{\rho} = \frac{1}{2} \left[\frac{\mu_r (k_o^2 I_o^r + I_1')}{\omega^2 \sum_{s=1}^{N} \rho_s I_o^s} \right] = \left(\frac{V_r}{E}\right)$$

$$\delta c = \sum_{r=1}^{N} \left[\left(\frac{\delta c}{\delta \rho_r}\right) \delta \rho_r + \left(\frac{\delta c}{\delta \mu_r}\right) \delta \mu_r \right]$$

$$\boxed{E\left(\frac{\delta\omega}{\omega}\right) = E\left(\frac{\delta c}{c}\right) = \sum_{r=1}^{N} \left[V_r \left(\frac{\delta \mu_r}{\mu_r}\right) - T_r \left(\frac{\delta \rho_r}{\rho_r}\right) \right]}$$

$$\boxed{E\left(\frac{\delta\omega}{\omega}\right) = E\left(\frac{\delta c}{c}\right) = \sum_{r=1}^{N} \left[(V_r - T_r) \frac{\delta \mu_r}{\mu_r} + 2 T_r \frac{\delta \beta_r}{\beta_r} \right]}$$

Note: $\sum_{r=1}^{N} V_r = \sum_{r=1}^{N} T_r = \frac{E}{2}$; $\sum_{r=1}^{N} \frac{V_r}{\mu_r} \neq \sum_{r=1}^{N} \frac{T_r}{\rho_r}$ in general

Fig. 7

related by a variation of the wave number k and frequency ω holding the elastic parameters constant. The result is as shown and was originally obtained by Meissner.

By defining the potential and kinetic energies for each layer of the medium we are able to show that the partial variations of interest may be expressed in terms of the kinetic or potential energy of the layer divided by the total energy of the system in the manner indicated.

Then using the first order relationship between the variations in the phase velocity or frequency and the variations in the elastic parameters, we obtain the final perturbation formulas for the toroidal oscillations or Love waves. Clearly the coefficients of the variations in the elastic parameters on the right are related by the condition noted and this relationship is important to the question of inversion, since these relationships will be used to solve for the variations in the elastic parameters in terms of δc and the energies at different frequencies.

Figure 8 indicates the perturbation formulas for the polynomial approximation as well. In this case the partial variations involving

Variational Partials (continued)

(b) POLYNOMIAL Approximation ($k = k_0 =$ constant)

$$\frac{1}{c}\left(\frac{\delta c}{\delta \alpha_i}\right)_\beta = \frac{1}{\omega}\left(\frac{\delta \omega}{\delta \alpha_i}\right)_\beta = -\frac{J_o^i}{2\sum_{j=1}^{k_1} \alpha_j J_o^j}$$

$$\frac{1}{c}\left(\frac{\delta c}{\delta \beta_i}\right)_\alpha = \frac{1}{\omega}\left(\frac{\delta \omega}{\delta \beta_i}\right)_\alpha = \frac{k_o^2 J_o^i + J_1^i}{2\omega^2 \sum_{j=1}^{k_1} \alpha_j J_o^j}$$

$$\delta c = \sum_{i=1}^{k} \left[\left(\frac{\delta c}{\delta \alpha_i}\right)_\beta \delta \alpha_i + \left(\frac{\delta c}{\delta \beta_i}\right)_\alpha \delta \beta_i\right] \; ; \quad k = Max.[k_1, k_2] << N$$

Note: $\sum_{i=1}^{k} \alpha_i \left(\frac{\delta c}{\delta \alpha_i}\right)_\beta = -\sum_{i=1}^{k} \beta_i \left(\frac{\delta c}{\delta \beta_i}\right)_\alpha \; ; \quad \sum_{i=1}^{k} \left(\frac{\delta c}{\delta \alpha_i}\right)_\beta \neq \sum_{i=1}^{k} \left(\frac{\delta c}{\delta \beta_i}\right)_\alpha$

FORMAL INVERSION

ω_s = Discrete set of frequencies, $s = 1, 2, \ldots M \geq \binom{2N}{2k}$

$$E(\omega_s)\left[\frac{\delta c(\omega_s)}{c(\omega_s)}\right] = b_s$$

$$\frac{V_r(\omega_s)}{\mu_r} = a_{sr}$$

$$-\frac{T_r(\omega_s)}{\rho_r} = a_{s,r+N}$$

$$\delta \mu_r = x_r$$

$$\delta \rho_r = x_{r+N}$$

MATRIX EQUATION

$$A_i x = b$$

$$x = (A_i^T A_i)^{-1} A_i^T b$$

(Best solved by Relaxation or Iterated)

Fig. 8

the polynomial coefficients cannot be connected to the energies of the system so directly as they were for the layered case. However,

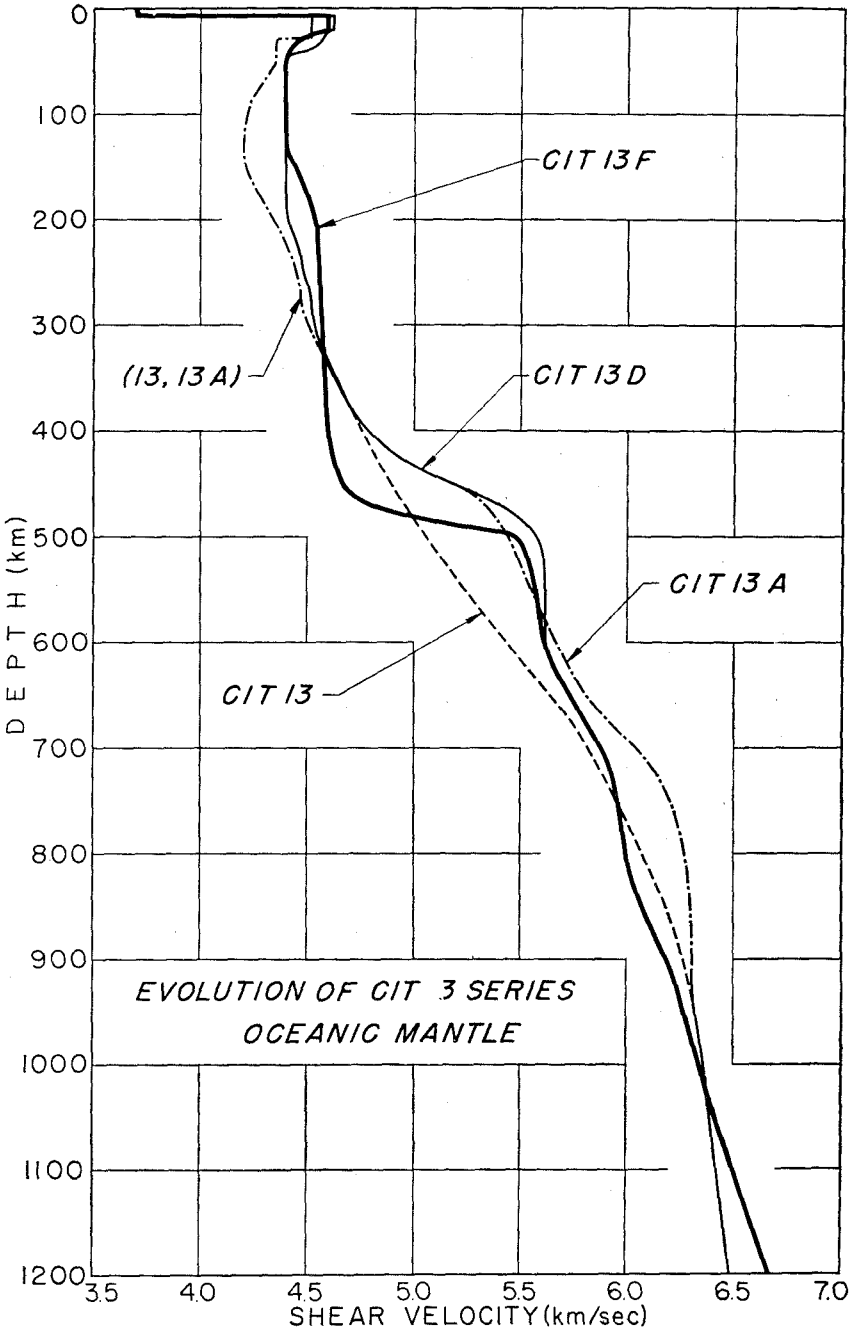
0
100
200
300
400
500
600
700
800
900
1000
1100
1200
DEPTH (km)
3.5
4.0
4.5
5.0
5.5
6.0
6.5
7.0
SHEAR VELOCITY (km/sec)
CIT 13F
(13, 13A)
CIT 13D
CIT 13A
CIT 13
EVOLUTION OF CIT 3 SERIES
OCEANIC MANTLE

Fig. 9

the partials can be shown to be related in a fashion similar to those for the layered approximation.

Now we can generate a set of perturbation equations corresponding to a set of observations at M distinct frequencies ω_s. This set of equations may obviously be expressed as a matrix equation of the form indicated for the layered approximation. The matrices A, b and x are defined as indicated and in this case we wish to obtain the required perturbations of the elastic parameters. Therefore, we must consider the inverse of the matrix A. Due to the relationship between the matrix elements of A, which is, in this case, that the sum of the layer potential energies is equal to the sum of the layer kinetic energies, we find that the matrix A is singular. Therefore, in order to achieve a solution for the $x's$, constraints are introduced so as to limit at least one of the layer parameters to have zero variation. In practice we have generally limited the rigidity in one or more crustal layers to be fixed. Under such constraints we denote the new conditioned A matrix by A_1, and it is easy to show that the inverse of A_1 exists. Therefore, the inversion may be accomplished in the least squares sense indicated or by simple relaxation or iteration methods. A similar situation holds for the method when the polynomial approximation is used.

Fig. 9 and 10 show various stages in the evolution of an earth structure and the resulting dispersion compared to longperiod Love wave data.

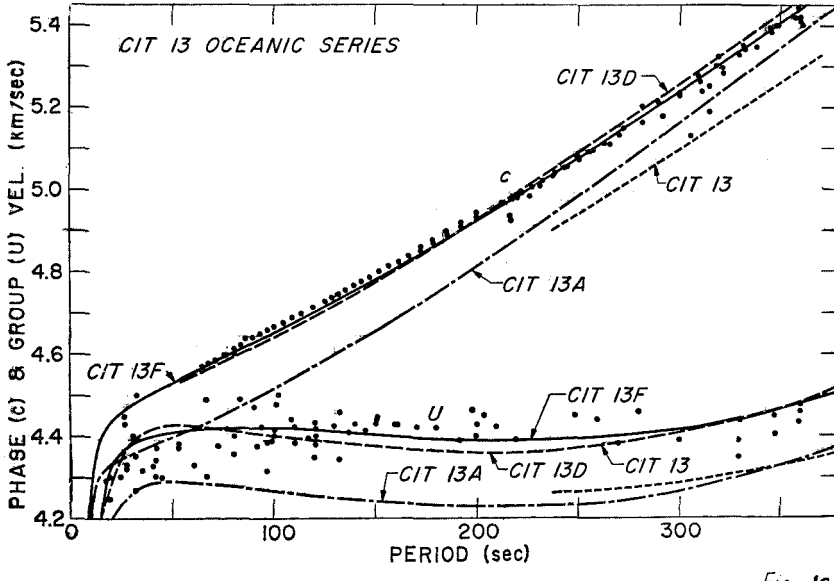

Fig 1c