Impulse Approximation

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The impulse approximation is discussed and simple applications are made to two problems of current interest, the determination of the ωNN coupling constant and the Regge pole predictions for high-energy scattering.

I. INTRODUCTION

THE impulse approximation plays a fundamental role in the S-matrix theory of strong interactions. It is involved whenever we deal with the scattering by composite systems. The usefulness of the impulse approximation in practice is well known, but its fundamental role was not clearly recognized until recently. Cutkosky[1] was among the first to realize its importance. He tried to justify the impulse approximation in terms of dispersion theory. One of the essential features in his approach is the construction of wave functions in terms of scattering amplitudes which somehow include the effects of anomalous thresholds.

The impulse approximation is usually derived in Born approximation from the principle of superposition of potentials,

$$V=\sum V_{ij}.$$

If one takes the S-matrix theory as more fundamental than potential theory, the "superposition of potentials" principle must, in fact, follow as a *consequence* of the impulse approximation, if at all. The impulse approximation is then to be regarded as a fundamental law of physics, and as such, its validity should be studied in detail by experiment, not left to the whim of the theorist. In the case of strong interactions, the Born approximation is not valid, but still the impulse approximation seems to hold.[2] On the other hand, the possibility exists that the assumption that the potentials superpose may no longer be valid. Attempts to "improve" the impulse approximation by using higher order terms of the Born series are then not of much help. The impulse approximation as usually given must, of course, be modified slightly to be consistent with unitarity. How this is to be achieved is not entirely clear, but this question can, in principle, be decided by experiment.

II. ELASTIC NUCLEON-NUCLEUS SCATTERING

By means of the impulse approximation and a simple model of the nucleus, we may express the scattering of a single nucleon on a large nucleus at high energies in terms of the fundamental nucleon-nucleon scattering matrix. Let us review briefly this well-known theory.[3]

Consider the elastic scattering of a proton with spin s_z and momentum $\mathbf{p}$ incident on a large nucleus of species Z^A with spin J_z, assumed to be in its ground state. The elastic scattering matrix is

$$S_{\rm elas}(E,\theta)=\langle Z^AJ_z';\,\mathbf{p}'s_z'\,\mathrm{out}\,|\,Z^AJ_z;\,\mathbf{p}s_z\,\mathrm{in}\rangle, \quad (2)$$

where

$$\cos\theta=\hat{p}'\cdot\hat{p}.$$

We write $S=1+iT$, where the transition matrix for proton-nucleus scattering is given in first Born approximation by

$$\langle\mathbf{p}'|T|\mathbf{p}\rangle=-(2m/4\pi)\langle\mathbf{p}'|V|\mathbf{p}\rangle. \quad (3)$$

Here the mass m is the reduced mass. If we set the nucleon mass equal to $\frac{1}{2}$, then $2m=A/(A+1)\approx 1$, for large A. It is not essential to refer to the Born approximation, as this equation may simply be taken as the definition of the potential V. The potential is introduced only as a means of visualizing the scattering matrix.

In describing the scattering matrix in the elastic channel by a Born-equivalent potential in this way, there is no need to use a many-body potential for V. One may remove any many-body complexity accidentally introduced by replacing V by $\mathcal{P}V\mathcal{P}$, where $\mathcal{P}$ is the projection operator for the elastic channel. This procedure yields what is commonly called the optical-model potential $V(\mathbf{J},\mathbf{S};\,\mathbf{p}',\mathbf{p})$. It is to be emphasized that the use of the optical-model potential is based directly on S-matrix theory, and in no way implies the existence of an ordinary potential. The optical potential may not be used in a Schrödinger equation, for example. In fact, as is well known, the optical model potential will even be non-Hermitian when competing channels are open.

III. IMPULSE APPROXIMATION

The impulse approximation states that the transition matrix T for proton-nucleus scattering is the superposition of the t matrices for the individual nucleon-nucleon encounters of the incident proton with the separate nucleons inside the nucleus.

The nucleon-nucleon transition matrix t is customarily given in momentum space. In the c.m. system it has the general form[4]

$$\langle\mathbf{p}'|t|\mathbf{p}\rangle=a+c(\sigma_{1N}+\sigma_{2N})+m\sigma_{1N}\sigma_{2N} +g(\sigma_{1P}\sigma_{2P}+\sigma_{1K}\sigma_{2K})+h(\sigma_{1P}\sigma_{2P}-\sigma_{1K}\sigma_{2K}), \quad (4)$$

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[1] R. E. Cutkosky, Phys. Rev. **125**, 745 (1962).
[2] H. A. Bethe, Ann. Phys. (New York) **3**, 190 (1958).
[3] W. B. Riesenfeld and K. M. Watson, Phys. Rev. **102**, 1157 (1956).
[4] M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, Ann. Rev. Nuclear Sci. **10**, 291 (1960).

where the coefficients are functions of energy and scattering angle. The vectors $\mathbf{p}$ and $\mathbf{p}'$ are the c.m. momenta of the incident and scattered proton. The particle "1" refers to the projectile proton, while "2" refers to any of the struck nucleons inside the target nucleus. Finally, σ_{1P} refers to the component of $\chi_{1'}^*\boldsymbol{\sigma}\chi_1$ in the direction of the unit vector $\hat{P}$. The unit vectors $\hat{K}$, $\hat{P}$, and $\hat{N}$ are parallel to the following:

$$\begin{aligned}\mathbf{P}&=\mathbf{p}'+\mathbf{p},\\ \mathbf{K}&=\mathbf{p}'-\mathbf{p},\\ \mathbf{N}&=4\omega\mathbf{p}\times\mathbf{p}'.\end{aligned} \tag{5}$$

In a spinless nucleus, as many nucleon spins are up as down. We may therefore average the t matrix over the spins of particle "2" or equivalently, perform a trace operation in the "2" spinor subspace. This spin average yields an effective nucleon-nucleon transition matrix,

$$\bar{t}=a+c\sigma_{1N}. \tag{6}$$

A significant cancellation has taken place. This effect enhances the polarization observed in proton-nucleus scattering as compared to the polarization observed in proton-proton and proton-neutron scattering. In a complex nucleus, the most important forces are, therefore, central and spin orbit, as required by the shell model. Our results have an important consequence for the meson-theory justification of the shell model of the nucleus, namely, that the shell-model potential is primarily due to ω-meson exchange.

We describe the distribution of nucleons within the nucleus by a dimensionless distribution function $\rho(\mathbf{x})$ which will be taken to have a Woods-Saxon shape,

$$\rho(r)=[1+\exp(r-r_0)/a]^{-1}. \tag{7}$$

The distribution function is approximately normalized to the nuclear volume,

$$\int \mathbf{d}^3x\,\rho(\mathbf{x})=\mathcal{V}=\frac{4\pi}{3}r_0^3. \tag{8}$$

The actual density of nucleons in the nucleus is thus $(A/\mathcal{V})\rho(r)$.

Since the density of nucleon is given in coordinate space, while the transition matrix is given in momentum space, the superposition postulated in the impulse approximation is somewhat ambiguous; the usual choice is

$$\langle\mathbf{p}'|T|\mathbf{p}\rangle=(A/\mathcal{V})\langle\mathbf{p}'|\rho|\mathbf{p}\rangle\langle\mathbf{p}'|\bar{t}|\mathbf{p}\rangle. \tag{9}$$

This may be justified by a classical optical analogy. We have set here

$$\langle\mathbf{p}'|\rho|\mathbf{p}\rangle=\int \mathbf{d}^3x\,\rho(\mathbf{x})e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}. \tag{10}$$

A certain number of physical assumptions have been made in this description. The energy must be sufficiently high so that the nucleons inside the nucleus can be treated as an unbound and stationary collection of protons and neutrons. The incident proton energy must therefore be large compared with the average kinetic (Fermi) energy of the nucleons within the nucleus. This sets a lower limit on the incident energy around 100 MeV. We have also neglected the effect of multiple scattering.

The nucleon-nucleon t matrix differs for p-p and n-p scattering. In general, we expect that the protons and neutrons in the nucleus may be described by the same distribution function for light nuclei. We may then simply write,

$$\bar{t}=(1/A)(Z\bar{t}_{pp}+N\bar{t}_{np}), \tag{11}$$

where N and Z are the number of neutrons and protons, respectively, in the given nucleus.

IV. THE OPTICAL-MODEL POTENTIAL

Let the momenta $\mathbf{p}$ and $\mathbf{p}'$ refer to the c.m. momenta of the individual nucleon-nucleon encounters, to be distinguished from the momenta $\mathbf{P}$ and $\mathbf{P}'$ of the incident and scattered nucleon in the actual nucleon-nucleus scattering process in its c.m. system. An approximate relation between these is

$$\hat{p}'\times\hat{p}=2[A/(A+1)]\hat{P}'\times\hat{P}. \tag{12}$$

More precisely, for small scattering angles, the relativistically correct relation is

$$|\mathbf{p}|\,\hat{p}'\times\hat{p}=|\mathbf{P}|\,\hat{P}'\times\hat{P}. \tag{13}$$

The spin of the incident proton is $\mathbf{S}=\frac{1}{2}\boldsymbol{\sigma}_1$. Thus,

$$\sigma_{1N}=\frac{2\mathbf{S}\cdot\hat{p}\times\hat{p}'}{|\hat{p}\times\hat{p}'|}. \tag{14}$$

The scattering at high energies is mainly forward or near-forward (diffraction scattering). For small angles we may write

$$\begin{aligned}a(p^2,\theta)&\approx M_0,\\ c(p^2,\theta)&\approx -4i(p^2/M_p^2)(\sin\theta)M_1.\end{aligned} \tag{15}$$

Here M_0 and M_1 are expected to be approximately constant. These quantities may be related to central and spin-orbit potentials, defined here

$$\begin{aligned}M_0&=-(1/4\pi)(\mathcal{V}/A)V_s,\\ M_1&=-(1/4\pi)(\mathcal{V}/A)V_s.\end{aligned} \tag{16}$$

The factors $\mathcal{V}/A$ are introduced for convenience in the final formula. We may write

$$\bar{t}=-(1/4\pi)(\mathcal{V}/A)\{V_c+iV_s\mathbf{P}'\times\mathbf{P}\cdot\mathbf{S}/M_p^2\}. \tag{17}$$

Using the impulse approximation, we obtain

$$V_{\text{opt}}=\{V_c+iV_s\mathbf{P}'\times\mathbf{P}\cdot\mathbf{S}/M_p^2\}\langle\mathbf{P}'|\rho|\mathbf{P}\rangle, \tag{18}$$

or, going over to coordinate space,

$$V(r)=V_c\rho(r)+(\hbar/M_pc)^2V_s(1/r)(d\rho/dr)\mathbf{L}\cdot\mathbf{S}. \quad (19)$$

To obtain the specifically nuclear part of the optical-model potential, the Coulomb effects must be subtracted out in the p-p matrix beforehand.

V. EXPERIMENTAL DATA

Hafner[5] has made a thorough experimental study of the scattering of 220-MeV polarized protons by complex nuclei, and analyzed these data in terms of an optical potential. Aside from this work, surprisingly little has been published on *high-energy* optical-model analyses. Only the results for p-C^{12} scattering are fully analyzed in Hafner's paper.

Hafner used a Woods-Saxon shape for $\rho(r)$, and writes

$$V_c=-V_1(0)-iV_2(0),\qquad V_s=V_3(0)+iV_4(0). \quad (20)$$

The best-fit analysis was made under the assumption that $V_4(0)=0$. The results for p-C^{12} scattering are

$$r_0=2.4\text{ F},\quad V_1(0)=10\text{ MeV}$$
$$a=0.1\text{ F},\quad V_2(0)=25\text{ MeV}$$
$$\mathcal{V}=58\text{ F}^3,\quad V_3(0)=225\text{ MeV}.$$

Note the large size of $V_3(0)$. The good fit of the experimental data with these parameters in part justifies the use of this type of phenomenological analysis.

VI. ESTIMATE OF THE ωNN COUPLING CONSTANT

We may use the above data to give an estimate of the ωNN coupling constant. One reason for believing that this method of determination to be at least as accurate as polological methods is that $V_3(0)$ is large. We are looking at the ω-exchange contribution where it appears to be dominant.

Pion exchange does not lead to any contribution to the spin-orbit force. Both the ω-meson and the ρ-meson exchange mechanisms lead to spin-orbit forces. In a pole approximation,

$$c\approx\frac{4ip^2\sin\theta}{8}\left\{\frac{3g_{\rho NN}^2}{2M_p^2}\frac{\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2}{m_\rho^2}+\frac{3g_{\omega NN}^2}{2M_p^2}\frac{1}{m_\omega^2}\right\}. \quad (21)$$

For the benefit of those who use a different notation, we give here the Born-equivalent Lagrangian term

$$\mathcal{L}=ig_{\omega NN}\bar{\psi}\gamma_\mu\psi\omega_\mu. \quad (22)$$

The average value of $\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2$ for a large nucleus is zero, since

$$(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)_{pp}=+1;\quad(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)_{pn}=-1. \quad (23)$$

Thus, when $Z=N$, the ρ-meson exchange contribution just cancels out in impulse approximation, leaving only the ω-meson exchange contribution. This cancellation phenomenon has important consequences for the shell model of the nucleus. It operates for central forces as well as the spin-orbit force, and in particular, the one-pion exchange force, which is proportional to $\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2$ averages to zero. This leaves the ω-meson exchange force to play the dominant role in the nuclear potential, both for the central and spin-orbit forces, ignoring correlation effects. This has also been emphasized by Sakurai[6] in many papers.

Using the formula,

$$c=4i(p^2/M_p^2)\sin\theta(\mathcal{V}/A)(1/4\pi)V_s, \quad (24)$$

we may solve for the coupling constant,

$$(g_{\omega NN}^2/4\pi\hbar c)=(m_\omega^2/3\pi^2)(\mathcal{V}/A)V_s. \quad (25)$$

We may insert the numerical data:

$$V_s=225\text{ MeV},\quad \mathcal{V}=58\text{ F}^3,$$
$$m_\omega=787\text{ MeV},\quad A=12,$$
$$\hbar c=197.33\text{ MeV F}.$$

This gives the estimate

$$(g_{\omega NN}^2/4\pi\hbar c)\approx2.9. \quad (26)$$

We must correct for the eclipse effect. Since the mean free path λ for the proton inside the nucleus is only 2 F (based on total neutron-C^{12} cross section), the target nucleons on the far side of the nucleus will be shielded and contribute less than those on the near side. This effectively decreases A by a factor of 4 or 5, and correspondingly increases the coupling constant.

We also remark here that we have omitted the contribution of the η meson as we believe its spin is zero.

VII. KINEMATICS

For small-angle scattering, we may consider the nucleus to be at rest both before and after scattering in the lab system, and that the projectile nucleon does not change its energy in the scattering.

In the lab frame, the momentum p_L of the incident nucleon is the same for the nucleon-nucleus scattering problem as for the individual nucleon-nucleon encounters. Neglecting binding effects and multiple scattering, as we may for high-energy grazing collisions, the elastic scattering angle θ_L is the same for both nucleon-nucleon and nucleon-nucleus problems in the lab system. Hence, in this limit, the momentum transfers in the two problems, t and T, respectively (not to be confused with the transition matrices) are approximately equal,

$$T\approx t=-(p_i-p_i')^2\approx-(p_L\theta_L)^2. \quad (27)$$

[5] E. M. Hafner, Phys. Rev. **111**, 297 (1958).

[6] J. J. Sakurai, Phys. Rev. **119**, 1784 (1960); G. Breit, *ibid.* **120**, 287 (1960). The nomenclature we use for mesons in the present paper differs from that used by Sakurai. The ρ meson is the 750-MeV, $T=J=1$ two-pion resonance, while the ω meson is the 787-MeV, $T=0$, $J=1$ three-pion resonance.

The general relation, for large angles, is much more complicated.

The invariant energy variables, s and S, respectively, are given in the two scattering problems by

$$s=-(p_i+p_t)^2=m_i^2+m_t^2+2m_t\omega_i, \qquad (28)$$

where "i" means "incident" and "t" means "target." The right side of this equation is written in the lab system. The only difference between s and S lies in the target masses to be used. We thus obtain

$$S=As+M_p^2(1+A^2), \qquad (29)$$

and at extremely high energies, we have simply $S=As$. Equivalently, we may say that the combination s/m_tm_i is the same for both problems,

$$S/m_tm_i \approx s/m_tm_i. \qquad (30)$$

Here m_t has a different meaning on the two sides of the equation.

VIII. REGGE POLES AND THE IMPULSE THEORY

The high-energy behavior of all scattering processes is currently thought to be dominated by the Pomeranchuk Regge trajectory in the crossed channel. Hence, the nucleon-nucleon scattering amplitude is given by[7]

$$M_{\text{nucleon-nucleon}}(s,t) \underset{s\to\infty}{\longrightarrow} b(t)(s/m_tm_i)^{\alpha(t)}. \qquad (31)$$

For small momentum transfers we may set

$$\langle \mathbf{p}'|\rho|\mathbf{p}\rangle \approx \mathcal{V}. \qquad (32)$$

Hence, the impulse approximation gives,

$$M_{\text{nucleon-nucleus}}=AM_{\text{nucleon-nucleon}}. \qquad (33)$$

[7] S. C. Frautschi, M. Gell-Mann, and F. Zachariasen, Phys. Rev. **126**, 2204 (1962).

We therefore obtain the following prediction for nucleon-nucleus scattering at high energies,

$$M_{\text{nucleon-nucleus}}(S,T) \underset{S\to\infty}{\longrightarrow} Ab(T)(S/m_tm_i)^{\alpha(T)}, \qquad (34)$$

Again, we neglect the eclipse effect for convenience.

We may state this result in the form of a theorem: "If a particle A scatters on a composite system consisting of n B particles, then the scattering amplitude for this process is equal to n times the amplitude for A-B scattering in the high-energy diffraction limit, provided we use the invariant variables s/m_tm_i and t." This theorem was first empirically discovered in applications of Regge analysis to high-energy scattering data.

One might speculate that the nucleon, at least for grazing collisions, may be considered as a composite system of a core and a pion cloud. For glancing collisions, the core is shielded, and the scattering of nucleons by anything reduces to the scattering of that thing by the pions in the cloud. In that case, pion-pion, pion-nucleon, and nucleon-nucleon scattering will all be governed by the same Regge pole, including the same $b(t)$, except for an over-all factor related to the number of pions effectively present in the cloud. This number is about two.

Finally we may remark that it is quite likely that the impulse approximation becomes an exact theorem in the high-energy limit. This may be an essential input in the S-matrix theory of many-particle systems.

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