with

$$P_{\gamma\alpha}=(1-\delta_{\gamma\alpha})+G_0W_{\gamma\alpha}, \quad \text{(A4)}$$

$$Q_{\beta\gamma}=(1-\delta_{\beta\gamma})+W_{\beta\gamma}G_0, \quad \text{(A5)}$$

$$Q_{\beta\gamma}{}^{(1)}=(1-\delta_{\beta\gamma})+W_{\beta\gamma}{}^{(1)}G_0. \quad \text{(A6)}$$

On using $T_\gamma=T_\gamma{}^{(1)}+T_\gamma{}^{(2)}$ [Eq. (2.22)], Eqs. (A1) and (A3) give

$$W_{\beta\alpha}-W_{\beta\alpha}{}^{(1)}=\sum_{\gamma\neq\alpha} Q_{\beta\gamma}{}^{(1)}T_\gamma{}^{(2)} +\sum_{\gamma\neq\alpha}(Q_{\beta\gamma}-Q_{\beta\gamma}{}^{(1)})T_\gamma. \quad \text{(A7)}$$

With the use of Eqs. (A5) and (A6), and writing

$$A_{\beta\alpha}=W_{\beta\alpha}-W_{\beta\alpha}{}^{(1)}, \quad \text{(A8)}$$

we obtain

$$A_{\beta\alpha}=\sum_{\gamma\neq\alpha} Q_{\beta\gamma}{}^{(1)}T_\gamma{}^{(2)}+\sum_{\gamma\neq\alpha} A_{\beta\gamma}G_0T_\gamma. \quad \text{(A9)}$$

Now, with the successive use of Eqs. (A4), (A2), (A9), and (A2) we obtain

$$\begin{aligned}\sum_{\gamma\neq\alpha} A_{\beta\gamma}G_0T_\gamma &=\sum_\gamma A_{\beta\gamma}G_0T_\gamma(P_{\gamma\alpha}-G_0W_{\gamma\alpha})\\ &=\sum_\gamma A_{\beta\gamma}G_0T_\gamma(P_{\gamma\alpha}-G_0\sum_{\gamma\neq\delta} T_\delta P_{\delta\alpha})\\ &=\sum_\delta (A_{\beta\delta}-\sum_{\gamma\neq\delta} A_{\beta\gamma}G_0T_\gamma)G_0T_\delta P_{\delta\alpha}\\ &=\sum_{\gamma\neq\delta}\sum Q_{\beta\gamma}{}^{(1)}T_\gamma{}^{(2)}G_0T_\delta P_{\delta\alpha}\\ &=\sum_\gamma Q_{\beta\gamma}{}^{(1)}T_\gamma{}^{(2)}G_0W_{\gamma\alpha}. \end{aligned} \quad \text{(A10)}$$

The result of substituting (A10) into (A9),

$$A_{\beta\alpha}=\sum_\gamma Q_{\beta\gamma}{}^{(1)}T_\gamma{}^{(2)}P_{\gamma\alpha},$$

is equivalent to Eq. (2.25).

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Low-Energy Theorems, Dispersion Relations, and Superconvergence Sum Rules for Compton Scattering*

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A derivation of low-energy theorems for Compton scattering from spin-0 and spin-$\frac{1}{2}$ targets is given within the framework of dispersion theory. We work exclusively with physical helicity amplitudes and utilize the zeros of these amplitudes forced by angular momentum conservation to write unsubtracted dispersion relations. The conventional requirement of gauge invariance is replaced in our work by Lorentz invariance together with the knowledge that the photon is a massless spin-1 particle. From the dispersion relations we extract a number of sum rules of the superconvergence type, one example of which reduces the Drell-Hearn result in the forward direction.

I. INTRODUCTION

THE amplitude for the scattering of low-energy photons by spin-$\frac{1}{2}$ systems has been given by Low[1] and by Gell-Mann and Goldberger.[2] Using the full machinery of quantum field theory and, in particular, the gauge invariance of photon emission and absorption matrix elements, the following theorem was proved: The Compton amplitude, regarded as a function of the photon energy, at fixed scattering angle (and given target-photon polarizations), can be exactly specified in terms of the static properties of the target (i.e., charge, mass, and magnetic moment) provided only terms of zero and first order in photon energy are retained. The feature which distinguishes this result from a number of low-energy theorems recently derived from current algebra[3] is that it yields the amplitude in the physical

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[1] F. E. Low, Phys. Rev. **96**, 1428 (1954).

[2] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **96**, 1433 (1954).

[3] A thorough exposition of the methods involved in obtaining such theorems and a critical analysis of the results will be found in the forthcoming book by S. L. Adler and R. F. Dashen, *Current Algebra* (W. A. Benjamin, Inc., New York, 1967).

region and involves no extrapolations from unphysical points.

There are a number of reasons for our taking up this almost classic result at such a late date. One is that it should be possible to derive the theorem, since it is true, using only physical, on-mass-shell quantities at every stage. Among other things, this means we deal only with amplitudes for the emission and absorption of photons with physical helicity.[4] Another reason for our interest is that we shall be able to formulate unsubtracted dispersion relations for Compton scattering, which to our knowledge has not been done. Finally, utilizing these dispersion relations we are able to derive a number of superconvergence relations.

Customarily in dispersion theory one works with the scalar invariant coefficients of a set of basic tensors in spin-polarization space. Instead, we work directly with helicity amplitudes[5] which are physical S-matrix elements. We exploit the property of such amplitudes which states that at scattering angle Θ equal to zero (or π) the net initial state helicity, λ, and the net final state helicity, μ, must be equal (or opposite). In general, near $\Theta=0$, helicity amplitudes vanish like $[\sin\frac{1}{2}\Theta]^{|\lambda-\mu|}$ (or faster) and near $\Theta=\pi$, like $[\cos\frac{1}{2}\Theta]^{|\lambda+\mu|}$ (or faster). These results follow from angular momentum conservation and the assumption of no long-range forces. The vanishing of the helicity amplitudes at specified points enables us to write subtracted dispersion relations with subtraction constants known to be zero. Alternatively, we may deal with the amplitudes divided by suitable powers of $\sin\frac{1}{2}\Theta$ and $\cos\frac{1}{2}\Theta$ which may, hopefully, satisfy unsubtracted dispersion relations. This same division removes most of the kinematical singularities of the helicity amplitudes.[6]

We shall see that the fixed-t and fixed-s dispersion relations for the appropriately modified helicity amplitudes allow an easy derivation of the low-energy theorem of Refs. 1 and 2. In particular, the one-particle-state contribution to these dispersion relations yield exactly the results of lowest-order perturbation theory which when expanded in powers of photon frequency is well known[2] to be equivalent to the low-energy theorem. The continuum contributions are at least second order in the frequency. Although to our knowledge the unsubtracted dispersion relations for spin zero targets have not been heretofore given, they are essentially trivial to deduce. This is not so for spin one-half systems. The work of Holliday[7] and Hearn and Leader[8] has shown that the standard dispersion relations for invariants cannot simultaneously reproduce the low-energy theorem and be free of subtractions.

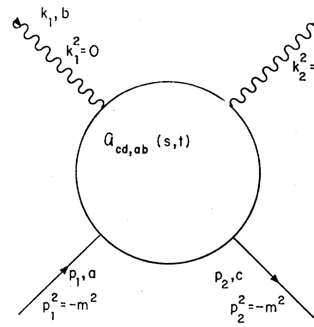

FIG. 1. The kinematics of Compton scattering.

Our unsubtracted dispersion relations give us a set of sum rules which in turn under some assumptions about high-energy behavior lead to superconvergence relations.[9] For example, the Drell-Hearn[10] sum rule and a direct generalization of it for nonforward scattering emerge as superconvergence relations. Similarly, we obtain a relation between the lifetime of the neutral pion and static parameters of the nucleon.

There are two interesting features about our derivation of the low-energy theorem. First, we make definite assumptions about high-energy behavior of amplitudes which are not connected in any obvious way to the tacit assumptions made in the conventional derivations. Second, we never use gauge invariance explicitly. This is because of our direct use of helicity amplitudes in which the two allowed photon helicities (± 1) are a consequence of masslessness and the Lorentz invariance of the theory, and the low-energy theorem follows from these properties of helicity amplitudes.[11]

In the next Section (II) we discuss some kinematical preliminaries which will be used extensively. In Sec. III, the low-energy theorem for Compton scattering from spinless targets is given as an example of our methods and to sharpen our tools for the more interesting but algebraically more involved case of spin-$\frac{1}{2}$ targets taken up in Sec. IV. Section V is devoted to sum rules and

[4] In conventional quantum field theory, one writes for the process: matter state a+photon $\rightarrow$ matter state b, the amplitude $\langle b|J_\alpha|a\rangle\epsilon_\alpha(k,\lambda)$, where k is the photon four-momentum, $\epsilon_\alpha(k,\lambda)$ is the polarization *four*-vector, and J_α is the current density operator of the matter system. Since photons have spin one, we may require $\epsilon_\alpha(k,\lambda)k_\alpha=0$ and further, from gauge invariance, that the amplitude be unchanged when $\epsilon_\alpha\rightarrow\epsilon_\alpha+\Lambda k_\alpha$ or $k_\alpha\langle b|J_\alpha|a\rangle=0$. The latter condition plays a key role in Low's (Ref. 1) derivation where he is able to concentrate on the evaluation of $\langle b|\rho|a\rangle$, $\rho=-iJ_4$. Such components of the current density, of course, do not really enter in physical matrix elements and our method will avoid ever talking about them. It is, of course, well known that the conditions $\epsilon_\alpha k_\alpha=0$ and invariance under $\epsilon_\alpha\rightarrow\epsilon_\alpha+\Lambda k_\alpha$ implies that there are only two helicity states.

From an S-matrix point of view D. Zwanziger, Phys. Rev. **133**, B1036 (1964), and S. Weinberg, *ibid.* **135**, B1049 (1964), have established that $k_\alpha\langle b|J_\alpha|a\rangle=0$ must be satisfied to guarantee the Lorentz invariance of the theory. The basic reason for this is that although $\epsilon_\alpha(k,\lambda)$ *looks* like a four-vector, when it describes a quantized massless photon field, it does not transform like a four-vector but acquires a component along its momentum k_α.

[5] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).

[6] M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. **133**, B145 (1964), Sec. 2. Also L. L. Wang, *ibid.* **142**, 1187 (1966).

[7] D. Holliday, Ann. Phys. (N. Y.) **24**, 289 (1964).

[8] A. C. Hearn and E. Leader, Phys. Rev. **126**, 789 (1962).

[9] For a survey of such relations, see F. E. Low, *Proceedings of the 13th International Conference on High-Energy Physics* (University of California Press, Berkeley, 1967).

[10] S. D. Drell and A. C. Hearn, Phys. Rev. Letters **16**, 908 (1966).

[11] A discussion with S. Adler helped clarify this point.

superconvergence relations. A summary of results, conclusions, and speculations is given in Sec. VI.

II. KINEMATICAL PRELIMINARIES

We consider the elastic scattering of a photon with (four-) momentum k_1, helicity b by a particle with momentum p_1, spin J, mass m, and helicity a leading to a particle of momentum p_2 and helicity c and a photon of momentum k_2, helicity d. See Fig. 1. The process will be described by a Lorentz-invariant helicity amplitude,[5] $\mathfrak{A}_{cd;ab}$, related to the S matrix as follows:

$$S(a+b\rightarrow c+d)=\delta_{cd;ab}+i(2\pi)^4\delta(p_2+k_2-p_1-k_1)N_aN_c\frac{\mathfrak{A}_{cd;ab}(s,t,u)}{(4k_{10}k_{20})^{1/2}}.$$

The normalization factors N_a, N_c are $(2p_{10})^{-1/2}$, $(2p_{20})^{-1/2}$ for boson targets or $(m/p_{10})^{1/2}$, $(m/p_{20})^{1/2}$ for fermions. The variables s, t, u are the usual ones: $s=-(p_1+k_1)^2$, $t=-(p_1-p_2)^2$, $u=-(p_1-k_2)^2$, and $s+t+u=2m^2$. We shall occasionally use the variable ν defined by $\nu=-\frac{1}{4}(p_1+p_2)\cdot(k_1+k_2)=\frac{1}{2}(s-m^2)+\frac{1}{4}t=\frac{1}{4}(s-u)$.

In the barycentric system of the scattering channel s is the square of the total energy, and $-t=2p^2(1-\cos\Theta_s)$ is the square of the momentum transfer with p the magnitude of the photon (or particle) three-momentum and Θ_s the scattering angle between the initial and final photon (or particle) directions. The particle energies $p_{10}=p_{20}=E$, the momentum p are given in terms of s by

$$p=(s-m^2)/2s^{1/2},\quad E=(s+m^2)/2s^{1/2};$$

some other useful kinematical relations are the following:

$$\cos\tfrac{1}{2}\Theta_s=[(4p^2+t)/4p^2]^{1/2}=[(s-m^2)^2+st]^{1/2}/(s-m^2)=(m^4-su)^{1/2}/(s-m^2)$$

$$\sin\tfrac{1}{2}\Theta_s=[-t/4p^2]^{1/2}=(-st)^{1/2}/(s-m^2).$$

We note in passing that for fixed $\Theta_s(\neq 0)$, t vanishes as the square of the photon momentum when the latter goes to zero.

The s-channel helicity amplitudes are denoted by $\mathfrak{A}_{cd;ab}(s,t)$ and we conventionally take the massive particle, a, to be incident in the positive z direction and imagine the scattering takes place in the x-z plane; the final momentum of the particle, c, is in the direction $(\sin\Theta_s,\,0,\,\cos\Theta_s)$. The projection of the total angular momentum of the initial (final) state is $\lambda\equiv a-b$ $(\mu=c-d)$ along the respective directions of motion [i.e., the z axis for the initial state and $(\sin\Theta_s, 0, \cos\Theta_s)$ for the final state]. It follows from angular momentum conservation that the helicity amplitudes vanish for $\Theta_s=0(\pi)$ unless $\lambda=\mu(\lambda=-\mu)$. Formally, this property of the helicity amplitudes emerges from the standard expansion.

$$\mathfrak{A}_{cd;ab}(s,t)=\sum_J(2J+1)\mathfrak{A}_{cd;ab}{}^J(s)d_{\lambda\mu}{}^J(\Theta_s)$$

when we use the fact that the rotation matrices $d_{\lambda\mu}{}^J(\Theta_s)$ may be written as $(\sin\frac{1}{2}\Theta_s)^{|\lambda-\mu|}(\cos\frac{1}{2}\Theta_s)^{|\lambda+\mu|}$ times a (Jacobi) polynomial in $\cos\Theta_s$. We define new helicity amplitudes $\bar{\mathfrak{A}}_{c\,d;ab}(s,t)$ by dividing the original ones by these factors of $\sin\frac{1}{2}\Theta_s$ and $\cos\frac{1}{2}\Theta_s$:

$$\bar{\mathfrak{A}}_{cd;ab}(s,t)=\mathfrak{A}_{cd;ab}/(\sin\tfrac{1}{2}\Theta_s)^{|\lambda-\mu|}(\cos\tfrac{1}{2}\Theta_s)^{|\lambda+\mu|}.$$

These new amplitudes are free of kinematical singularities in t (and u)[6] and have better large t behavior than the original set since $\sin\frac{1}{2}\Theta_s$ and $\cos\frac{1}{2}\Theta_s$ are, for fixed s, proportional to $t^{1/2}$ for large t.

We shall also have occasion to refer to the crossed (t) channel where the square of the center-of-mass energy is t. We take this reaction to be: $\gamma(d')+\gamma(b')\rightarrow$ particle $(c')+$(anti-) particle (a'). The photons have three-momenta of magnitude $k_t=\frac{1}{2}t^{1/2}$ and the particles $p_t=(\frac{1}{4}t-m^2)^{1/2}$. The scattering angle, Θ_t, is measured between the photon of helicity d' and the particle of helicity c'. Some relevant kinematic relations are

$$\cos\Theta_t=\nu/p_tk_t=(s-m^2+\tfrac{1}{2}t)/2p_tk_t,$$

$$p_t^2k_t^2\sin^2\Theta_t=p_t^2k_t^2-\nu^2=-\tfrac{1}{4}[(s-m^2)^2+st]=-\tfrac{1}{4}(s-m^2)^2\cos^2(\tfrac{1}{2}\Theta_s).$$

Just as in the s channel, we remove from the t-channel helicity amplitudes, $A_{c'a';d'b'}(s,t)$, kinematical singularities [this time in the s (and u) variables] by defining new amplitudes $\bar{A}_{c'a';d'b'}(s,t)$:

$$\bar{A}_{c'a';d'b'}(s,t)=A_{c'a';d'b'}/(\sin\tfrac{1}{2}\Theta_t)^{|\lambda'-\mu'|}(\cos\tfrac{1}{2}\Theta_t)^{|\lambda'+\mu'|},$$

where $\lambda'=d'-b'$, $\mu'=c'-a'$. Our notation and phase conventions are those given in Ref. 6.

III. SPIN-ZERO TARGETS

We consider first Compton scattering from spin-zero targets which we refer to as pions. There are two independent transitions in this process (corresponding to electric and magnetic multipoles which, because the target is spinless, cannot mix) which we represent by the helicity amplitudes $\mathfrak{A}_{1;1}(s,t)$ and $\mathfrak{A}_{1;1}(s,t)$, suppressing the (zero) pion helicities.

In lowest-order perturbation theory (Fig. 2), these amplitudes are (e is the pion charge and $e^2/4\pi=1/137$):

$$\mathfrak{A}_{1;1}(s,t)=+2e^2[(s-m^2)^2+st/(m^2-s)(m^2-u)]=-2e^2[\tfrac{1}{2}(1+\cos\Theta_s)/1-p(1-\cos\Theta_s)s^{1/2}],$$

$$\mathfrak{A}_{1;1}(s,t)=-2e^2[m^2t/(m^2-s)(m^2-u)]=-e^2[1-\cos\theta_s/1-p(1-\cos\Theta_s)/s^{1/2}].$$

The second forms for $\mathfrak{A}_{1;\pm1}$ are obtained using the previously given kinematical relations together with

$$m^2-u=s-m^2+t=(s-m^2)[1-p(1-\cos\Theta_s)/s^{1/2}].$$

We note that in the forward direction $\mathfrak{A}_{1,-1}$ vanishes while $\mathfrak{A}_{1;1}$ yields the Thomson limit $-2e^2$.[12] The low-energy theorem tells us that this is correct to order p^2 as $p\to 0$. Actually, the theorem tells us even more: For fixed $\Theta_s(\neq 0)$, both the zero- and first-order terms in p are given exactly by these Born approximation expressions. It is our purpose to show how this result may be derived from dispersion theory.

If one attempts to approach this problem by writing unsubtracted, fixed-t dispersion relations for $\mathfrak{A}_{1;\pm 1}(s,t)$, he quickly encounters a contradiction. In particular, from such an expression for $\mathfrak{A}_{1;1}(s,0)$ one finds that an integral over the total photon-pion cross section is negative.[13] Thus, a dispersion relation for $\mathfrak{A}_{1;1}(s,t)$ may be expected to require a subtraction. The fact that $\mathfrak{A}_{1;1}$ must vanish at $\Theta_s=\pi$, since it contains a factor of $\cos^2(\frac{1}{2}\Theta_s)$, enables us to make such a subtraction. In fact, using the previously noted fact that $\cos^2(\frac{1}{2}\Theta_s)\sim\sin^2\Theta_t$ is zero at $\nu=\pm p_t k_t$, we may make two subtractions at points where $\mathfrak{A}_{1;1}$ is zero without introducing extraneous quantities.

The precise manner in which we capitalize on this kinematical fact is as follows: We note from the crossing relations[14,15] that to within a phase (± 1) (restoring the pion helicities momentarily)

$$\mathfrak{A}_{01;01}\propto A_{00;1-1},$$

which we write as

$$(s-m^2)^2\cos^2(\tfrac{1}{2}\Theta_s)\left(\frac{\mathfrak{A}_{01;01}}{(s-m^2)^2\cos^2(\frac{1}{2}\Theta_s)}\right)\propto p_t^2k_t^2\sin^2\Theta_t\left(\frac{A_{00;1-1}}{p_t^2k_t^2\sin^2\Theta_t}\right).$$

The quantity in parentheses on the left-hand side (right-hand side) is free of t-(s-) kinematical singularities, and since $p_t^2k_t^2\sin^2\Theta_t=-\frac{1}{4}(s-m^2)^2\cos^2(\frac{1}{2}\Theta_s)$, we may conclude that

$$\mathfrak{A}_{01;01}/(s-m^2)^2\cos^2(\tfrac{1}{2}\Theta_s)=\mathfrak{A}_{01;01}/(s-m^2)^2+st\equiv\mathcal{A}_{1;1}(s,t)$$

is free of both s *and* t singularities. The amplitude $\mathcal{A}_{1;1}$ is a likely candidate for a fixed-t unsubtracted dispersion relation by virtue of the factor s^{-2} we have been able to introduce. Indeed, if one may appeal to Regge-pole theory to predict high-energy Compton scattering,[16,14] we may expect $\mathcal{A}_{1;1}\sim s^{\alpha(t)-2}$ for large s where for $t\leq 0$, $\alpha(t)\leq 1$, which is more than adequate for an unsubtracted dispersion relation.

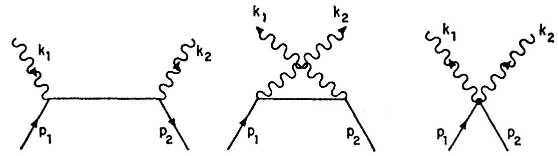

FIG. 2. The lowest-order perturbation-theory contributions to Compton scattering on a spinless target.

We assume, then, the legitimacy of writing $\mathcal{A}_{1;1}(s,t)$ as

$$\mathcal{A}_{1;1}(s,t)=\frac{1}{\pi}\int_{m^2}^{\infty}\frac{ds'}{s'-s}\mathcal{A}_{1,1}{}^{s}(s',t)+\frac{1}{\pi}\int_{m^2}^{\infty}ds'\frac{\mathcal{A}_{1,1}{}^{u}(u',t)}{u'-u}.$$

The quantities $\mathcal{A}_{1,1}{}^{s,u}$ are the absorptive parts of $\mathcal{A}_{1,1}$ in the s and u channels, respectively. Since $\mathcal{A}_{1,1}$ is even under interchange of s and u at fixed t (this follows from the relation between $\mathcal{A}_{1,1}$ and the t-channel amplitude $A_{00;1-1}$ which contains only even angular momenta in its partial-wave expansion; it therefore does not change when $\cos\Theta_t\rightleftarrows-\cos\Theta_t$ or $s\rightleftarrows u$), we may write

$$\mathcal{A}_{1;1}(s,t)=\frac{1}{\pi}\int_{m^2}^{\infty}ds'\mathcal{A}_{1;1}{}^{u}(s',t)\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right].$$

The absorptive part of $\mathcal{A}$ from the one-pion intermediate state can be computed from the generalized unitarity relation or more simply from the Born amplitude previously given. We find for the one-particle state,

$$\mathcal{A}_{1;1}{}^{u}(s',t)\,|_{\text{Born}}=\frac{2\pi e^2}{t}\delta(m^2-s'),$$

so that our dispersion relation becomes [using $(m^2-s)^{-1}+(m^2-u)^{-1}=t(m^2-s)^{-1}(m^2-u)^{-1}$]

$$\mathcal{A}_{1;1}(s,t)=\frac{2e^2}{(m^2-s)(m^2-u)}+\frac{1}{\pi}\int_{s_0}^{\infty}ds'\mathcal{A}_{1;1}{}^{u}(s',t)\times\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right],$$

where s_0 is the inelastic threshold; since we work to lowest order in e^2, $s_0=4m^2$. Finally, using the relation $\mathfrak{A}_{1;1}=[(s-m^2)^2+st]\mathcal{A}_{1;1}$, we have

$$\mathfrak{A}_{1;1}(s,t)=\frac{2e^2[(s-m^2)^2+st]}{(m^2-s)(m^2-u)}+\frac{[(s-m^2)^2+st]}{\pi}\times\int_{s_0}^{\infty}ds'\frac{\mathfrak{A}_{1;1}{}^{u}(s',t)}{(s'-m^2)^2+s't}\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right].$$

This is precisely the desired result. The first term is just the Born approximation given earlier while the second term, for fixed Θ_s, is proportional to p^2 since $(s-m^2)^2+st=\frac{1}{2}(2ps^{1/2})^2(1+\cos\Theta_s)$. The exact terms of order p^0 and p^1 are given by the Born approximation.

[12] The scattering amplitude, f, defined so $d\sigma/d\Omega=|f|^2$ is $f_{cd,ab}(s,t)=(1/8\pi\sqrt{s})\times\mathfrak{A}_{cd,ab}(s,t)$ and becomes $-\alpha/m$ as $p^2\to 0$ for $\Theta_s=0$.

[13] M. Gell-Mann, M. L. Goldberger, and W. E. Thirring, Phys. Rev. **95**, 1612 (1954).

[14] H. D. I. Abarbanel and S. Nussinov, Phys. Rev. **158**, 1462 (1967).

[15] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) **26**, 322 (1964).

[16] V. D. Mur, Zh. Eksperim. i Teor. Fiz. 44, 2173 (1963); **45**, 1051 (1964) [English transls.: Soviet Phys.—JETP **17**, 1458 (1963); **18**, 727 (1964)].

This is half of the story, since we must still obtain $\mathfrak{A}_{1;-1}(s,t)$. To do this, we remark that $\mathfrak{A}_{1;-1}$ is proportional (to within a phase) to the t-channel amplitude A_{11}.[14,15] Since the latter is free of s-kinematical singularities ($\lambda'=\mu'=0$, so there are no factors of $\sin\Theta_t/2$ or $\cos\Theta_t/2$ to be removed) we may divide by t and introduce none. Thus $\mathfrak{A}_{1;-1}/t=\mathfrak{A}_{1;-1}/-2p^2(1-\cos\Theta_s)$ is free of *both* s- and t- kinematical singularities; further $\mathfrak{A}_{1;1}/t$ clearly is sufficiently well behaved for large t that we may write a fixed s unsubtracted dispersion relation for it. We return to this point below.

We write a fixed-s dispersion relation for $\mathcal{A}_{1;-1}=\mathfrak{A}_{1;-1}/t$:

$$\mathcal{A}_{1;-1}(s,t)=\frac{1}{\pi}\int_{t_0}^{\infty}\frac{dt'}{t'-t}\mathcal{A}_{1;-1}{}^{t}(s,t')+\frac{1}{\pi}\int_{m^2}^{\infty}\frac{du'}{u'-u}\mathcal{A}_{1,-1}{}^{u}(s,u'),$$

where $t_0=4m^2$. There is a one-particle contribution in the u channel which one may compute from unitarity or from the Born approximation, namely,

$$\mathcal{A}_{1;-1}{}^{u}(s,u)=\frac{-2\pi e^2m^2}{m^2-s}\delta(m^2-u').$$

Using this in our dispersion relation, we find ($u_0=4m^2$)

$$\mathcal{A}_{1;-1}(s,t)=-\frac{2e^2m^2}{(m^2-s)(m^2-u)}+\frac{1}{\pi}\int_{t_0}^{\infty}\frac{dt'}{t'-t}\mathcal{A}_{1;-1}{}^{t}(s,t')+\frac{1}{\pi}\int_{u_0}^{\infty}\frac{du'}{u'-u}\mathcal{A}_{1;1}{}^{u}(s,u'),$$

$$\mathfrak{A}_{1;-1}(s,t)=-\frac{2e^2m^2t}{(m^2-s)(m^2-u)}+\frac{t}{\pi}\int_{t_0}^{\infty}\frac{dt'}{t'-t}\frac{\mathfrak{A}_{1;-1}{}^{t}(s,t')}{t'}+\frac{t}{\pi}\int_{u_0}^{\infty}\frac{du'}{u'-u}\frac{\mathfrak{A}_{1;-1}{}^{u}(s,u')}{2m^2-s-u'}.$$

The first term is the full Born approximation while the integrals are multiplied by $t=-2p^2(1-\cos\Theta_s)$ and thus goes zero like p^2 for fixed Θ_s. The zeroth and first powers of p are again given exactly by the Born term as required by the low-energy theorem.

The assumption that $\mathcal{A}_{1;-1}$ satisfies a fixed-s dispersion relation may be based again on Regge arguments which lead to an asymptotic behavior $t^{\alpha(s)-1}$ or $u^{\alpha(s)-1}$ and we may reasonably expect a substantial range in s, say $s<4m^2$ where $\alpha(s)<1$, which is all we need.

Before considering the algebraically more complicated case of a spin-$\frac{1}{2}$ target, let us summarize the important steps in our derivation. The first is the recognition and removal of kinematic zeros from the helicity amplitudes. Since one-photon helicity always flips in crossing from the s to the t channel, there must be a channel in which there is helicity flip. Holding fixed the variable in the helicity flip channel (s ot t), we may hope to write an unsubtracted dispersion relation in the other variable (t or s). Thus, since $\mathfrak{A}_{1,1}(s,t)$ is proportional to $A_{1-1}(s,t)$, we are led to consider a fixed-t dispersion relation for $\mathfrak{A}_{1,1}$ after removal of a factor of $p_t^2k_t^2\sin^2\Theta_t$. Stated otherwise, we may make two subtractions (at the points $\nu=\pm p_tk_t$) where $\mathfrak{A}_{1,1}$ is known to vanish. Similarly, after removal of a factor of t we may write a fixed-s dispersion relation for $\mathfrak{A}_{1;-1}$. The next step is the assumption, with possible justification from Regge theory, that the helicity amplitudes, with appropriate kinematic factors removed, satisfy unsubtracted dispersion relations. The fact that we have in addition worked with amplitudes which are free of *both* s- and t-kinematical singularities means that our subsequent discussion of the limit $p\to 0$ or $s\to m^2$ is legitimate once we have isolated the dynamical singularity at $s=m^2$, the single-particle state.

The no-subtraction philosophy is not crucial to the part of our program which involves establishing the low-energy theorems. Consider, for example, a once subtracted dispersion relation for $\mathcal{A}_{1;1}(s,t)$:

$$\mathcal{A}_{1;1}(s,t)=\mathcal{A}_{1;1}(s_1,t)+2e^2\left[\frac{1}{(m^2-s)(m^2-u)}-\frac{1}{(m^2-s_1)(m^2-u_1)}\right]+\frac{(s-s_1)}{\pi}\int_{s_0}^{\infty}\mathcal{A}_{1;1}{}^{u}(s',t)\times\left[\frac{1}{(s'-s)(s'-s_1)}-\frac{1}{(s'-u)(s'-u_1)}\right].$$

Because $\mathcal{A}_{1;1}(s,t)$ is free of s- *and* t-kinematic singularities, the possible contribution to the low-energy theorem $[st+(s-m^2)^2]\mathcal{A}_{1;1}(s_1,t)$ will vanish in the limit as s approaches m^2 with Θ_s held fixed. This property may easily be seen by making a power series expansion of $\mathcal{A}_{1;1}(s_1,t)$ around $t=0$ utilizing the analyticity of $\mathcal{A}_{1;1}(s,t)$ in t for fixed s. As far as establishing the low-energy theorems is concerned, then, we are free to make as many subtractions as we choose, since the simultaneous analyticity in s and t assures us that the contributions of the subtraction functions will still be killed by factors such as $st+(s-m^2)^2$ when the appropriate limit is taken. The low-energy theorems are thus liberated from any assumptions about high-energy behavior.

It is, of course, possible that subtractions are required. Such a circumstance would vitiate the general usefulness of our representations, but we cannot claim to have proved the high-energy behavior required for no subtractions. The assumed behavior can only be said to be a sufficient condition and obviously represents the most beautiful way for the low-energy theorems to emerge.

IV. SPIN-$\frac{1}{2}$ TARGETS

We now address ourselves to Compton scattering from spin-$\frac{1}{2}$ targets. The essential ingredients for the deduction of the low-energy theorem are just those illustrated in the preceding section, but there are a number of nontrivial algebraic complications with which we must deal.

We know from the work of Ref. 2. that the lowest-order Born approximation, when expanded in powers of the photon momentum, gives the exact Compton amplitude up to order (p^2) where p is the photon momentum, as before. We shall show that it is possible to construct single-variable dispersion relations (with definite high-energy assumptions) which reproduce precisely the Born terms plus terms definitely of order p^2 or higher so that the low-energy theorem is explicitly obtained.

The predictions of lowest-order perturbation theory for the six independent helicity amplitudes corresponding to the Feynman diagrams of Fig. 3 are as follows:

$$\mathfrak{A}_{\frac{1}{2}1;\frac{1}{2}1}=\frac{2s^{1/2}p}{m^2-u}\left\{\frac{-e^2}{m}-\frac{2\mu^2 s^{1/2}p}{m}+\sin^2(\tfrac{1}{2}\Theta_s)\times\left[\frac{e^2m}{s}-\frac{2\mu m}{s}\left(\frac{e}{m}+\mu\right)(s-m^2)+\frac{\mu^2}{m}(s-m^2)\right]\right\}\times\cos(\tfrac{1}{2}\Theta_s)\,,$$

$$\mathfrak{A}_{\frac{1}{2}1;\frac{1}{2}-1}=\frac{-p}{m^2-u}\left\{\frac{2e^2m}{s^{1/2}}+4\mu m p\left(\frac{e}{m}+\mu\right)\right\}\times\sin^2(\tfrac{1}{2}\Theta_s)\cos(\tfrac{1}{2}\Theta_s)\,,$$

$$\mathfrak{A}_{\frac{1}{2}-1;\frac{1}{2}-1}=\frac{2s^{1/2}p}{m^2-u}\left\{-\frac{e^2}{m}+\frac{2\mu^2 s^{1/2}p}{m}\right\}\cos^3(\tfrac{1}{2}\Theta_s)\,,$$

$$\mathfrak{A}_{\frac{1}{2}1;-\frac{1}{2}1}=\frac{p}{m^2-u}\left\{-2a^2-\frac{4\mu a s^{1/2}p}{m}+4\mu\left(\frac{e}{m}+\mu\right)(s+m^2)+2\mu^2(s-m^2)\right\}\sin(\tfrac{1}{2}\Theta_s)\cos^2(\tfrac{1}{2}\Theta_s)\,,$$

$$\mathfrak{A}_{\frac{1}{2}-1;-\frac{1}{2}1}=\frac{-2p}{m^2-u}\left\{e^2+\frac{2\mu m p}{s^{1/2}}(\mu m+2e)\right\}\sin^3(\tfrac{1}{2}\Theta_s)\,,$$

$$\mathfrak{A}_{\frac{1}{2}1;-\frac{1}{2}-1}=\frac{p}{m^2-u}\left\{4\mu\left(\frac{e}{m}+\mu\right)(s-m^2)-\frac{2}{s}\sin^2(\tfrac{1}{2}\Theta_s)\times\left[e^2m^2+\frac{2\mu e}{m}(s-m^2)^2+\mu^2(s-m^2)(2s-m^2)\right]\right\}\sin(\tfrac{1}{2}\Theta_s)\,.$$

The charge of the particle is e, its anomalous magnetic

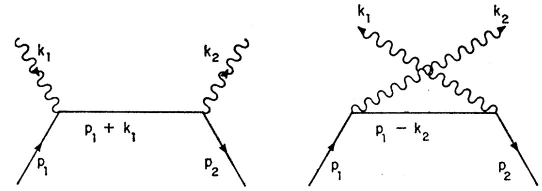

FIG. 3. The lowest-order perturbation-theory contributions to Compton scattering on a spin-$\frac{1}{2}$ target.

is μ, and we have introduced the quantity $a\equiv e+2m\mu$. These results simplify dramatically at $\Theta_s=0$, where only $\mathfrak{A}_{\frac{1}{2}1;\frac{1}{2}1}$ and $\mathfrak{A}_{\frac{1}{2}-1;\frac{1}{2}-1}$ are different from zero. In the first of these, the photon and particle spins are antiparallel and we call the amplitude $\mathfrak{A}_a$, while for the second the spins are parallel and we call the amplitude $\mathfrak{A}_p$. We have then

$$\mathfrak{A}_{\frac{1}{2}1;\frac{1}{2}1}(s,0)\equiv\mathfrak{A}_a(s,0)=-\frac{e^2}{m}-\frac{2\mu^2 s^{1/2}p}{m}\,,$$

$$\mathfrak{A}_{\frac{1}{2}-1;\frac{1}{2}-1}(s,0)\equiv\mathfrak{A}_p(s,0)=-\frac{e^2}{m}+\frac{2\mu^2 s^{1/2}p}{m}\,.$$

In working out the Born approximation corresponding to the diagrams of Fig. 3, it is necessary to specify the nonunique photon-particle vertex $\Gamma_\lambda(p',p)$. We have taken this to be $\Gamma_\lambda(p',p)=ie\gamma_\lambda-i\mu\sigma_{\lambda\nu}(p'-p)_\nu$, which is the choice made in Ref. 2, consistent with the generalized Ward identity. In our dispersion theoretic approach, we encounter only $\bar{u}(p')\Gamma_\lambda(p',p)u(p)$ where the u's are mass-shell spinors satisfying the Dirac equations $(i\gamma\cdot p+m)u(p)=0$ and $\bar{u}(p')(i\gamma\cdot p'+m)=0$, so any vertex ambiguity disappears. The point here is whereas

$$\bar{u}(p')\Gamma_\lambda(p',p)u(p)=\bar{u}(p')[ia\gamma_\lambda-\mu(p'+p)_\lambda]u(p)\,,$$

we have more generally

$$\begin{aligned}\Gamma_\lambda(p',p)&=ia\gamma_\lambda-\mu(p'+p)_\lambda-i\mu\gamma_\lambda(i\gamma\cdot p+m)\\&=ia\gamma_\lambda-\mu(p'+p)_\lambda-i\mu(i\gamma\cdot p'+m)\gamma_\lambda\,.\end{aligned}$$

The use of the possible vertex $\Gamma_\lambda'(p',p)$ given by $\Gamma_\lambda'(p',p)=ia\gamma_\lambda-\mu(p'+p)_\lambda$ would yield Born approximation amplitudes different from those given above. Exploitation of the relation between $\Gamma_\lambda(p',p)$ and $\Gamma_\lambda'(p',p)$, which is, of course, the familiar Gordon decomposition of the current in Dirac theory, simplifies the evaluation of the Born approximation.

We turn now to dispersion theory and our presentation of dispersion relations which are unsubtracted and which reproduce the Born terms given above, together with continuum contributions which are negligible in the low-energy limit. We shall start by considering fixed-s dispersion relations and later work with fixed t after discussing the rather involved s-t crossing relations. According to the discussion given in the preceding sections, the following helicity amplitudes divided by their kinematic zeroes are candidates for unsubtracted fixed s-dispersion relations (they will in addition, as we shall see, lead to a simple fashion to the low-energy theorems):

$$\bar{\mathfrak{A}}_{\frac12 1;\frac12 -1}(s,t)\equiv \mathfrak{A}_{\frac12 1;\frac12 -1}[\sin^2(\tfrac12\Theta_s)\cos(\tfrac12\Theta_s)]^{-1},$$
$$\bar{\mathfrak{A}}_{\frac12 -1;\frac12 -1}(s,t)\equiv \mathfrak{A}_{\frac12 -1;\frac12 -1}[\cos^3(\tfrac12\Theta_s)]^{-1},$$
$$\bar{\mathfrak{A}}_{\frac12 1;-\frac12 1}(s,t)\equiv \mathfrak{A}_{\frac12 1;-\frac12 1}[\sin(\tfrac12\Theta_s)\cos^2(\tfrac12\Theta_s)]^{-1},$$
$$\bar{\mathfrak{A}}_{\frac12 -1;-\frac12 1}(s,t)\equiv \mathfrak{A}_{\frac12 -1;-\frac12 1}[\sin^3(\tfrac12\Theta_s)]^{-1}.$$

In order to illustrate our method and show how the low-energy theorem emerges, we discuss in detail the amplitude $\bar{\mathfrak{A}}_{\frac12 -1;\frac12 -1}\equiv\bar{\mathfrak{A}}_p$; the dispersion relation is

$$\bar{\mathfrak{A}}_p(s,t)=\frac{1}{\pi}\int_{t_0}^{\infty}\frac{dt'}{t'-t}\bar{\mathfrak{A}}_p{}^t(s,t')+\frac{1}{\pi}\int_{m^2}^{\infty}\frac{du'}{u'-u}\bar{\mathfrak{A}}_p{}^u(s,u'),$$

where the t and u discontinuities of $\bar{\mathfrak{A}}_p$ have been designated by $\bar{\mathfrak{A}}_p{}^t$ and $\bar{\mathfrak{A}}_p{}^u$; the t-channel threshold is t_0. The one-particle intermediate state contribution to $\bar{\mathfrak{A}}_p{}^u$ may be calculated explicitly using the on-mass-shell current matrix element given previously $(a=e+2m\mu)$:

$$\left(\frac{p_0p_0'}{m^2}\right)^{1/2}\langle p'|J_\lambda|p\rangle=\bar{u}(p')[ia\gamma_\lambda-\mu(p'+p)_\lambda]u(p);$$

if $(p'-p)^2=0$, as in our applications, the above current matrix element is the same as that of the proper vertex function $\bar{u}\Gamma_\lambda(p',p)u$. We find

$$\bar{\mathfrak{A}}_p{}^u=-\pi\delta(m^2-u)\left\{4\mu m\left(\frac{e}{m}+\mu\right)t+\frac{2ps^{1/2}}{m}[\mu^2t+a^2]\right\},$$

which we rewrite for fixed s, $u=m^2$, and $t=m^2-s$, as

$$\bar{\mathfrak{A}}_p{}^u=-\pi\delta(m^2-u)\left\{(2ps^{\frac12})\left(\frac{a^2}{m}-4\mu e-4m\mu^2\right)-(2ps^{\frac12})^2\frac{\mu^2}{m}\right\}$$
$$=\pi\delta(m^2-u)2ps^{1/2}\left\{-\frac{e^2}{m}+\frac{2ps^{1/2}\mu^2}{m}\right\}.$$

Inserting this single-particle contribution, our dispersion relation becomes

$$\bar{\mathfrak{A}}_p(s,t)=\frac{2ps^{1/2}}{m^2-u}\left[-\frac{e^2}{m}+\frac{2ps^{1/2}\mu^2}{m}\right]+\frac{1}{\pi}\int_{t_0}^{\infty}\frac{dt'}{t'-t}\bar{\mathfrak{A}}_p{}^t(s,t')+\frac{1}{\pi}\int_{u_0}^{\infty}\frac{du'}{u'-u}\bar{\mathfrak{A}}_p{}^u(s,u').$$

It is convenient to rewrite this expression using various of the formulas

$$(s-m^2)^2\cos^2(\tfrac12\Theta_s)=(s-m^2)^2+st=m^4-su;$$

we find then [remembering that in the u' integral, for example, the $\cos^3(\frac12\Theta_s)$ implicitly contained in the definition of $\bar{\mathfrak{A}}$ must be written as a function of u' at fixed s]

$$\mathfrak{A}_p(s,t)=\frac{2ps^{1/2}\cos^3(\frac12\Theta_s)}{m^2-u}\left(-\frac{e^2}{m}+\frac{2ps^{1/2}\mu^2}{m}\right)+(s-m^2)^3\times\cos^3(\tfrac12\Theta_s)\left\{\frac{1}{\pi}\int_{t_0}^{\infty}\frac{dt'}{t'-t}\left[\frac{\mathfrak{A}_p(s,t')}{[st'+(s-m^2)^2]^{3/2}}\right]^t+\frac{1}{\pi}\int_{u_0}^{\infty}\frac{du'}{u'-u}\left[\frac{\mathfrak{A}_p(s,u')}{(m^2-su')^{3/2}}\right]^u\right\}.$$

The notation $[\ ]^t$, $[\ ]^u$ means the t, u absorption part of the quantities $[\ ]$.

The first term is precisely the previously calculated Born contribution and the continuum integrals appear superficially to go to zero as p^3 since $p\sim(s-m^2)$. This would be very surprising since we expect the deviations from the low-energy theorem to be $O(p^2)$. We shall see below from a study of the crossing relations that $\mathfrak{A}_p/[st+(s-m^2)^2]^{3/2}$ contains a factor of $(s-m^2)^{-1}$, and in this manner our expectation obtains. The reason this complication did not appear in the spin-zero case is that the crossing matrix there involves just constants—$\mathfrak{A}_{1;1}\sim A_{1-1}$, etc.

For completeness, we record the one-particle-state contributions to the remaining three amplitudes for which we expect unsubtracted fixed-s dispersion relations:

$$\bar{\mathfrak{A}}_{\frac12 1;\frac12 -1}{}^u=\pi\delta(m^2-u)\left\{-8m\mu\left(\frac{e}{m}+\mu\right)p^2\cos\Theta_s+4e\left(\frac{e}{m}+\mu\right)p^2-\frac{2ps^{1/2}}{m}[4\mu^2p^2\cos^2(\tfrac12\Theta_s)+a^2]\right\},$$

$$\bar{\mathfrak{A}}_{\frac12 1;-\frac12 1}{}^u=-\pi\delta(m^2-u)\left\{4(s^{1/2}-p)t\mu\left(\frac{e}{m}+\mu\right)+\frac{4\mu ap^2s^{1/2}}{m}+2a^2p+2\mu^2pt\right\},$$

$$\bar{\mathfrak{A}}_{\frac12 -1;-\frac12 1}{}^u=-\pi\delta(m^2-u)\left\{2a^2p+8\mu^2p^3\cos^2(\tfrac12\Theta_s)+16\mu\left(\frac{e}{m}+\mu\right)(s^{1/2}-p)p^2\cos^2(\tfrac12\Theta_s)\right\}.$$

In these absorptive parts we must express t and $\cos\Theta_s$ in terms of u and s; since $u=m^2$ we find $t=m^2-s$, $\cos\Theta_s=(m^2+s)/(m^2-s)$, $\cos^2(\frac12\Theta_s)=m^2/(m^2-s)$. When these results are substituted into fixed-s dispersion relations like the one written for $\mathfrak{A}_p$, above, we reproduce the full Born approximation together with continuum contributions which superficially go like

p^3 as $p\to 0$ but which, as for $\mathfrak{A}_p$ in fact go like p^2:

$$\mathfrak{A}_{\frac12 1;\frac12-1}=\frac{-p\sin^2(\frac12\Theta_s)\cos(\frac12\Theta_s)}{m^2-u}\left\{\frac{2e^2m}{s^{1/2}}+4\mu m p\left(\frac{e}{m}+\mu\right)\right\}+\frac{(s-m^2)^3}{s}\sin^2(\tfrac12\Theta_s)\cos(\tfrac12\Theta_s)$$

$$\times\left\{\frac1\pi\int_{t_0}^\infty\frac{dt'}{t'-t}\left[\frac{\mathfrak{A}_{\frac12 1;\frac12-1}(s,t')}{(-t')[st'+(s-m^2)^2]^{1/2}}\right]^t+\frac1\pi\int_{u_0}^\infty\frac{du'}{u'-u}\left[\frac{\mathfrak{A}_{\frac12 1;\frac12-1}(s,u')}{(s+u'-2m^2)(m^4-su')^{1/2}}\right]^u\right\},$$

$$\mathfrak{A}_{\frac12 1;-\frac12 1}=\frac{p\sin(\frac12\Theta_s)\cos^2(\frac12\Theta_s)}{m^2-u}\left\{-2a^2-\frac{4\mu a s^{1/2}p}{m}+4\mu\left(\frac{e}{m}+\mu\right)(s+m^2)+2\mu^2(s-m^2)\right\}+\frac{(s-m^2)^3}{s^{1/2}}\sin(\tfrac12\Theta_s)\cos(\tfrac12\Theta_s)$$

$$\times\left\{\frac1\pi\int_{t_0}^\infty\frac{dt'}{t'-t}\left[\frac{\mathfrak{A}_{\frac12 1;-\frac12 1}(s,t')}{(-t')^{1/2}[st'+(s-m^2)^2]}\right]^t+\frac1\pi\int_{u_0}^\infty\frac{du'}{u'-u}\left[\frac{\mathfrak{A}_{\frac12 1;-\frac12 1}(s,u')}{(s+u'-2m^2)^{1/2}(m^4-su')}\right]^u\right\},$$

$$\mathfrak{A}_{\frac12-1;-\frac12 1}=\frac{-2p\sin^3(\frac12\Theta_s)}{m^2-u}\left\{e^2+\frac{2\mu m p}{s^{1/2}}(\mu m+2e)\right\}+\frac{(s-m^2)^3}{s^{3/2}}\sin^3(\tfrac12\Theta_s)$$

$$\times\left\{\frac1\pi\int_{t_0}^\infty\frac{dt'}{t'-t}\left[\frac{\mathfrak{A}_{\frac12-1;-\frac12 1}(s,t')}{(-t')^{3/2}}\right]^t+\frac1\pi\int_{u_0}^\infty\frac{du'}{u'-u}\left[\frac{\mathfrak{A}_{\frac12-1;-\frac12 1}(s,u')}{(s+u'-2m^2)^{3/2}}\right]^u\right\}.$$

If we imagine that our spin-$\frac12$ target is a nucleon in the real world, we may adduce Regge asymptotic arguments in support of our unsubtracted dispersion relations for the $\mathfrak{A}$'s. Alternatively, one may regard the reproduction of the low-energy theorem as an expression of the consistency of the no-subtraction hypothesis.

We now have established the low-energy theorem for four of the six Compton amplitudes as well as presenting unsubtracted fixed-s dispersion relations for them. To obtain the remaining two we turn to the t channel. We form linear combinations of s-channel amplitudes which are proportional, via crossing, to t-channel amplitudes with known kinematical zeroes. These linear combinations, divided by kinematical zeroes, are taken to satisfy unsubtracted fixed-t dispersion relations.

We discuss the t channel at some length. Recall from Sec. II that we take the t-channel reaction to be $\gamma(d')+\gamma(b')\to$ particle (c')+antiparticle (a') and that the scattering angle is that between $\mathbf{p}_{d'}$ and $\mathbf{p}_{c'}$. First we note in Table I the allowed physical partial-wave amplitudes in the t channel. The angular momentum and parity of the transitions follow from the identity of the photons and charge conjugation invariance. It is convenient to work with the following combinations of t-channel helicity amplitudes (the helicity labels are $A_{c'a';d'b'}$):

$$A_1\equiv A_{\frac12\frac12;11}+A_{-\frac12-\frac12;11}=\sum_J(2J+1)a_1{}^J(t)d_{00}{}^J(\Theta_t),$$

$$A_2\equiv A_{\frac12\frac12;1-1}+A_{-\frac12-\frac12;1-1}=\sum_J(2J+1)a_2{}^J(t)d_{20}{}^J(\Theta_t),$$

$$A_3\equiv A_{\frac12-\frac12;11}-A_{-\frac12\frac12;11}=\sum_J(2J+1)a_3{}^J(t)d_{01}{}^J(\Theta_t),$$

$$\tfrac12(A_4+A_5)\equiv A_{\frac12-\frac12;1-1}=\sum_J(2J+1)\frac{a_4{}^J(t)+a_5{}^J(t)}{2}d_{21}{}^J(\Theta_t),$$

$$\tfrac12(A_4-A_5)\equiv A_{-\frac12\frac12;1-1}=\sum_J(2J+1)\frac{a_4{}^J(t)-a_5{}^J(t)}{2}d_{2-1}{}^J(\Theta_t),$$

$$A_6\equiv A_{\frac12\frac12;11}-A_{-\frac12-\frac12;11}=\sum(2J+1)a_6{}^J(t)d_{00}{}^J(\Theta_t).$$

The behavior of the amplitudes under $s-u$ crossing at fixed t can be easily deduced using Table I for the allowed J values, together with the properties of the d^J's; we use the fact that $p_tk_t\cos\Theta_t=\nu=\frac14(s-u)$ and spell out the results in terms of $\nu\to-\nu$ which is equivalent to $s\rightleftarrows u$. We find four amplitudes which are even under $\nu\to-\nu$,

$$A_{\frac12\frac12;11}\pm A_{-\frac12-\frac12;11}=A_{1,6},$$

$$(A_{\frac12\frac12;1-1}+A_{-\frac12-\frac12;1-1})/p_t{}^2k_t{}^2\sin^2\Theta_t=A_2/p_t{}^2k_t{}^2\sin^2\Theta_t,$$

$$\left(\frac{A_{\frac12-\frac12;1-1}}{1+\cos\Theta_t}+\frac{A_{-\frac12\frac12;1-1}}{1-\cos\Theta_t}\right)\Big/p_t{}^2k_t{}^2\sin\Theta_t\equiv A_+,$$

TABLE I. Allowed transitions in the t channel $\gamma+\gamma\to$ fermion +antifermion. The states of definite angular momentum $|\lambda_1\lambda_2\rangle_\pm$ are $(1/\sqrt2)[|\lambda_1\lambda_2\rangle\pm|-\lambda_1-\lambda_2\rangle]$ and have the J suppressed.

Transition	J	P
$\mathfrak{A}_1{}^J\equiv\langle\frac12\frac12\|T^J\|11\rangle_+$	even	even
$\mathfrak{A}_2{}^J\equiv\langle\frac12\frac12\|T^J\|1-1\rangle_+$	even	even
$\mathfrak{A}_3{}^J\equiv\langle\frac12-\frac12\|T^J\|11\rangle_+$	even	even
$\mathfrak{A}_4{}^J\equiv\langle\frac12-\frac12\|T^J\|1-1\rangle_+$	even	even
$\mathfrak{A}_5{}^J\equiv\langle\frac12-\frac12\|T^J\|1-1\rangle_-$	odd	even
$\mathfrak{A}_6{}^J\equiv\langle\frac12\frac12\|T^J\|11\rangle_-$	even	odd

and two odd ones,

$$(A_{\frac{1}{2}-\frac{1}{2};11}-A_{-\frac{1}{2}\frac{1}{2};11})/p_tk_t\sin\Theta_t=A_3/p_tk_t\sin\Theta_t,$$

$$\left(\frac{A_{\frac{1}{2}-\frac{1}{2};1-1}}{1+\cos\Theta_t}-\frac{A_{-\frac{1}{2}\frac{1}{2};1-1}}{1-\cos\Theta_t}\right)\Big/ p_t^2k_t^2\sin\Theta_t\equiv A_-.$$

Furthermore, $A_\pm$, $A_2(p_t^2k_t^2\sin^2\Theta_t)^{-1}$, and $A_3(p_tk_t\times\sin\Theta_t)^{-1}$ are free of kinematical singularities in s and are candidates for unsubtracted dispersion relations which because of factors like $p_t^2k_t^2\sin^2\Theta_t$, say, will also yield low-energy theorems as in the scalar target case.

It is for these four amplitudes (or more precisely for the combination of s-channel amplitudes they cross into) that we will write fixed-t dispersion relations. We will then have eight ways of representing six quantities. There is a redundancy because some of the s-channel amplitudes entering the fixed-t dispersion relations also satisfy fixed-s ones by themselves. In principle one can extract two sum rules involving mixtures of fixed-s and -t dispersion integrals and we will use these in connection with superconvergence relations in the next section.

The next step in our procedure is to give the crossing relations between s- and t-channel helicity amplitudes and write fixed-t dispersion relations for $A_\pm$, A_2, and A_3 (the latter two divided by the appropriate factors). To do this, we evaluate the absorptive parts of (s-channel) $\mathfrak{A}$'s in the s-channel center-of-mass system. With the crossing relations given below the absorptive parts of the A's are then obtained. We may solve for the individual s-channel $\mathfrak{A}$'s if we choose to, and all reference to the t channel disappears. This rather tortuous path is followed to gain algebraic sign security.

The crossing relations between the $\mathfrak{A}$'s and the A's almost have to be derived by the individual reader. With our conventions as spelled out in Appendix B, we find the following results:

$$\frac{\mathfrak{A}_a+\mathfrak{A}_p}{\cos\frac{1}{2}\Theta_s}=\frac{m}{p_t}A_2-\frac{(s+m^2)}{2p_t^2\sin\Theta_t}A_5,$$

$$\frac{\mathfrak{A}_a-\mathfrak{A}_p}{\cos\frac{1}{2}\Theta_s}=\frac{(s-m^2)}{2p_tk_t\sin\Theta_t}A_4,$$

$$\frac{\mathfrak{A}_{\frac{1}{2}1;\frac{1}{2}-1}}{\cos\frac{1}{2}\Theta_s}=\frac{m}{2p_t}A_1-\frac{s+m^2}{4p_t^2\sin\Theta_t}A_3,$$

$$\frac{\mathfrak{A}_{\frac{1}{2}1;-\frac{1}{2}1}}{\sin\frac{1}{2}\Theta_s}=\frac{s^{1/2}-p}{2p_t}A_2-\frac{m[st+(s-m^2)^2]}{8s^{1/2}p_t^2k_t^2\sin\Theta_t}A_5,$$

$$\frac{\mathfrak{A}_{\frac{1}{2}1;-\frac{1}{2}-1}+\mathfrak{A}_{\frac{1}{2}-1;-\frac{1}{2}1}}{\sin\frac{1}{2}\Theta_s}=\frac{s^{1/2}-p}{p_t}A_1-\frac{m[st+(s-m^2)^2]}{4s^{1/2}p_t^2k_t^2\sin\Theta_t}A_3,$$

$$\frac{\mathfrak{A}_{\frac{1}{2}-1;-\frac{1}{2}1}-\mathfrak{A}_{\frac{1}{2}1;-\frac{1}{2}-1}}{\sin\frac{1}{2}\Theta_s}=\frac{p}{k_t}A_6.$$

In these formulas, $p_t=(\frac{1}{4}t-m^2)^{1/2}$, $k_t=\frac{1}{2}t^{1/2}$, and $\mathfrak{A}_a=\mathfrak{A}_{\frac{1}{2}1;\frac{1}{2}1}$, $\mathfrak{A}_p=\mathfrak{A}_{\frac{1}{2}-1;\frac{1}{2}-1}$. Any reader concern about what we mean by p_t when $t<0$ should be restrained; these quantities will shortly disappear from view.

We must digress briefly to clear up a point mentioned in connection with our fixed-s dispersion relations having to do with possible s-dependent kinematic singularities. If we solve for $\mathfrak{A}_p$ we find

$$\frac{2\mathfrak{A}_p}{\cos\frac{1}{2}\Theta_s}=\frac{m}{p_t}A_2-\frac{s+m^2}{2p_t^2\sin\Theta_t}A_5-\frac{s-m^2}{2p_tk_t\sin\Theta_t}A_4.$$

If this expression is divided by $p_t^2k_t^2\sin^2\Theta_t$ on both sides, we obtain on the right quantities guaranteed free of s-kinematical singularities, e.g., $A_2/p_t^2k_t^2\sin^2\Theta_t$ and and on the left we have $-8[\mathfrak{A}_p/\cos^3(\frac{1}{2}\Theta_s)](s-m^2)^{-2}$. The quantity in square brackets is guaranteed free of t-kinematical singularities and thus the whole thing is free of both s- and t-kinematical singularities. The important point is that we have the factor $(s-m^2)^{-2}$ instead of the naively expected $(s-m^2)^{-3}$. Stated otherwise, the proper amplitude for a fixed-s dispersion relation is $(s-m^2)\{\mathfrak{A}_p/[st+(s-m^2)^2]^{3/2}\}$.

Returning now to the fixed-t dispersion relations we find, using the $s-u$ crossing symmetries previously noted and our old calculation of the u-channel absorptive parts, the following (after some rather tedious algebra):

$$A_2(s,t)=\frac{-2[st+(s-m^2)^2]}{p_t}\left[e^2+\tfrac{1}{2}t\mu\left(\frac{e}{m}+\mu\right)\right]\frac{1}{(m^2-s)(m^2-u)}+\frac{st+(s-m^2)^2}{\pi}\int_{s_0}^{\infty}\frac{ds'A_2{}^u(s',t)}{s't+(s'-m^2)^2}\left(\frac{1}{s'-s}+\frac{1}{s'-u}\right),$$

$$A_3(s,t)=\frac{-(u-s)[st+(s-m^2)^2]^{1/2}}{(m^2-s)(m^2-u)}\mu m\left(\frac{e}{m}+\mu\right)\frac{(-t)^{1/2}}{p_t}+\frac{(u-s)[st+(s-m^2)^2]^{1/2}}{\pi}\times\int_{s_0}^{\infty}\frac{ds'A_3{}^u(s',t)}{[s't+(s'-m^2)^2]^{1/2}}\frac{1}{(s'-u)(s'-s)},$$

$$A_+(s,t)=\frac{-2tp_tk_t}{m(m^2-u)(m^2-s)}(ea+\tfrac{1}{2}\mu^2t)+\frac{1}{\pi}\int_{s_0}^{\infty}ds'A_+{}^u(s',t)\left(\frac{1}{s'-u}+\frac{1}{s'-s}\right),$$

$$A_-(s,t)=\frac{4(s-u)p_t^2k_t^2\mu^2}{m(m^2-s)(m^2-u)}+\frac{u-s}{\pi}\int_{s_0}^{\infty}ds'\frac{A_-{}^u(s',t)}{(s'-u)(s'-s)}.$$

We are now in position to write dispersion relations for the two remaining s-channel amplitudes, namely $\mathfrak{A}_{\frac{1}{2}1;\frac{1}{2}1}$ (called $\mathfrak{A}_a$ sometimes) and $\mathfrak{A}_{\frac{1}{2}1;-\frac{1}{2}-1}$. Using the above fixed-$t$ dispersion relation for A_2 and our earlier fixed-s relations for $\mathfrak{A}_{\frac{1}{2}1;-\frac{1}{2}1}$, we may solve for A_5 from the fourth of the crossing relations and then with the fixed-s dispersion relation for $\mathfrak{A}_p$ solve the first crossing relation for $\mathfrak{A}_a$. Similarly, from fixed-s dispersion relations for $\mathfrak{A}_{\frac{1}{2}-1;-\frac{1}{2}1}$ and $\mathfrak{A}_{\frac{1}{2}1;\frac{1}{2}-1}$, and a fixed-$t$ representation for A_3, we solve the third and fifth crossing relations for $\mathfrak{A}_{\frac{1}{2}1;-\frac{1}{2}-1}$. We find:

$$\mathfrak{A}_{\frac{1}{2}1;\frac{1}{2}1}=\frac{2s^{1/2}p\cos\frac{1}{2}\Theta_s}{m^2-u}\left\{\left(-\frac{e^2}{m}-\frac{2\mu s^{1/2}p}{m}\right)+\sin^2(\tfrac{1}{2}\Theta_s)\left[\frac{e^2m}{s}-\frac{2\mu m}{s}\left(\frac{e}{m}+\mu\right)(s-m)+\frac{\mu^2}{m}(s-m^2)\right]\right\}-\frac{[st+(s-m^2)^2]^{3/2}}{\pi}$$

$$\times\left\{\int_{t_0}^{\infty}\frac{dt'}{t'-t}\left[\frac{\mathfrak{A}_{\frac{1}{2}-1;\frac{1}{2}-1}(s,t')}{[st'+(s-m^2)^2]^{3/2}}\right]^t+\int_{u_0}^{\infty}\frac{du'}{u'-u}\left[\frac{\mathfrak{A}_{\frac{1}{2}-1;\frac{1}{2}-1}(s,u')}{(m^4-su')^{3/2}}\right]^u\right\}-\frac{(s-m^2)[st+(s-m^2)^2]^{1/2}}{\pi}$$

$$\times\int_{s_0}^{\infty}\frac{ds'}{s't+(s'-m^2)^2}\left(\frac{1}{s'-s}+\frac{1}{s'-u}\right)\left[\frac{[s't+(s'-m^2)^2]^{1/2}}{s'-m^2}[\mathfrak{A}_{\frac{1}{2}1;\frac{1}{2}1}{}^u(s',t)+\mathfrak{A}_{\frac{1}{2}-1;\frac{1}{2}-1}{}^u(s't)]+(-t)^{1/2}\left(\frac{s'+m^2}{s'-m^2}\right)\right.$$

$$\left.\times\mathfrak{A}_{\frac{1}{2}1;-\frac{1}{2}1}{}^u(s',t)\right]+\frac{(s+m^2)t[st+(s-m^2)^2]^{1/2}}{m\pi}\left\{\int_{t_0}^{\infty}\frac{dt'}{t'-t}\left[\frac{\mathfrak{A}_{\frac{1}{2}1;-\frac{1}{2}1}(s,t')}{(-t')^{1/2}[st'+(s-m^2)^2]}\right]^t\right.$$

$$\left.+\int_{u_0}^{\infty}\frac{du'}{u'-u}\left[\frac{\mathfrak{A}_{\frac{1}{2}1;-\frac{1}{2}1}(s,u')}{(s+u'-2m^2)^{1/2}(m^4-su')}\right]^u\right\},$$

$$\mathfrak{A}_{\frac{1}{2}1;-\frac{1}{2}-1}=\frac{p\sin(\frac{1}{2}\Theta_s)}{m^2-u}\left\{4\mu\left(\frac{e}{m}+\mu\right)(s-m^2)-\frac{2}{s}\sin^2(\tfrac{1}{2}\Theta_s)\left[m^2e^2+\frac{2\mu e}{m}(s-m^2)^2+\mu^2(s-m^2)(2s-m^2)\right]\right\}-\frac{(-t)^{3/2}}{\pi}$$

$$\times\left\{\int_{t_0}^{\infty}\frac{dt'}{t'-t}\left[\frac{\mathfrak{A}_{\frac{1}{2}-1;-\frac{1}{2}1}(s,t')}{(-t')^{3/2}}\right]^t+\int_{u_0}^{\infty}\frac{du'}{u'-u}\left[\frac{\mathfrak{A}_{\frac{1}{2}-1;-\frac{1}{2}1}(s,u')}{(s+u'-2m^2)^{3/2}}\right]^u\right\}$$

$$+\frac{(u-s)(s-m^2)}{\pi}\int_{s_0}^{\infty}\frac{ds'}{(s'-s)(s'-u)}\frac{1}{[s't+(s'-m^2)^2]^{1/2}}\left\{\frac{[s't+(s'-m^2)^2]^{1/2}}{s'-m^2}\right.$$

$$\left.\times[\mathfrak{A}_{\frac{1}{2}1;-\frac{1}{2}-1}{}^u(s',t)+\mathfrak{A}_{\frac{1}{2}-1;-\frac{1}{2}1}{}^u(s',t)]-\frac{s'+m^2}{s'-m^2}(-t)^{1/2}\mathfrak{A}_{\frac{1}{2}1;\frac{1}{2}-1}{}^u(s',t)\right\}+\frac{s+m^2}{m}\frac{(-t)^{3/2}}{\pi}$$

$$\times\left\{\int_{t_0}^{\infty}\frac{dt'}{t'-t}\left[\frac{\mathfrak{A}_{\frac{1}{2}1;\frac{1}{2}-1}(s,t')}{(-t')[st'+(s-m^2)^2]^{1/2}}\right]^t+\int_{u_0}^{\infty}\frac{du'}{u'-u}\left[\frac{\mathfrak{A}_{\frac{1}{2}1;\frac{1}{2}-1}(s,u')}{(u'+s-2m^2)(m^4-su')^{1/2}}\right]^u\right\}.$$

The only immediate consequence of these rather ferocious expressions for $\mathfrak{A}_{\frac{1}{2}1;\frac{1}{2}1}$ and $\mathfrak{A}_{\frac{1}{2}1;-\frac{1}{2}-1}$ is that they indeed yield the low-energy theorems as advertised.

It is also possible and perhaps useful to note that fixed-t dispersion relations for $\mathfrak{A}_a\pm\mathfrak{A}_p$ may be derived by virtue of the fact A_2 and $A_\pm$ (and hence A_4 and A_5) satisfy such. It is convenient to use a rather widely mixed notation involving s, t, $\nu=\frac{1}{2}(s-m^2+\frac{1}{2}t)$, $p_t{}^2=\frac{1}{4}t-m^2$, and $k_t{}^2=\frac{1}{4}t$:

$$\frac{\mathfrak{A}_a+\mathfrak{A}_p}{\cos\frac{1}{2}\Theta_s}=\frac{2ps^{1/2}}{m^2-u}\left\{-\frac{2e^2}{m}+\frac{1}{s}\sin^2(\tfrac{1}{2}\Theta_s)\left[\frac{e^2}{m}(s+m^2)-2\mu m\left(\frac{e}{m}+\mu\right)(s-m^2)\right]\right\}$$

$$+\frac{4(\nu^2-p_t{}^2k_t{}^2)}{\pi(4m^2-t)}\int_{\nu_0(t)}^{\infty}\frac{d\nu'}{\nu'^2-\nu^2}\frac{\mathrm{Im}[\mathfrak{A}_a(\nu',t)+\mathfrak{A}_p(\nu',t)]}{(\nu'-k_t{}^2)(\nu'^2-p_t{}^2k_t{}^2)^{1/2}}\left[2m^2\nu'-\frac{\nu t(\nu-p_t{}^2)(\nu'-p_t{}^2)}{\nu^2-p_t{}^2k_t{}^2}\right]$$

$$-\frac{2m(-t)^{1/2}}{\pi}\int_{\nu_0(t)}^{\infty}\frac{d\nu'}{\nu'-k_t{}^2}\mathrm{Im}\mathfrak{A}_{\frac{1}{2}1;-\frac{1}{2}1}(\nu',t)\left[k_t{}^2-\frac{\nu\nu'+p_t{}^2k_t{}^2}{\nu+\nu'}\right],$$

$$\frac{\mathfrak{A}_a-\mathfrak{A}_p}{\cos\frac{1}{2}\Theta_s}=\frac{2ps^{1/2}}{m^2-u}\left\{-\frac{4\mu^2ps^{1/2}}{m}+\frac{(s-m^2)\sin^2(\frac{1}{2}\Theta_s)}{s}\left[-\frac{ea}{m}+\frac{2\mu^2(s-m^2)}{m}\right]\right\}+\frac{2ps^{1/2}}{\pi}(\nu^2-p_t^2k_t^2)$$

$$\times\int_{\nu_0(t)}^{\infty}\frac{\nu'd\nu'}{\nu'^2-\nu^2}\,\mathrm{Im}\left[\frac{\mathfrak{A}_a(\nu',t)-\mathfrak{A}_p(\nu',t)}{(\nu'^2-k_t^2p_t^2)^{3/2}}\right]+\frac{2ps^{1/2}m(-t)^{1/2}}{\pi}\int_{\nu_0(t)}^{\infty}\frac{d\nu'}{\nu'^2-p_t^2k_t^2}\,\frac{\mathrm{Im}\mathfrak{A}_{\frac{1}{2}1;-\frac{1}{2}1}(\nu',t)}{\nu'-k_t^2}$$

$$-\frac{2ps^{1/2}k_t^2}{\pi}\int_{\nu_0(t)}^{\infty}\frac{d\nu'(\nu'-p_t^2)}{\nu'-k_t^2}\,\mathrm{Im}\left[\frac{\mathfrak{A}_a(\nu',t)+\mathfrak{A}_p(\nu',t)}{(\nu'^2-p_t^2k_t^2)^{3/2}}\right],$$

where $\nu_0(t)=\frac{1}{2}(s_0-m^2)+\frac{1}{4}t$. Therse complicated dispersion relations are generalizations to nonforward scattering of those given by Gell-Mann, Goldberger, and Thirring[17] many years ago and reduce to them at $t=0$:

$$\mathfrak{A}_a(\nu,0)+\mathfrak{A}_p(\nu,0)=-\frac{2e^2}{m}+\frac{2\nu^2}{\pi}\int_{\nu_0(0)}^{\infty}\frac{d\nu'}{\nu'^2-\nu^2}\,\frac{\mathrm{Im}[\mathfrak{A}_a(\nu',0)+\mathfrak{A}_p(\nu',0)]}{\nu'},$$

$$\mathfrak{A}_a(\nu,0)-\mathfrak{A}_p(\nu,0)=-\frac{4\mu^2\nu}{m}+\frac{2\nu^3}{\pi}\int_{\nu_0(0)}^{\infty}\frac{d\nu'}{\nu'^2-\nu^2}\,\frac{\mathrm{Im}[\mathfrak{A}_a(\nu',0)-\mathfrak{A}_p(\nu',0)]}{\nu'^2}.$$

(In the ancient notation, $2f\equiv\mathfrak{A}_a+\mathfrak{A}_p$, $2f_2\equiv\mathfrak{A}_a-\mathfrak{A}_p$.)

We have now presented unsubtracted dispersion representations for each of the six independent s-channel helicity amplitudes for Compton scattering on a spin-$\frac{1}{2}$ particle. These representations reproduce the Born approximation exactly and have the property that the continuum contributions are explicitly proportional to p^2 as $p\to 0$. We have thus given a dispersion theoretic deviation of the low-energy theorems and this completes the main task of this work. In the next section we shall explore a few consequences of our unsubtracted dispersion relations.

V. SUM RULES AND SUPERCONVERGENCE RELATIONS

There is a class of what have come to be called superconvergence relations which can be extracted from single variable dispersion relations. As an example, consider the fixed-s dispersion relation for the amplitude $\mathfrak{A}_{1;-1}(s,t)$ encountered in pion scattering:

$$\frac{\mathfrak{A}_{1;-1}(s,t)}{t}=-\frac{2e^2m^2}{(m^2-s)(s+t-m^2)}+\frac{1}{\pi}\int_{t_0}^{\infty}\frac{dt'}{t'-t}\,\frac{\mathfrak{A}_{1;-1}{}^t(s,t')}{t'}$$

$$+\frac{1}{\pi}\int_{u_0}^{\infty}\frac{du'}{u'-2m^2+s+t}\,\frac{\mathfrak{A}_{1;-1}{}^u(s,u')}{(2m^2-s-u')}.$$

In a truly Reggeistic world, the large-t behavior of $\mathfrak{A}_{1;-1}(s,t)$ is given by $t^{\alpha(s)}$ where the trajectory $\alpha(s)$ is the one for which $\alpha(m^2)=0$. Since one expects the slope of α to be positive, $\alpha(s)<0$ for $s<m^2$, and thus $\mathfrak{A}_{1;1}/t$ should decrease faster than t^{-1} for large t. Imposing this condition on our represenation leads to the sum rule (superconvergence relation)

$$\frac{2\pi e^2m^2}{m^2-s}=\int_{4m^2}^{\infty}du'\frac{\mathfrak{A}_{1;-1}{}^u(s,u')}{(2m^2-s-u')}-\int_{4m^2}^{\infty}dt'\frac{\mathfrak{A}_{1;-1}{}^t(s,t')}{t'}.$$

This relation is not too easy to check; one might imagine, as is fashionable nowadays, saturating the integrals with resonances: vector mesons for the u integral and $C=1$, $G=+1$, J even, partity even states (like the f_0) for the t integral. A characteristic problem encountered in trying to saturate superconvergence relations is the choice of the parameter s. We shall not discuss this particular relation further.

Let us turn to the case of Compton scattering by nucleons and consider the fixed-t dispersion relation for $\mathfrak{A}_a(s,t)-\mathfrak{A}_p(s,t)$. From our crossing relations we see that this combination is proportional to t-channel amplitude, $A_4(s,t)$. On the basis of Regge asymptotic arguments we would expect that for fixed $t\leq 0$, $\mathfrak{A}_a-\mathfrak{A}_p\sim s^{\alpha(t)}$ with $\alpha(t)<1$, *assuming* that there are no fixed poles in the complex angular momentum plane at $J=1$. The latter assumption is a rather violent one[18] and we shall return to this point in Appendix A. We shall show in fact that the sum rule we are about to obtain is the residue of a fixed pole at $J=1$. If, however, we demand that $(\mathfrak{A}_a-\mathfrak{A}_p)/2ps^{1/2}\cos\frac{1}{2}\Theta\to 0$ as $s\to\infty$ for fixed $t<0$, we are led to the following superconvergence relation from the fixed-t dispersion relation given at the end of Sec. IV:

$$\frac{2\mu^2}{m}=\frac{m(-t)^{1/2}}{\pi}\int_{\nu_0(t)}^{\infty}d\nu'\frac{\mathrm{Im}\mathfrak{A}_{\frac{1}{2}1;-\frac{1}{2}1}(\nu',t)}{(\nu'^2-p_t^2k_t^2)(\nu'-k_t^2)}$$

$$-\frac{t}{4\pi}\int_{\nu_0(t)}^{\infty}d\nu'\frac{\nu'-p_t^2}{\nu'-k_t^2}\,\mathrm{Im}\left[\frac{\mathfrak{A}_a(\nu',t)+\mathfrak{A}_p(\nu',t)}{(\nu'^2-p_t^2k_t^2)^{3/2}}\right]$$

$$-\frac{1}{\pi}\int_{\nu_0(t)}^{\infty}\frac{\nu'd\nu}{(\nu'^2-p_t^2k_t^2)^{3/2}}\,\mathrm{Im}[\mathfrak{A}_a(\nu',t)-\mathfrak{A}_p(\nu',t)].$$

[17] M. Gell-Mann, M. L. Goldberger, and W. Thirring, Phys. Rev. 95, 1612 (1954).

[18] H. D. I. Abarbanel *et al.*, Phys. Rev. 160, 1329 (1967).

This is a sum rule on the anomalous magnetic moment of the target which must hold for all t. It is traditional to evaluate such superconvergence relations at $t=0$ and we obtain

$$\frac{2\mu^2}{m}=\frac{1}{\pi}\int_{\nu_0(0)}^{\infty}\frac{d\nu'}{\nu'^2}\,\mathrm{Im}[\mathfrak{A}_p(\nu',0)-\mathfrak{A}_a(\nu',0)];$$

upon using $\mu^2\equiv(e^2/4m^2)\kappa^2=\pi(e^2/4\pi)\kappa^2/m^2=\pi\alpha\kappa^2/m^2$ and

$$\sigma_{a,p}=(4\pi/p)(m/4\pi s^{1/2})\mathrm{Im}\mathfrak{A}_{a,p}(\nu,0)=(m/\nu)\mathrm{Im}\mathfrak{A}_{a,p},$$

where $\sigma_{a,p}$ are the total nuclear cross sections for photons circularly polarized antiparallel or parallel to target spin, we find

$$\kappa^2=\frac{m^2}{2\pi^2\alpha}\int_{\nu_0(0)}^{\infty}\frac{d\nu'}{\nu'}[\sigma_p(\nu')-\sigma_a(\nu')].$$

This is the so-called Drell-Hearn sum rule[10] and we see that our original superconvergence relation is the generalization of it to $t\neq 0$.

In view of the fact that this result is so directly subject to experimental test, it is worth being very explicit about the assumption being made to derive it. One may start directly with the $t=0$ relation

$$\mathfrak{A}_a(\nu,0)-\mathfrak{A}_p(\nu,0)=\frac{-4\mu^2\nu}{m}+\frac{2\nu^3}{\pi}\int_{\nu_0}^{\infty}\frac{d\nu'}{\nu'^2-\nu^2}\,\frac{\mathrm{Im}[\mathfrak{A}_a(\nu',0)-\mathfrak{A}_p(\nu',0)]}{\nu'^2},$$

which may be derived from the usual analyticity requirements together with the assertion that $[\mathfrak{A}_a(\nu,0)-\mathfrak{A}_p(\nu,0)]/\nu$ approach a constant or go to zero as $\nu\to\infty$, in addition to the low-energy theorem. The crucial assumption is that $(\mathfrak{A}_a-\mathfrak{A}_p)/\nu\to 0$, *not a constant* as $\nu\to\infty$. If there is a fixed pole at $J=1$ in the odd signature part of t-channel amplitude $A_4(s,t)$ it will contribute a constant to $(\mathfrak{A}_a-\mathfrak{A}_p)/\nu$. Thus a failure of the Drell-Hern sum rule would be a direct experimental proof of the presence of a fixed singularity in the angular momentum plane in an electromagnetic process.

Let us turn now to our fixed-s dispersion relations. We can extract some interesting information by noting that the amplitude $\mathfrak{A}_{\frac{1}{2}-1;-\frac{1}{2}1}$ has a contribution from the exchange of a neutral pion in the t channel. To discuss this we write a fixed-s dispersion relation for $\mathfrak{A}_{\frac{1}{2}-1;-\frac{1}{2}1}/\sin^3(\frac{1}{2}\Theta_s)$ (which we have given previously) and write explicitly the contribution from the single neutral pion t-channel intermediate state:

$$\frac{\mathfrak{A}_{\frac{1}{2}-1;-\frac{1}{2}1}}{\sin^3(\frac{1}{2}\Theta_s)}=\frac{p}{m^2-\mu}\{-2e^2-p(4m^2\mu^2+8\mu em)/s^{1/2}\}+\frac{2p^3F_\pi g}{m_\pi^2-t}+\frac{8p^3}{\pi}\int_{t_0}^{\infty}\frac{dt'}{t'-t}\left[\frac{\mathfrak{A}_{\frac{1}{2}-1;-\frac{1}{2}1}(s,t')}{(-t')^{3/2}}\right]^t+\frac{8p^3}{\pi}\int_{u_0}^{\infty}\frac{du'}{u'-u}\left[\frac{\mathfrak{A}_{\frac{1}{2}-1;-\frac{1}{2}1}(s,u')}{(s+u'-2m^2)^{3/2}}\right]^u,$$

where g is the pion nucleon coupling constant ($g^2/4\pi\sim 15$) and F_π is $\pi^0\to 2\gamma$ amplitude factor defined by Goldberger and Treiman.[19] From Regge asymptotic arguments we expect that $\mathfrak{A}_{\frac{1}{2}-1;-\frac{1}{2}1}(\sin\frac{1}{2}\Theta_s)^{-3}$ behaves like $t^{\alpha(W)-3/2}$ for large t ($W=s^{1/2}$) and $\alpha(m)=\frac{1}{2}$, so that for $s<m^2$ we have superconvergence relation:

$$0=-2e^2s-\tfrac{1}{2}(s-m^2)(4m^2\mu^2+8\mu em)-\frac{(s-m^2)^2}{2m}F_\pi g-\frac{2(s-m^2)^2}{\pi}\int_{t_0}^{\infty}dt'\left[\frac{\mathfrak{A}_{\frac{1}{2}-1;-\frac{1}{2}1}(s,t')}{(-t')^{3/2}}\right]^t+\frac{2(s-m^2)^2}{\pi}\int_{u_0}^{\infty}du'\left[\frac{\mathfrak{A}_{\frac{1}{2}-1;-\frac{1}{2}1}(s,u')}{(s+u'-2m^2)^{3/2}}\right]^u.$$

If we imagine that at $s=0$ the contribution of the integrals is negligible and further take the t-channel isospin to be 1 (by subtracting from the above relation for a proton target the corresponding one for a neutron) we find (the κ's are anomalous moments in units of $e/2m$)

$$\frac{F_\pi g}{e^2}=\frac{\kappa_p^2-\kappa_n^2+4\kappa_p}{2m}.$$

This differs from the result of Goldberger and Treiman as corrected by Pagels[20] in that the coefficient of κ_p found by those authors was 2. The π^0 lifetime from our formula is $\tau\approx 0.5\times10^{-16}$ sec, which agrees quite well with experiment. If the t-channel isospin were zero we would encounter an η-meson pole (take the sum of proton and neutron scattering for this) and we find, at $s=0$,

$$\frac{F_\eta g_\eta}{e^2}=\frac{\kappa_p^2+\kappa_n^2+4\kappa_p}{2m},$$

where g_η is the ηNN coupling contrast.

We present these calculations of decay amplitudes, F, only as examples and not because we can give compelling reasons for the neglect of the continuum integrals at $s=0$. If one neglects the integrals, he may study F as a function of s and find that it is a slowly varying function in the region $-m^2/2\leq s\leq\frac{1}{2}m^2$ but that as $s\to m^2$ the neglect of the integrals is unacceptable—they in fact diverge there.

VI. CONCLUSIONS

We have given a derivation of the low-energy theorems for Compton scattering from spin-zero and spin-$\frac{1}{2}$ targets from the standpoint of dispersion theory. In addition we have presented unsubtracted single-variable dispersion relations for all of the physical amplitudes. To accomplish this we have exploited the

[19] M. L. Goldberger and S. B. Treiman, Nuovo Cimento **9**, 451 (1958).
[20] H. Pagels, Phys. Rev. **158**, 1566 (1967).

fact that because the transverse character of photons it is inevitably true that in either the s or t channel, one has enough helicity flip to exploit the kinematical zeros required by angular momentum conservation to make free subtractions in dispersion relations. We make the additional assumption, which we have based on Regge-pole ideas, that this minimal number of subtractions is adequate. We have been unable to trace in detail the corresponding assumptions on high-energy behavior which go into the conventional derivation of the low-energy theorems.

The unsubtracted dispersion relations for the Compton amplitudes may be exploited to derive superconvergence relations, one of which, for the pion lifetime, is new. Further, the relation of such sum rules to the presence or absence of fixed singularities in the angular momentum plane is clarified by our approach.

Using the techniques given here it is possible to write unsubtracted dispersion relations for the Compton effect on a target of any spin. Further, one may discuss low-energy theorems for any process involving the emission or absorption of massless particles of arbitrary spin (e.g., neutrinos or gravitons). A typical example is the Kroll-Ruderman theorem on meson photoproduction which one can prove utilizing our methods; considering the fixed-s dispersion relations for this process as well as for electroproduction of mesons, one may derive some new superconvergence sum rules.

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APPENDIX A

We wish to discuss the relation between the Drell-Hearn sum rule and the presence or absence of fixed poles in the angular momentum plane. The critical point for our discussion is that the combination of s-channel amplitudes entering that relation, namely $\mathfrak{A}_a-\mathfrak{A}_p$, is to within a constant the t-channel amplitude $A_{\frac{1}{2}-\frac{1}{2};1-1}+A_{-\frac{1}{2}\frac{1}{2};1-1}$; the latter in turn gets contributions from partial-wave transitions with J, P odd, even (like the A_1 meson) and J, P even, even (like the vacuum trajectory). As we shall see, A_1-type trajectories are dominant here and a fixed pole at $J=1$, corresponding as it does to physical signature, leads to an asymptotic form $\mathfrak{A}_a-\mathfrak{A}_p\sim\nu$ which would invalidate the Drell-Hearn sum rule.

We begin by constructing the parity-conserving helicity amplitudes[6] we have used elsewhere in this paper, namely

$$A_\pm=\frac{A_{\frac{1}{2}-\frac{1}{2};1-1}}{(1+z)\sin\Theta}\pm\frac{A_{-\frac{1}{2}+\frac{1}{2};1,-1}}{(1-z)\sin\Theta},$$

where $z=\cos\Theta$ and Θ is the t-channel scattering angle. The partial-wave expansions of these quantiites are given by

$$A_\pm=\sum_J(2J+1)[\tfrac{1}{2}a(J,t)+\tfrac{1}{2}b(J,t)][e_{21}{}^{J+}(z)+e_{21}{}^{J-}(z)]$$
$$\pm\sum_J(2J+1)[\tfrac{1}{2}a(J,t)-\tfrac{1}{2}b(J,t)][e_{21}{}^{J+}(z)-e_{21}{}^{J-}(z)]$$
$$=\sum_J(2J+1)[a(J,t)e_{21}{}^{J\pm}(z)+b(J,t)e_{21}{}^{J\mp}(z)],$$

where $a(J,t)$ and $b(J,t)$ are what are called $a_4{}^J$ and $a_3{}^J$ in Table I and correspond to the physical partial-wave transitions

$$a(J,t)=\langle\tfrac{1}{2}-\tfrac{1}{2}|T^J(t)|1-1\rangle_+,\quad J\text{ even},\quad P\text{ even},$$
$$b(J,t)=\langle\tfrac{1}{2}-\tfrac{1}{2}|T^J(t)|1-1\rangle_-,\quad J\text{ odd},\quad P\text{ even}.$$

We note then that

$$2(A_{\frac{1}{2}-\frac{1}{2};1-1}+A_{-\frac{1}{2}\frac{1}{2};1-1})=\sin\Theta\,A_++z\sin\Theta\,A_-,$$

and that $e_{21}{}^{J+}\sim z^{J-2}$, $e_{21}{}^{J-}\sim z^{J-3}$. Thus a Pomeranchuk trajectory contributes for large z [remember that this trajectory is in $a(J,t)$] a term $\sim z^{\alpha_P-2}$ to A_+ and a term $\sim z^{\alpha_P-3}$ to A_-, while the A_1 type leads to z^{α_1-3} to A_+ and z^{α_1-2} to A_-. It is the latter, therefore, which leads to the dominant asymptotic behavior of $A_{\frac{1}{2}-\frac{1}{2};1-1}+A_{-\frac{1}{2}\frac{1}{2};1-1}$, namely $\sim z^{\alpha_1}$; a fixed singularity at $J=1$ in $b(J,t)$ would lead to the z^1 referred to above. Finally, we remark in passing that because $a(J,t)$ and $b(J,t)$ are nonvanishing for even and odd J, respectively, A_+ (A_-) is an even (odd) function of z.

We may invert the partial-wave expansions and solve for $a(J,t)$ and $b(J,t)$:

$$a(J,t)=\frac{1}{2}\int_{-1}^{+1}dz[C_{21}{}^{J+}(z)A_+(z,t)+C_{12}{}^{J-}A_-(z,t)],$$

$$b(J,t)=\frac{1}{2}\int_{-1}^{+1}dz[C_{21}{}^{J+}(z)A-(z,t)+C_{21}{}^{J-}(z)A_+(z,t)].$$

For the purpose of discussing $a(J,t)$ and $b(J,t)$ in the complex angular momentum plane, we introduce fixed-t dispersion relations for $A_\pm(z,t)$, namely

$$A_\pm(z,t)=\frac{1}{\pi}\int_{z_0}^{\infty}dz'\,\mathrm{Im}A_\pm(z',t)\left(\frac{1}{z'-z}\pm\frac{1}{z'+z}\right),$$

which incorporate the known even-odd properties. Using the well-known relations (true for integral l)

$$Q_l(z')=\frac{1}{2}\int_{-1}^{+1}dz\frac{P_l(z)}{z'-z},\quad Q_l(-z)=-(-1)^lQ_l(z),$$

where P_l (Q_l) are Legendre functions of the first (second) kind together with the definitions of the

$C_{21}{}^{J\pm}(z)$, namely

$$C_{21}{}^{J+}=\frac{-[(J-1)(J+2)]^{1/2}}{(2J-1)(2J+1)(2J+3)}[(J+1)(2J+3)P_{J-2}(z)-3(2J+1)P_J-J(2J-1)P_{J+2}],$$

$$C_{21}{}^{J-}=\frac{-[(J-1)(J+2)]^{1/2}}{2J+1}[P_{J-1}-P_{J+1}],$$

we find for $a(J,t)$ and $b(J,t)$ the expressions

$$\frac{\begin{Bmatrix}a(J,t)\\ b(J,t)\end{Bmatrix}}{[(J-1)(J+2)]^{1/2}}=\frac{-2}{\pi}\left\{\int_{z_0}^{\infty}\frac{\operatorname{Im}A_\pm(z)}{(2J-1)(2J+1)(2J+3)}[(J+1)(2J+3)\times Q_{J-2}(z)-3(2J+1)Q_J(z)-J(2J-1)Q_{J+2}]+\int_{z_0}^{\infty}dz\frac{\operatorname{Im}A_\mp(z)}{2J+1}(Q_{J-1}-Q_{J+1})\right\}\left[\frac{1\pm(-1)^J}{2}\right].$$

In the usual way, we define partial-wave amplitudes of definite signature by taking the coefficient of the factor $[1\pm(-1)^J]/2$ for the extension into the complex J plane. Assuming that for the relevant t values, the signatured amplitudes $a^{(+)}$, $b^{(-)}$ may be defined by these integrals in the neighborhood of $J=1$, we may ask whether there is a fixed singularity at $J=1$ coming from the fact that

$$Q_{J-2}(z)\approx\frac{P_0(z)}{J-1}\quad\text{near}\quad J=1.$$

We have

$$\lim_{J\to1}(J-1)\frac{\begin{Bmatrix}a^{(+)}(J,t)\\ b^{(-)}(J,t)\end{Bmatrix}}{[(J-1)(J+2)]^{1/2}}=-\frac{4}{3\pi}\int_{z_0}^{\infty}dz\,\operatorname{Im}A_\pm(z,t)\equiv-p_t{}^2k_t{}^2\frac{4}{3}\begin{Bmatrix}R_a\\ R_b\end{Bmatrix},$$

or in terms of $\nu=p_tk_tz$,

$$\begin{Bmatrix}R_a(t)\\ R_b(t)\end{Bmatrix}=\frac{1}{\pi}\int_0^{\infty}d\nu\,\frac{\operatorname{Im}A_\pm(\nu,t)}{p_t{}^2k_t{}^2}.$$

The Born contributions to $\operatorname{Im}A_+$ are easily deduced from calculations carried out in the text and they are

$$\frac{\operatorname{Im}A_+(\nu,t)}{p_t{}^2k_t{}^2}=-\frac{2\pi}{mp_tk_t}(ea+\mu^2t/2)[\delta(\tfrac{1}{2}t-2\nu)-\delta(\tfrac{1}{2}t+2\nu)],$$

$$\frac{\operatorname{Im}A_-}{p_t{}^2k_t{}^2}=\frac{4\pi\mu^2}{m}[\delta(\tfrac{1}{2}t-2\nu)+\delta(\tfrac{1}{2}t+2\nu)],$$

and it should be remembered that to insure against subtractions we want to take $t<0$, so that it is $\delta(\frac{1}{2}t+2\nu)$ that contributes. We have then the following relations for $R_a(t)$ and $R_b(t)$:

$$R_a(t)=\frac{(ea+\mu^2t/2)}{mp_tk_t}+\frac{1}{\pi}\int_{\nu_0(t)}^{\infty}d\nu\frac{\operatorname{Im}A_+(\nu,t)}{p_t{}^2k_t{}^2},$$

$$R_b(t)=\frac{2\mu^2}{m}+\frac{1}{\pi}\int_{\nu_0(t)}^{\infty}d\nu\frac{\operatorname{Im}A_-(\nu,t)}{p_t{}^2k_t{}^2}.$$

We shall concentrate on $R_b(t)$ here although there are interesting aspects of $R_a(t)$ and we return to them elsewhere. To proceed, we relate A_- to s-channel quantities using the crossing relations given in the text:

$$\frac{A_-}{p_t{}^2k_t{}^2}(p_t{}^2k_t{}^2-\nu^2)=\frac{2k_t{}^2m}{p^2s^{1/2}}\frac{\mathfrak{A}_{\frac{1}{2}1;-\frac{1}{2}1}}{\sin\frac{1}{2}\Theta_s}-\frac{k_t{}^2(s^{1/2}-p)}{p^2s^{1/2}\cos\frac{1}{2}\Theta_s}\times(\mathfrak{A}_a+\mathfrak{A}_p)+\frac{\nu}{ps^{1/2}\cos\frac{1}{2}\Theta_s}(\mathfrak{A}_p-\mathfrak{A}_a).$$

Substituting this into $R_b(t)$ we find

$$R_b(t)=\frac{2\mu^2}{m}+\frac{1}{\pi}\int_{\nu_0(t)}^{\infty}\frac{d\nu}{p_t{}^2k_t{}^2-\nu^2}(-t)^{1/2}m\times\operatorname{Im}\left[\frac{\mathfrak{A}_{\frac{1}{2}1;-\frac{1}{2}1}(\nu,t)}{\nu-t/4}\right]-\tfrac{1}{4}t\int_{\nu_0(t)}^{\infty}d\nu\frac{\nu-p_t{}^2}{\nu-t/4}\frac{1}{p_t{}^2k_t{}^2-\nu^2}\times\operatorname{Im}\left[\frac{\mathfrak{A}_a(2,t)+\mathfrak{A}_p(p,t)}{(\nu^2-p_t{}^2k_t{}^2)^{1/2}}\right]+\int_{\nu_0(t)}^{\infty}d\nu\frac{\nu}{p_t{}^2k_t{}^2-\nu^2}\times\operatorname{Im}\left[\frac{\mathfrak{A}_p(\nu,t)-\mathfrak{A}_a(\nu,t)}{(\nu^2-p_t{}^2k_t{}^2)^{1/2}}\right],$$

which is just the Drell-Hearn relation for $t\neq0$ *if* $R_b(t)=0$. At $t=0$, our result reduces to

$$R_b(0)=\frac{2\mu^2}{m}-\frac{1}{m\pi}\int_{\nu_0(0)}^{\infty}\frac{d\nu}{\nu}[\sigma_p(\nu)-\sigma_a(\nu)]$$

upon using the optical theorem $\sigma_{p,a}(\nu)=(m/\nu)\operatorname{Im}\mathfrak{A}_{p,a}(\nu,0)$. Thus we see that a failure of the Drell-Hearn sum rule would be an *experimental* proof that $R_b(0)\neq0$, that there is a fixed pole in the angular momentum plane.

APPENDIX B

In evaluating the Born approximation and the absorptive parts of the Compton scattering amplitude as well as in the deduction of the crossing matrix, it was necessary to choose certain phase conventions, etc. We have carried out these calculations in a very pedestrian way, using explicit single-particle wave functions which we now present in detail.

Two-particle helicity states in the barycentric coordinate system are defined with the Jacob-Wick phase.[5] A photon moving along the positive z axis with polarization λ is described by a polarization vector

$$\boldsymbol{\varepsilon}=-\frac{\lambda}{\sqrt{2}}(\mathbf{e}_x+i\lambda\mathbf{e}_y),$$

where $\mathbf{e}_x$, $\mathbf{e}_y$ are unit vectors along the x and y axes, and λ takes on the values ±1. For a photon moving in the direction $\hat{k}=(\sin\Theta,\,0,\,\cos\Theta)$, i.e., in the x-z plane, we have

$$\boldsymbol{\varepsilon}(\hat{k})=-\frac{\lambda}{\sqrt{2}}(\mathbf{e}_x\cos\Theta-\mathbf{e}_z\sin\Theta+i\lambda\mathbf{e}_y).$$

If the photon is the "target," or what is called particle 2 by Jacob and Wick, and $\hat{p}$ is the projectile, or particle 1 direction specified by $\hat{p}=(\sin\Theta,\,0,\,\cos\Theta)$,

$$\boldsymbol{\varepsilon}(\hat{k})\rightarrow\boldsymbol{\varepsilon}(-\hat{p})=-\frac{\lambda}{\sqrt{2}}(-\mathbf{e}_x\cos\Theta+\mathbf{e}_z\sin\Theta+i\lambda\mathbf{e}_y)$$

obtained simply by replacing Θ in $\epsilon(\hat{k})$ by $\pi+\Theta$; the phase factor $(-1)^{s_2-\lambda_2}$ is unity in this case. We have taken the photon to be the "target" in defining our s-channel helicity amplitudes. The corresponding initial state "projectile" nucleon spinor is:

$$u(p_1,\lambda_a)=N_1\left(1+\frac{2\lambda_a p_1\rho_1}{E_1+m}\right)X_a,$$

where $E_1=(m^2+p_1^2)^{1/2}$, $N_1=[(E_1+m)/2m]^{1/2}$, ρ_1 is the 4×4 matrix

$$\rho_1=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix},$$

with I a 2×2 unit matrix, and X_a is a four-component spinor

$$X_a=\begin{pmatrix}X_{\lambda_a}\\ 0\end{pmatrix},$$

with X_{λ_a} a two-component spinor satisfying $\sigma_z X_{\lambda_a}=2\lambda_a X_{\lambda_a}$. The final-state spinor is given by

$$u(p_2,\lambda_c)=N_2\left(1+\frac{2\lambda_c p_2\rho_1}{E_2+m}\right)e^{-i\sigma_2\Theta/2}X_c,$$

where Θ is the scattered direction in the x-z plane, and

$$\bar{u}(p_2,\lambda_c)=N_2X_c^{\dagger}e^{i\sigma_2\Theta/2}\left(1-\frac{2\lambda_c p_2\rho_1}{E_2+m}\right).$$

Here σ_2 is the 4×4 matrix

$$\sigma_2=\begin{pmatrix}\sigma_y & 0\\ 0 & \sigma_y\end{pmatrix},$$

with σ_y the usual Pauli matrix, and X_c is a four-component spinor just like X_a such that $\sigma_z X_c=2\lambda_c X_c$.

The Born-approximation invariant amplitude for the s-channel process $p_1(a)+k_1(b)\rightarrow p_2(c)+k_2(d)$, where the helicity labels are in parentheses, is, for spin-zero targets,

$$\mathcal{B}_{\lambda_d;\lambda_c}=e^2\left\{\frac{\epsilon_d^*\cdot(2p_2+k_2)(2p_1+k_1)\cdot\epsilon_b}{m^2-s}+\frac{\epsilon_d^*\cdot(2p_1-k_2)(2p_2-k_1)\cdot\epsilon_b}{m^2-u}-2\epsilon_d^*\cdot\epsilon_b\right\},$$

where in the s-channel barycentric system the fourth components of the polarization vectors are zero,

$$\mathbf{e}_b=-\lambda_b(-\mathbf{e}_x+i\lambda\mathbf{e}_y)/\sqrt{2},$$
$$\mathbf{e}_d^*=-\lambda_d(-\mathbf{e}_x\cos\Theta_s+\mathbf{e}_z\sin\Theta_s-i\lambda_d\mathbf{e}_y)/\sqrt{2},$$

and $s=-(p_1+k_1)^2$, $u=-(p_1-k_2)^2$. The physical scattering amplitude f is related to the invariant amplitude by $f=[8\pi s^{1/2}]^{-1}\times$invariant. The t-channel Born amplitude B corresponding to the process $k_2'(D)+k_1(b)\rightarrow p_2(c)+p_1'(A)$ is *defined* to be that obtained from the above $\mathcal{B}$ by the substitution $k_2\rightarrow-k_2'$, $p_1\rightarrow-p_1'$, $\boldsymbol{\varepsilon}_d^*\rightarrow\boldsymbol{\varepsilon}_D=-\lambda_D(\mathbf{e}_x+i\lambda_D\boldsymbol{\varepsilon}_y)/\sqrt{2}$, where the transverse polarizations of the photons are taken to have zero fourth component in the t-barycentric system. One finds by explicit evaluation that the relation between s- and t-channel Born terms is

$$\mathcal{B}_{1;1}=-B_{;1-1},\quad \mathcal{B}_{1;-1}=-B_{;11}.$$

This crossing relation is, of course, true for the entire amplitudes as may be immediately verified from the Trueman-Wick crossing relations[16] in a more elementary fashion by writing the general amplitude in terms of invariants and eliminating the latter from the helicity amplitudes in the two channels. A convenient representation for this exercise is

$$T=A\{q_1\cdot q_2\epsilon_2\cdot\epsilon_1-\epsilon_2\cdot q_1\epsilon_1\cdot q_2\}+B\{q_1\cdot q_2\epsilon_2\cdot P\epsilon_1\cdot P-P\cdot K[\epsilon_2\cdot P\epsilon_1\cdot q_2+\epsilon_2\cdot q_1\epsilon_1\cdot P]+\epsilon_2\cdot\epsilon_1(P\cdot K)^2\},$$

where A, B are scalar functions of s, t and in the s-channel, $q_1=k_1$, $q_2=k_2$, $\epsilon_2=\epsilon_d^*$, $\epsilon_1=\epsilon_c$, $P=\frac{1}{2}(p_1+p_2)$, $K=\frac{1}{2}(k_1+k_2)$; in the t-channel $\epsilon_2=\epsilon_D$, $\epsilon_1=\epsilon_c$, $q_1=k_1$, $q_2=-k_2'$, $P=\frac{1}{2}(p_2-p_1')$, $K=\frac{1}{2}(k_1-k_2')$. Recall that according to our convention, $\cos\Theta_t=\hat{k}_2'\cdot\hat{p}_2$. One finds

$$\mathfrak{A}_{1;1}=\tfrac{1}{4}[st+(s-m^2)^2]B=-A_{;1-1},$$
$$\mathfrak{A}_{1;-1}=\tfrac{1}{4}t[2A+(\tfrac{1}{4}t-m^2)B]=-A_{;11}.$$

The minus signs are, of course, totally without significance since the relative phase of the *physical* s- and t-channel amplitudes is not measurable. One does indeed get a unique, continuation-path-dependent, answer by going from one channel to another, but this unique answer may differ by a unimodular factor from one's definition of the physical amplitude.

For completeness we give the Born approximation invariant amplitude for a spin-$\frac{1}{2}$ target and the s-channel process $p_1(a)+k_1(b) \to p_2(c)+k_2(d)$:

$$\mathcal{B}_{\lambda_c\lambda_d;\lambda_a\lambda_b}=\bar{u}(p_2,\lambda_c)\left\{\epsilon_d^*\cdot\Gamma(p_2,p_2+k_2)\frac{1}{i\gamma\cdot(p_1+k_1)+m}\right.$$
$$\times\epsilon_b\cdot\Gamma(p_1+k_1,\,p_1)+\epsilon_b\cdot\Gamma(p_2,\,p_2-k_1)$$
$$\left.\times\frac{1}{i\gamma\cdot(p_1-k_2)+m}\epsilon_d^*\cdot\Gamma(p_1-k_2,\,p_1)\right\}u(p_1,\lambda_a),$$

where ϵ_b and ϵ_d^* are as given above for the spin-zero case, and

$$\Gamma_\lambda(p',p)=ie\gamma_\lambda-i\mu\sigma_{\lambda\nu}(p'-p)_\nu,$$

with μ the anomalous magnetic moment. Our γ-matrices are Hermitian and satisfy $\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2\delta_{\mu\nu}$, $\sigma_{\lambda\nu}=[\gamma_\lambda,\gamma_\nu]/2i$. The standard representation, the one in which our explicit spinors are given, is

$$\gamma_4=\rho_3\begin{pmatrix}I & 0\\ 0 & I\end{pmatrix},\quad \gamma=\rho_2\boldsymbol{\sigma}=\begin{pmatrix}0 & -i\boldsymbol{\sigma}\\ i\boldsymbol{\sigma} & 0\end{pmatrix}.$$

The physical scattering amplitude is obtained from the invariant amplitude by multiplying the latter by $m/4\pi s^{1/2}$.

The s,t-crossing relations in the spin-$\frac{1}{2}$ target problem may be taken from the appropriate zero-mass limit of the Trueman-Wick relations or, as in the spin-zero case, be deduced by brute strength by introducing and eliminating invariants. The precise form of the results depends in an unessential way on conventions made at various stages. Ours were the following: The invariant amplitude in the s channel [the reaction $p_1(a)+k_1(b) \to p_2(c)+k_2(d)$] is written as

$$\mathfrak{A}_{\lambda_c\lambda_d;\lambda_a\lambda_b}=e_\mu^*(k_2,\lambda_d)\bar{u}(p_2,\lambda_c)\mathfrak{M}_{\mu\nu}(p_2k_2;\,p_1k_1)$$
$$\times u(p_1,\lambda_a)\epsilon_\nu(k_1,\lambda_b),$$

where $\mathfrak{M}_{\mu\nu}$ is a 4×4 Dirac matrix and a tensor in Minkowski space. The t-channel invariant amplitude corresponding to the process $k_2'(D)+k_1(b) \to p_2(c)+p_1'(A)$ where particle A is an antiparticle is taken to be

$$A_{\lambda_c\lambda_A;\lambda_D\lambda_b}=\bar{u}(p_2,\lambda_c)\mathfrak{M}_{\mu\nu}(p_2,-k_2';\,-p_1'k_1)v(p_1',\lambda_A)$$
$$\times\epsilon_\mu(k_2',\lambda_D)\epsilon_\nu(k_1,\lambda_b),$$

where $v(p_1',\lambda_A)$ is a negative-energy spinor to be defined in a moment.

We evaluate the helicity amplitudes in the center-of-mass systems of the respective channels and the photon polarization vectors have zero fourth components in each system. The polarization vectors are exactly the same as those used for the spin-zero case. The antiparticle of momentum p_1', helicity λ_A is the "target" or particle "2" in the Jacob-Wick sense and the spinor $v(p_1',\lambda_A)$ is

$$v(p_1',\lambda_A)=\left(\frac{E_1'+m}{2m}\right)^{1/2}\left(1-\frac{2\lambda_A p_1'\rho_1}{E_1'+m}\right)e^{-i\sigma_2\Theta_t/2}i\sigma_2\rho_1 X_A,$$

where

$$X_A=\begin{pmatrix}X_{-\lambda_A}\\ 0\end{pmatrix},$$

and Θ_t is, as before, the angle between $\hat{k}'_2$ and $\hat{p}'_2$. The appearance of the Pauli spinor $X_{-\lambda_A}$ rather than X_{λ_A} is a consequence of the fact that the antiparticle is particle "2."

Since the invariants are to be eliminated between the helicity amplitudes *any* choice may be made. We found it convenient to use the following:

$$\mathfrak{M}_{\mu\nu}(p_2k_2;p_1k_1)$$
$$=M_1P_\mu'P_\nu'+M_2L_\mu L_\nu+M_3(P_\mu'L_\nu-L_\mu P_\nu')i\gamma_5$$
$$+M_4P_\mu'P_\nu'i\gamma\cdot K+M_5L_\mu L_\nu i\gamma\cdot K$$
$$+M_6(P_\mu'L_\nu+L_\mu P_\nu')i\gamma_5 i\gamma\cdot K,$$

where

$$L_\mu=i\epsilon_{\mu\nu\lambda\sigma}P_\nu'K_\lambda Q_\sigma,\quad P'=P-(P\cdot K/K^2)K,$$
$$P=\tfrac{1}{2}(p_1+p_2),\quad K=\tfrac{1}{2}(k_1+k_2),\quad Q=\tfrac{1}{2}(k_1-k_2).$$