

A simple efficient algorithm in frustration-free one-dimensional gapped systems

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We propose an efficient algorithm for the ground state of frustration-free one-dimensional gapped Hamiltonians. This algorithm is much simpler than the original one by Landau et al., and thus may be easily accessible to a general audience in the community. We present all the details in two pages.

Computing the ground state (energy) of local Hamiltonians is a fundamental problem in condensed matter physics and the emerging area of Hamiltonian complexity [4, 8]. In a recent remarkable paper, Landau et al. [7] proposed a randomized polynomial-time algorithm for the (unique) ground state of frustration-free one-dimensional (1D) gapped Hamiltonians. Huang [5] extended it to general 1D gapped systems. Chubb and Flammia [3] studied gapped spin chains with degenerate ground states.

This line of research is very technical. Here we significantly simplify the method in the hope that the results are easily accessible to a general audience. For this purpose, we will not use tricks that are not essential at a high level, even if they can improve the performance of the algorithm. The new ingredients of our approach allow us to get rid of many technical tools in [7]. We present all the details in two pages.

Consider a chain of n spins (qudits), where the local dimension d of each spin is an absolute constant. Let \mathcal{H}_i be the Hilbert space of the spin i ; define $\mathcal{H}_{[i,j]} = \bigotimes_{k=i}^j \mathcal{H}_k$ as the Hilbert space of the spins with indices in $[i, j]$. Since the standard bra-ket notation can be cumbersome, in most but not all cases quantum states and their inner products are denoted by $\psi, \phi \dots$ and $\langle \psi, \phi \rangle$, respectively. All states are normalized unless otherwise noted. Let $H = \sum_{i=1}^{n-1} H_i$ be a 1D Hamiltonian with H_i acting on $\mathcal{H}_{[i,i+1]}$ (nearest-neighbor interaction). Suppose H has a unique ground state Ψ_0 and a constant energy gap ϵ . The goal is to find an efficient matrix product state (MPS) approximation to Ψ_0 . The existence of such an MPS is a by-product of the proof of the area law for entanglement.

Lemma 1 ([1]). *There exists an MPS Ψ of bond dimension $n^{o(1)}$ such that $|\langle \Psi, \Psi_0 \rangle| \geq 1 - n^{-\omega(1)}$.*

The best known algorithm is

Theorem 1 ([5]). *There is a deterministic $n^{O(1)}$ -time algorithm that outputs an MPS Ψ such that $|\langle \Psi, \Psi_0 \rangle| \geq 1 - 1/\text{poly } n$.*

Suppose H is frustration-free, i.e., Ψ_0 is in the ground space of each H_i . Assume without loss of generality that each H_i is a projector, i.e., $H_i^2 = H_i$, so that the ground-state energy of H is zero. Let $\tilde{O}(x) := O(x \text{ poly log } x)$ hide a polylogarithmic factor. We give a simple proof of

Theorem 2. *There is a randomized $n^{\tilde{O}(1/\epsilon)}$ -time algorithm that outputs an MPS Ψ such that $|\langle \Psi, \Psi_0 \rangle| \geq$*

$1 - 1/\text{poly } n$ with probability at least $1 - 1/\text{poly } n$.

We begin by recalling some known facts and/or tools.

Lemma 2. $|\langle \psi, \phi \rangle| \geq \Omega(n^{-2}/\sqrt{d})$ with probability $1 - O(n^{-2})$ for two random states $\psi, \phi \in \mathbf{C}^d$.

Fix a cut $i|i+1$ separating the spins i and $i+1$.

Lemma 3 ([2]). *A matrix product operator A_i of bond dimension $2^{\tilde{O}(1/\epsilon)}$ can be efficiently constructed such that (i) $A_i \Psi_0 = \Psi_0$; (ii) $A_i \Psi_1 \perp \Psi_0$ and $\|A_i \Psi_1\|^2 \leq \Delta$ for any $\Psi_1 \perp \Psi_0$; (iii) A_i^l has Schmidt rank $2^{\tilde{O}(l/\epsilon^3)} D^l$ across the cut $i|i+1$; (iv) $D\Delta \leq 1/2$ with $D = 2^{\text{poly log}(1/\epsilon)}$.*

We can get $D^2\Delta \leq 1/2$ by modifying some unimportant constants in the construction of A_i .

Definition 1. A state $\psi_l \in \mathcal{H}_{[1,i]}$ is a (i, δ, b) -left state if (i) there exists a state $\psi_r \in \mathcal{H}_{[i+1,n]}$ such that $|\langle \Psi_0, \psi_l \otimes \psi_r \rangle| \geq \delta$; (ii) ψ_l is an MPS of bond dimension b .

Definition 2. Let $\psi = \sum_{j \geq 1} \lambda_j l_j \otimes r_j$ be the Schmidt decomposition of a state across the cut $i|i+1$, where the Schmidt coefficients are in descending order: $\lambda_1 \geq \lambda_2 \geq \dots > 0$. Define $\text{trunc}_i^D \psi = \sum_{j=1}^D \lambda_j l_j \otimes r_j / \sqrt{\sum_{i=1}^D \lambda_j^2}$.

The next lemma is an immediate corollary of the fact that the best rank- D approximation to ψ is $\text{trunc}_i^D \psi$.

Lemma 4 ([7]). *Suppose ϕ is a state of Schmidt rank D across the cut $i|i+1$. Then, $|\langle \text{trunc}_i^{D/\eta} \psi, \phi \rangle| \geq |\langle \psi, \phi \rangle| - \eta$ for any $\eta > 0$.*

The algorithm proceeds by iteratively constructing an (i, δ, b) -left state for $i = 1, 2, \dots, n$, where $b = n^{1+o(1)}/\delta$ with δ to be specified later. Each iteration has one random step that succeeds with probability $1 - O(n^{-2})$. Thus, the overall failure probability is $O(n^{-1})$.

Suppose we have an $(i-1, \delta, b)$ -left state ψ_l . By definition 1, there exists a state $\psi_r = \sum_{j=1}^d \lambda_j |j\rangle_i \otimes r_j$ such that $|\langle \Psi_0, \psi_l \otimes \psi_r \rangle| \geq \delta$, where $\{|j\rangle_i\}_{j=1}^d$ is the computational basis of \mathcal{H}_i and $r_j \in \mathcal{H}_{[i+1,n]}$. Lemma 3 implies that

A_i^l can be decomposed as $A_i^l = \sum_{J=1}^{2^{\tilde{O}(1/\epsilon^3)} D^l} \Lambda_J L_J \otimes R_J$, where L_J (R_J) is a matrix product operator of bond dimension $2^{\tilde{O}(1/\epsilon)}$ acting on $\mathcal{H}_{[1,i]}$ ($\mathcal{H}_{[i+1,n]}$). We have

$$\phi_1 \propto A_i^l \psi_l \otimes \psi_r = \sum_{J=1}^{2^{\tilde{O}(1/\epsilon^3)} D^l} \sum_{j=1}^d \Lambda_J \lambda_j L_J \psi_l |j\rangle_i \otimes R_J r_j,$$

$$|\langle \Psi_0, \phi_1 \rangle|^2 \geq \delta^2 / (\delta^2 + \Delta^l (1 - \delta^2)) \geq 1 - \Delta^l / \delta^2. \quad (1)$$

Let ϕ_2 be a random state in $\text{span}\{L_J\psi_l \otimes |j\rangle_i\}$. With probability $1 - O(n^{-2})$, there exists $\psi'_r \in \text{span}\{R_{Jr}j\}$ such that $|\langle \phi_1, \phi_2 \otimes \psi'_r \rangle| \geq \Omega(n^{-2}D^{-l/2})$. Hence,

$$|\langle \Psi_0, \phi_2 \otimes \psi'_r \rangle| \geq \Omega(n^{-2}D^{-l/2}) - O(\Delta^{l/2}/\delta). \quad (2)$$

Choosing $l = O(\log n)$ and $\delta = n^{\text{poly} \log(1/\epsilon)}$ suitably, we have $\Delta^l = \Theta(\delta^4)$ and that the right-hand side of (2) is greater than 4δ .

We obtain ϕ_3 by truncating each bond (in whatever order) of ϕ_2 to $n^{1+o(1)}/\delta$. Lemma 1 implies an MPS Ψ of bond dimension $n^{o(1)}$ such that $1 - |\langle \Psi, \Psi_0 \rangle| \ll \delta^2$. Hence $|\langle \Psi, \phi_2 \otimes \psi'_r \rangle| \geq 3\delta$. Lemma 4 implies $|\langle \Psi, \phi_3 \otimes \phi'_r \rangle| \geq 2\delta$. Finally, $|\langle \Psi_0, \phi_3 \otimes \phi'_r \rangle| \geq \delta$, and ϕ_3 is an (i, δ, b) -left state.

The final output of the algorithm is ϕ_1 in the last iteration with the error estimate $|\langle \Psi_0, \phi_1 \rangle| \geq 1 - O(\delta^2)$. It is an MPS of bond dimension $b2^{l\tilde{O}(1/\epsilon)} = n^{\tilde{O}(1/\epsilon)}$. It is easy to see that the running time of the algorithm is $n^{\tilde{O}(1/\epsilon)}$.

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my approach has some overlap with theirs. Although the present work is independent, IA, ZL, UV, & TV deserve the credit of the overlapping part because they had the ideas since May 2015. They will post their results soon.

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- [1] I. Arad, A. Kitaev, Z. Landau, and U. Vazirani. An area law and sub-exponential algorithm for 1D systems. arXiv:1301.1162v1.
- [2] I. Arad, Z. Landau, and U. Vazirani. Improved one-dimensional area law for frustration-free systems. *Physical Review B*, 85(19):195145, 2012.
- [3] C. T. Chubb and S. T. Flammia. Computing the degenerate ground space of gapped spin chains in polynomial time. arXiv:1502.06967.
- [4] S. Gharibian, Y. Huang, Z. Landau, and S. W. Shin. Quantum Hamiltonian complexity. *Foundations and Trends in Theoretical Computer Science*, 10(3):159–282, 2015.
- [5] Y. Huang. A polynomial-time algorithm for the ground state of one-dimensional gapped Hamiltonians. arXiv:1406.6355.
- [6] Z. Landau. Local algorithms for 1D quantum systems. <https://simons.berkeley.edu/talks/zeph-landau-2015-05-05>. The slides were posted in mid-October 2015.
- [7] Z. Landau, U. Vazirani, and T. Vidick. A polynomial time algorithm for the ground state of one-dimensional gapped local Hamiltonians. *Nature Physics*, 11(7):566–569, 2015.
- [8] T. J. Osborne. Hamiltonian complexity. *Reports on Progress in Physics*, 75(2):022001, 2012.