Differential rotation of the unstable nonlinear r-modes

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At second order in perturbation theory, the r-modes of uniformly rotating stars include an axisymmetric part that can be identified with differential rotation of the background star. If one does not include radiation reaction, the differential rotation is constant in time and has been computed by Sá. It has a gauge dependence associated with the family of time-independent perturbations that add differential rotation to the unperturbed equilibrium star: For stars with a barotropic equation of state, one can add to the time-independent second-order solution arbitrary differential rotation that is stratified on cylinders (that is a function of distance ϖ to the axis of rotation). We show here that the gravitational radiation-reaction force that drives the r-mode instability removes this gauge freedom; the exponentially growing differential rotation of the unstable second-order r-mode is unique. We derive a general expression for this rotation law for Newtonian models and evaluate it explicitly for slowly rotating models with polytropic equations of state.

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I. INTRODUCTION

Unstable r-modes [1,2] may limit the angular velocity of old neutron stars spun up by accretion and may contribute to the spin down of nascent neutron stars (see Refs. [3–6] for references and reviews). Spruit [7] argued that angular momentum loss from the star would generate differential rotation, because the loss rate depends on the mode shape and varies over the star. Growing differential rotation winds up and amplifies the star's magnetic field, and Rezzolla and collaborators [8–10] studied the possibility that the energy lost to the magnetic field would damp out the r-mode instability. (In Spruit's scenario, a buoyancy instability of the greatly enhanced magnetic field could power a γ-ray burst.) To estimate the magnetic-field wind-up, Rezzolla *et al.* used a drift velocity of a fluid element; this is second order in perturbation theory, but because the second-order velocity field had not been computed, they estimated it by integrating the first-order velocity field. Subsequently, Cuofano *et al.* [11,12] used this estimate of drift velocity to study the evolution of the r-mode instability damped by magnetic field wind-up.[1]

Following Spruit's work, Levin and Ushomirsky found the differential rotation of the unstable r-mode in a toy model of a spherical shell of fluid [14]. Sá [15] then carried out the first computation of the differential rotation associated with a stable r-mode of uniformly rotating barotropic Newtonian stellar models and, with collaborators, looked at implications of the calculation for the unstable mode [16,17]. The differential rotation arises at second order in perturbation theory as a time-independent, axisymmetric part of the solution to the perturbed Euler equations; for the r-mode whose linear part is associated with the angular harmonic $Y^{\ell\ell}$, Sá's solution has the form

$$\delta^{(2)}\Omega = \alpha^2\Omega C_\Omega\left(\frac{z}{R}\right)^2\left(\frac{\varpi}{R}\right)^{2\ell-4} + \alpha^2\delta_N^{(2)}\Omega(\varpi). \qquad (1)$$

Here α measures the amplitude of the first-order perturbation, C_Ω is dimensionless and of order unity, the z axis is the axis of rotation, and ϖ is the distance from the axis. The function $\delta_N^{(2)}\Omega(\varpi)$ is arbitrary. This ambiguity in the rotation law is present for the following reason. One can perturb a uniformly rotating barotropic star by adding differential rotation, changing the angular velocity from Ω to $\Omega+\delta\Omega(\varpi)$. If $\delta\Omega(\varpi)$ is chosen to be quadratic in α, $\delta\Omega(\varpi)=\alpha^2\delta_N^{(2)}\Omega(\varpi)$, it and the corresponding time-independent perturbations of density, pressure, and gravitational potential Φ constitute a solution to the

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[1]Work by Abbassi *et al.* [13] also looks at the damping of r-modes due to a magnetic field; here, however, the magnetic dissipation arises from magnetic diffusivity in a linearized magnetohydrodynamics treatment.

time-independent second-order perturbation equations. Cao *et al.* [18] use a particular choice of $\delta^{(2)}\Omega$ to recompute the magnetic damping.

In the present paper, we show that the second-order radiation-reaction force removes the ambiguity in the differential rotation associated with the Newtonian r-modes. In effect, the degeneracy in the space of zero-frequency solutions is broken by the radiation-reaction force, which picks out a unique differential rotation law that depends on the neutron-star equation of state. We find an explicit formula for that rotation law for the unstable r-modes of slowly rotating stars.

To lowest nonvanishing post-Newtonian order, the growth time τ of the radiation-reaction driven Chandrasekhar-Friedman-Schutz instability instability of an r-mode is given by

$$\beta \equiv \frac{1}{\tau} = C_\beta \frac{G}{c^{2\ell+3}} M R^{2\ell} \Omega^{2\ell+2},$$

where C_β is a dimensionless constant that depends on the equation of state. In using the Newtonian Euler equation together with the radiation-reaction force at lowest nonvanishing post-Newtonian order, we are neglecting radiation-reaction terms smaller by factors of $\mathcal{O}(R\Omega/c)$ and $\mathcal{O}(GM/Rc^2)$; this means, in particular, that we keep only terms linear in the dimensionless parameter β/Ω.

Three small parameters appear in the paper: the amplitude α of the perturbation; the dimensionless growth rate β/Ω; and, in the final, slow-rotation part of the paper, the angular velocity Ω. For the logic of the paper, it is helpful to note that these three parameters can be regarded as independent of one another. The growth rate β can be varied by changing the equation of state of the material while keeping α and Ω fixed; for example, in polytropes (stars based on the polytropic equation of state $p = K\rho^n$), one can change β by changing the polytropic constant K.

The plan of the paper is as follows. Section II lists the equations governing a Newtonian star acted on by a post-Newtonian radiation-reaction force, with the star modeled as a self-gravitating perfect fluid. In Sec. III, we discuss first- and second-order perturbations of a uniformly rotating star. From the second-order equations, we obtain a formal expression for the unique differential rotation law of an unstable r-mode in terms of the first-order perturbations and second-order contributions that will turn out to be of higher order in Ω. Up to this point in the paper, the analysis holds for rapidly rotating stars. In Sec. IV, we specialize to a slowly rotating background, keeping terms of lowest nonvanishing order in Ω and thereby obtaining an explicit formula for the radiation-reaction induced differential rotation. Finally, a discussion section briefly comments on the validity of the results for an accreting neutron star, when one includes magnetic fields, nonzero initial data for other modes, and viscosity.

Our notation for fluid perturbations is chosen to make explicit the orders of the expansions in the amplitude α and angular velocity Ω. The notation is defined as it is introduced in Secs. II and III, but, for easy reference, we also provide a table that summarizes the notation in Appendix A. We use gravitational units, setting $G = c = 1$.

II. NEWTONIAN STELLAR MODELS

Let $Q = \{\rho, v^a, p, \Phi\}$ denote the collection of fields that determine the state of the fluid in a self-gravitating Newtonian stellar model. The quantity ρ represents the mass density, v^a the fluid velocity, p the pressure, and Φ the gravitational potential. For a barotropic equation of state $p = p(\rho)$, the specific enthalpy h of the fluid is

$$h = \int_0^p \frac{dp}{\rho}, \tag{2}$$

and we define a potential U by

$$U = h + \Phi. \tag{3}$$

The evolution of the fluid is determined by Euler's equation, the mass-conservation law, and the Poisson equation for the Newtonian gravitational potential. These equations may be written as

$$E^a \equiv \partial_t v^a + v^b \nabla_b v^a + \nabla^a U = f^a_{GR}, \tag{4}$$

$$0 = \partial_t \rho + \nabla_a(\rho v^a), \tag{5}$$

$$\nabla^2 \Phi = 4\pi\rho. \tag{6}$$

The version of the Euler equation that we use, Eq. (4), includes $\vec{f}_{GR}$, the post-Newtonian gravitational radiation-reaction force (per unit mass). This force plays a central role in the nonlinear evolution of the r-modes that is the primary focus of our paper. It is given by

$$\vec{f}_{GR} = \sum_{l\geq 2} \sum_{|m|\leq l} \frac{(-1)^{\ell+1} N_\ell}{32\pi} \Re\left\{ \frac{\vec{\nabla}(r^\ell Y^{\ell m})}{\sqrt{\ell}} \frac{d^{2\ell+1} I^{\ell m}}{dt^{2\ell+1}} - \frac{2 r^\ell \vec{Y}^{\ell m}_B}{\sqrt{\ell+1}} \frac{d^{2\ell+2} S^{\ell m}}{dt^{2\ell+2}} - \frac{2\vec{v}\times\vec{\nabla}(r^\ell Y^{\ell m})}{\sqrt{\ell}} \frac{d^{2\ell+1} S^{\ell m}}{dt^{2\ell+1}} \right\}, \tag{7}$$

where $\Re(Z)$ denotes the real part of a complex quantity Z. The quantities $I^{\ell m}$ and $S^{\ell m}$ are the complex mass and current multiple moments of the fluid source [cf. Thorne [19] Eqs. (5.18a) and (5.18b)] defined by

$$I^{\ell m} = \frac{N_\ell}{\sqrt{\ell}} \int \rho r^\ell Y^{*\ell m} d^3x, \tag{8}$$

$$S^{\ell m} = \frac{2N_\ell}{\sqrt{\ell+1}} \int \rho r^\ell \vec{v} \cdot \vec{Y}^{*\ell m}_B d^3x, \tag{9}$$

with N_ℓ the constant,

$$N_\ell = \frac{16\pi}{(2\ell+1)!!}\sqrt{\frac{(\ell+2)(\ell+1)}{2(\ell-1)}}. \qquad (10)$$

The functions $Y^{\ell m}$ are the standard spherical harmonics, while the $\vec{Y}_B^{\ell m}$ are the magnetic-type vector harmonics defined by

$$\vec{Y}_B^{\ell m} = \frac{\vec{r}\times\vec{\nabla}Y^{lm}}{\sqrt{\ell(\ell+1)}}. \qquad (11)$$

We use the normalizations $1 = \int |Y^{\ell m}|^2 d\cos\theta d\phi$ and $1 = \int |\vec{Y}_B^{\ell m}|^2 d\cos\theta d\phi$ for these spherical harmonics. In Cartesian coordinates, $\vec{r}$ is given by $\vec{r} = (x, y, z)$. We point out that this expression for the gravitational radiation-reaction force, Eq. (7), agrees with the mass-multipole part of the force given by Ipser and Lindblom [20]. It also agrees with the current-multipole part of the force given by Lindblom, *et al.* [21] (following Blanchet [22] and Rezzolla *et al.* [23]) for the $\ell = 2$ and $m = 2$ case. The general form of the force given in Eq. (7), however, is new.

The post-Newtonian radiation-reaction force is gauge dependent, so the expression for it is not unique. We derived the expression for the force given in Eq. (7) by requiring that it implies a time-averaged (over several oscillation periods) power $\langle\langle dE/dt\rangle\rangle|_{GR}$ (which is gauge invariant) and angular momentum flux $\langle\langle d\vec{J}/dt\rangle\rangle|_{GR}$ lost to gravitational waves that agree with the standard post-Newtonian expressions, cf. Thorne [19]. We present expressions for these flux quantities in Appendix B that are equivalent to, but are somewhat simpler than, the standard ones.

We consider small perturbations of rigidly rotating, axisymmetric, barotropic equilibrium models (models with a barotropic equation of state). The fluid velocity in these equilibria is denoted

$$\vec{v} = \Omega\vec{\phi}, \qquad (12)$$

where $\vec{\phi}$ generates rotations about the z axis; in Cartesian coordinates, $\vec{\phi} = (-y, x, 0)$. For barotropic equilibria, Euler's equation reduces to

$$0 = \nabla_a\left(h + \Phi - \frac{1}{2}\varpi^2\Omega^2\right), \qquad (13)$$

where h is the specific enthalpy of the fluid and ϖ is the cylindrical radial coordinate, $\varpi^2 = x^2 + y^2$. The surface of the star is the boundary where the pressure and the enthalpy vanish: $p = h = 0$.

III. PERTURBED STELLAR MODELS

We denote by $Q(\alpha, t, \vec{x})$ a one-parameter family of stellar models. For each value of the parameter α, $Q(\alpha, t, \vec{x})$ satisfies the full nonlinear time-dependent Eqs. (4)–(6). We assume that the model with $\alpha = 0$ is an axisymmetric equilibrium model, as described in Eqs. (12) and (13). The exact perturbation δQ, defined as the difference between $Q(\alpha)$ and $Q(0)$, is defined everywhere on the intersection of the domains where $Q(\alpha)$ and $Q(0)$ are defined:

$$\delta Q(\alpha, t, \vec{x}) \equiv Q(\alpha, t, \vec{x}) - Q(0, t, \vec{x}). \qquad (14)$$

It is also useful to define $\delta^{(n)}Q$, the derivatives of the one-parameter family $Q(\alpha)$ evaluated at the unperturbed stellar model, where $\alpha = 0$:

$$\delta^{(n)}Q(t, \vec{x}) = \frac{1}{n!}\frac{\partial^n Q(\alpha, t, \vec{x})}{\partial\alpha^n}\bigg|_{\alpha=0}. \qquad (15)$$

These derivatives can be used to define a formal power series expansion for δQ:

$$\delta Q(\alpha, t, \vec{x}) = \alpha\delta^{(1)}Q(t, \vec{x}) + \alpha^2\delta^{(2)}Q(t, \vec{x}) + \mathcal{O}(\alpha^3). \qquad (16)$$

Each point in the interior of the unperturbed star is, for sufficiently small α, in the interior of the perturbed star; the derivatives $\delta^{(n)}Q$ defined in Eq. (15) and the formal power series expansion in Eq. (16) are thus well defined at all points of the interior of the unperturbed star, but may diverge at the surface. We consider constant-mass sequences of stellar models, i.e., models whose exact mass perturbations, $\delta M = M(\alpha) - M(\alpha = 0)$, vanish identically for all values of α. The integrals of the nth-order density perturbations therefore vanish identically for these models:

$$0 = \frac{1}{n!}\frac{d^n M(\alpha)}{d\alpha^n}\bigg|_{\alpha=0} = \int \delta^{(n)}\rho\sqrt{g}d^3x. \qquad (17)$$

The exact (to all orders in the perturbation parameter α) perturbed evolution equations for these stellar models can be written in the form

$$\delta E^a = (\partial_t + \Omega \pounds_\phi)\delta v^a + 2\Omega\delta v^b\nabla_b\phi^a + \nabla^a\delta U + \delta v^b\nabla_b\delta v^a = \delta f^a_{GR}, \qquad (18)$$

$$0 = (\partial_t + \Omega\pounds_\phi)\delta\rho + \nabla_a(\rho\delta v^a + \delta\rho\delta v^a), \qquad (19)$$

$$\nabla^2\delta\Phi = 4\pi\delta\rho, \qquad (20)$$

where $\pounds_\phi$ is the Lie derivative along the vector field $\vec{\phi}$ and ρ is the density of the unperturbed star. The exact perturbed gravitational radiation-reaction force $\delta\vec{f}_{GR}$ that appears in

Eq. (18) is given by

$$\delta\vec{f}_{GR} = \sum_{l\geq 2}\sum_{|m|\leq l}\frac{(-1)^{\ell+1}N_\ell}{32\pi}\Re\left\{\frac{\vec{\nabla}(r^\ell Y^{\ell m})}{\sqrt{\ell}}\frac{d^{2\ell+1}\delta I^{\ell m}}{dt^{2\ell+1}} - \frac{2r^\ell\vec{Y}_B^{\ell m}}{\sqrt{\ell+1}}\frac{d^{2\ell+2}\delta S^{\ell m}}{dt^{2\ell+2}} - \frac{2\Omega\vec{\phi}\times\vec{\nabla}(r^\ell Y^{\ell m})}{\sqrt{\ell}}\frac{d^{2\ell+1}\delta S^{\ell m}}{dt^{2\ell+1}} - \frac{2\delta\vec{v}\times\vec{\nabla}(r^\ell Y^{\ell m})}{\sqrt{\ell}}\frac{d^{2\ell+1}\delta S^{\ell m}}{dt^{2\ell+1}}\right\}, \tag{21}$$

where

$$\delta I^{\ell m} = \frac{N_\ell}{\sqrt{\ell}}\int \delta\rho r^\ell Y^{*\ell m} d^3x, \tag{22}$$

$$\delta S^{\ell m} = \frac{2N_\ell}{\sqrt{\ell+1}}\int r^\ell[\rho\delta\vec{v} + \delta\rho(\Omega\vec{\phi} + \delta\vec{v})]\cdot\vec{Y}_B^{*\ell m} d^3x. \tag{23}$$

It is convenient to decompose the perturbations δQ into parts $\delta_N Q$ that satisfy the pure Newtonian evolution equations and parts $\delta_R Q$ caused by the addition of the gravitational radiation-reaction force. In particular, the nonradiative stellar perturbations $\delta_N Q$ satisfy the perturbed Euler equation:

$$\delta\vec{E} = 0. \tag{24}$$

When the effects of gravitational radiation-reaction are included, the complete perturbation, δQ, satisfies the Euler equation driven by the gravitational radiation-reaction force

$$\delta\vec{E} = \delta\vec{f}_{GR}. \tag{25}$$

A. First-order perturbations

The classical first-order (in powers of α) r-modes have angular and temporal dependence [4,24]

$$\delta_N^{(1)}\rho = \delta_N^{(1)}\hat{\rho}_- \sin\psi_N, \tag{26}$$

$$\delta_N^{(1)}v^a = \varpi^{-2}\phi^a\phi_b\delta_N^{(1)}\hat{v}_+^b\sin\psi_N + P^a{}_b\delta_N^{(1)}\hat{v}_+^b\cos\psi_N, \tag{27}$$

$$\delta_N^{(1)}U = \delta_N^{(1)}\hat{U}_-\sin\psi_N, \tag{28}$$

$$\delta_N^{(1)}\Phi = \delta_N^{(1)}\hat{\Phi}_-\sin\psi_N, \tag{29}$$

where $\psi_N = \omega_N t + m\phi$, with $m \neq 0$. The tensor

$$P^a{}_b \equiv \delta^a{}_b - \varpi^{-2}\phi^a\phi_b \tag{30}$$

is the projection operator orthogonal to ϕ^a, and $\delta_N^{(1)}\hat{Q} = \delta_N^{(1)}\hat{Q}(\varpi, z)$ depends on the cylindrical coordinates ϖ and z, but not on ϕ or t. The origin of time has been chosen to give the perturbations definite parity under the diffeomorphism $\phi\to-\phi$ at $t=0$. We use the term ϕ- *parity* to mean parity under this transformation. The subscripts $\pm$ indicate that $\delta_N^{(1)}\hat{\rho}_-$, $\delta_N^{(1)}\hat{U}_-$, and $\delta_N^{(1)}\hat{\Phi}_-$ are parts of odd ϕ-parity scalars, while $\delta_N^{(1)}\hat{v}_+^a$ is part of an even ϕ-parity vector field.

When gravitational radiation reaction is included, the Euler equation is altered by the relatively weak radiation-reaction force $\vec{f}_{GR}$. The first-order radiation-reaction force can be written in the form

$$\delta^{(1)}\vec{f}_{GR} = \beta\delta_N^{(1)}\vec{v}_+ + \delta_\perp^{(1)}\vec{f}_{GR+}, \tag{31}$$

where β is the growth rate of the r-mode instability, and $\delta_\perp^{(1)}\vec{f}_{GR+}$ is (by definition) the even ϕ-parity part of the radiation-reaction force that is orthogonal to $\delta_N^{(1)}\vec{v}_+$ and that therefore does not contribute directly to the energy evolution of the mode. Equation (21) implies that the odd ϕ-parity part of the radiation-reaction force, $\delta_\perp^{(1)}\vec{f}_{GR-}$, vanishes when the classical r-mode is chosen to have the ϕ-parity given in Eqs. (26)–(29). The gravitational radiation-reaction force causes an instability by introducing an imaginary part β to the frequency of the mode. The overall structure of the modes is therefore changed in the following way (schematically),

$$\delta^{(1)}\rho = (\delta_N^{(1)}\hat{\rho}_- + \delta_R^{(1)}\hat{\rho}_-)\sin\psi e^{\beta t} + \delta_R^{(1)}\hat{\rho}_+\cos\psi e^{\beta t}, \tag{32}$$

$$\delta^{(1)}v^a = \delta_R^{(1)}\hat{v}_-^b[\varpi^{-2}\phi^a\phi_b\cos\psi + P^a{}_b\sin\psi]e^{\beta t} + (\delta_N^{(1)}\hat{v}_+^b + \delta_R^{(1)}\hat{v}_+^b) \times[\varpi^{-2}\phi^a\phi_b\sin\psi + P^a{}_b\cos\psi]e^{\beta t}, \tag{33}$$

$$\delta^{(1)}U = (\delta_N^{(1)}\hat{U}_- + \delta_R^{(1)}\hat{U}_-)\sin\psi e^{\beta t} + \delta_R^{(1)}\hat{U}_+\cos\psi e^{\beta t}, \tag{34}$$

$$\delta^{(1)}\Phi = (\delta_N^{(1)}\hat{\Phi}_- + \delta_R^{(1)}\hat{\Phi}_-)\sin\psi e^{\beta t} + \delta_R^{(1)}\hat{\Phi}_+\cos\psi e^{\beta t}, \tag{35}$$

where $\psi = \psi_N + \psi_R = \omega_N t + \omega_R t + m\phi$. The radiative corrections $\delta_R^{(1)}\hat{Q}$ are smaller than the nonradiative perturbations $\delta_N^{(1)}\hat{Q}$ by terms of order $\mathcal{O}(\beta/\omega_N)$. The radiative correction ω_R to the frequency is smaller than

ω_N by a term of order $\mathcal{O}(\beta/\omega_N)^2$, so we ignore that change here, setting $\psi = \psi_N$.[2]

The radiative corrections to the r-mode, $\delta_R^{(1)}Q$, are determined by substituting Eqs. (32)–(35) into the first-order perturbed mass conservation and Euler equations. After applying the equations satisfied by the nonradiative parts of the perturbations, $\delta_N^{(1)}Q$, the resulting system of equations can be divided into parts proportional to $\sin\psi_N$ and $\cos\psi_N$ respectively, each of which must vanish separately. The resulting equations can be divided further into a set that determines $\delta_R^{(1)}\hat\rho_-$, $\delta_R^{(1)}\hat U_-$, and $\delta_R^{(1)}\hat v_+^a$ and another that determines $\delta_R^{(1)}\hat\rho_+$, $\delta_R^{(1)}\hat U_+$, and $\delta_R^{(1)}\hat v_-^a$.

The equations that determine the radiative corrections having the same ϕ-parity as the classical nonradiative r-modes are then

$$(\omega_N + m\Omega)\delta_R^{(1)}\hat\rho_- + m\rho\varpi^{-2}\phi_a\delta_R^{(1)}\hat v_+^a + \nabla_a(\rho P^a{}_b\delta_R^{(1)}\hat v_+^b) = 0, \tag{36}$$

$$[(\omega_N + m\Omega)\phi_a + 2\varpi\Omega\nabla_a\varpi]\delta_R^{(1)}\hat v_+^a = -m\delta_R^{(1)}\hat U_-, \tag{37}$$

$$\left[(\omega_N + m\Omega)P^a{}_b + \frac{2}{\varpi}\Omega\nabla^a\varpi\phi_b\right]\delta_R^{(1)}\hat v_+^b = P^{ab}\nabla_b\delta_R^{(1)}\hat U_-. \tag{38}$$

These equations are homogeneous and are identical to those satisfied by the classical r-modes. The solutions for $\delta_R^{(1)}\hat\rho_-$, $\delta_R^{(1)}\hat U_-$, and $\delta_R^{(1)}\hat v_+^a$ are therefore proportional to the classical r-modes: $\delta_N^{(1)}\hat\rho_-$, $\delta_N^{(1)}\hat U_-$, and $\delta_N^{(1)}\hat v_+^a$. The effect of adding these radiative corrections to the classical r-modes is simply to rescale its amplitude. We choose to keep the amplitude, α, of the mode fixed, and therefore without loss of generality, we set

$$0 = \delta_R^{(1)}\hat\rho_- = \delta_R^{(1)}\hat U_- = \delta_R^{(1)}\hat v_+^a. \tag{39}$$

It follows that the first-order radiative corrections have ϕ-parity opposite to that of the classical r-modes: $\delta_R^{(1)}\hat\rho = \delta_R^{(1)}\hat\rho_+$, $\delta_R^{(1)}\hat U = \delta_R^{(1)}\hat U_+$, and $\delta_R^{(1)}\hat v^a = \delta_R^{(1)}\hat v_-^a$. They are determined by the equations

$$(\omega_N + m\Omega)\delta_R^{(1)}\hat\rho + m\rho\varpi^{-2}\phi_a\delta_R^{(1)}\hat v^a - \nabla_a(\rho P^a{}_b\delta_R^{(1)}\hat v^b) = \beta\delta_N^{(1)}\rho, \tag{40}$$

$$[(\omega_N + m\Omega)\phi_a - 2\varpi\Omega\nabla_a\varpi]\delta_R^{(1)}\hat v^a + m\delta_R^{(1)}\hat U = \phi_b\delta_\perp^{(1)}\hat f_{GR}^b, \tag{41}$$

$$[(\omega_N + m\Omega)P^a{}_b - \frac{2}{\varpi}\Omega\nabla^a\varpi\phi_b]\delta_R^{(1)}\hat v^b + P^{ab}\nabla_b\delta_R^{(1)}\hat U = P^a{}_b\delta_\perp^{(1)}\hat f_{GR}^b. \tag{42}$$

The general solution to the inhomogeneous system, Eqs. (40)–(42), for $\delta_R^{(1)}\hat\rho$, $\delta_R^{(1)}\hat U$, and $\delta_R^{(1)}\hat v^a$ consists of an arbitrary solution to the homogeneous equations (obtained by setting $\beta\delta_N^{(1)}\hat\rho = \delta_\perp^{(1)}f_{GR}^a = 0$) plus a particular solution. These homogeneous equations are identical to Eqs. (36)–(38), so their general solution is a multiple of the classical r-modes. Because their ϕ-parity is opposite to that of the classical r-modes, the effect of the homogeneous contributions $\delta_R^{(1)}\hat\rho$, $\delta_R^{(1)}\hat U$, and $\delta_R^{(1)}\hat v^a$ is to change the overall phase of the mode. We choose (by appropriately adjusting the time that we label $t = 0$) to keep this phase unchanged, and we can therefore, without loss of generality, set to zero the homogeneous parts of the solutions to Eqs. (36)–(38). The inhomogeneous terms on the right sides of Eqs. (40)–(42), $\beta\delta_N^{(1)}\hat\rho$ and $\delta_\perp^{(1)}\hat f_{GR}^a$, are all of order β. Thus, the particular solution to Eqs. (40)–(42) must also be of order β as well. It follows that the radiation-reaction corrections to the first-order r-modes $\delta_R^{(1)}Q$ are smaller than the classical r-modes $\delta_N^{(1)}Q$ by terms of order $\mathcal{O}(\beta/\omega)$. To lowest order in β, therefore, the corrections to the first-order r-modes in Eqs. (32)–(35) simply change the overall scale of the mode by the factor $e^{\beta t}$: $\delta^{(1)}Q = \delta_N^{(1)}Qe^{\beta t}$.

B. Second-order perturbations

The second-order perturbation equations are a sum of terms linear in $\delta^{(2)}Q$ and terms quadratic in $\delta^{(1)}Q$. For example, the second-order perturbation of the Euler equation, $\delta^{(2)}E^a = \frac{1}{2}\frac{d^2}{d\alpha^2}E^a\big|_{\alpha=0}$, includes the term $\delta^{(1)}v^b\nabla_b\delta^{(1)}v^a$, which serves as an effective source term for the second-order perturbations $\delta^{(2)}v^a$ and $\delta^{(2)}U$. In the absence of gravitational radiation reaction, it follows that the second-order Newtonian r-mode $\delta_N^{(2)}Q$ is a sum of terms of three kinds: a term with angular and temporal dependence $\cos(2\psi_N)$, where $\psi_N = m\phi + \omega_N t$, a term with dependence $\sin(2\psi_N)$, and a term that is time independent and axisymmetric. This time-independent axisymmetric part of the velocity perturbation can be regarded as differential rotation. As we have emphasized in the Introduction, the second-order Newtonian r-modes are not determined uniquely; given a particular solution

[2] Friedman and Schutz [25] derive the following general expression for the frequencies of the modes of Lagrangian systems (including Newtonian fluids with gravitational radiation-reaction forces): $0 = A(\omega + i\beta)^2 - (B + iD)(\omega + i\beta) - C$, where A, B, C, and D are real. The term D vanishes for nondissipative Newtonian fluid stars. When D is small, it is straightforward to show that the real part of the frequency, ω, differs from the frequency of the nondissipative $D = 0$ system, ω_N, by terms of order D^2: $\omega = \omega_N + \mathcal{O}(D^2)$. It is also easy to show that the imaginary part of the frequency β is proportional to D for a mode with $\beta_N = 0$.

$\delta^{(2)}_{NP}Q$ to the second-order Newtonian perturbation equations with perturbed velocity field $\delta^{(2)}_{NP}v^a$, there is a family of solutions $\delta^{(2)}_{N}Q$ with perturbed velocity field $\delta^{(2)}_{N}v^a = \delta^{(2)}_{NP}v^a + \delta^{(2)}_{N}\Omega(\varpi)\phi^a$, where $\delta^{(2)}_{N}\Omega(\varpi)$ is arbitrary. This degeneracy is broken by the gravitational radiation reaction. The presence of the radiation-reaction force picks out a unique $\delta^{(2)}v^a$ that displays the gravitational radiation driven growth of the second-order r-modes: $\delta^{(2)}v^a \propto e^{2\beta t}$.

To find this differential rotation law, one must solve the second-order axisymmetric perturbation equations with the radiation-reaction force for the axisymmetric parts of the second-order r-modes. Denote the axisymmetric part of a perturbation δQ by $\langle\delta Q\rangle$, and denote by $\delta^{(2)}\Omega$ the exponentially growing differential rotation of the unstable r-mode:

$$\delta^{(2)}\Omega \equiv \langle\delta^{(2)}_{N}v^\phi\rangle e^{2\beta t} = [\langle\delta^{(2)}_{NP}v^\phi\rangle + \delta^{(2)}_{N}\Omega(\varpi)]e^{2\beta t}. \qquad (43)$$

Without solving the full system, however, one can obtain a formal expression for $\delta^{(2)}\Omega$ in terms of the known first-order perturbation together with other parts of the second-order axisymmetric perturbation. As we will see in the next section, this expression is all that is needed to find $\delta^{(2)}\Omega$ to lowest nonvanishing order in Ω: The other parts of the second-order perturbation give only higher-order contributions. Finding this formal expression for $\delta^{(2)}\Omega$ and showing that it is unique are the goals of the present section.

We now turn our attention to solving the perturbation equations for the axisymmetric parts of the second-order r-modes. The axisymmetric parts of the second-order perturbations can be written in terms of their radiative and nonradiative pieces:

$$\langle\delta^{(2)}\rho\rangle = (\langle\delta^{(2)}_{N}\rho\rangle + \langle\delta^{(2)}_{R}\rho\rangle)e^{2\beta t}, \qquad (44a)$$

$$\langle\delta^{(2)}v^a\rangle = (\langle\delta^{(2)}_{N}v^a\rangle + \langle\delta^{(2)}_{R}v^a\rangle)e^{2\beta t}, \qquad (44b)$$

$$\langle\delta^{(2)}U\rangle = (\langle\delta^{(2)}_{N}U\rangle + \langle\delta^{(2)}_{R}U\rangle)e^{2\beta t}, \qquad (44c)$$

$$\langle\delta^{(2)}\Phi\rangle = (\langle\delta^{(2)}_{N}\Phi\rangle + \langle\delta^{(2)}_{R}\Phi\rangle)e^{2\beta t}, \qquad (44d)$$

$$\langle\delta^{(2)}f^a_{GR}\rangle = \langle\delta^{(2)}_{R}f^a_{GR}\rangle e^{2\beta t}. \qquad (44e)$$

These quantities are determined by the second-order axisymmetric parts of the perturbed stellar evolution equations:

$$2\beta\langle\delta^{(2)}v^a\rangle + 2\Omega\langle\delta^{(2)}v^b\rangle\nabla_b\phi^a + \nabla^a\langle\delta^{(2)}U\rangle = \langle\delta^{(2)}f^a_{GR}\rangle - \langle\delta^{(1)}v^b\nabla_b\delta^{(1)}v^a\rangle, \qquad (45)$$

$$2\beta\langle\delta^{(2)}\rho\rangle + \nabla_a[\rho\langle\delta^{(2)}v^a\rangle + \langle\delta^{(1)}\rho\delta^{(1)}v^a\rangle] = 0, \qquad (46)$$

$$\nabla^2\langle\delta^{(2)}\Phi\rangle = 4\pi\langle\delta^{(2)}\rho\rangle. \qquad (47)$$

The uniqueness of the second-order differential rotation $\delta^{(2)}\Omega$ can be seen as follows. Let $\langle\delta^{(2)}Q\rangle$ and $\langle\delta^{(2)}\tilde{Q}\rangle$ be two solutions to the second-order perturbation equations, Eqs. (45)–(47), associated with the same time dependence $e^{2\beta t}$ and with the same first-order solution $\delta^{(1)}Q$. The difference $\langle\delta^{(2)}Q\rangle - \langle\delta^{(2)}\tilde{Q}\rangle$ of the two solutions then satisfies the *linearized* Poisson equation and the linearized Euler and mass conservation equations obtained by setting to zero the terms involving $\delta^{(1)}v^a$ and $\delta^{(2)}f^a_{GR}$ in Eqs. (45) and (46). That is, $(\langle\delta^{(2)}Q\rangle - \langle\delta^{(2)}\tilde{Q}\rangle)e^{2\beta t}$ is an axisymmetric solution to the first-order Newtonian perturbation equations. But the Newtonian star has no such solution, no mode with growth rate 2β. Thus, $(\langle\delta^{(2)}Q\rangle - \langle\delta^{(2)}\tilde{Q}\rangle)e^{2\beta t} = 0$, implying that $\delta^{(2)}\Omega$ is unique. [Note, however, that the decomposition (43) is not unique; the arbitrariness in the differential rotation of the Newtonian r-mode means that one is free to add to $\langle\delta^{(2)}_{NP}v^\phi\rangle$ an arbitrary function $f(\varpi)$ if one simultaneously changes $\delta^{(2)}_{N}\Omega(\varpi)$ to $\delta^{(2)}_{N}\Omega(\varpi) - f(\varpi)$.]

We now obtain equations for $\delta^{(2)}_{N}Q$ and $\delta^{(2)}_{R}Q$. Keeping terms to first order in β, the terms quadratic in first-order perturbed quantities that appear in Eqs. (45) and (46) have the forms

$$\langle\delta^{(1)}v^b\nabla_b\delta^{(1)}v^a\rangle = (\langle\delta^{(1)}_{N}v^b\nabla_b\delta^{(1)}_{N}v^a\rangle + \beta\langle\delta^{(2)}_{R}V^a\rangle)e^{2\beta t}, \qquad (48)$$

$$\langle\delta^{(1)}\rho\delta^{(1)}v^a\rangle = (\langle\delta^{(1)}_{N}\rho\delta^{(1)}_{N}v^a\rangle + \beta\langle\delta^{(2)}_{R}W^a\rangle)e^{2\beta t}, \qquad (49)$$

where

$$\beta\langle\delta^{(2)}_{R}V^a\rangle = \langle\delta^{(1)}_{R}v^b\nabla_b\delta^{(1)}_{N}v^a\rangle + \langle\delta^{(1)}_{N}v^b\nabla_b\delta^{(1)}_{R}v^a\rangle, \qquad (50)$$

$$\beta\langle\delta^{(2)}_{R}W^a\rangle = \langle\delta^{(1)}_{R}\rho\delta^{(1)}_{N}v^a\rangle + \langle\delta^{(1)}_{N}\rho\delta^{(1)}_{R}v^a\rangle. \qquad (51)$$

The nonradiative parts $\langle\delta^{(2)}_{N}Q\rangle$ of the perturbations are determined, up to a perturbation that adds differential rotation $\delta^{(2)}_{N}\Omega(\varpi)$, by the axisymmetric parts of the Newtonian Euler and mass-conservation equations:

$$2\Omega\langle\delta^{(2)}_{N}v^b\rangle\nabla_b\phi^a + \nabla^a\langle\delta^{(2)}_{N}U\rangle = -\langle\delta^{(1)}_{N}v^b\nabla_b\delta^{(1)}_{N}v^a\rangle, \qquad (52)$$

$$\nabla_a[\rho\langle\delta^{(2)}_{N}v^a\rangle + \langle\delta^{(1)}_{N}\rho\delta^{(1)}_{N}v^a\rangle] = 0. \qquad (53)$$

Given a particular solution $\delta^{(2)}_{NP}Q$ to these equations, we want to find the remaining contribution $\delta^{(2)}_{N}\Omega(\varpi)$ to the differential rotation of Eq. (43) that is picked out by the radiation reaction.

We define the radiative part of the perturbation, $\langle\delta_R^{(2)}Q\rangle$, by requiring that it be created entirely by the radiation-reaction forces; $\langle\delta_R^{(2)}Q\rangle$ is therefore proportional to the radiation-reaction rate β. When $\langle\delta_N^{(2)}Q\rangle$ satisfies the Newtonian equations (52) and (53), the axisymmetric parts of the full perturbed Euler and mass-conservation equations with radiation reaction have at $\mathcal{O}(\beta)$ the form

$$2\beta\langle\delta_N^{(2)}v^a\rangle + 2\Omega\langle\delta_R^{(2)}v^b\rangle\nabla_b\phi^a + \nabla^a\langle\delta_R^{(2)}U\rangle = \langle\delta_R^{(2)}f^a_{GR}\rangle - \beta\langle\delta_R^{(2)}V^a\rangle, \tag{54}$$

$$\nabla_a(\rho\langle\delta_R^{(2)}v^a\rangle) = -2\beta\langle\delta_N^{(2)}\rho\rangle - \beta\nabla_a\langle\delta_R^{(2)}W^a\rangle. \tag{55}$$

To find an expression for $\delta_N^{(2)}\Omega(\varpi)$, we first write $\langle\delta_N^{(2)}v^a\rangle$ as $\langle\delta_{NP}^{(2)}v^a\rangle + \delta_N^{(2)}\Omega(\varpi)\phi^a$ and move the term involving $\langle\delta_{NP}^{(2)}v^a\rangle$ to the right side of Eq. (55),

$$2\beta\delta_N^{(2)}\Omega(\varpi)\phi^a + 2\Omega\langle\delta_R^{(2)}v^b\rangle\nabla_b\phi^a + \nabla^a\langle\delta_R^{(2)}U\rangle = \beta\langle\delta_R^{(2)}F^a\rangle, \tag{56}$$

where

$$\beta\langle\delta_R^{(2)}F^a\rangle = \langle\delta_R^{(2)}f^a_{GR}\rangle - 2\beta\langle\delta_{NP}^{(2)}v^a\rangle - \beta\langle\delta_R^{(2)}V^a\rangle. \tag{57}$$

We next write the components of the axisymmetric part of the second-order perturbed Euler equation, Eq. (56), in cylindrical coordinates:

$$2\beta\varpi\delta_N^{(2)}\Omega(\varpi) + 2\Omega\langle\delta_R^{(2)}v^\varpi\rangle = \beta\varpi\langle\delta_R^{(2)}F^\phi\rangle, \tag{58a}$$

$$-2\Omega\varpi\langle\delta_R^{(2)}v^\phi\rangle = -\partial_\varpi\langle\delta_R^{(2)}U\rangle + \beta\langle\delta_R^{(2)}F^\varpi\rangle, \tag{58b}$$

$$0 = -\partial_z\langle\delta_R^{(2)}U\rangle + \beta\langle\delta_R^{(2)}F^z\rangle. \tag{58c}$$

Using Eq. (58a) to determine $\langle\delta_R^{(2)}v^\varpi\rangle$, the axisymmetric part of the second-order mass conservation Eq. (55) can be written as

$$\frac{\beta}{2\Omega\varpi}\partial_\varpi[\rho\varpi^2(\langle\delta_R^{(2)}F^\phi\rangle - 2\delta_N^{(2)}\Omega(\varpi))] + \partial_z[\rho\langle\delta_R^{(2)}v^z\rangle] = -2\beta\langle\delta_N^{(2)}\rho\rangle - \beta\nabla_a\langle\delta_R^{(2)}W^a\rangle. \tag{59}$$

The star's surface is defined as the $p = 0$ surface. Because $\delta^{(2)}\rho$ is a derivative evaluated at $\alpha = 0$, it has support on the unperturbed star. While the density perturbation $\delta^{(2)}\rho$ is not finite for some equations of state at the surface of the star, it is integrable in the sense that $\delta^{(2)}\int\rho dz$ is finite, as one would expect from the integrability of the mass-conservation condition in Eq. (17). In particular, for polytropes with fractional polytropic index $0 < n < 2$, $\delta^{(2)}\rho$ diverges at $z = z_S$, but, as we show in Appendix C, $\delta^{(2)}\int\rho dz$ is finite. Here, we denote by $z_S(\varpi)$ the value of z (the Cartesian coordinate axis parallel to the rotation axis) at the surface of the unperturbed star.

We now multiply the second-order mass conservation equation, Eq. (59), by $2\varpi\Omega/\beta$ and integrate with respect to z over the support of the star. It will be convenient to extend the domain of integration to extend slightly beyond the surface of the unperturbed star. Because each integrand has support on the unperturbed star, we simply take the integrals to extend from $-\infty$ to ∞ instead of $-z_S$ to z_S. We then have

$$0 = 4\varpi\Omega\int_{-\infty}^{\infty}dz\langle\delta_N^{(2)}\rho\rangle + \int_{-\infty}^{\infty}dz\partial_\varpi[\rho\varpi^2(\langle\delta_R^{(2)}F^\phi\rangle - 2\delta_N^{(2)}\Omega(\varpi))] + 2\varpi\Omega\int_{-\infty}^{\infty}dz\nabla_a\langle\delta_R^{(2)}W^a\rangle. \tag{60}$$

The second integral on the right side of Eq. (60) can be rewritten as

$$\int_{-\infty}^{\infty}dz\partial_\varpi[\rho\varpi^2(\langle\delta_R^{(2)}F^\phi\rangle - 2\delta_N^{(2)}\Omega(\varpi))] = \partial_\varpi\int_{-\infty}^{\infty}dz\rho\varpi^2(\langle\delta_R^{(2)}F^\phi\rangle - 2\delta_N^{(2)}\Omega(\varpi)). \tag{61}$$

The expression in Eq. (60) can then be integrated from $\varpi = 0$ to ϖ, using Eq. (61), to obtain an expression for $\delta_N^{(2)}\Omega(\varpi)$:

$$2\varpi^2\delta_N^{(2)}\Omega(\varpi)\int_{-\infty}^{\infty}dz\rho = \varpi^2\int_{-\infty}^{\infty}dz\rho\langle\delta_R^{(2)}F^\phi\rangle + 4\Omega\int_0^\varpi d\varpi'\varpi'\int_{-\infty}^{\infty}dz\langle\delta_N^{(2)}\rho\rangle + 2\Omega\int_0^\varpi d\varpi'\varpi'\int_{-\infty}^{\infty}dz\nabla_a\langle\delta_R^{(2)}W^a\rangle. \tag{62}$$

Because of the axisymmetry of its integrand, the third term on the right side of Eq. (62) is, up to a factor of 2π, the volume integral of a divergence. The boundary of the three-dimensional region of integration has two parts: One is outside the surface of the star, where $\delta_R^{(2)}W^a$ vanishes; the second is the cylinder at constant ϖ from $-z_S$ to z_S, with outward normal $\nabla_a\varpi$ and element of area $\varpi d\phi dz$. The term is then given by

$$\int_0^\varpi d\varpi'\varpi'\int_{-\infty}^{\infty}dz\nabla_a\langle\delta_R^{(2)}W^a\rangle = \varpi\int_{-\infty}^{\infty}dz\langle\delta_R^{(2)}W^\varpi\rangle. \tag{63}$$

With this simplification, Eq. (62) can be written in the form

$$
\begin{aligned}
2\varpi^2\delta_N^{(2)}\Omega(\varpi)\int_{-\infty}^{\infty}dz\rho &= \varpi^2\int_{-\infty}^{\infty}dz\rho\langle\delta_R^{(2)}F^\phi\rangle \\
&+4\Omega\int_0^{\varpi}d\varpi'\varpi'\int_{-\infty}^{\infty}dz\langle\delta_N^{(2)}\rho\rangle \\
&+2\varpi\Omega\int_{-\infty}^{\infty}dz\langle\delta_R^{(2)}W^\varpi\rangle. \qquad (64)
\end{aligned}
$$

This provides a formal expression for $\delta_N^{(2)}\Omega(\varpi)$ in terms of the first-order perturbations that comprise $\langle\delta_R^{(2)}F^\phi\rangle$ and $\langle\delta_R^{(2)}W^\varpi\rangle$ and the second-order perturbation $\langle\delta_N^{(2)}\rho\rangle$.[3] Together with $\langle\delta_{NP}^{(2)}v^\phi\rangle$, it determines the differential rotation of the unstable r-mode.

We conclude this section with a discussion of two simplifications in evaluating $\delta_N^{(2)}\Omega(\varpi)$, one from the fact that we work to first order in the growth rate β and the second from the slow-rotation approximation of the next section. The first is a simplification of the expression for the radiation-reaction force. The integrand of the first term in Eq. (64), $\rho\langle\delta_R^{(2)}F^\phi\rangle$, is given by the ϕ-component of Eq. (57):

$$
\beta\langle\delta_R^{(2)}F^\phi\rangle = \langle\delta_R^{(2)}f_{GR}^\phi\rangle - 2\beta\langle\delta_N^{(2)}v^\phi\rangle - \beta\langle\delta_R^{(2)}V^\phi\rangle. \qquad (65)
$$

To evaluate $\langle\delta_R^{(2)}f_{GR}^\phi\rangle$, we must find the axisymmetric, second-order, part of the expression for $\delta\vec{f}_{GR}$ on the right side of Eq. (21). Recall that the axisymmetric parts of any second-order quantity have time dependence $e^{2\beta t}$. The first three terms in the bracketed expression in Eq. (21) involve high-order time derivatives of $\delta^{(2)}I^{\ell 0}$ or $\delta^{(2)}S^{\ell 0}$ and are therefore proportional to high powers of β and can be neglected. We are left with only the fourth term,

$$
\langle\delta_R^{(2)}\vec{f}_{GR}\rangle = \frac{(-1)^\ell N_\ell}{8\pi\sqrt{\ell}} \times\Re\left\langle\delta_N^{(1)}\vec{v}\times\vec{\nabla}(r^\ell Y^{\ell\ell})\frac{d^{2\ell+1}\delta_N^{(1)}S^{\ell\ell}}{dt^{2\ell+1}}\right\rangle. \qquad (66)
$$

The second simplification involves the quantities $\langle\delta_R^{(2)}V^a\rangle$ and $\langle\delta_R^{(2)}W^a\rangle$ that appear in Eq. (64). They are defined in Eqs. (50) and (51). Using the general expressions for the first-order perturbations given in Eqs. (32)–(35), we can express these quantities in terms of the first-order perturbations:

$$
\langle\beta\delta_R^{(2)}W^a\rangle = \frac{1}{2}P^a{}_b(\delta_R^{(1)}\hat{\rho}\delta_N^{(1)}\hat{v}^b + \delta_N^{(1)}\hat{\rho}\delta_R^{(1)}\hat{v}^b), \qquad (67)
$$

$$
\begin{aligned}
\langle\beta\delta_R^{(2)}V^a\rangle &= \frac{1}{2}\varpi^{-2}\phi^a[\delta_R^{(1)}\hat{v}^k\nabla_k(\delta_N^{(1)}\hat{v}^b\phi_b) \\
&+\delta_N^{(1)}\hat{v}^k\nabla_k(\delta_R^{(1)}\hat{v}^b\phi_b)]. \qquad (68)
\end{aligned}
$$

As we will see in the following section, these terms and the term involving $\delta_N^{(2)}\rho$ in Eq. (64) are higher order in Ω than the first two terms of Eq. (65) and can therefore be neglected when evaluating $\delta_N^{(2)}\Omega(\varpi)$ for slowly rotating stars using Eq. (64). This fact is essential, because $\delta_N^{(2)}\rho$ itself depends on $\delta_N^{(2)}\Omega$.

This discussion has been somewhat abstract but quite general. Apart from assuming the integrability of the perturbed density so that mass conservation, Eq. (17), can be enforced, no assumption has been made up to this point about the particular equation of state of the matter in these stellar models, nor has any assumption been made about the magnitude of the angular velocity of the star. In order to proceed further, however, we will need to assume that the stellar model is slowly rotating in a suitable sense. To find an explicit solution for $\delta_N^{(2)}\Omega(\varpi)$, we will also need to make some choice for the equation of state for the stellar matter. The slow rotation assumption and its implications are discussed in Sec. IV, while the complete solution for $\delta^{(2)}\Omega$, the second-order r-mode angular velocity that is driven by gravitational radiation reaction, is determined in Sec. V for the case of stars composed of matter with a range of polytropic equations of state.

IV. SLOW ROTATION EXPANSION

We consider the one-parameter families of stars $Q = Q(\Omega)$ composed of matter with a fixed equation of state and having masses that are independent of the angular velocity: $M(\Omega) = M_0$. The structures of slowly rotating stellar models in these families are conveniently written as expansions in the dimensionless angular velocity,

$$
\tilde{\Omega} = \frac{\Omega}{\Omega_0}, \qquad (69)
$$

where $\Omega_0 = \sqrt{M_0/R^3}$, and M_0 is the mass and R the radius of the nonrotating star in the sequence. The slow rotation expansion of these stellar models is denoted

$$
Q = \sum_{n=0} Q_n\tilde{\Omega}^n = Q_0 + Q_1\tilde{\Omega} + Q_2\tilde{\Omega}^2 + \mathcal{O}(\tilde{\Omega}^3). \qquad (70)
$$

For equilibrium rotating stars, these expansions of the basic fluid variables have the forms

[3] As mentioned above, Appendix C shows that assuming smoothness of the displacement of the surface as a function of α and $\vec{x}$ implies integrability of $\delta_N^{(2)}\rho$. A simpler way to see that the right side of Eq. (63) is finite is to note that smoothness of the displacement of the surface implies one-sided differentiability of $\delta^{(2)}\vec{v}$ at the surface. The perturbed mass conservation equation, Eq. (55), then implies that the combination $2\langle\delta_N^{(2)}\rho\rangle + \nabla_a\langle\delta_R^{(2)}W^a\rangle$ is finite at the surface and hence integrable. This is enough to imply that the expression in Eq. (64) for $\delta_N^{(2)}\Omega(\varpi)$ is finite.

$$\rho = \rho_0 + \rho_2\tilde{\Omega}^2 + \mathcal{O}(\Omega^4), \tag{71}$$

$$v^a = \Omega\phi^a, \tag{72}$$

$$p = p_0 + p_2\tilde{\Omega}^2 + \mathcal{O}(\Omega^4), \tag{73}$$

$$\Phi = \Phi_0 + \Phi_2\tilde{\Omega}^2 + \mathcal{O}(\Omega^4). \tag{74}$$

We will represent the perturbations of these stellar models δQ as dual expansions in the mode amplitude α and the angular velocity parameter $\tilde{\Omega}$:

$$\delta Q = \sum_{n,k} \alpha^n \tilde{\Omega}^k \delta^{(n)} Q_k. \tag{75}$$

Our main goal here is to determine to lowest order in angular velocity the axisymmetric part of the second-order perturbations of the r-mode angular velocity field $\langle \delta_R^{(2)} v^\phi \rangle$ that is driven by the gravitational-radiation instability. Doing this requires the explicit slow-rotation forms of the first- and the second-order perturbations. These slow-rotation expansions are described in the remainder of this section.

A. First-order perturbations

The effect of the first-order gravitational radiation-reaction force $\delta^{(1)}\vec{f}_{GR}$ on the structure of the classical r-mode (beyond its overall effect on its amplitude) was first studied (for $\ell = 2$) by Dias and Sá [17]. We agree with the results they obtain but will need to clarify their meaning. We also extend the calculation to general values of ℓ.

To first order in mode amplitude α and lowest nontrivial order in angular velocity $\tilde{\Omega}$, the classical r-modes with the ϕ-parity described in Sec. III A can be written the form

$$\delta_N^{(1)} p_1 = \delta_N^{(1)} \rho_1 = \delta_N^{(1)} \Phi_1 = 0, \tag{76}$$

$$\delta_N^{(1)} \vec{v}_1 = \Im\left[\frac{R\Omega_0}{\ell}\left(\frac{r}{R}\right)^\ell \vec{r} \times \vec{\nabla}(\sin^\ell\theta e^{i\ell\phi + i\omega t})\right], \tag{77}$$

where $\Im(Z)$ is the imaginary part of a quantity Z. An equivalent expression for the classical r-mode velocity in terms of vector spherical harmonics is

$$\delta_N^{(1)} \vec{v}_1 = \Im(A_\ell r^\ell \vec{Y}_B^{\ell\ell} e^{i\omega t}), \tag{78}$$

$$= -\frac{iA_\ell r^\ell}{2}[\vec{Y}_B^{\ell\ell} e^{i\omega t} - (-1)^\ell \vec{Y}_B^{\ell -\ell} e^{-i\omega t}], \tag{79}$$

where A_ℓ is given by

$$A_\ell = (-1)^\ell 2^\ell (\ell-1)! \sqrt{\frac{4\pi\ell(\ell+1)}{(2\ell+1)!}} R^{-\ell+1}\Omega_0. \tag{80}$$

The frequencies of these classical r-modes have the form

$$\omega_N = -\frac{(\ell-1)(\ell+2)}{\ell+1}\Omega + \mathcal{O}(\Omega^3). \tag{81}$$

At this order in Ω, the r-modes do not affect the fluid variables $\delta\rho$ and δp, which are $\mathcal{O}(\Omega^2)$. Because of this, the r-mode velocity field at order Ω does not depend on the equation of state.

Four features of the gravitational radiation-reaction force are important in determining the way it alters each r-mode: *a*) The ϕ-parity of $\delta^{(1)}\vec{f}_{GR}$, as shown in the last section, is opposite to that of the classical mode; *b*) its magnitude, as shown below, is dominated by the current current multipole $S^{\ell\ell}$; *c*) it can be decomposed in the manner

$$\delta^{(1)}\vec{f}_{GR} = \beta\delta_N^{(1)}\vec{v} + \delta_\perp^{(1)}\vec{f}_{GR}, \tag{82}$$

where the two terms in the decomposition are orthogonal with respect to a density-weighted inner product, $\int \sqrt{g} d^3x \rho_0 \delta_N^{(1)}\vec{v} \cdot \delta_\perp^{(1)}\vec{f}_{GR} = 0$; and *d*) as we show below, $\delta_\perp^{(1)}\vec{f}_{GR}$ is a gradient, $\delta_\perp^{(1)}\vec{f}_{GR} = \vec{\nabla}\delta_\perp^{(1)}\mathcal{F}$.

It is straightforward to evaluate the multipole moments of the r-modes using Eqs. (22) and (23) and the expressions for the classical r-modes from Eqs. (76) and (77). The expressions for the nonvanishing multipole moments of the r-modes can be written in the form

$$\begin{aligned}\delta_N^{(1)} S^{\ell\ell} &= (-1)^\ell \delta_N^{(1)} S^{*\ell -\ell} \\ &= -i\frac{A_\ell N_\ell e^{i\omega t}}{\sqrt{\ell+1}} \int_0^R r^{2\ell+2}\rho_0 dr.\end{aligned} \tag{83}$$

Inserting these expressions into the formula for the gravitational radiation-reaction force, Eq. (21), we find

$$\begin{aligned}\delta_N^{(1)}\vec{f}_{GR} = \frac{(-1)^\ell N_\ell}{8\pi}\Re\Bigg\{\Bigg[\frac{i\omega}{\sqrt{\ell+1}} r^\ell \vec{Y}_B^{\ell\ell} \\ + \frac{\Omega}{\sqrt{\ell}}\vec{\phi} \times \vec{\nabla}(r^\ell Y^{\ell\ell})\Bigg]\frac{d^{2\ell+1}\delta S^{\ell\ell}}{dt^{2\ell+1}}\Bigg\}.\end{aligned} \tag{84}$$

This expression can be rewritten as a linear combination of $r^\ell \vec{Y}_B^{\ell\ell}$ and $\vec{\nabla}(r^\ell Y^{\ell\ell})$ using the identity

$$\vec{\phi} \times \vec{\nabla}(r^\ell Y^{\ell\ell}) = i\sqrt{\ell(\ell+1)} r^\ell \vec{Y}_B^{\ell\ell} - z\vec{\nabla}(r^\ell Y^{\ell\ell}). \tag{85}$$

The resulting expression for $\delta_N^{(1)}\vec{f}_{GR}$ can therefore be written in the following way,

$$\delta^{(1)}\vec{f}_{GR} = \beta\delta_N^{(1)}\vec{v} + \delta_\perp^{(1)}\vec{f}_{GR}, \tag{86}$$

where β is given by

$$\beta = \frac{N_\ell^2 \omega^{2\ell+2}}{4\pi(\ell^2-1)(\ell+2)} \int_0^R r^{2\ell+2}\rho_0 dr \tag{87}$$

and where $\delta_\perp^{(1)}\vec{f}_{GR}$ is defined by

$$\delta_\perp^{(1)}\vec{f}_{GR} = -\frac{N_\ell^2\omega^{2\ell+1}\Omega}{8\pi}\int_0^R r^{2\ell+2}\rho dr \times \left\{\frac{\delta_N^{(1)}\vec{v}}{\ell+1} + \frac{\Re[zA_\ell\vec{\nabla}(r^\ell Y^{\ell\ell})e^{i\omega t}]}{\sqrt{\ell(\ell+1)}}\right\}. \tag{88}$$

This expression for $\delta_\perp^{(1)}\vec{f}_{GR}$ can be rewritten as a gradient,

$$\delta_\perp^{(1)}\vec{f}_{GR} = \Im\left\{i\beta A_\ell\frac{\sqrt{\ell(\ell+1)}}{2}\vec{\nabla}[r^{\ell+1}\cos\theta Y^{\ell\ell}e^{i\omega t}]\right\} =: \vec{\nabla}\delta_\perp^{(1)}\mathcal{F}. \tag{89}$$

Equations (86) and (89) give the decomposition of Eq. (82), and the orthogonality of the two parts,

$$\int \rho_0\delta_N^{(1)}\vec{v}\cdot\delta_\perp^{(1)}\vec{f}_{GR}\sqrt{g}d^3x = 0, \tag{90}$$

is implied by the relation

$$\int \epsilon^{abc}\nabla_a(\cos\theta Y^{\ell\ell})\nabla_b r\nabla_c\bar{Y}^{\ell\ell}\sqrt{g}d^2x = -\int \epsilon^{abc}\cos\theta Y^{\ell\ell}\nabla_b r\nabla_a\nabla_c\bar{Y}^{\ell\ell}\sqrt{g}d^2x = 0, \tag{91}$$

where $\sqrt{g}d^2x$ is the volume element on the sphere: $\sqrt{g}d^2x \equiv -r^2 d\cos\theta d\phi$. At this order in Ω, the density ρ_0 plays no role in the orthogonality, but it is with respect to the density-weighted inner product that the operators appearing in the perturbed Euler equation are formally self-adjoint.

It follows that $\delta_\perp^{(1)}\vec{f}_{GR}$ is the part of the gravitational radiation-reaction force that does not contribute directly to the exponential growth of the classical r-mode instability and that the coefficient β is the growth rate of the gravitational radiation driven instability in the r-modes. Substituting into Eq. (87) the expressions for N_ℓ from Eq. (10) and the r-mode frequency ω from Eq. (81) gives

$$\beta = \frac{32\pi\Omega^{2\ell+2}(\ell-1)^{2\ell}}{[(2\ell+1)!!]^2}\left(\frac{\ell+2}{\ell+1}\right)^{2\ell+2}\int_0^R r^{2\ell+2}\rho_0 dr, \tag{92}$$

which agrees with the expression for the gravitational radiation growth rate of the r-mode instability given by Lindblom *et al.* [3].

These expressions for the slow rotation limits of the radiation-reaction force confirm the general expressions, e.g. Eq. (31), used in our discussion of the general properties of the first-order r-modes in Sec. III A. It follows from that discussion that the general form of the first-order r-mode velocity, to lowest order in the angular velocity of the star, is given by

$$\delta^{(1)}\vec{v} = \tilde{\Omega}\delta_N^{(1)}\vec{v}_1 e^{\beta t}. \tag{93}$$

To evaluate $\delta_N^{(2)}\Omega$ using Eq. (64), we need to determine $\delta_R^{(1)}\rho$ and $\delta_R^{(1)}\vec{v}$, or at least to show that they are negligibly small compared to other terms in the equation. We show in the heuristic argument below that $\delta_R^{(1)}\rho = \mathcal{O}(\beta\Omega)$ and $\delta_R^{(1)}\vec{v} = \mathcal{O}(\beta\Omega^2)$, which will allow us to neglect them in our slow rotation expansion. A more precise version of the argument is given in Appendix D. The fact that $\delta_R^{(1)}\vec{v}$ is higher order in Ω than $\delta_R^{(1)}\rho$ is the reverse of their relation in the classical r-modes. This reversal depends on the appearance of the gradient $\vec{\nabla}\delta_\perp^{(1)}\mathcal{F}$ in the decomposition of the gravitational radiation-reaction force $\delta_R^{(1)}\vec{f}_{GR}$.

The equations that determine $\delta_R^{(1)}Q$, Eqs. (40)–(42), can be written more compactly as

$$(\omega_N + \ell\Omega)\delta_R^{(1)}\hat{\rho} + \vec{\nabla}\cdot(\rho\delta_R^{(1)}\hat{\vec{v}}) = \beta\delta_N^{(1)}\rho, \tag{94}$$

$$(\omega_N + \ell\Omega)\delta_R^{(1)}\hat{\vec{v}} + 2\Omega\delta_R^{(1)}\hat{\vec{v}}\cdot\nabla\vec{\phi} = -\vec{\nabla}(\delta_R^{(1)}\hat{U} - \delta_\perp^{(1)}\mathcal{F}). \tag{95}$$

The value of $\delta_R^{(1)}\hat{\vec{v}}$ is fixed by the curl of the perturbed Euler equation (95),

$$\vec{\nabla}\times[(\omega_N+\ell\Omega)\delta_R^{(1)}\hat{\vec{v}} + 2\Omega\delta_R^{(1)}\hat{\vec{v}}\cdot\nabla\vec{\phi}] = 0, \tag{96}$$

which involves only $\delta_R^{(1)}\hat{\vec{v}}$. Its two independent components give two relations for the three components of $\delta_R^{(1)}\hat{\vec{v}}$, in which all coefficients are $\mathcal{O}(\Omega)$. All components of $\delta_R^{(1)}\hat{\vec{v}}$ are therefore of the same order in Ω. Similarly, the two relations among $\delta_R^{(1)}U$, $\delta_R^{(1)}\Phi$, and $\delta_R^{(1)}\rho$ given by the equation of state and the Poisson equation imply that $\delta_R^{(1)}U$ and $\delta_R^{(1)}\rho$ are of the same order in Ω. The continuity equation (94) then implies that $\delta_R^{(1)}\vec{v} = \mathcal{O}(\Omega\delta_R^{(1)}\rho)$. Finally, the ϕ-component of the Euler equation gives, to lowest order in Ω,

$$\delta_R^{(1)}U = \delta_\perp^{(1)}\mathcal{F} + \mathcal{O}(\Omega^2\delta_R^{(1)}\rho). \tag{97}$$

From its definition in Eq. (89), it follows that $\delta_\perp^{(1)}\mathcal{F} = \mathcal{O}(\Omega\beta)$, which then implies that $\delta_R^{(1)}\rho = \mathcal{O}(\beta\Omega)$ and $\delta_R^{(1)}\vec{v} = \mathcal{O}(\beta\Omega^2)$.

Dias and Sá [17] find, for an $\ell = 2$ perturbation, a solution $\delta_R^{(1)}\vec{v}, \delta_R^{(1)}U$ that is a sum of a) our solution with

$\delta_R^{(1)}U$ given by Eq. (97) and b) a solution to the homogeneous equations with ϕ-parity opposite to that of the Newtonian r-mode $\delta_N^{(1)}Q$. As noted above, adding part b of their solution is equivalent to changing the initial phase of the perturbation.

B. Second-order axisymmetric perturbations

In computing the quadratic terms that enter the second-order perturbation equations, it will be useful to have explicit expressions for the classical r-mode $\delta_N^{(1)}v_1^a$ in cylindrical coordinates (ϖ, z, ϕ),

$$\delta_N^{(1)}v_1^\varpi = -\Omega_0 z\left(\frac{\varpi}{R}\right)^{\ell-1}\cos(\ell\phi+\omega t), \tag{98a}$$

$$\delta_N^{(1)}v_1^z = \Omega_0 R\left(\frac{\varpi}{R}\right)^{\ell}\cos(\ell\phi+\omega t), \tag{98b}$$

$$\delta_N^{(1)}v_1^\phi = \Omega_0\frac{z}{R}\left(\frac{\varpi}{R}\right)^{\ell-2}\sin(\ell\phi+\omega t). \tag{98c}$$

From these, one finds explicit expressions for the cylindrical components of the quadratic term $\langle\delta_N^{(1)}v_1^b\nabla_b\delta_N^{(1)}v_1^a\rangle$, which appears as a source in the second-order Euler equation, Eq. (45):

$$\langle\delta_N^{(1)}\vec{v}_1\cdot\vec{\nabla}\delta_N^{(1)}v_1^\varpi\rangle = \frac{\Omega_0^2}{2R}[2(\ell-1)z^2-\varpi^2]\left(\frac{\varpi}{R}\right)^{2\ell-3}, \tag{99a}$$

$$\langle\delta_N^{(1)}\vec{v}_1\cdot\vec{\nabla}\delta_N^{(1)}v_1^z\rangle = -\ell\Omega_0^2 z\left(\frac{\varpi}{R}\right)^{2\ell-2}, \tag{99b}$$

$$\langle\delta_N^{(1)}\vec{v}_1\cdot\vec{\nabla}\delta_N^{(1)}v_1^\phi\rangle = 0. \tag{99c}$$

The axisymmetric parts of the nonradiative second-order perturbations $\langle\delta_N^{(2)}v^a\rangle$ and $\langle\delta_N^{(2)}U\rangle$ are determined by solving the perturbed Euler equation, Eq. (52), and the perturbed mass conservation equation, Eq. (53). The contributions to each component of Euler's equation at lowest order in angular velocity are given by,

$$0 = \langle\delta_N^{(2)}E_\varpi\rangle = -2\varpi\Omega_0\langle\delta_N^{(2)}v_1^\phi\rangle + \partial_\varpi\langle\delta_N^{(2)}U_2\rangle + [2(\ell-1)z^2-\varpi^2]\frac{\Omega_0^2}{2R}\left(\frac{\varpi}{R}\right)^{2\ell-3}, \tag{100a}$$

$$0 = \langle\delta_N^{(2)}E_z\rangle = \partial_z\langle\delta_N^{(2)}U_2\rangle - \ell z\Omega_0^2\left(\frac{\varpi}{R}\right)^{2\ell-2}, \tag{100b}$$

$$0 = \langle\delta_N^{(2)}E_\phi\rangle = 2\varpi\Omega_0\langle\delta_N^{(2)}v_1^\varpi\rangle. \tag{100c}$$

The integrability conditions for these equations, $\langle\delta_N^{(2)}E_a\rangle = 0$, are given by $\nabla_{[a}\langle\delta_N^{(2)}E_{b]}\rangle = 0$. In cylindrical coordinates, these integrability conditions, at lowest order in angular velocity, are

$$0 = \nabla_{[z}\langle\delta_N^{(2)}E_{\varpi]}\rangle = -\varpi\Omega_0\partial_z\langle\delta_N^{(2)}v_1^\phi\rangle + (\ell^2-1)\frac{\Omega_0^2 z}{R}\left(\frac{\varpi}{R}\right)^{2\ell-3}, \tag{101a}$$

$$0 = \nabla_{[z}\langle\delta_N^{(2)}E_{\phi]}\rangle = \Omega_0\partial_z\langle\delta_N^{(2)}v_1^\varpi\rangle, \tag{101b}$$

$$0 = \nabla_{[\varpi}\langle\delta_N^{(2)}E_{\phi]}\rangle = \Omega_0\partial_\varpi(\varpi\langle\delta_N^{(2)}v_1^\varpi\rangle). \tag{101c}$$

These conditions, together with the requirement that the solution is nonsingular on the rotation axis, determine $\langle\delta_N^{(2)}v_1^\varpi\rangle$ and $\langle\delta_N^{(2)}v_1^\phi\rangle$, up to the time-independent differential rotation $\delta_N^{(2)}\Omega(\varpi)$ As before, we denote a particular choice by $\delta_{NP}^{(2)}v^\phi$:

$$\langle\delta_N^{(2)}v_1^\varpi\rangle = 0, \tag{102}$$

$$\langle\delta_{NP}^{(2)}v_1^\phi\rangle = (\ell^2-1)\frac{\Omega_0 z^2}{2R^2}\left(\frac{\varpi}{R}\right)^{2\ell-4}. \tag{103}$$

The remaining component, $\langle\delta_N^{(2)}v_1^z\rangle$, is determined from the lowest order in the angular velocity piece of the perturbed mass conservation equation [cf. Eq. (53)],

$$\nabla_a(\rho\langle\delta_N^{(2)}v_1^a\rangle) = 0. \tag{104}$$

This equation, together with Eq. (102), shows that the only nonsingular solution for $\langle\delta_N^{(2)}v_1^z\rangle$ is

$$\langle\delta_N^{(2)}v_1^z\rangle = 0. \tag{105}$$

The scalar parts of the second-order nonradiative r-mode, $\langle\delta_N^{(2)}\rho\rangle$ and $\langle\delta_N^{(2)}\Phi\rangle$, are determined by completing the solution to the perturbed Euler equation $\langle\delta_N^{(2)}E_a\rangle = 0$ and then solving the perturbed gravitational potential equation. The potential $\langle\delta_N^{(2)}U\rangle$ is determined by integrating the perturbed Euler Eqs. (100a) and (100b). Using Eqs. (43) and (103), we obtain the following expression for the axisymmetric part of the solution, to lowest order in angular velocity,

$$\langle\delta_N^{(2)}U_2\rangle = \frac{\Omega_0^2R^2}{4\ell}\left(\frac{\varpi}{R}\right)^{2\ell} + \frac{\ell\Omega_0^2z^2}{2}\left(\frac{\varpi}{R}\right)^{2\ell-2} + 2\Omega_0\int_0^\varpi \varpi'\delta_N^{(2)}\Omega(\varpi')d\varpi' + \delta_N^{(2)}C_2, \tag{106}$$

where $\delta_N^{(2)}C_2$ is a constant.

The pressure as well as the density perturbations, $\delta^{(2)}p$ and $\delta^{(2)}\rho$, are related to $\delta^{(2)}U$ as follows,

$$\delta^{(2)}U = \delta^{(2)}\Phi + \frac{1}{\rho}\delta^{(2)}p - \frac{1}{2\rho^2}\delta^{(1)}p\delta^{(1)}\rho$$
$$= \delta^{(2)}\Phi + \frac{\gamma p}{\rho^2}\delta^{(2)}\rho + \frac{p}{2\rho^2}\left[\frac{\gamma(\gamma-2)}{\rho} + \frac{d\gamma}{d\rho}\right](\delta^{(1)}\rho)^2, \tag{107}$$

where $\gamma = d\log p/d\log\rho$ is the adiabatic index. For the r-modes, the first-order perturbations $\delta^{(1)}p$ and $\delta^{(1)}\rho$ are $\mathcal{O}(\Omega^2)$. So at lowest order in angular velocity, the relation between $\delta^{(2)}U$ and $\delta^{(2)}\rho$ simplifies to

$$\delta^{(2)}U_2 = \delta^{(2)}\Phi_2 + \frac{\gamma p}{\rho^2}\delta^{(2)}\rho_2. \tag{108}$$

The gravitational potential $\delta^{(2)}\Phi$ is determined by solving the perturbed gravitational potential equation,

$$\nabla^2\delta^{(2)}\Phi = 4\pi\delta^{(2)}\rho. \tag{109}$$

For the r-modes, to lowest order in the angular velocity, this equation my be rewritten as

$$\nabla^2\delta^{(2)}\Phi_2 + \frac{4\pi\rho^2}{\gamma p_0}\delta^{(2)}\Phi_2 = \frac{4\pi\rho^2}{\gamma p_0}\delta^{(2)}U_2. \tag{110}$$

Using the expression derived in Eq. (106) for the axisymmetric part of $\delta_N^{(2)}U_2$, we find the general equation for $\langle\delta_N^{(2)}\Phi_2\rangle$:

$$\nabla^2\langle\delta_N^{(2)}\Phi_2\rangle + \frac{4\pi\rho^2}{\gamma p_0}\langle\delta^{(2)}\Phi_2\rangle$$
$$= \frac{4\pi\rho^2}{\gamma p_0}\left\{\frac{\Omega_0^2R^2}{4\ell}\left(\frac{\varpi}{R}\right)^{2\ell} + \frac{\ell\Omega_0^2z^2}{2}\left(\frac{\varpi}{R}\right)^{2\ell-2}\right.$$
$$\left. + 2\Omega_0\int_0^{\varpi}\varpi'\delta_N^{(2)}\Omega(\varpi')d\varpi' + \delta_N^{(2)}C_2\right\}. \tag{111}$$

Finally, we use Eq. (64) to obtain an explicit formula for the second-order differential rotation, $\delta_N^{(2)}\Omega(\varpi)$, in terms of the second-order radiation-reaction force and the second-order velocity perturbation $\delta_N^{(2)}v^a$. Of the three terms on the right side of that equation, we will see that the second and third are higher order in Ω than the first, and we will evaluate the first term to leading order in Ω.

We first use Eq. (66) to find an explicit form for the second-order radiation-reaction force $\langle\delta_R^{(2)}\vec{f}_{GR}\rangle$. From Eqs. (98) and (83) for $\delta_N^{(1)}v^\theta$ and $\delta_N^{(1)}S^{\ell\ell}$, we find

$$\langle\delta_R^{(2)}\vec{f}_{GR}\rangle = -\frac{(\ell+1)^2}{4}\beta\Omega\left(\frac{\varpi}{R}\right)^{2\ell-2}\vec{\phi}. \tag{112}$$

The second term $\delta_N^{(2)}v^\phi$ in Eq. (65) is given by Eq. (103). In the final term, $\delta_R^{(2)}V^\phi$, by its definition (50), is proportional to a product of components of $\delta_N^{(1)}\vec{v}$ and $\delta_R^{(1)}\vec{v}$. By our initial normalization, $\delta_N^{(1)}\vec{v} = \mathcal{O}(\Omega)$, and we found in Sec. IV A that $\delta_R^{(1)}\vec{v}$ is $\mathcal{O}(\Omega\delta_R^{(1)}\vec{f}_{GR}) = \mathcal{O}(\beta\Omega^2)$.

From Eqs. (65), (112), and (103), we have

$$\langle\delta_R^{(2)}F^\phi\rangle = -\Omega\left(\frac{\varpi}{R}\right)^{2\ell-4}\left[\frac{(\ell+1)^2}{4}\left(\frac{\varpi}{R}\right)^2\right.$$
$$\left. + (\ell^2-1)\left(\frac{z}{R}\right)^2\right]. \tag{113}$$

Equation (113) implies $\langle\delta_R^{(2)}F^\phi\rangle = \mathcal{O}(\Omega)$. The second term in Eq. (64) has an integrand proportional to $\langle\delta_N^{(2)}\rho\rangle$. Because $\delta_N^{(2)}\rho = \mathcal{O}(\Omega^2)$, the integrand is $\mathcal{O}(\Omega^2)$, and the term itself is $\mathcal{O}(\Omega^3)$, two orders higher than $\langle\delta_R^{(2)}F^\phi\rangle$. Finally, the last term in (64) is proportional to $\Omega\langle\delta_R^{(2)}W^a\rangle$. Equation (51) implies $\langle\delta_R^{(2)}W^a\rangle = \mathcal{O}(\Omega^2)$, whence the last term is again $\mathcal{O}(\Omega^3)$.

With the dominant term in Eq. (64) determined by $\langle\delta_R^{(2)}F^\phi\rangle$, we have

$$\delta_N^{(2)}\Omega(\varpi) = \frac{\int_{-z_S}^{z_S}dz\rho\langle\delta_R^{(2)}F^\phi\rangle}{2\int_{-z_S}^{z_S}dz\rho}. \tag{114}$$

This integrand can be rewritten in a more explicit form using Eqs. (113) and (103),

$$\delta_N^{(2)}\Omega(\varpi) = -\Omega\left(\frac{\varpi}{R}\right)^{2\ell-4}\left[\frac{(\ell+1)^2}{8}\left(\frac{\varpi}{R}\right)^2\right.$$
$$\left. + \frac{(\ell^2-1)}{2}\Upsilon(\varpi)\right], \tag{115}$$

where $\Upsilon(\varpi)$ is the equation of state-dependent, mass-weighted average of $(z/R)^2$,

$$\Upsilon(\varpi) = \frac{\int_{-z_S}^{z_S}dz\rho z^2}{R^2\int_{-z_S}^{z_S}dz\rho}. \tag{116}$$

The limits of integration, $\pm z_S(\varpi)$, in this expression are the ϖ-dependent values of z at the surface of the equilibrium star. To lowest order in Ω, these limits are the same as those in a spherical nonrotating star:

$$z_S(\varpi) = \sqrt{R^2 - \varpi^2}. \tag{117}$$

The part of the second-order differential rotation that is not explicitly caused by the radiation-reaction force, $\langle\delta_{NP}^{(2)}v_1^\phi\rangle$, is given in Eq. (103):

$$\langle\delta_{NP}^{(2)}v^\phi\rangle = (\ell^2-1)\frac{\Omega}{2}\left(\frac{z}{R}\right)^2\left(\frac{\varpi}{R}\right)^{2\ell-4}. \tag{118}$$

Together, Eqs. (115) and (118) determine (to lowest order in Ω) the time-dependent differential-rotation induced by the gravitational-radiation reaction:

$$\delta^{(2)}\Omega = [\langle \delta^{(2)}_{NP} v^\phi \rangle + \delta^{(2)}_N \Omega(\varpi)] e^{2\beta t}. \tag{119}$$

The key result of this section is the derivation of an explicit expression (114) for $\delta^{(2)}_N \Omega(\varpi)$ in terms of the first-order r-mode. An expression of this kind exists because the rest of the second-order perturbation, the perturbed density, pressure, and potential, are higher order in Ω. Like the velocity field of the first-order r-mode, the second-order differential rotation of the unstable r-mode can be found without simultaneously solving for the perturbed density and pressure.

This separation of orders also leads to an iterative method for solving the second-order Newtonian perturbation equations at successive orders in Ω that mirrors the method we have just used to determine the axisymmetric parts of $\delta^{(2)}_N v^a$ at $\mathcal{O}(\Omega)$ and $\delta^{(2)}_N \rho$, $\delta^{(2)}_N p$, and $\delta^{(2)}_N \Phi$ at $\mathcal{O}(\Omega^2)$. At each order, the ambiguity in the Newtonian differential rotation is resolved by using Eq. (64). We assume that the first-order Newtonian perturbation equations have been solved to the desired order in Ω. We suppose one has found the perturbed Newtonian velocity $\delta^{(2)}_N v^a$ to $\mathcal{O}(\Omega^{2k-1})$ and the scalar quantities in $\delta^{(2)}_N Q$ to $\mathcal{O}(\Omega^{2k})$, and we list the steps to obtain the next-order correction, to find $\delta^{(2)}_N v^a_{2k+1}$ and the scalar quantities to $\mathcal{O}(\Omega^{2k+2})$:

(1) Because $\delta^{(2)}_N v^a_{2k-1}$ is known, and the integrability conditions $\nabla_{[a}\delta^{(2)}_N E_{b]} = 0$ have an additional power of Ω in each term, they are satisfied at $\mathcal{O}(\Omega^{2k})$. One can then integrate the ϖ or z component of the perturbed Newtonian Euler equation (52) to find $\delta^{(2)}_N U_{2k+2}$ up to a constant $\delta^{(2)}_N C_{2k+2}$.

(2) Equation (107) determines $\delta^{(2)}_N \rho_{2k+2}$ up to the ambiguity associated with $\delta^{(2)}_N C_{2k+2}$. The Poisson equation, Eq. (47), with the conditions that $\delta^{(2)}_N \Phi_{2k+2}$ vanish at infinity and have no monopole part (no change in mass), determines both $\delta^{(2)}_N \Phi_{2k+2}$ and the constant $\delta^{(2)}_N C_{2k+2}$.

(3) Equation (107) (or, alternatively, the Poisson equation) gives $\delta^{(2)}_N \rho_{2k+2}$, and the equation of state determines $\delta^{(2)}_N p_{2k+2}$.

(4) Finally, one uses the known first-order perturbation $\delta^{(1)}_N v^a$ to solve two independent components of the curl of the Euler equation, $\delta^{(2)}_N E_a = 0$ for $\delta^{(2)}_N v^\phi_{2k+1}$ and $\delta^{(2)}_N v^\varpi_{2k+1}$; $\langle \delta^{(2)}_N v^\phi_{2k+1} \rangle$ has an $f(\varpi)$ ambiguity that is resolved by Eq. (64). The final component $\delta^{(2)}_N v^z_{2k+1}$ is found from the second-order mass-conservation equation.

C. Secular drift of a fluid element

The differential rotation we have found for the unstable r-mode extends the work of Sá and collaborators [15–17] to obtain the differential rotation of the unstable second-order r-mode. The studies of magnetic-field wind-up by Rezzolla *et al.* [8–10], which predated this work, explicitly omitted the form of the second-order perturbation to the velocity field that we have computed here. These authors obtained a secular drift $\phi(t)$ in the position of a fluid element by integrating the $\ell = 2$ form of the equations for the position $\phi(t)$ and $\theta(t)$ of a particle whose perturbed velocity field is found solely from the first-order perturbation $\delta^{(1)}_N v^a$ of Eq. (77), from the equations

$$\frac{d\theta}{dt} = \alpha \delta^{(1)}_N v^\theta[\theta(t), \phi(t)], \tag{120a}$$

$$\frac{d\phi}{dt} = \alpha \delta^{(1)}_N v^\phi[\theta(t), \phi(t)]. \tag{120b}$$

The equations are nonlinear in $\theta(t)$, $\phi(t)$, and the solution is written to $\mathcal{O}(\alpha^2)$. The axisymmetric part of the solution is again the part that is not oscillatory in time; in our notation, it has the form

$$\langle \theta(t) \rangle = 0, \quad \langle \phi(t) \rangle = \alpha^2 \frac{3}{4}\left[\left(\frac{\varpi}{R}\right)^2 - 2\left(\frac{z}{R}\right)^2\right]\Omega t. \tag{121}$$

A secular drift obtained in this way has been used in subsequent papers by Cuofano *et al.* [11,12] and by Cao *et al.* [18].

When one includes the second-order differential rotation $\delta^{(2)}\Omega$ of the unstable $\ell = 2$ r-mode from Eqs. (119), additional terms are added to the secular drift $\phi(t)$ of a fluid element's position. The resulting expression is given for $t \ll 1/\beta$ by

$$\langle \phi(t) \rangle = \alpha^2 \left\{ \frac{3}{4}\left[\left(\frac{\varpi}{R}\right)^2 - 2\left(\frac{z}{R}\right)^2\right]\Omega + \delta^{(2)}\Omega|_{t=0} \right\} t. \tag{122}$$

Using the expression for $\delta^{(2)}\Omega$ in Eq. (119), with Eqs. (115) and (118), we obtain the following explicit form for the second-order drift of an unstable $\ell = 2$ r-mode:

$$\langle \phi(t) \rangle = -\frac{3}{2}\alpha^2 \Omega \left[\frac{1}{4}\left(\frac{\varpi}{R}\right)^2 + \Upsilon(\varpi)\right] t. \tag{123}$$

This expression for the drift $\langle \phi(t) \rangle$ is independent of z and therefore describes a drift that is constant on $\varpi =$ constant cylinders. The analogous expression for the drift found previously by Sá [15] has this same feature, and Chugunov [26] observes that the drift in these modes can therefore be completely eliminated in the pure Newtonian case by appropriately choosing the arbitrary second-order angular velocity perturbation.

For long times (that is, for βt arbitrary but $\beta \ll \Omega$), the time dependence t in Eq. (123) is replaced by $(e^{2\beta t} - 1)/2\beta$. This expression is not of order $1/\beta$ but satisfies the bound

$$\frac{e^{2\beta t} - 1}{2\beta} < t\frac{e^{2\beta t} + 1}{2}, \tag{124}$$

for $t > 0$.

V. POLYTROPIC STELLAR MODELS

In this section, we evaluate Eq. (119), to determine the changes in the rotation laws of uniformly rotating polytropes that are induced by the gravitational-radiation driven instability in the r-modes. Polytropic stellar models (polytropes) are stars composed of matter whose equation of state has the form

$$p = K\rho^{1+1/n}, \tag{125}$$

where K and n, the *polytropic index*, are constants. We start with the simplest case, $n = 0$, the uniform-density models. The only dependence of the differential rotation $\delta^{(2)}\Omega$ on the equation of state is in $\Upsilon(\varpi)$, the mass-weighted average of $(z/R)^2$ at fixed ϖ defined in Eq. (116). This average can be evaluated analytically in the uniform-density case:

$$\Upsilon(\varpi) = \frac{R^2 - \varpi^2}{3R^2} = \frac{z_S^2(\varpi)}{3R^2}. \tag{126}$$

Combining this result with Eqs. (115), (118), and (119) gives

$$\delta^{(2)}\Omega = \Omega\left(\frac{\varpi}{R}\right)^{2\ell-4}\left[\frac{(\ell+1)(\ell-7)}{24}\left(\frac{\varpi}{R}\right)^2 + \frac{(\ell^2-1)}{6}\left(3\frac{z^2}{R^2} - 1\right)\right]e^{2\beta t}. \tag{127}$$

In particular, for the $\ell = 2$ r-mode, the radiation-reaction induced differential rotation has the form

$$\delta^{(2)}\Omega = \Omega\left[\frac{3}{2}\left(\frac{z}{R}\right)^2 - \frac{5}{8}\left(\frac{\varpi}{R}\right)^2 - \frac{1}{2}\right]e^{2\beta t}, \tag{128}$$

which is positive in a neighborhood of the poles and negative near the equatorial plane. Figure 1 illustrates the gravitational-radiation driven differential rotation $\delta^{(2)}\Omega/\Omega$ from the $\ell = 2$ r-mode instability of a slowly rotating uniform-density star. This figure shows contours of constant $\delta^{(2)}\Omega/\Omega$, on a cross section of the star that passes through the rotation axis. For example, this figure illustrates that $\delta^{(2)}\Omega/\Omega \approx -9/8$ near the surface of the star at the equator. This indicates that the angular velocity of the star is reduced by an amount $\approx -(9/8)\Omega\alpha^2 e^{2\beta t}$ in this region, where $\alpha e^{\beta t}$ is the amplitude of the r-mode and Ω is the

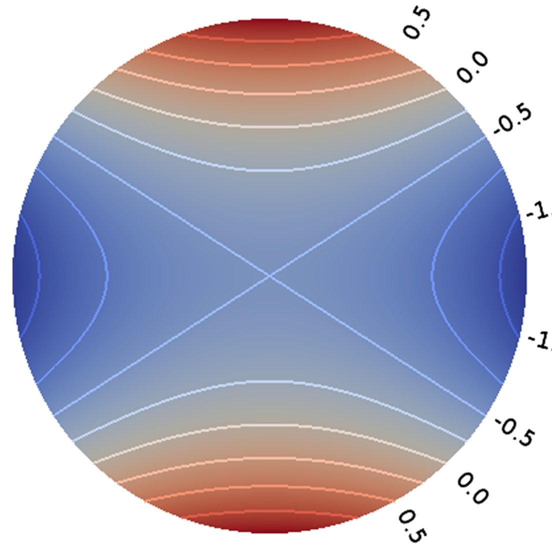

FIG. 1. Differential rotation $\delta^{(2)}\Omega/\Omega$ from the $\ell = 2$ r-mode instability evaluated on a cross section through the rotation axis of a slowly rotating uniform-density star. The solution scales with time as $e^{2\beta t}$.

angular velocity of the unperturbed star. Similarly, this figure illustrates that $\delta^{(2)}\Omega/\Omega \approx 1$ near the poles. The angular velocity of the star is enhanced by the r-mode instability in these regions.

The equilibrium structures of $n = 1$ polytropes can also be expressed in terms of simple analytical functions, but the integrals that determine $\Upsilon(\varpi)$ in Eq. (116) cannot. We therefore evaluate these quantities for all the $n \neq 0$ polytropes numerically.

The structures of the nonrotating Newtonian polytropes are determined by the Lane-Emden equations, which are generally written in the form

$$\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\xi^2\theta^n, \tag{129}$$

where θ is related to the density by $\rho = \rho_c\theta^n$, with $\theta = 1$ at the center of the star and $\theta = 0$ at its surface. The variable ξ is the scaled radial coordinate, $r = a\xi$, with

$$a^2 = \frac{(n+1)K\rho_c^{(1-n)/n}}{4\pi G}. \tag{130}$$

We solve Eq. (129) numerically to determine the Lane-Emden functions $\theta(\xi)$; use them to evaluate the density profiles of these stars, $\rho(r) = \rho_c\theta^n$; and finally perform the integrals numerically in Eq. (116) that determine the mass-weighted average $\Upsilon(\varpi)$ of $(z/R)^2$ for spherical polytropes. Figure 2 illustrates the results for a range of polytropic indices. Because they are more centrally condensed, stars with softer equations of state, i.e. polytropes with larger values of n, have smaller $\Upsilon(\varpi)$. This is most pronounced near the rotation axis of the star where $\varpi = 0$ and values of

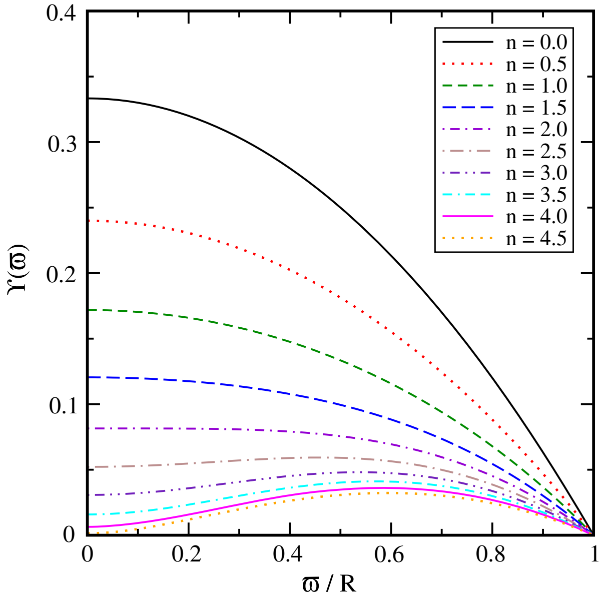

FIG. 2. Dimensionless ratio of the integrals $\Upsilon(\varpi)$ defined in Eq. (116) that determines the gravitational-radiation induced differential rotation in polytropic stellar models having a range of polytropic indices n.

z^2 in the dense core dominate the average. Figure 3 illustrates $\delta_N^{(2)}\Omega/\Omega$ from Eq. (115), the differential rotation induced by the gravitational-radiation driven instability in the $\ell = 2$ r-modes for polytropes having a range of polytropic indices n. This graph shows that the equatorial surface value ($\varpi = R$) of $\delta_N^{(2)}\Omega/\Omega$ is the same for all the polytropes. This is not a surprise, because $\Upsilon(\varpi) = 0$ there for all equations of state. Stars composed of fluid having stiffer equations of state, i.e. smaller values of n, have larger

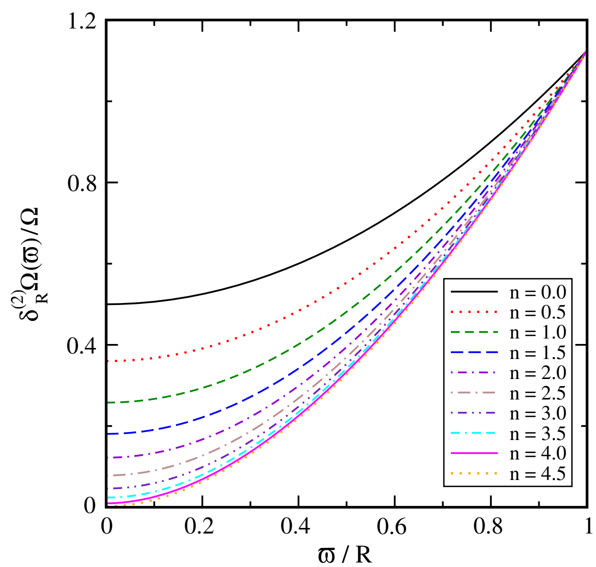

FIG. 3. Differential rotation induced by the gravitational-radiation instability in the $\ell = 2$ r-modes for a range of polytropic indices n.

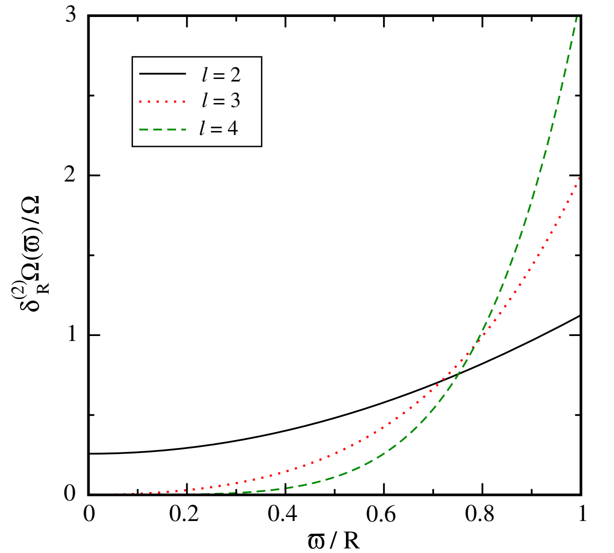

FIG. 4. Differential rotation induced by the gravitational-radiation instability in various r-modes of $n = 1$ polytropes for a range of spherical harmonic mode index ℓ values.

values of $|\delta_N^{(2)}\Omega/\Omega|$ near the rotation axis where $\varpi = 0$. Figure 4 illustrates the differential rotation induced by the gravitational-radiation induced instability in the r-modes of $n = 1$ polytropes having a range of different spherical harmonic mode index ℓ values. The figure portrays a differential rotation $\delta_N^{(2)}\Omega/\Omega$ induced by gravitational radiation that, like the magnitude of the linear mode, is more narrowly confined to the equatorial region near the surface of the star as the r-mode harmonic index ℓ is increased.

VI. DISCUSSION

The radiation-reaction force uniquely determines the exponentially growing differential rotation of the unstable, nonlinear r-mode. We have found expressions for the rotation law and for the corresponding secular drift of a fluid element and have obtained their explicit forms for slowly rotating polytropes. The formalism presented here describes an r-mode, driven by a gravitational radiation reaction, at second order in its amplitude α, and restricted to a perfect-fluid Newtonian model. We now comment briefly on the meaning of the work within a broader physical context.

First, a realistic evolution involves coupling to other modes, because realistic initial data have small, nonzero initial amplitudes for all modes and, at higher orders in α, other modes are excited by the r-mode itself. As a result of the couplings, the r-mode amplitude will saturate, and studies of its nonlinear evolution (see Refs. [5,6] and references therein) suggest a saturation amplitude of order 10^{-4} or smaller. By the time the mode reaches saturation, the amplitude of daughter modes may be large enough that their own second-order axisymmetric parts contribute significantly to the differential rotation law.

Second, when there is a background magnetic field, the growing axisymmetric magnetic field generated by the r-mode's secular drift can change the profile of the growing differential rotation [26]. The second-order Euler equation (45) is altered by the second-order Lorentz force per unit mass, given in an ideal magnetohydrodynamics approximation by $\alpha^2\langle\delta^{(2)} f_{\mathrm{magnetic}}\rangle = \alpha^2\langle\delta^{(2)}[\frac{1}{4\pi\rho}(\nabla\times\vec{B})\times\vec{B}]\rangle$. This will be of order the radiation-reaction force after an amplitude-independent time[4]

$$t \sim \beta t_A^2 \sim 10^6\ \mathrm{s}\,\frac{\rho}{10^{15}\ \mathrm{g/cm^3}}\frac{\beta}{10^{-6}\ \mathrm{s^{-1}}}\left(\frac{10^8\ \mathrm{G}}{B_0}\frac{R}{10^6\ \mathrm{cm}}\right)^2, \tag{131}$$

where t_A is the Alfvén time associated with the background field, $t_A = R\sqrt{4\pi\rho}/B_0$. After this time and until the mode reaches its nonlinear saturation amplitude, we expect that the radiation-reaction force will continue to drive growing differential rotation. The functional form of this differential rotation, however, will be determined by both $\delta^{(2)} f_{GR}$ and $\langle\delta^{(2)} f_{\mathrm{magnetic}}\rangle$.

After nonlinear saturation, we expect the growth of differential rotation and of the magnetic field to stop within a time on the order of the Alfvén time. This is because (1) the radiation-reaction force is now time independent and (2), with a background magnetic field, there should no longer be a zero-frequency subspace of modes associated with adding differential rotation. Reason 2 means that the differential rotation and the magnetic field at the time of mode saturation become initial data for a set of modes whose frequencies are of order the Alvén frequency. The second-order axisymmetric part of the r-mode after saturation becomes effectively a system of stable oscillators driven by a constant force. Such systems have no growing modes and therefore no secularly growing magnetic field.

The explicit form of the secular drift we obtain is new, but its magnitude is consistent with that used in earlier work [8–12,18] that examines the damping of the unstable r-mode by this energy transfer mechanism. This damping mechanism becomes important whenever the rate of energy transfer to the magnetic field (by winding up magnetic-field lines or, for a superconducting region in a neutron-star interior, by stretching magnetic-flux tubes or other mechanisms) is comparable to the growth rate of the unstable r-mode. Assuming the energy transferred to the magnetic field is not returned to the r-mode and that a large fraction of the core is a type-II superconductor, Rezzolla *et al.* [8] estimate that the instability will be magnetically damped for a magnetic field of order 10^{12} G. As noted above, we expect this magnetic damping mechanism to play a role only if the magnetic field reaches this 10^{12} G range prior to the nonlinear saturation of the r-mode. We think it likely that a limit on magnetic-field growth imposed by saturation means that this field strength can be reached only if the initial field is not far below 10^{12} G. In addition, for an initial field of order $B \geq 10^{12}$ G or larger, if all axisymmetric perturbations that wind up the magnetic field have frequency higher than or of order the Alfvén frequency, we conjecture (based on the toy model mentioned in Footnote 4) that the enhancement of the magnetic field will be too small to damp the r-mode.

Finally, if the magnetic field is large enough to significantly modify the structure of the first-order r-modes, all of the calculations here would need to be modified. Previous studies, however [13,29–34], find that field strength $B \gtrsim 10^{14} - 10^{15}$ G is needed to significantly alter the linear r-mode of a star with spin greater than 300 Hz. When the viscous damping time is comparable to the gravitational-wave growth time, one would also need to include viscosity in the second-order equations that determine the differential rotation.

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[4]For a magnetic field that grows linearly in time, we have

$$\alpha^2\langle\delta^{(2)} f_{\mathrm{magnetic}}\rangle \sim \alpha^2\frac{1}{4\pi\rho R}B_0^2\Omega t.$$

The second-order radiation-reaction force is given by $\alpha^2\delta^{(2)} f_{GR} \sim \alpha^2\beta\Omega R$, implying that the Lorentz force $\alpha^2\langle\delta^{(2)} f_{\mathrm{magnetic}}\rangle$ has comparable magnitude after a time given in Eq. (131). Here, we follow Chugunov [26]. Chugunov uses this argument to conclude that the magnetic field will not be significantly enhanced after it reaches $B \sim 10^8(\alpha/10^{-4})^2$ G, but his analysis is restricted to the case where the gravitational radiation-reaction force on the r-mode is negligible. We have checked the conclusion of continued growth for Shapiro's model of a uniform-density cylinder with an initial magnetic field [27], by adding a forcing term of the form of the second-order axisymmetric radiation-reaction force [28]. We expect the amplification factor of the magnetic field to be limited by the value of the mode amplitude, $\alpha e^{\beta t}$, at nonlinear saturation, not by the value of the field, unless the initial magnetic field is of order 10^{12} G or larger.

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APPENDIX A: NOTATION

The symbols in Table I are listed by order of appearance in the paper, *starting with Sec. II. We omit a few symbols that are used only where they are defined.*

TABLE I. Notation.

Symbol	Meaning
Q	The set of variables $\{\rho, \vec{v}, p, \Phi\}$
ρ	Mass density
$\vec{v}$	Fluid velocity
p	Fluid pressure
Φ	Newtonian gravitational potential
h	Fluid specific enthalpy
U	Effective potential
E^a	$E^a = 0$ is the Newtonian Euler equation
$\vec{f}_{GR}$	Radiation-reaction force
$I^{\ell m}, S^{\ell m}$	Mass and current multipoles
N_ℓ	A constant defined in Eq. (10)
Ω	Fluid angular velocity
$\vec{\phi}$	Rotational symmetry vector $x\hat{y} - y\hat{x}$
α	Dimensionless amplitude of r-mode
$\delta^{(1)}Q$	First-order perturbation of Q: $\partial_\alpha Q\|_{\alpha=0}$
$\delta^{(2)}Q$	Second-order perturbation of Q: $\frac{1}{2}\partial_\alpha^2 Q\|_{\alpha=0}$
$\delta_N^{(1)}, \delta_N^{(2)}$	First- and second-order Newtonian perturbations (no radiation reaction)
$\delta_N^{(1)}\hat{Q}$	ϖ, z dependence of perturbation: Eqs. (32)–(35)
$\delta_R^{(1)}\hat{Q}$	A correction in first-order perturbation due to radiation reaction
$\delta^{(1)}Q_\pm$	Subscript $\pm$ denotes even $(+)$ or odd $(-)$ ϕ-parity under the diffeomorphism $\phi \to 2\pi - \phi$
ω_N	Frequency of Newtonian r-mode
ψ_N	$\psi_N \equiv \omega_N t + m\phi$
$P^a{}_b$	Projection operator orthogonal to $\vec{\phi}$: Eq. (30)
β	Imaginary part of frequency of unstable r-mode
$\langle \delta Q \rangle$	Axisymmetric part of δQ
$\delta_{NP}^{(2)}Q$	Second-order Newtonian perturbation with a Particular choice of $\delta_N^{(2)}\Omega(\varpi)$
$\delta_N^{(2)}\Omega(\varpi)$	Arbitrary function of ϖ in second-order Newtonian differential rotation
$\delta^{(2)}\Omega$	Second-order differential rotation, $\langle \delta^{(2)} v^\phi \rangle$
$\delta_R^{(2)}Qe^{2\beta t}$	Radiative part of second-order perturbation
$\delta_R^{(2)}\vec{V}$, $\delta_R^{(2)}\vec{W}$	Defined in Eqs. (50) and (51)
$\langle \delta_R^{(2)}\vec{F} \rangle$	Effective driving force for $\langle \delta_R^{(2)}\vec{v} \rangle$: Eq. (57)
M_0, R	Mass and radius of spherical stellar model
Ω_0	$\sqrt{M_0/R^3}$
$\tilde{\Omega}$	Dimensionless angular velocity, Ω/Ω_0
Q_n	Part of Q that is nth order in $\tilde{\Omega}$: Eq. (70)
$\delta_\perp^{(1)}\vec{f}_{GR}$	Part of $\delta^{(1)}\vec{f}_{GR}$ orthogonal to $\delta_N^{(1)}\vec{v}$
$\delta_\perp^{(1)}\mathcal{F}$	Function for which $\delta_\perp^{(1)}\vec{f}_{GR} = \vec{\nabla}\delta_\perp^{(1)}\mathcal{F}$

APPENDIX B: GRAVITATIONAL WAVE ENERGY AND ANGULAR MOMENTUM FLUXES

The expression for the radiation-reaction force $\vec{f}_{GR}$ given in Eq. (7) was derived by constructing a force that reproduces the standard expressions for the time averaged gravitational wave energy and angular momentum fluxes:

$$\left\langle\!\!\left\langle \frac{dE}{dt} \right\rangle\!\!\right\rangle\bigg|_{GR} = \left\langle\!\!\left\langle \int \rho\vec{v}\cdot\vec{f}_{GR} d^3x \right\rangle\!\!\right\rangle, \\ = -\sum_{\ell\geq 2}\sum_{|m|\leq\ell} \frac{1}{32\pi}\left\langle\!\!\left\langle \left|\frac{d^{\ell+1}I^{\ell m}}{dt^{\ell+1}}\right|^2 + \left|\frac{d^{\ell+1}S^{\ell m}}{dt^{\ell+1}}\right|^2 \right\rangle\!\!\right\rangle, \tag{B1}$$

$$\left\langle\!\!\left\langle \frac{d\vec{J}}{dt} \right\rangle\!\!\right\rangle\bigg|_{GR} = \left\langle\!\!\left\langle \int \rho\vec{r}\times\vec{f}_{GR} d^3x \right\rangle\!\!\right\rangle, \\ = \sum_{\ell\geq 2}\sum_{|m|\leq\ell} \frac{1}{32\pi}\Re\left\langle\!\!\left\langle \frac{d^\ell I^{*\ell m}}{dt^\ell}\frac{d^{\ell+1}\vec{I}_B^{\ell m}}{dt^{\ell+1}} + \frac{d^\ell S^{*\ell m}}{dt^\ell}\frac{d^{\ell+1}\vec{S}_B^{\ell m}}{dt^{\ell+1}} \right\rangle\!\!\right\rangle. \tag{B2}$$

The expression given here for the angular momentum flux, Eq. (B2), is somewhat more compact than the standard post-Newtonian expression [cf. Thorne [19], Eq. (4.23)]. We express this flux in terms of the magnetic-type mass and current-multipole moments, $\vec{I}_B^{\ell m}$ and $\vec{S}_B^{\ell m}$, which we define as

$$\vec{I}_B^{\ell m} = N_\ell\sqrt{\ell+1}\int \rho r^\ell \vec{Y}_B^{\ell m} d^3x, \tag{B3}$$

$$\vec{S}_B^{\ell m} = \frac{2N_\ell}{\sqrt{\ell+1}}\int \rho r^\ell (\vec{v}\cdot\vec{r}\times\vec{\nabla})\vec{Y}_B^{\ell m} d^3x. \tag{B4}$$

These magnetic-type mass and current mutipole moments can be expressed in terms of the standard $I^{\ell m}$ and $S^{\ell m}$,

$$\vec{I}_B^{\ell m} = -\frac{i}{2}\sqrt{(\ell-m)(\ell+m+1)}I^{\ell m+1}(\hat{x}+i\hat{y}) \\ -\frac{i}{2}\sqrt{(\ell+m)(\ell-m+1)}I^{\ell m-1}(\hat{x}-i\hat{y}) - imI^{\ell m}\hat{z}, \tag{B5}$$

$$\vec{S}_B^{\ell m} = -\frac{i}{2}\sqrt{\frac{(\ell-m)(\ell+m+1)}{\ell(\ell+1)}}S^{\ell m+1}(\hat{x}+i\hat{y}) \\ -\frac{i}{2}\sqrt{\frac{(\ell+m)(\ell-m+1)}{\ell(\ell+1)}}S^{\ell m-1}(\hat{x}-i\hat{y}) \\ -\frac{im}{\sqrt{\ell(\ell+1)}}S^{\ell m}\hat{z}, \tag{B6}$$

where $\hat{x}$, $\hat{y}$, and $\hat{z}$ are unit vectors. Both of these expressions are based on the following identity for vector spherical harmonics:

$$\vec{Y}_B^{\ell m} = \frac{i}{2}\sqrt{\frac{(\ell-m)(\ell+m+1)}{\ell(\ell+1)}}Y^{\ell m+1}(\hat{x}-i\hat{y}) + \frac{i}{2}\sqrt{\frac{(\ell+m)(\ell-m+1)}{\ell(\ell+1)}}Y^{\ell m-1}(\hat{x}+i\hat{y}) + \frac{im}{\sqrt{\ell(\ell+1)}}Y^{\ell m}\hat{z}. \tag{B7}$$

Using this transformation, Eq. (B2) reproduces the standard post-Newtonian expression [cf. Thorne [19], Eq. (4.23)]. The calculation needed to verify that the expression for the radiation-reaction force $\vec{f}_{GR}$ given in Eq. (7) satisfies the time averaged gravitational wave energy and angular momentum flux expressions given in Eqs. (B1) and (B2) is straightforward but lengthy.

APPENDIX C: INTEGRATING $\delta\rho$

For rotating equilibrium stellar models having polytropic equations of state with polytropic index n, the density $\rho \propto$ (distance to the surface)^n near the star's surface. We assume here that the surface of the perturbed star is smooth as a function of α and $\vec{x}$. Although the surface itself is smooth, the behavior of ρ near $\rho = 0$ implies that $\nabla_a\rho$ diverges for $n < 1$ and $\nabla_a\nabla_b\rho$ diverges for $n < 2$. It follows that $\delta^{(1)}\rho$ and $\delta^{(2)}\rho$ diverge because they involve first and second derivatives, respectively, of the unperturbed density. We show, however, that continuity and differentiability of the star's surface as a function of $\vec{x}$ and α imply finiteness of the integrals $\int_{-\infty}^{\infty}\delta^{(1)}\rho dz$ and $\int_{-\infty}^{\infty}\delta^{(2)}\rho dz$, when $\delta^{(2)}\rho$ is regarded as a distribution.

We first verify the claimed behavior of ρ for the unperturbed polytrope and then use the form of the Lagrangian perturbation of the enthalpy to deduce the behavior of $\delta^{(1)}\rho$ and $\delta^{(2)}\rho$ near the surface. Denote by $z_S^{\pm}(\alpha, t, \varpi, \phi)$ the values of z at the top and bottom parts of the surface of the perturbed star. We again introduce the polytropic function θ, related to the specific enthalpy by $\theta = \rho_o/[(n+1)p_o]h$. Then, $\rho = \rho_o\theta^n\Theta(z_S^+ - z)\Theta(z - z_S^-)$, where $\Theta(z_S^+ - z) = 1$ for $z_S^+ > z$ and $\Theta(z_S^+ - z) = 0$ for $z_S^+ < z$. For the unperturbed rotating polytrope, θ is finite with finite derivatives at the surface of the star.[5] The lack of smoothness in ρ at the surface thus arises from the fact that n is not an integer. We now show for the perturbed polytrope that ρ is again proportional to (distance to the surface)^n to second order in α.

The vanishing of θ at the surface of the perturbed star is equivalent to the vanishing of the Lagrangian perturbation of θ at the unperturbed surface,

$$\Delta\theta = 0, \tag{C1}$$

where

$$\Delta\theta := \theta(\alpha, t, \vec{x} + \vec{\xi}) - \theta(0, t, \vec{x}), \tag{C2}$$

with $\vec{\xi}(\alpha, t, \vec{x})$ the exact Lagrangian displacement—a vector from the position $\vec{x}$ of each fluid element in the unperturbed star to its position $\vec{x} + \vec{\xi}$ in the perturbed fluid. Our assumption that the surface changes smoothly as a function of α and $\vec{x}$ is then the requirement that $\vec{\xi}$ and its derivatives are smooth at the surface of the star. Writing

$$\vec{\xi} = \alpha\vec{\xi}^{(1)} + \alpha^2\vec{\xi}^{(2)} + \mathcal{O}(\alpha^3) \tag{C3}$$

and taking derivatives of (C2) with respect to α, we have

$$\delta^{(1)}\theta = \Delta^{(1)}\theta - \xi^{(1)a}\nabla_a\theta, \tag{C4a}$$

$$\delta^{(2)}\theta = \Delta^{(2)}\theta - \xi^{(2)a}\nabla_a\theta - \xi^{(1)a}\nabla_a\delta^{(1)}\theta - \frac{1}{2}\xi^{(1)a}\xi^{(1)b}\nabla_a\nabla_b\theta. \tag{C4b}$$

Then, $\delta^{(1)}\theta$ and $\delta^{(2)}\theta$ are finite at the unperturbed surface, and, to second order in α, we can write for θ the Taylor expansion

$$\theta(\alpha, z, \varpi) = \partial_z\theta|_{z_S^+}(z - z_S^+) + \mathcal{O}(z - z_S^+)^2, \tag{C5}$$

for $z < z_S^+$. The corresponding expansion for $\rho = \rho_0\theta^n$ is thus

$$\rho(\alpha, z, \varpi) = \rho_0(-\partial_z\theta|_{z_S^+})^n(z_S^+ - z)^n + \mathcal{O}(z_S^+ - z)^{n+1}. \tag{C6}$$

We can now show directly that the integrals $\int_{-\infty}^{\infty}\delta^{(1)}\rho dz$ and $\int_{-\infty}^{\infty}\delta^{(2)}\rho dz$ are finite for polytropic equations of state with any polytropic index $n > 0$ for which the equilibrium star has a finite surface. More precisely, they are finite everywhere except the equator, where the range of integration vanishes.

For a given value of ϖ, we choose $Z^{\pm}$ with $0 < Z^+ < z_S^+$ and $0 > Z^- > z_S^-$ for all $\alpha < \epsilon$, for some finite $\epsilon > 0$. We write the integral as a sum of three parts,

$$\int_{-\infty}^{\infty}\delta\rho dz = \int_{Z^-}^{Z^+}\delta\rho dz + \int_{Z^+}^{\infty}\delta\rho dz + \int_{-\infty}^{Z^-}\delta\rho dz. \tag{C7}$$

[5] For the unperturbed star, Eq. (13) implies $\theta = \rho_o/[(n+1)p_o](\mathcal{E} - \Phi + \frac{1}{2}\varpi^2\Omega^2)$, where $\mathcal{E}$ is the constant injection energy per unit mass. Caffarelli and Friedman prove that ρ is Hölder continuous, $\rho \in C^{0,\alpha}(\mathbb{R}^3)$ [35], and the Poisson equation then implies $\Phi \in C^{2,\alpha}(\mathbb{R}^3)$. Thus, θ has one-sided first and second derivatives at the surface.

In the first integral on the right side, $\delta^{(1)}\rho$ and $\delta^{(2)}\rho$ are finite, so we need only to consider $\int_{Z^+}^{\infty}\delta^{(1)}\rho dz$, $\int_{Z^+}^{\infty}\delta^{(2)}\rho dz$, and the corresponding integrals near the bottom part of the surface. Because the finiteness argument is identical for the integrals near z_S^- and z_S^+, we consider the integrals near z_S^+.

We have

$$\begin{aligned}\partial_\alpha\rho &= \partial_\alpha\{[\rho_0(-\partial_z\theta|_{z_S^+})^n(z_S^+-z)^n \\ &\quad +\mathcal{O}(z_S^+-z)^{n+1}]\,\Theta\,(z_S^+-z)\} \\ &= \rho_0(-\partial_z\theta|_{z_S^+})^n\partial_\alpha[(z_S^+-z)^n\,\Theta\,(z_S^+-z)]+\mathcal{O}(z_S^+-z)^n \\ &= -\rho_0\partial_\alpha z_S^+(-\partial_z\theta|_{z_S^+})^n\partial_z[(z_S^+-z)^n\,\Theta\,(z_S^+-z)] \\ &\quad +\mathcal{O}(z_S^+-z)^n, \end{aligned}\tag{C8}$$

implying

$$\begin{aligned}\delta^{(1)}\rho &= -\rho_0[\xi^{(1)z}(-\partial_z\theta)^n]_{z_S}\partial_z[(z_S-z)^n\,\Theta\,(z_S-z)] \\ &\quad +\mathcal{O}(z_S-z)^n,\end{aligned}\tag{C9}$$

where we have used the relation $\partial_\alpha z_S^+|_{\alpha=0}=\xi^{(1)z}|_{z_S}$. From Eq. (C8), we have

$$\begin{aligned}\partial_\alpha^2\rho &= -\rho_0\partial_\alpha z_S^+(-\partial_z\theta|_{z_S^+})^n\partial_\alpha\partial_z[(z_S^+-z)^n\,\Theta\,(z_S^+-z)] \\ &\quad +\mathcal{O}(z_S^+-z)^{n-1} \\ &= \rho_0(\partial_\alpha z_S^+)^2(-\partial_z\theta|_{z_S^+})^n\partial_z^2[(z_S^+-z)^n\,\Theta\,(z_S^+-z)] \\ &\quad +\mathcal{O}(z_S^+-z)^{n-1},\end{aligned}\tag{C10}$$

implying

$$\begin{aligned}\delta^{(2)}\rho &= \frac{1}{2}\rho_0[(\xi^{(1)z})^2(-\partial_z\theta)^n]_{z_S}\partial_z^2[(z_S-z)^n\,\Theta\,(z_S^+-z)] \\ &\quad +\mathcal{O}(z_S-z)^{n-1}.\end{aligned}\tag{C11}$$

Finiteness of $\int\delta^{(1)}\rho dz$ is immediate from the integrability of $(z_S-z)^{n-1}$ for $n>0$. For $\delta^{(2)}\rho$, we had to retain the factor $\Theta\,(z_S-z)$, and we kept it for $\delta^{(1)}\rho$ as well to display pairs of analogous equations. From Eqs. (C9) and (C11), the leading term in each of $\delta^{(1)}\rho$ and $\delta^{(2)}\rho$ is a z-derivative, and we immediately obtain the integrals

$$\int_{Z^+}^{\infty}\delta^{(1)}\rho dz = \rho(Z)\xi^{(1)z}|_{z_S}+\mathcal{O}(z_S-Z^+)^{n+1},\tag{C12}$$

$$\int_{Z^+}^{\infty}\delta^{(2)}\rho dz = \frac{n}{2}\frac{\rho(Z)}{z_S-Z^+}(\xi^{(1)z}|_{z_S})^2+\mathcal{O}(z_S-Z^+)^n.\tag{C13}$$

The integrals $\int_{-\infty}^{\infty}\delta^{(1)}\rho dz$ and $\int_{-\infty}^{\infty}\delta^{(2)}\rho dz$ are therefore finite as claimed.

APPENDIX D: ORDERING IN Ω OF $\delta_R^{(1)}Q$

To make the heuristic argument of Sec. IV A more precise, we use the two-potential formalism of Ipser and Lindblom [36] to write an explicit form for $\delta_R^{(1)}v^a$ in terms of $\delta_R^{(1)}U$ and $\delta_R^{(1)}\mathcal{F}$. Because that formalism uses the complex version of a perturbation, we write $\delta_R^{(1)}Q=\Im(\tilde{\delta}_R^{(1)}Q)$. The perturbed Euler equation, Eq. (95), with radiation-reaction force then has the form

$$\begin{aligned}Q_{ab}^{-1}\tilde{\delta}_R^{(1)}v^b &\equiv [(\omega_N+\ell\Omega)g_{ab}+2i\Omega\nabla_a\phi_b]\tilde{\delta}_R^{(1)}v^b \\ &= i\nabla_a(\tilde{\delta}_R^{(1)}U-\tilde{\delta}_R^{(1)}\mathcal{F}).\end{aligned}\tag{D1}$$

Using the slow-rotation form (81) of ω_N and Eq. (13) of Ref. [36], we write the solution to this equation for $\tilde{\delta}_R^{(1)}v^a$ as

$$\tilde{\delta}_R^{(1)}v^a = iQ^{ab}\nabla_b(\tilde{\delta}_R^{(1)}U-\tilde{\delta}_\perp^{(1)}\mathcal{F}),\tag{D2}$$

where the inverse of Q_{ab}^{-1} is the tensor $Q^{ab}=\Omega^{-1}\tilde{Q}^{ab}$, with

$$\begin{aligned}\tilde{Q}^{ab} &= -\frac{\ell+1}{2\ell(\ell+2)} \\ &\quad\times[g^{ab}-(\ell+1)^2\nabla^a z\nabla^b z-i(\ell+1)\nabla^a\phi^b].\end{aligned}\tag{D3}$$

With $\tilde{\delta}_R^{(1)}v^a$ replaced by the expression on the right side of Eq. (D2), the mass conservation equation, Eq. (94) becomes an elliptic equation for $\tilde{\delta}_R^{(1)}U-\delta_\perp^{(1)}\mathcal{F}$, namely

$$\begin{aligned}&\nabla_a[\rho\tilde{Q}^{ab}\nabla_b(\tilde{\delta}_R^{(1)}U-\delta_\perp^{(1)}\mathcal{F})] \\ &\quad+\frac{2}{\ell+1}\Omega^2\rho\frac{d\rho}{dp}(\tilde{\delta}_R^{(1)}U-\tilde{\delta}_R^{(1)}\Phi)=i\Omega\beta\tilde{\delta}_N^{(1)}\rho.\end{aligned}\tag{D4}$$

The potentials $\tilde{\delta}_R^{(1)}U$ and $\tilde{\delta}_R^{(1)}\Phi$ are determined by this equation, together with the Poisson equation,

$$\nabla^2\tilde{\delta}_R^{(1)}\Phi = 4\pi\rho\frac{d\rho}{dp}(\tilde{\delta}_R^{(1)}U-\tilde{\delta}_R^{(1)}\Phi),\tag{D5}$$

and the two boundary conditions,

$$\lim_{r\to\infty}\tilde{\delta}_R^{(1)}\Phi=0\tag{D6}$$

and

$$\Delta^{(1)}h=(\tilde{\delta}_R^{(1)}U|_S-\tilde{\delta}_R^{(1)}\Phi+\tilde{\xi}_R^{(1)b}\nabla_b h)|_S=0;\tag{D7}$$

here, S is the surface of the unperturbed star, and the Lagrangian displacement $\tilde{\xi}_R^{(1)a}$ is defined by

$$\tilde{\xi}_R^{(1)a}=\frac{1}{i(\omega_N+\ell\Omega)}\tilde{\delta}_R^{(1)}v^a.\tag{D8}$$

Using the value of ω_N from Eq. (81), and Eq. (D2), we can write the second boundary condition as

$$\tilde{Q}^{ab}\nabla_a h\nabla_b(\tilde{\delta}_R^{(1)}U - \tilde{\delta}_\perp^{(1)}\mathcal{F}) + \frac{2}{\ell+1}\Omega^2(\tilde{\delta}_R^{(1)}U - \tilde{\delta}_R^{(1)}\Phi) = 0. \tag{D9}$$

To find the orders in Ω of $\delta_R^{(1)}\vec{v}$, $\delta_R^{(1)}U$, and $\delta_R^{(1)}\Phi$, we begin with the relations $\delta_N^{(1)}\rho = \mathcal{O}(\Omega^2)$ and, from Eq. (89), $\tilde{\delta}_\perp^{(1)}\mathcal{F} = \mathcal{O}(\Omega\beta)$. From the Poisson equation (D5), $\delta_R^{(1)}U$ and $\delta_R^{(1)}\Phi$ are the same order in Ω. From Eq. (D4), we then have $\delta_R^{(1)}U - \delta_\perp^{(1)}\mathcal{F} = \mathcal{O}(\Omega^2\delta_\perp^{(1)}\mathcal{F}) + \mathcal{O}(\Omega^3\beta) = \mathcal{O}(\Omega^3\beta)$. Then,

$$\delta_R^{(1)}\Phi = \mathcal{O}(\delta_R^{(1)}U) = \mathcal{O}(\delta_\perp^{(1)}\mathcal{F}) = \mathcal{O}(\Omega\beta). \tag{D10}$$

Finally, Eq. (D2) implies

$$\delta_R^{(1)}v^a = \mathcal{O}(\Omega^{-1}\Omega^3\beta) = \mathcal{O}(\Omega^2\beta). \tag{D11}$$

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