Erratum: Post-Newtonian, Quasi-Circular Binary Inspirals in Quadratic Modified Gravity

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In this erratum, we point out that the correction to the gravitational energy flux from compact binary inspirals in dynamical Chern-Simons gravity enters at third post-Newtonian order instead of second post-Newtonian order. We also correct the scalar energy flux at second post-Newtonian order by considering terms that we missed in the original paper.

I. GRAVITATIONAL RADIATION

In [1], we claimed that the correction to the gravitational energy flux, i.e. the energy flux from binary inspirals induced by the metric perturbation in dynamical Chern-Simons (dCS) gravity, enters at second post-Newtonian (PN) order. Here, we point out that there are terms that we overlooked, and if we take these terms into account properly, the 2PN terms cancel and the correction enters at 3PN order.[1]

First, let us remind ourselves what we did in [1]. Our starting point was perturbing the metric around a flat background as

$$g_{\mu\nu} = g_{\mu\nu}^{\rm GR} + \mathfrak{h}_{\mu\nu}\,, \qquad g_{\mu\nu}^{\rm GR} = \eta_{\mu\nu} + h_{\mu\nu}\,, \qquad (1)$$

where $h_{\mu\nu}$ and $\mathfrak{h}_{\mu\nu}$ are the General Relativity (GR) and dCS metric perturbations respectively. We worked in harmonic gauge, $\bar{h}^{\mu\nu}{}_{,\nu} = 0 = \bar{\mathfrak{h}}^{\mu\nu}{}_{,\nu}$, where we defined the metric perturbations via

$$\bar{h}^{\mu\nu} \equiv \eta^{\mu\nu} - \sqrt{-g_{\rm GR}}g_{\rm GR}^{\mu\nu}\,, \qquad (2)$$
$$\bar{\mathfrak{h}}^{\mu\nu} \equiv \left(\eta^{\mu\nu} - \sqrt{-g}g^{\mu\nu}\right) - \bar{h}^{\mu\nu}\,. \qquad (3)$$

We then expanded the modified Einstein equations, focusing on the dCS correction and keeping terms up to linear order in $h_{\mu\nu}$ and $\bar{\mathfrak{h}}_{\mu\nu}$:

$$\kappa\Box_\eta\bar{\mathfrak{h}}_{\mu\nu} = 2\alpha_4\tilde{\mathcal{K}}_{\mu\nu}^{(1)} - T_{\mu\nu}^{(\vartheta)}\,, \qquad (4)$$

where $\Box_\eta$ is the d'Alembertian operator of flat spacetime and

$$\tilde{\mathcal{K}}_{\mu\nu}^{(1)} = \vartheta^{,\delta}{}_{,\sigma}\eta_{\nu\alpha}\epsilon^{\alpha\sigma\beta\gamma}\left(h_{\mu[\gamma,\beta]\delta} + h_{\delta[\beta,\gamma]\mu}\right)$$
$$-2\vartheta^{,\delta}\epsilon_{\delta\sigma\chi\mu}h^{\sigma}{}_{[\alpha,}{}^{\alpha\chi}{}_{\nu]} + \mathcal{O}(h^2) + (\mu \leftrightarrow \nu)\,, \quad (5)$$
$$T_{\mu\nu}^{(\vartheta)} = \beta\left(\vartheta_{,\mu}\vartheta_{,\nu} - \frac{1}{2}\eta_{\mu\nu}\vartheta_{,\delta}\vartheta^{,\delta}\right) + \mathcal{O}(h)\,. \quad (6)$$

We then solved this wave equation in the far zone (FZ), for field points much larger than the gravitational wave (GW) wavelength of the system, using a Green's function. The Green's function solution requires the integration of $\tilde{\mathcal{K}}_{\mu\nu}^{(1)}$ and $T_{\mu\nu}^{(\vartheta)}$, and thus of the fields $h_{\mu\nu}$ and ϑ evaluated in the near zone (NZ), i.e. at field point distances smaller than the GW wavelength of the system, but far from the location of the objects. To leading PN order, the NZ fields are given by

$$h_{00}^{\rm NZ} = 2\frac{m_1}{r_1} + \mathcal{O}\left(\frac{m_1^2}{r_1^2}\right) + (1 \leftrightarrow 2)\,, \qquad (7)$$

$$h_{ij}^{\rm NZ} = 2\frac{m_1}{r_1}\delta_{ij} + \mathcal{O}\left(\frac{m_1^2}{r_1^2}\right) + (1 \leftrightarrow 2)\,, \qquad (8)$$

$$\vartheta^{\rm NZ} = -\mu_1^i\partial_i\frac{1}{r_1} + \mathcal{O}\left(\frac{m_1^3}{r_1^3}\right) + (1 \leftrightarrow 2)$$
$$= \frac{\mu_1^i n_i^i}{r_1^2} + \mathcal{O}\left(\frac{m_1^3}{r_1^3}\right) + (1 \leftrightarrow 2)\,, \qquad (9)$$

where μ_A^i is the NZ scalar dipole charge, with

$$\mu_A^i \equiv \frac{5}{2}\frac{\alpha_4}{\beta}\chi_A\hat{S}_A^i \qquad (10)$$

for a black hole, where χ_A and $\hat{S}_A^i$ represent respectively the dimensionless spin parameter and the unit spin angular momentum vector of the Ath binary constituent. The dominant contribution to the FZ solution comes from the integration of the $T_{\mu\nu}^{(\vartheta)}$ term in Eq. (4), which is roughly of the following PN order:

$$\bar{\mathfrak{h}}_{ij}^{\rm FZ} \sim \delta_{ij}\int\vartheta_{,k}^{\rm NZ}\vartheta_{,k}^{\rm NZ}\frac{d^3x'}{|\boldsymbol{x} - \boldsymbol{x}'|}$$
$$\propto \frac{1}{r}\frac{1}{b^3}\frac{1}{b^3}b^3 \propto \frac{v^6}{r}\,, \qquad (11)$$

where b is the binary's separation, v is the binary's (relative) velocity and r is the FZ field point distance. Comparing this with the GR, leading PN contribution, $\bar{h}_{ij}^{\rm FZ} \sim v^2/r$, we concluded that the dCS correction first

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[1] Recall that a term in an expression is said to be of Nth PN order if it is proportional to v^{2N} relative to the leading-order term in that same expression.

enters at 2PN order, i.e. at $\mathcal{O}(v^4/c^4)$ relative to the leading PN order GR contribution.

Now, let us look at terms that we neglected in Eq. (4). Let us begin by expanding out this equation to find

$$\kappa \Box_\eta \bar{\mathfrak{h}}_{\mu\nu} = S_{\mu\nu} , \tag{12}$$
$$S_{\mu\nu} \equiv 2\alpha_4 \tilde{\mathcal{K}}^{(1)}_{\mu\nu} - T^{(\vartheta)}_{\mu\nu} - \mathfrak{h} T^{\mathrm{mat}}_{\mu\nu} - t^{(1)}_{\mu\nu} + \mathcal{O}(\mathfrak{h}^2) . \tag{13}$$

One of the terms we neglected in Eq. (4) is the third one in Eq. (13), which comes from expanding the metric determinant in the term $(g T^{\mathrm{mat}}_{\mu\nu})$, where $T^{\mathrm{mat}}_{\mu\nu}$ is the matter stress-energy tensor (see the relaxed Einstein equations in GR given by Eqs. (2.4) and (2.5) in [2]). Another term we neglected is the fourth term in Eq. (13), which is the linear order in $\bar{\mathfrak{h}}$ piece of $t_{\mu\nu}$, where the latter nonlinear tensor is defined by

$$t_{\mu\nu} \equiv (-g) t^{\mathrm{LL}}_{\mu\nu} + \kappa \left(\tilde{h}_{\mu\alpha}{}^{,\beta} \tilde{h}_{\nu\beta}{}^{,\alpha} - \tilde{h}^{\alpha\beta} \tilde{h}_{\mu\nu,\alpha\beta} \right) , \tag{14}$$

with $t^{\mathrm{LL}}_{\mu\nu}$ the Landau-Lifshitz pseudotensor, which depends on $\tilde{h}_{\mu\nu} \equiv \bar{h}_{\mu\nu} + \bar{\mathfrak{h}}_{\mu\nu}$.

Clearly then, in order to solve Eq. (12) in terms of Green's functions, we need to integrate $(\mathfrak{h} T^{\mathrm{mat}}_{\mu\nu})$ and $t^{(1)}_{\mu\nu}$ in the FZ using the NZ solution for $\mathfrak{h}$. The latter is given by [3, 4]

$$\mathfrak{h}^{\mathrm{NZ}} \sim \zeta_4 \frac{m_1^3}{r_1^3} \chi_1^2 + (1 \leftrightarrow 2) . \tag{15}$$

Denoting the contribution of the third and fourth terms in Eq. (13) to $\bar{\mathfrak{h}}^{\mathrm{FZ}}_{ij}$ as $\bar{\mathfrak{h}}^{\mathrm{FZ},T^{\mathrm{mat}}_{ij}}_{ij}$ and $\bar{\mathfrak{h}}^{\mathrm{FZ},t_{ij}}_{ij}$ respectively, one then finds

$$\bar{\mathfrak{h}}^{\mathrm{FZ},T^{\mathrm{mat}}_{ij}}_{ij} \sim \int \mathfrak{h}^{\mathrm{NZ}} T^{\mathrm{mat}}_{ij} \frac{d^3 x'}{|\boldsymbol{x} - \boldsymbol{x}'|}$$
$$\propto \frac{1}{r} \frac{1}{b^3} v^2 \propto \frac{v^8}{r} , \tag{16}$$
$$\bar{\mathfrak{h}}^{\mathrm{FZ},t_{ij}}_{ij} \sim \int \mathfrak{h}^{\mathrm{NZ}}_{kl} h^{\mathrm{NZ}}_{ij,kl} \frac{d^3 x'}{|\boldsymbol{x} - \boldsymbol{x}'|}$$
$$\propto \frac{1}{r} \frac{1}{b^3} \frac{1}{b^3} b^3 \propto \frac{v^6}{r} , \tag{17}$$

where we used that $T^{\mathrm{mat}}_{ij} \sim \rho v_i v_j$ with ρ representing the energy density of one of the bodies. From the above equations, we see that the $(\mathfrak{h} T^{\mathrm{mat}}_{\mu\nu})$ and $t^{(1)}_{\mu\nu}$ terms give 3PN and 2PN corrections to the gravitational radiation respectively. Notice that the latter is of the same order as the contribution from $T^{(\vartheta)}_{\mu\nu}$ that we calculated in [1].

Given this, what is the correct dCS corrections to the gravitational energy flux? One can answer this question by using a very useful (and standard) trick employed regularly in the PN expansion of GR. Due to the gauge condition $\bar{\mathfrak{h}}_{\mu\nu}{}^{,\nu} = 0$, the tensor $S_{\mu\nu}$ in Eq. (13) is conserved, i.e. $S_{\mu\nu}{}^{,\nu} = 0$. Thanks to this conservation law, one can write (see e.g. [5])

$$\bar{\mathfrak{h}}^{\mathrm{FZ}}_{ij} \sim \frac{1}{r} \int S_{ij} d^3 x' \sim \frac{1}{r} \frac{\partial^2}{\partial t^2} \int S_{00} x'^i x'^j d^3 x' , \tag{18}$$

after integrating by parts a few times. Then, denoting the contribution of $T^{(\vartheta)}_{00}$, T^{mat}_{00} and t_{00} to $\bar{\mathfrak{h}}^{\mathrm{FZ}}_{ij}$ as $\bar{\mathfrak{h}}^{\mathrm{FZ},T^{(\vartheta)}_{00}}_{ij}$, $\bar{\mathfrak{h}}^{\mathrm{FZ},T^{\mathrm{mat}}_{00}}_{ij}$ and $\bar{\mathfrak{h}}^{\mathrm{FZ},t_{00}}_{ij}$ respectively, one finds

$$\bar{\mathfrak{h}}^{\mathrm{FZ},T^{(\vartheta)}_{00}}_{ij} \sim \frac{1}{r} \delta_{00} \frac{\partial^2}{\partial t^2} \int \vartheta^{\mathrm{NZ}}_{,k} \vartheta^{\mathrm{NZ}}_{,k} x'^i x'^j d^3 x'$$
$$\propto \frac{1}{r} \omega^2 \frac{1}{b^3} \frac{1}{b^3} b^2 b^3 \propto \frac{v^8}{r} , \tag{19}$$
$$\bar{\mathfrak{h}}^{\mathrm{FZ},T^{\mathrm{mat}}_{00}}_{ij} \sim \frac{1}{r} \frac{\partial^2}{\partial t^2} \int \mathfrak{h}^{\mathrm{NZ}} T^{\mathrm{mat}}_{00} x'^i x'^j d^3 x'$$
$$\propto \frac{1}{r} \omega^2 \frac{1}{b^3} b^2 \propto \frac{v^8}{r} , \tag{20}$$
$$\bar{\mathfrak{h}}^{\mathrm{FZ},t_{00}}_{ij} \sim \frac{1}{r} \frac{\partial^2}{\partial t^2} \int \mathfrak{h}^{\mathrm{NZ}}_{kl} h^{\mathrm{NZ}}_{00,kl} x'^i x'^j d^3 x'$$
$$\propto \frac{1}{r} \omega^2 \frac{1}{b^3} \frac{1}{b^3} b^2 b^3 \propto \frac{v^8}{r} , \tag{21}$$

where we used $T^{\mathrm{mat}}_{00} \sim \rho$. Observe that all of these contributions enter at 3PN order relative to the leading GR expression, and not at 2PN order. This implies that certain terms we neglected actually cancel the 2PN correction to the metric perturbation, forcing the correction to enter at 3PN order. A similar result can be obtained in GR when deriving an expression for the FZ metric perturbation in terms of the source multipole moments. Using the conservation law of $S_{\mu\nu}$ in GR, one can show that $t_{\mu\nu}$ in GR sources higher PN contributions compared to the leading quadrupolar term sourced by the matter stress-energy tensor.

A very important consequence of all of this is that the contribution of the dCS correction to the metric perturbation to the energy flux is now smaller (in a PN sense) than that induced by the scalar field. Therefore, one does not need to include the dCS correction to the metric perturbation when computing the energy flux to leading dCS and PN order in compact binary inspirals [6] and in the orbital decay rate of binary pulsars [4].

II. SCALAR RADIATION

We now look at the scalar energy flux. The scalar field evolution equation is

$$\Box \vartheta = -\alpha_4 R\,{}^*R , \tag{22}$$

where the source is the Pontryagin density (see e.g. Eq. (8) in [1]). When one solves this equation through a Green's function approach, one must employ finite part regularization to avoid divergences introduced by the point-particle approximation. A matching calculation gives rise to an effective source term to the scalar field evolution equation that gives the correct scalar field solution. To leading PN order and in *covariant form*, the

evolution equation then becomes

$$\Box\vartheta = 4\pi \sum_{A=1,2} \int d\tau \mu_A^\alpha \left(\frac{\delta^{(4)}[x^\mu - x_A^\mu(\tau)]}{\sqrt{-g}}\right)_{,\alpha} - \alpha_4 R\,{}^*R,$$

(23)

where x_A^μ is the trajectory of body A. We will neglect $-\alpha_4 R\,{}^*R$ on the right-hand side because it only sources higher PN order contributions to the FZ scalar field, as shown in [1]. In [1], we did not use the covariant form of the effective source term (compare Eq. (23) to Eq. (74) in [1]); as we will see, there are terms that arise from the covariant form of the source that enter at the same PN order as those computed in [1].

As we are interested in ϑ^{FZ}, μ_A^α in Eq. (23) corresponds to the FZ scalar dipole moment. In principle, this moment must be determined by matching it to the NZ moment in the buffer zone, i.e. where the FZ and the NZ overlap. As in the case of the (source and radiative) multipole moments of metric perturbations in GR, the NZ scalar dipole moment is equal to the FZ scalar dipole moment up to non-linear terms. These correspond, for example, to the propagation of the scalar field on a non-flat background, leading to scalar field scattering analogous to the "tail" terms of the metric perturbation in GR. As such, these non-linear terms enter at higher PN order, and thus, they can be neglected to the PN order we consider here. The FZ scalar dipole moment is then equal to the NZ scalar dipole moment, which was related in [1] to the inner zone (IZ) integral of the self interaction contribution of both the Pontryagin density and the $\Box\vartheta$ terms in the scalar field equation. Therefore, the scalar

dipole moment μ_A^α can be obtained by promoting the spin 3-vector $\hat{S}^i$ in Eq. (10) to a spin 4-vector $\hat{S}^\alpha$, where the condition that determines μ_A^0 will be discussed later.

Let us now manipulate Eq. (23) so that one can solve it using the Green's function approach for a flat metric. We decompose the left-hand side of Eq. (23) as $\Box = \Box_\eta + \Box_h$, where $\Box_\eta$ is the flat space d'Alembertian operator. Then, Eq. (23) becomes

$$\Box_\eta \vartheta = 4\pi \sum_{A=1,2} \int d\tau \left\{ \mu_A^i \left(\frac{\delta^{(4)}[x^\mu - x_A^\mu(\tau)]}{\sqrt{-g}}\right)_{,i} \right.$$
$$\left. + \mu_A^0 \left(\frac{\delta^{(4)}[x^\mu - x_A^\mu(\tau)]}{\sqrt{-g}}\right)_{,0} \right\} - \Box_h \vartheta$$
$$\approx 4\pi \sum_{A=1,2} \left\{ \mu_A^i \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_A)_{,i} \right.$$
$$\left. + \int d\tau \mu_A^0 \delta^{(4)}[x^\mu - x_A^\mu(\tau)]_{,0} \right\} - \Box_h \vartheta, \quad (24)$$

where we have only kept leading PN order terms. In [1], we only included the first term on the right-hand side of Eq. (24), which gives a quadrupolar scalar radiation term from a cross interaction contribution:

$$\vartheta_{(1)}^{\mathrm{FZ,cross}} = \frac{1}{r}\ddot{\mu}_{ij}n^{ij} = -\frac{1}{r}\omega^2\mu_{ij}n^{ij} \sim \frac{1}{r}v^6 b \sim \frac{1}{r}v^4, \quad (25)$$

where $\mu_{ij} \equiv x_1^{(i}\mu_1^{j)} + (1 \leftrightarrow 2)$. Below, we look at the contribution from the other terms that we did not consider in [1].

We first look at the FZ scalar field $\vartheta_{(2)}^{\mathrm{FZ}}$ sourced by the second term in Eq. (24):

$$4\pi \sum_{A=1,2} \int d\tau \mu_A^0 \delta^{(4)}[x^\mu - x_A^\mu(\tau)]_{,0} = 4\pi \sum_{A=1,2} \int d\tau \mu_A^0 \delta^{(3)}[x^i - x_A^i(\tau)]\delta[t - t_A(\tau)]_{,0}$$

$$= -4\pi \sum_{A=1,2} \int d\tau \mu_A^0 \delta^{(3)}[x^i - x_A^i(\tau)]\delta[t - t_A(\tau)]_{,\tau} \left(\frac{dt_A}{d\tau}\right)^{-1}$$

$$\approx -4\pi \sum_{A=1,2} \int d\tau \mu_A^0 \delta^{(3)}[x^i - x_A^i(\tau)]\delta[t - \tau]_{,\tau} + \mathcal{O}(v^2)$$

$$\approx 4\pi \sum_{A=1,2} \int d\tau \left\{ \mu_A^0 \delta^{(3)}[x^i - x_A^i(\tau)] \right\}_{,\tau} \delta[t - \tau]$$

$$\approx 4\pi \sum_{A=1,2} \left\{ \mu_A^0 \delta^{(3)}[x^i - x_A^i(t)] \right\}_{,0}$$

$$\approx 4\pi \sum_{A=1,2} \left\{ \frac{d\mu_A^0}{dt}\delta^{(3)}[x^i - x_A^i(t)] - \mu_A^0 \delta^{(3)}[x^i - x_A^i(t)]_{,j}\, v_A^j(t) \right\}, \quad (26)$$

where in the third line, we used $t_A \approx \tau$ to leading PN order. To obtain μ_A^0, we follow [7] and assume that the spin vector is purely spatial in the stellar rest frame. Then,

in any frame, one has

$$\mu_A^\mu u_\mu^A = 0, \quad (27)$$

because the spin vector is assumed orthogonal to the four-

velocity and such an assumption gives $\mu_a^0 \propto S^0 = 0$ in the rest frame; this leads to

$$\mu_A^0 = \mu_A^i v_i^A \left[1 + \mathcal{O}\left(v^2\right)\right], \qquad (28)$$

where we used $u_0^A = -1$ to leading PN order. Substituting this and Eq. (26) into Eq. (24), the field equation for $\vartheta_{(2)}^{\mathrm{FZ}}$ becomes

$$\Box_\eta \vartheta_{(2)}^{\mathrm{FZ}} = 4\pi \sum_{A=1,2} \left[\frac{d}{dt}\left(\mu_A^i v_i^A\right)\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_A)\right.$$
$$\left. - \mu_A^i v_i^A v_A^j \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_A)_{,j}\right]. \qquad (29)$$

When solving this equation for the FZ solution, the first term gives a monopolar contribution while the second term gives a dipolar one, and hence, the former is dominant. One then finds

$$\vartheta_{(2)}^{\mathrm{FZ}} = -\frac{1}{r}\ddot{\mu}_{ii}. \qquad (30)$$

We next look at the FZ scalar field $\vartheta_{(3)}^{\mathrm{FZ}}$ sourced by the third term in Eq. (24). Since

$$\Box\vartheta = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,\partial^\mu\vartheta\right)$$
$$= \left(1-\frac{h}{2}\right)\partial_\mu\left[\left(1+\frac{h}{2}\right)(\eta^{\mu\nu}-h^{\mu\nu})\partial_\nu\vartheta\right] + \mathcal{O}(h^2\vartheta)$$
$$= \partial_\mu\left[\left(\eta^{\mu\nu}-h^{\mu\nu}+\frac{h}{2}\eta^{\mu\nu}\right)\partial_\nu\vartheta\right] - \frac{h}{2}\eta^{\mu\nu}\partial_{\mu\nu}\vartheta$$
$$+\mathcal{O}(h^2\vartheta)$$
$$= \eta^{\mu\nu}\partial_{\mu\nu}\vartheta - (\partial_\mu h^{\mu\nu})(\partial_\nu\vartheta) - h^{\mu\nu}\partial_{\mu\nu}\vartheta$$
$$+\frac{1}{2}\eta^{\mu\nu}(\partial_\mu h)(\partial_\nu\vartheta) + \mathcal{O}(h^2\vartheta), \qquad (31)$$

one finds

$$\Box_\eta\vartheta_{(3)}^{\mathrm{FZ}} = -\Box_h\vartheta$$
$$= (\partial_\mu h^{\mu\nu})(\partial_\nu\vartheta) + h^{\mu\nu}\partial_{\mu\nu}\vartheta - \frac{1}{2}\eta^{\mu\nu}(\partial_\mu h)(\partial_\nu\vartheta). \qquad (32)$$

The leading contribution comes from spatial derivatives (not time derivatives). When solving the above wave equation by using the Green's function method and performing the volume integral of the source terms on the right-hand side, the contribution from the first and second terms to the FZ solution vanish upon integration by parts. Therefore, one is left with

$$\Box_\eta\vartheta_{(3)}^{\mathrm{FZ}} = -\frac{1}{2}(\partial_i h)(\partial_i\vartheta). \qquad (33)$$

Let us first comment on the self-interaction contribution of the source term in Eq. (33) to the FZ scalar field solution. To leading PN order, one finds

$$\vartheta_{(3)}^{\mathrm{FZ,self}} = -\frac{1}{4\pi}\frac{1}{r}\int\left(-\frac{1}{2}h_{1,i}^{\mathrm{NZ}}\vartheta_{1,i}^{\mathrm{NZ}}\right)d^3x + (1\leftrightarrow 2)$$

$$= -\frac{1}{2\pi}\frac{1}{r}m_1\mu_1^j\int\partial_i\left(\frac{1}{r_1}\right)\partial_{ij}\left(\frac{1}{r_1}\right)d^3x + (1\leftrightarrow 2). \qquad (34)$$

Such an integral diverges at the body's position. The effective source term in Eq. (23) captures the regular part of this contribution.

Let us then focus on the cross-interaction contribution of Eq. (33). The monopole radiation piece of $\vartheta_{(3)}^{\mathrm{FZ}}$ is given by

$$\vartheta_{(3)}^{\mathrm{FZ}} = -\frac{1}{4\pi}\frac{1}{r}\int\left(-\frac{1}{2}h_{1,i}^{\mathrm{NZ}}\vartheta_{2,i}^{\mathrm{NZ}}\right)d^3x + (1\leftrightarrow 2)$$
$$= -\frac{1}{2\pi}\frac{1}{r}m_1\mu_2^j\int\partial_i\left(\frac{1}{r_1}\right)\partial_{ij}\left(\frac{1}{r_2}\right)d^3x + (1\leftrightarrow 2)$$
$$= \frac{1}{2\pi}\frac{1}{r}m_1\mu_2^j\partial_i^{(1)}\partial_{ij}^{(2)}\int\frac{1}{r_1}\frac{1}{r_2}d^3x + (1\leftrightarrow 2), \qquad (35)$$

where $\partial_i^{(A)}$ is the particle derivative with respect to the Ath body. Since the finite part of the remaining integral is given by $\int 1/(r_1 r_2)d^3x = -2\pi b$ [8], one finds

$$\vartheta_{(3)}^{\mathrm{FZ}} = -\frac{1}{r}m_1\mu_2^j\partial_i^{(1)}\partial_{ij}^{(2)}b + (1\leftrightarrow 2)$$
$$= -\frac{1}{r}m_1\mu_2^j\partial_{jii}^{(1)}b + (1\leftrightarrow 2). \qquad (36)$$

and since

$$\partial_{jii}^{(1)}b = \partial_{ji}^{(1)}n_{12i} = \partial_j^{(1)}\left(\frac{\delta_{ii}-n_{12ii}}{b}\right)$$
$$= 2\partial_j^{(1)}\left(\frac{1}{b}\right) = -\frac{2}{b^2}n_{12j}, \qquad (37)$$

one finally finds

$$\vartheta_{(3)}^{\mathrm{FZ}} = \frac{2}{r}m_1\mu_2^j\frac{n_{12}^j}{b^2} + (1\leftrightarrow 2)$$
$$= \frac{2}{r}(m_1\mu_2^j - m_2\mu_1^j)\frac{n_{12}^j}{b^2} = \frac{2}{r}\ddot{\mu}_{ii}. \qquad (38)$$

Combining Eqs. (25), (30) and (38), the new FZ scalar field solution is given by

$$\vartheta^{\mathrm{FZ}} = \sum_{k=1}^{3}\vartheta_{(k)}^{\mathrm{FZ}} = \frac{1}{r}\ddot{\mu}_{ij}\left(n^{ij}+\delta^{ij}\right). \qquad (39)$$

We now derive the new scalar energy flux by substituting Eq. (39) into the formula

$$\delta\dot{E}^{(\vartheta)} = -\beta\lim_{r\to\infty}\int_{S_r^2}\left\langle\left(\dot{\vartheta}^{\mathrm{FZ}}\right)^2\right\rangle_\omega r^2 d\Omega. \qquad (40)$$

Doing so, one finds

$$\delta\dot{E}^{(\vartheta)} = -\beta\left\langle\dddot{\mu}_{ij}\dddot{\mu}_{kl}\right\rangle_\omega\int_{S_\infty^2}\left(n^{ij}+\delta^{ij}\right)\left(n^{kl}+\delta^{kl}\right)d\Omega. \qquad (41)$$

Using the angle-averaging formulas

$$\int_{S^2_\infty} n^{ij} d\Omega = \frac{4\pi}{3} \delta^{ij} \,, \qquad (42)$$

$$\int_{S^2_\infty} n^{ijkl} d\Omega = \frac{4\pi}{15} \left(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} \right) \,, \quad (43)$$

we find

$$
\begin{aligned}
\delta\dot{E}^{(\vartheta)} &= -\frac{4\pi}{15}\beta \left\langle \dddot{\mu}_{ij}\dddot{\mu}_{kl} \right\rangle_\omega \left(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} \right. \\
&\qquad \left. + 5\delta^{ij}\delta^{kl} + 5\delta^{ij}\delta^{kl} + 15\delta^{ij}\delta^{kl} \right) \\
&= -\frac{4\pi}{15}\beta \left\langle \dddot{\mu}_{ij}\dddot{\mu}_{kl} \right\rangle_\omega \left(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} + 26\delta^{ij}\delta^{kl} \right) \\
&= -\frac{4\pi}{15}\beta \left\langle \left[\dddot{\mu}_{ij}\dddot{\mu}^{ij} + 27\left(\dddot{\mu}^i{}_i\right)^2 \right] \right\rangle_\omega \,.
\end{aligned}
\qquad (44)
$$

Equation (136) of [1] gives

$$\dddot{\mu}^{ij} = \frac{1}{b^3} \left(m_1 v_{12}^{(i}\mu_2^{j)} - m_2 v_{12}^{(i}\mu_1^{j)} \right) \,, \qquad (45)$$

and substituting Eq. (10) into this equation, one finds

$$\dddot{\mu}^{ij} = -\frac{5}{2}\frac{\alpha_4}{\beta}\frac{m}{b^3} v_{12}^{(i}\bar{\Delta}^{j)} \,, \qquad (46)$$

where

$$\bar{\Delta} \equiv \frac{m_2}{m}\chi_1 \hat{S}_1^i - \frac{m_1}{m}\chi_2 \hat{S}_2^i \,. \qquad (47)$$

Substituting this equation into Eq. (44), we finally find the new scalar energy flux:

$$\delta\dot{E}^{(\vartheta)} = -\frac{5}{48}\zeta_4 \left[\bar{\Delta}^2 + 27\left\langle (\bar{\Delta}\cdot\hat{v}_{12})^2 \right\rangle_\omega \right] v^{14} \,. \qquad (48)$$

This expression is very similar to Eq. (137) in [1]. The only difference is that the factor of "2" in front of $\left\langle (\bar{\Delta}\cdot\hat{v}_{12})^2 \right\rangle_\omega$ has now become "27". We remind the reader that Eq. (27) does not imply that $\left\langle (\bar{\Delta}\cdot\hat{v}_{12})^2 \right\rangle_\omega$ vanishes because the former is a condition on the 4-velocity. Observe, however, that $\left\langle (\bar{\Delta}\cdot\hat{v}_{12})^2 \right\rangle_\omega$ does vanish for spin-aligned binaries, and hence, this correction does not affect the main results in [6].

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Post-Newtonian, Quasi-Circular Binary Inspirals in Quadratic Modified Gravity

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We consider a general class of quantum gravity-inspired, modified gravity theories, where the Einstein-Hilbert action is extended through the addition of all terms quadratic in the curvature tensor coupled to scalar fields with standard kinetic energy. This class of theories includes Einstein-Dilaton-Gauss-Bonnet and Chern-Simons modified gravity as special cases. We analytically derive and solve the coupled field equations in the post-Newtonian approximation, assuming a comparable-mass, spinning black hole binary source in a quasi-circular, weak-field/slow-motion orbit. We find that a naive subtraction of divergent piece associated with the point-particle approximation is ill-suited to represent compact objects in these theories. Instead, we model them by appropriate effective sources built so that known strong-field solutions are reproduced in the far-field limit. In doing so, we prove that black holes in Einstein-Dilaton-Gauss-Bonnet and Chern-Simons theory can have hair, while neutron stars have no scalar monopole charge, in diametrical opposition to results in scalar-tensor theories. We then employ techniques similar to the direct integration of the relaxed Einstein equations to obtain analytic expressions for the scalar field, metric perturbation, and the associated gravitational wave luminosity measured at infinity. We find that scalar field emission mainly dominates the energy flux budget, sourcing electric-type (even-parity) dipole scalar radiation and magnetic-type (odd-parity) quadrupole scalar radiation, correcting the General Relativistic prediction at relative −1PN and 2PN orders. Such modifications lead to corrections in the emitted gravitational waves that can be mapped to the parameterized post-Einsteinian framework. Such modifications could be strongly constrained with gravitational wave observations.

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I. INTRODUCTION

The validity of Einstein's theory in the strong-gravity regime will soon be put to the most stringent tests yet, through the observation of gravitational waves (GWs) from compact object binary inspirals [1–3]. Such waves carry detailed information about their source and the underlying gravitational theory in play. This information is primarily encoded in the evolution of the GW frequency, which in turn depends directly on the rate of energy transport away from the binary [4]. In general relativity (GR), this transport is performed exclusively by GWs. In modified gravity theories, however, additional (scalar, vectorial or tensorial) degrees of freedom can also carry energy and angular momentum away as they propagate.

Calculating how gravitational waves are corrected in modified gravity theories can be a gargantuan task as the modification can increase the number of propagating degrees of freedom and the non-linearity of the equations that control their propagation. For example, the amount of energy-momentum transported away from a binary system must be computed both from the GWs excited by the corresponding sources, as well as any additional waves associated with extra degrees of freedom [5]. The sources that drive such waves can depend both on derivatives of the metric perturbation and the extra degrees of freedom, which, in turn, are specified by the solution to their own equations of motion. The situation worsens if these are non-linearly coupled, e.g. a scalar field equation of motion that depends on the metric tensor, whose evolution in turn depends on derivatives of the scalar field.

Such calculations, however, are feasible if one treats any GR deviations as *small deformations* [6], which can be formalized through the *small-coupling approximation*, a common technique in perturbation theory to isolate physically relevant solutions in higher-derivative theories [7–9]. This is a reasonable approximation given that GR has passed a large number of tests, albeit in the weak-gravity regime. Even in the GW regime, signals will slowly transition from sampling weak fields to moderately strong fields during a full binary inspiral. The strongest GW events will not be able to sample anywhere close to the Plank regime, where one would expect completely new physics. The largest gravitational fields experienced by binaries occur when these merge, and even then, the metric curvature cannot exceed m^{-2}, where m is the total mass of the binary. Earth-based detectors, such as LIGO [10], VIRGO [11] and LCGT [12], and future space-borne detectors, such as LISA [13], will only be able to sample gravitational fields up to this strength.

Of the plethora of modified gravity theories, we choose to focus on a general class that is characterized by the addition of quadratic curvature invariants to the action, coupled to scalar fields with standard kinetic terms (see e.g. Eq. (1). Such theories are motivated from loop quantum gravity [14, 15] and heterotic string theory [16], arising generically upon four-dimensional com-

pactification in the low-energy limit. Disjoint sub-classes of quadratic theories reduce to Einstein-Dilaton-Gauss-Bonnet (EDGB) theory [17, 18] and Dynamical Chern-Simons (CS) modified gravity [19, 20].

From a phenomenological standpoint, such quadratic gravity theories are also interesting as straw-men to study small deviations from GR. This is because the new quadratic terms are always small relative to the Einstein-Hilbert term when considering merging binaries. In such systems, the minimum radius of curvature is always larger than the new scale introduced by the scalar fields. If this were not the case, astrophysical observations would already have constrained quadratic gravity deviations.

Quadratic gravity introduces an equation of motion for the scalar field and modifies the metric field equations. The former is a driven wave equation, whose sources are quadratic curvature invariants. The latter contains new terms that depend on the product of the scalar field and its derivatives with the Riemann tensor, Ricci tensor, Ricci scalar and their derivatives. As such, one might worry that higher derivative terms in the field equations could render the system unstable. One must remember, however, that the action is a truncation (at quadratic order in the present case) of an *effective theory* derived by integrating out heavy degrees of freedom contained in a more complete theory. Since we truncate the effective action, its validity is limited only to leading-order in the coupling parameters. Accounting for higher-order terms in the coupling would require the inclusion of higher-order terms (cubic, quartic, etc.) in the action [8]. Therefore, the modified field equations should not be considered as an exact system, but rather as an effective one.

Given the above and using the *small-coupling* approximation, the field equations become driven differential equations for the metric deformation and the scalar field. The source of the latter depends only on derivatives of the GR metric perturbation, while the source of the former depends both on the GR metric perturbation and the scalar field. We solve these equations in the post-Newtonian (PN) limit, where in particular we consider comparable-mass, spinning black hole (BH) binaries (electromagnetically uncharged), spiraling in a quasi-circular orbit. This forces the driven differential equations into driven wave equations, which can be studied with PN techniques [21–27] and then solved via retarded Green function methods.

A complication arises when attempting to solve these equations, as one must choose a prescription to describe BHs and neutron stars (NSs). In standard PN theory and up to a certain high PN order, one can choose a point-particle prescription, essentially because the exterior gravitational field of a compact object is the same as that induced by a point-particle. In modified quadratic gravity, however, both non-spinning [28] and spinning [6], strong-field BH solutions differ from that generated by simple point particles with a mass-

monopole and a current-dipole moment; BHs in these theories have additional scalar multipole moments. (See Refs. [29, 30] for similar discussions on NSs in CS gravity.) One can take these effects into account by constructing an *effective* point-particle source that reproduces known, strong-field solutions to leading order in the weak-field region, sufficiently far away from the compact objects. With this effective point-particle prescription, we can then evaluate the source of the driven wave equations and analytically solve them to find the radiative part of the scalar field and metric perturbation.

Executive Summary of Results

Given the length of this paper, let us summarize the main results. We have devised a framework in the small-coupling approximation to solve for compact binary inspirals in modified quadratic gravity theories. One of the key ingredients in this framework is the calculation of effective source terms that allow us to use the point-particle approximation even for theories where such approximation is not valid. We applied this to modified quadratic gravity to find that both NSs and BHs have scalar hair, which leads to dipolar emission. EDGB and CS gravity are exceptions, where although BHs retain scalar monopole and dipole charge, respectively, NSs shed the scalar monopole charge. Therefore, BHs in EDGB generically contains dipolar GW emission, while CS gravity leads to modified quadrupolar emission.

The presence of scalar monopole and dipole hair, and in particular the flux of energy-momentum carried by this hair, leads to a modification in the rate of change of the binary's binding energy. The even-parity sector of the theory leads to scalar hair, which modifies the energy flux at -1PN order relative to the GR quadrupole flux. Of course, such a modification is proportional to the coupling parameter of the theory, which is assumed small. The odd-parity sector leads to dipole hair for spinning BH binaries, which modifies the energy flux at 2PN relative order. If the BH binary components are non-spinning, they have no dipole hair but the binary orbital interaction generates a modification in the energy flux that enters at relative 7PN order. Figure 1 shows the energy flux carried by the even-parity scalar field (long dashed line), odd-parity scalar field (dot-dashed for spinning binaries and short dashed line for non-spinning binaries), and the GR quadrupole flux (solid line) as a function of orbital velocity. Observe that when one assumes that BHs are non-spinning, the scalar emission is greatly suppressed.

These energy flux corrections translate into changes to the waveform observables. We explicitly calculate these and map them to the parametrized post-Einsteinian (ppE) framework [31, 32]. Using the results of Cornish et al. [32] we estimate that GW observations could constrain the new length scale introduced in quadratic gravity (related to the coupling constants of the the-
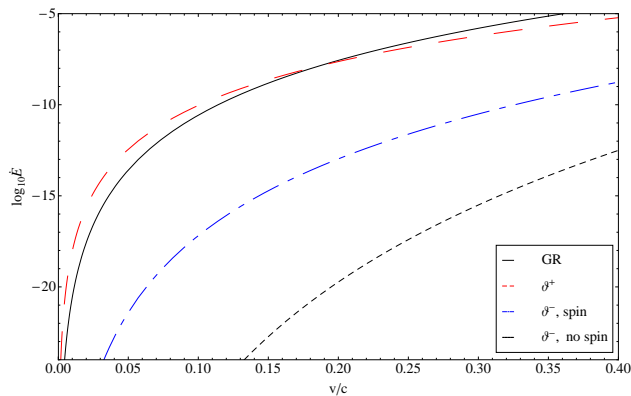
FIG. 1: Comparison of the energy flux carried by scalar fields of even-parity (dashed red), odd-parity and sourced by spinning BHs (blue dot-dashed) and odd-parity and sourced by non-spinning BHs (short dashed) relative to the GR prediction (solid black) as a function of orbital velocity. We here consider a quasi-circular, BH inspiral with $(m_1, m_2) = (8, 20)M_\odot$, normalized spins $\hat{S}_1^i \equiv |S_1^i|/m_1^2 = -\hat{S}_2^i \equiv -|S_2^i|/m_2^2$ perpendicular to the orbital plane, $|S_A^i| = m_A^2$ and coupling constants $\zeta_3 = 6.25 \times 10^{-3} = \zeta_4$.

ory) to roughly the BH horizon scale. With a typical Ad. LIGO stellar-mass BH inspiral observation, one should be able to constrain the even-parity sector to roughly $\mathcal{O}(10)$ km. With a typical LISA extreme-mass ratio inspiral (EMRI) observation, one should be able to constrain the odd-parity sector to roughly $\mathcal{O}(100)$ km. Such projected constraints are much stronger than current Solar System bounds [30, 33–35].

This paper is organized as follows: Section II describes the action that will be considered in this paper and reviews the associated modified field equations and the scalar field equation of motion. Section III expands the field equations in the small-deformation approximation. Sections IV and V study the scalar field and metric deformation evolution, analytically solving the modified field equations. Section VI computes the energy flux carried by the scalar field and the metric deformation. Section VII considers the impact that such fluxes would have on gravitational waveform phase. Section VIII concludes and points to future research.

We have deferred many details of the computational techniques to the appendices. Appendix A shows the NSs in EDGB theory have no scalar monopole charge. Appendix B discusses specific integration techniques. Appendix C estimates the order of the metric correction from the regularized contribution for non-spinning BHs in the odd-parity sector of the modified theory. Appendix D discusses particular integrals that appear when solving the field equations.

Henceforth, we follow mostly the conventions of Misner, Thorne and Wheeler [5]: Greek letters stand for spacetime indices; Latin letters in the middle of the alphabet $i, j, \ldots$, stand for spatial indices only. Parenthesis, square brackets and angled brackets in index lists de-

note symmetrization, antisymmetrization and the symmetric and trace free (STF) operator, respectively. Capital Latin letters usually refer to a multi-index, such as $x^Q = x^{ijk\cdots}$, where $x^{ijk\cdots} = x^i x^j x^k \ldots$. Partial derivatives are denoted with $\partial_i A = A_{,i} = \partial A/\partial x^i$, while covariant derivatives are denoted with the nabla $\nabla_i A$, for any quantity A. Deformations are labeled with the order-counting parameter ς. Finally, we use geometric units, where $G = c = 1$, except when denoting the order of certain terms in the PN approximation. Throughout, we performed analytic calculations with the xTENSOR package for MATHEMATICA [36, 37].

II. MODIFIED GRAVITY THEORIES

In this Section, we introduce the class of modified gravity theories that we study, by writing down its action and equations of motion. We then proceed to define the small deformation approximation more precisely.

A. ABC of quadratic gravity

Consider the following 4-dimensional effective action:

$$
\begin{aligned}
S \equiv \int d^4 x \sqrt{-g} & \left\{ \kappa R + \alpha_1 f_1(\vartheta) R^2 + \alpha_2 f_2(\vartheta) R_{\mu\nu} R^{\mu\nu} \right. \\
& + \alpha_3 f_3(\vartheta) R_{\mu\nu\delta\sigma} R^{\mu\nu\delta\sigma} + \alpha_4 f_4(\vartheta) R_{\mu\nu\delta\sigma} {}^* R^{\mu\nu\delta\sigma} \\
& \left. - \frac{\beta}{2} \left[\nabla_\mu \vartheta \nabla^\mu \vartheta + 2V(\vartheta) \right] + \mathcal{L}_{\text{mat}} \right\} .
\end{aligned}
\tag{1}
$$

Here, g stands for the determinant of the metric $g_{\mu\nu}$. R, $R_{\mu\nu}$, $R_{\mu\nu\delta\sigma}$ and $^* R_{\mu\nu\delta\sigma}$ are the Ricci scalar and tensor, the Riemann tensor and its dual [38], respectively, with the latter defined as[1] $^* R^\mu{}_{\nu\delta\sigma} = (1/2)\varepsilon_{\delta\sigma}{}^{\alpha\beta} R^\mu{}_{\nu\alpha\beta}$ and with $\varepsilon^{\mu\nu\delta\sigma}$ the Levi-Civita tensor. The quantity $\mathcal{L}_{\text{mat}}$ is the external matter Lagrangian, ϑ is a field, (α_i, β) are coupling constants and $\kappa = (16\pi)^{-1}$. This action contains all possible quadratic, algebraic curvature scalars with running (i.e. non-constant) couplings, where we assumed that all quadratic terms are coupled to the *same* field. All other quadratic curvature terms are linearly dependent, such as the Weyl tensor squared.

The theory defined by the action above is different from $f(R)$ theories on several counts. First, $f(R)$ theories depend only on the Ricci scalar, while the action above depends on the Ricci tensor, the Riemann tensor and a dynamical field ϑ. Second, $f(R)$ theories are usually treated as exact, while the action presented above is an *effective theory*, truncated to quadratic order in the Riemann tensor. The consequence of this is insisting on the use of order-reduction in the field equations, where

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[1] This definition is correct, in agreement with [38], and fixing an inconsequential typo in [39].

we treat all quantities that depend on α_i perturbatively. Such order reduction then leads to the absence of additional polarization modes [39, 40], such as the longitudinal scalar mode that arises in $f(R)$ theories.

The field equations of dynamical quadratic gravity can be obtained by varying the action with respect to all fields. For simplicity, we restrict attention to coupling functions $f_i(\vartheta)$ that admit the Taylor expansion $f_i(\vartheta) = f_i(0) + f_i'(0)\vartheta + \mathcal{O}(\vartheta^2)$ about small ϑ, where $f_i(0)$ and $f_i'(0)$ are constants, and we assume that the asymptotic value of θ at spatial infinity vanishes. Let us further reabsorb $f_i(0)$ into the coupling constants $\alpha_i^{(0)} \equiv \alpha_i f_i(0)$ and $f_i'(0)$ into the constants $\alpha_i^{(1)} \equiv \alpha_i f_i'(0)$. Equation (1) then becomes $S = S_{\mathrm{GR}} + S_0 + S_1$:

$$S_{\mathrm{GR}} \equiv \int d^4x \sqrt{-g}\, \{\kappa R + \mathcal{L}_{\mathrm{mat}}\}\,, \tag{2}$$

$$S_0 \equiv \int d^4x \sqrt{-g}\, \Big\{\alpha_1^{(0)} R^2 + \alpha_2^{(0)} R_{\mu\nu}R^{\mu\nu}$$
$$+ \alpha_3^{(0)} R_{\mu\nu\delta\sigma}R^{\mu\nu\delta\sigma}\Big\}\,, \tag{3}$$

$$S_1 \equiv \int d^4x \sqrt{-g}\, \Big\{\alpha_1^{(1)}\vartheta R^2 + \alpha_2^{(1)}\vartheta R_{\mu\nu}R^{\mu\nu}$$
$$+ \alpha_3^{(1)}\vartheta R_{\mu\nu\delta\sigma}R^{\mu\nu\delta\sigma} + \alpha_4^{(1)}\vartheta R_{\mu\nu\delta\sigma}{}^* R^{\mu\nu\delta\sigma}$$
$$- \frac{\beta}{2}\left[\nabla_\mu\vartheta\nabla^\mu\vartheta + 2V(\vartheta)\right]\Big\}\,, \tag{4}$$

where clearly S_{GR} is the Einstein-Hilbert plus matter action. Notice that S_0 defines a GR correction that is decoupled from θ. The term proportional to $\alpha_4^{(0)}$ can not affect the classical field equations since it is topological, i.e. the second Chern form, so we have omitted it. Similarly, if $\alpha_i^{(0)}$ are chosen to reconstruct the Gauss-Bonnet invariant, $(\alpha_1^{(0)}, \alpha_2^{(0)}, \alpha_3^{(0)}) = (1, -4, 1)\alpha_{\mathrm{GB}}$, then these will not modify the field equations. On the other hand, S_1 defines a modification to GR with a direct (non-minimal) scalar field coupling, such that as the field goes to zero, the modified theory reduces to GR. We here restrict attention to the case $\alpha_i^{(0)} = 0$. From this point forward, we will drop the superscript from $\alpha_i^{(1)}$.

The action above defines a class of modified gravity theories that contains well-known GR extensions. For example, when $\alpha_4 = -\frac{1}{4}\alpha_{\mathrm{CS}}$ and all other $\alpha_i = 0$, quadratic gravity reduces to dynamical CS gravity, where α_{CS} is the CS coupling parameter (see e.g. [38]). Alternatively, when $\alpha_4 = 0$, while $(\alpha_1, \alpha_2, \alpha_3) = (1, -4, 1)\alpha_{\mathrm{EDGB}}$, quadratic gravity reduces to Einstein-Dilaton-Gauss-Bonnet theory (see e.g. [18]). Both of these theories are motivated from fundamental physics; they unavoidably arise as low-energy expansions of heterotic string theory [41–44]. Dynamical CS gravity also arises in loop quantum gravity when the Barbero-Immirzi parameter is promoted to a field in the presence of fermions [45–47].

Variation of the action with respect to the metric yields the modified field equations:

$$G_{\mu\nu} + \frac{\alpha_1\vartheta}{\kappa}\mathcal{H}_{\mu\nu}^{(0)} + \frac{\alpha_2\vartheta}{\kappa}\mathcal{I}_{\mu\nu}^{(0)} + \frac{\alpha_3\vartheta}{\kappa}\mathcal{J}_{\mu\nu}^{(0)}$$
$$+ \frac{\alpha_1}{\kappa}\mathcal{H}_{\mu\nu}^{(1)} + \frac{\alpha_2}{\kappa}\mathcal{I}_{\mu\nu}^{(1)} + \frac{\alpha_3}{\kappa}\mathcal{J}_{\mu\nu}^{(1)} + \frac{\alpha_4}{\kappa}\mathcal{K}_{\mu\nu}^{(1)}$$
$$= \frac{1}{2\kappa}\left(T_{\mu\nu}^{\mathrm{mat}} + T_{\mu\nu}^{(\vartheta)}\right)\,, \tag{5}$$

where we have defined the short-hands[2]

$$\mathcal{H}_{\mu\nu}^{(0)} \equiv 2RR_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^2 - 2\nabla_{\mu\nu}R + 2g_{\mu\nu}\Box R\,, \tag{6a}$$

$$\mathcal{I}_{\mu\nu}^{(0)} \equiv \Box R_{\mu\nu} + 2R_{\mu\delta\nu\sigma}R^{\delta\sigma} - \frac{1}{2}g_{\mu\nu}R^{\delta\sigma}R_{\delta\sigma}$$
$$+ \frac{1}{2}g_{\mu\nu}\Box R - \nabla_{\mu\nu}R\,, \tag{6b}$$

$$\mathcal{J}_{\mu\nu}^{(0)} \equiv 8R^{\delta\sigma}R_{\mu\delta\nu\sigma} - 2g_{\mu\nu}R^{\delta\sigma}R_{\delta\sigma} + 4\Box R_{\mu\nu}$$
$$- 2R\,R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R^2 - 2\nabla_{\mu\nu}R\,, \tag{6c}$$

$$\mathcal{H}_{\mu\nu}^{(1)} \equiv -4(\nabla_{(\mu}\vartheta)\nabla_{\nu)}R - 2R\nabla_{\mu\nu}\vartheta$$
$$+ g_{\mu\nu}\left[2R\Box\vartheta + 4(\nabla^\delta\vartheta)\nabla_\delta R\right]\,, \tag{6d}$$

$$\mathcal{I}_{\mu\nu}^{(1)} \equiv -(\nabla_{(\mu}\vartheta)\nabla_{\nu)}R - 2\nabla^\delta\vartheta\nabla_{(\mu}R_{\nu)\delta}$$
$$+ 2\nabla^\delta\vartheta\nabla_\delta R_{\mu\nu} + R_{\mu\nu}\Box\vartheta - 2R_{\delta(\mu}\nabla^\delta\nabla_{\nu)}\vartheta$$
$$+ g_{\mu\nu}\left(\nabla^\delta\vartheta\nabla_\delta R + R^{\delta\sigma}\nabla_{\delta\sigma}\vartheta\right)\,, \tag{6e}$$

$$\mathcal{J}_{\mu\nu}^{(1)} \equiv -8\left(\nabla^\delta\vartheta\right)\left(\nabla_{(\mu}R_{\nu)\delta} - \nabla_\delta R_{\mu\nu}\right) + 4R_{\mu\delta\nu\sigma}\nabla^{\delta\sigma}\vartheta\,, \tag{6f}$$

$$\mathcal{K}_{\mu\nu}^{(1)} \equiv -4\left(\nabla^\delta\vartheta\right)\varepsilon_{\delta\sigma\chi(\mu}\nabla^\chi R_{\nu)}{}^\sigma + 4(\nabla_{\delta\sigma}\vartheta)^* R_{(\mu}{}^\delta{}_{\nu)}{}^\sigma\,, \tag{6g}$$

where ∇_μ is the covariant derivative, $\nabla_{\mu\nu} \equiv \nabla_\mu\nabla_\nu$, and $\Box = \nabla_\mu\nabla^\mu$ is the d'Alembertian operator. The ϑ field's stress-energy tensor is

$$T_{\mu\nu}^{(\vartheta)} = \beta\left[(\nabla_\mu\vartheta)(\nabla_\nu\vartheta) - \frac{1}{2}g_{\mu\nu}\left(\nabla_\delta\vartheta\nabla^\delta\vartheta - 2V(\vartheta)\right)\right]\,. \tag{7}$$

Variation of the action with respect to ϑ yields the ϑ equation of motion:

$$\beta\Box\vartheta - \beta\frac{dV}{d\vartheta} = -\alpha_1 R^2 - \alpha_2 R_{\mu\nu}R^{\mu\nu}$$
$$- \alpha_3 R_{\mu\nu\delta\sigma}R^{\mu\nu\delta\sigma} - \alpha_4 R_{\mu\nu\delta\sigma}{}^* R^{\mu\nu\delta\sigma}\,. \tag{8}$$

Notice that when the spacetime is curved by some mass distribution, the right-hand side will be proportional to density squared.

The parity of the field ϑ can be inferred from its equation of motion. Since terms of the form R^2 are even-parity, while terms of the form $R_{\mu\nu\delta\sigma}{}^* R^{\mu\nu\delta\sigma}$ are odd-parity, the field ϑ is of mixed parity. Note however that the even and odd-parity couplings tend to have different

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[2] This corrects an error in Eq. (5b) of [28].

origins from an underlying theory. In this paper we will consider the even and odd-parity cases separately.

The inclusion of dynamics for the ϑ field in the action guarantees that the field equations are covariantly conserved without having to include any additional constraints, i.e. the covariant divergence of Eq. (5) identically vanishes, upon imposition of Eq. (8). This is a consequence of the action being diffeomorphism invariant. Such invariance is in contrast to the preferred-frame effects present in a non-dynamical theory [19], i.e. in the theory defined by the action in Eq. (4) but with $\beta = 0$. In the latter, the field ϑ must be prescribed *a priori*. Moreover, the theory requires the existence of an additional constraint (the right-hand side of (8) to vanish), which is an unphysical consequence of treating ϑ as prior structure [48, 49].

Before proceeding, let us further discuss the scalar field potential $V(\vartheta)$. This potential allows us to introduce additional couplings, such as a mass term, to drive the evolution in Eq. (8). However, there are reasons one might restrict such a potential. If the mass is much larger than the inverse length scale of the system that we concern, the effect of such a field on the dynamics of binaries is strongly suppressed. To the contrary, if the mass is much smaller, the presence of mass does not give any significant effects. Therefore we cannot expect to observe the effects of a finite mass without fine tuning. No mass term may appear in a theory with a shift symmetry, which is invariance under $\vartheta \to \vartheta + \text{const}$. Such theories are common in 4D, low-energy, effective string theories [41, 42, 44, 50, 51], such as dynamical CS and EDGB. For these reasons, and because the assumption makes the resulting equations analytically tractable, we will henceforth assume $V(\vartheta) = 0$.

B. Small deformations

The "unreasonable" accuracy of GR to explain all experimental data to date suggests that it is an excellent approximation to nature in situations where the gravitational field is very weak and velocities are very small relative to the speed of light. GW detectors will be sensitive to events in situations where the field is stronger than ever previously sampled. This, however, does not imply that GWs will ever sample the Planck/string regime, where one could expect large deviations from GR.

We will here be interested in binary compact object coalescences up until the binary reaches the innermost stable circular orbit (ISCO). Even during merger, the largest curvature that GWs will sample will be limited to the scale determined by the horizon sizes, proportional to m^{-2}. Such scales are far removed from high-energy ones, like the electroweak one, as GW detectors will not be sensitive to mergers of compact objects with masses below a solar mass. Even then, however, GWs can and will probe the *strong field*, which has not been tested

before. One is then justified in modeling GWs that may contain deviations from GR as *small deformations*.

The small deformation scheme is also appealing for theoretical reasons. As mentioned earlier, the theories we consider are effective, valid only up to the truncation order. There are higher-order terms that we have here neglected in the action, such as cubic and quartic curvature combinations. Thus, one should not treat these theories as exact nor insist on solving the equations of motion to higher orders in α_i. If this is desired, then higher-order curvature terms should also be included in the action.

One might be worried that such effective theories are unstable, since they lead to field equations with derivatives higher than second order. Such derivatives could lead to instabilities or ghost modes if the Hamiltonian is not bounded from below. Linearization in the coupling parameter, however, has the effect of recasting the field equations in Einstein form with an effective stress-energy tensor that depends on the GR solution, thus stabilizing the differential equations [7]. Linearization removes modes besides the two that arise in GR [39, 40].

Small deformations can be treated similarly to how one models BH perturbations. That is, we expand the metric as

$$g_{\mu\nu} = g_{\mu\nu}^{\text{GR}} + \varsigma\, \mathfrak{h}_{\mu\nu} + O(\varsigma^2)\,, \qquad (9)$$

where the GR superscript is to remind us that this quantity is a GR solution, while $\mathfrak{h}_{\mu\nu}$ is a metric deformation away from GR. The order-counting parameter ς is kept around only for book-keeping purposes and is to be set to unity in the end.

Applying such an expansion to Eq. (8), one finds

$$\beta\Box\vartheta = -\alpha_i\, \mathcal{S}(R_{\text{GR}}^2) + \mathcal{O}(\varsigma)\,, \qquad (10)$$

where $\mathcal{S}(R_{\text{GR}}^2)$ stands for all source terms evaluated on the GR background $g_{\mu\nu}^{\text{GR}}$. The solution to this equation will obviously scale as $\vartheta \propto \alpha_i/\beta$. Applying the decomposition and expansion of Eq. (9) to Eq. (5) in vacuum, one finds

$$G_{\mu\nu}[\mathfrak{h}_{\mu\nu}] = -\frac{\alpha_i}{\kappa} C_{\mu\nu}[\vartheta, g_{\mu\nu}^{\text{GR}}] + \frac{1}{2\kappa} T_{\mu\nu}^{(\vartheta)}[\vartheta]\,, \qquad (11)$$

where the $\mathcal{O}(\varsigma^0)$ terms automatically vanish, as $g_{\mu\nu}^{\text{GR}}$ satisfies the Einstein equations, and we have grouped modifications into the tensor $C_{\mu\nu}$. This tensor and $T_{\mu\nu}^{(\vartheta)}$ are to be evaluated on the GR metric and act as sources for the metric deformation. Notice that, as a differential operator acting on $\mathfrak{h}_{\mu\nu}$, the principal part of these differential equations continues to be strongly hyperbolic, as it is still given by the $G_{\mu\nu}$ differential operator, with the higher derivatives in $C_{\mu\nu}$ and the $T_{\mu\nu}^{(\vartheta)}$ acting as sources. Given this, the metric deformation is proportional to $\xi_i \equiv \alpha_i^2/(\beta\kappa)$, which is our actual perturbation parameter.

Proper perturbation or deformation parameters should be dimensionless, but the ξ_i are dimensional. The dimensions of α and β, of course, depend on the choice of dimensions for the scalar field. We here take the viewpoint that ϑ is dimensionless, which then forces β to be dimensionless as well as κ, and α to have dimensions of length squared. Then, the deformation parameter ξ has units of length to the fourth power, which is why we define the dimensionless

$$\zeta_i \equiv \xi_i/m^4 = \mathcal{O}(\varsigma)\,, \tag{12}$$

as our proper deformation parameter. One could choose different units for the scalar field, but in all cases one arrives at the conclusion that ζ_i is the proper deformation parameter [6].

III. EXPANSION OF THE FIELD EQUATIONS

Let us decompose the GR metric tensor into a flat background plus a metric perturbation:

$$g_{\mu\nu}^{\mathrm{GR}} = \eta_{\mu\nu} + h_{\mu\nu}\,. \tag{13}$$

We emphasize here that throughout this paper, $h_{\mu\nu}$ denotes the metric perturbation in GR while $\mathfrak{h}_{\mu\nu}$ is the metric deformation away from GR.

In expanding the modified field equations, we will also find it useful to define the standard trace-reversed metric perturbation in GR as

$$\bar{h}^{\mu\nu} \equiv \eta^{\mu\nu} - \sqrt{-g_{\mathrm{GR}}}\,g_{\mathrm{GR}}^{\mu\nu}\,. \tag{14}$$

In particular, notice that when the background is flat $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}$ and $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}\eta_{\mu\nu}$ to linear order in GR. We also define the deformed trace reversed metric perturbation as

$$\bar{\mathfrak{h}}^{\mu\nu} \equiv \left(\eta^{\mu\nu} - \sqrt{-g}\,g^{\mu\nu}\right) - \bar{h}^{\mu\nu}\,. \tag{15}$$

The harmonic gauge condition reduces to $\bar{h}^{\mu\nu}{}_{,\nu} = 0$ and $\bar{\mathfrak{h}}^{\mu\nu}{}_{,\nu} = 0$. Throughout this paper, we only study the GR deformation up to $\mathcal{O}(\alpha_i/\beta)$ for ϑ and $\mathcal{O}(\zeta_i)$ for $\mathfrak{h}_{\mu\nu}$.

A. Scalar field

The evolution equation for the scalar field at leading order in the metric perturbation becomes

$$\begin{aligned}
\Box_\eta \vartheta = &-\frac{\alpha_1}{\beta}\left(\frac{1}{2\kappa}\right)^2 T_{\mathrm{mat}}^2 - \frac{\alpha_2}{\beta}\left(\frac{1}{2\kappa}\right)^2 T_{\mathrm{mat}}^{\mu\nu} T_{\mu\nu}^{\mathrm{mat}} \\
&-\frac{2\alpha_3}{\beta}\left(h_{\alpha\beta,\mu\nu}h^{\alpha[\beta,\mu]\nu} + h_{\alpha\beta,\mu\nu}h^{\mu[\nu,\alpha]\beta}\right) \\
&-\frac{2\alpha_4}{\beta}\epsilon^{\alpha\beta\mu\nu}h_{\alpha\delta,\gamma\beta}h_\nu{}^{[\gamma,\delta]}{}_\mu
\end{aligned} \tag{16}$$

with relative remainders of $\mathcal{O}(h)$. Here, $\epsilon^{\mu\nu\delta\sigma}$ is the Levi-Civita symbol with convention $\epsilon^{0123} = +1$ in an orthonormal, positively oriented frame, and we have used the harmonic gauge condition.

B. Metric perturbation

Let us now perturb the metric field equations [Eq. (5)] about $\varsigma = 0$. The deformed metric wave equation at linear order in $\mathfrak{h}_{\mu\nu}$ becomes

$$\begin{aligned}
\frac{\kappa}{2}\Box_\eta \mathfrak{h}_{\mu\nu} = &\,\alpha_1 \vartheta \tilde{\mathcal{H}}_{\mu\nu}^{(0)} + \alpha_2 \vartheta \tilde{\mathcal{I}}_{\mu\nu}^{(0)} + \alpha_3 \vartheta \tilde{\mathcal{J}}_{\mu\nu}^{(0)} \\
&+ \alpha_1 \tilde{\mathcal{H}}_{\mu\nu}^{(1)} + \alpha_2 \tilde{\mathcal{I}}_{\mu\nu}^{(1)} + \alpha_3 \tilde{\mathcal{J}}_{\mu\nu}^{(1)} + \alpha_4 \tilde{\mathcal{K}}_{\mu\nu}^{(1)} \\
&- \frac{1}{2}\delta T_{\mu\nu}^{\mathrm{mat}} - \frac{1}{2}T_{\mu\nu}^{(\vartheta)}\,,
\end{aligned} \tag{17}$$

where the tensors on the right-hand side are given by

$$\tilde{\mathcal{H}}_{\mu\nu}^{(0)} = -4\left(h_\rho{}^{[\sigma,\rho]}{}_{\sigma\mu\nu} - \eta_{\mu\nu}\Box_\eta h_\rho{}^{[\sigma,\rho]}{}_\sigma\right)\,, \tag{18}$$

$$\begin{aligned}
\tilde{\mathcal{I}}_{\mu\nu}^{(0)} = &\,\Box_\eta h_{\nu[\rho,\mu]}{}^\rho - \Box_\eta h^\rho{}_{[\rho,\mu]\nu} - 2h_\rho{}^{[\sigma,\rho]}{}_{\sigma\mu\nu} \\
&+\eta_{\mu\nu}\Box_\eta h_\rho{}^{[\sigma,\rho]}{}_\sigma\,,
\end{aligned} \tag{19}$$

$$\tilde{\mathcal{J}}_{\mu\nu}^{(0)} = 4\left(-\Box_\eta h_{\nu[\mu,\rho]}{}^\rho - \Box_\eta h^\rho{}_{[\rho,\mu]\nu} - h_\rho{}^{[\sigma,\rho]}{}_{\sigma\mu\nu}\right)\,, \tag{20}$$

$$\begin{aligned}
\tilde{\mathcal{H}}_{\mu\nu}^{(1)} = &-8h_\rho{}^{[\sigma,\rho]}{}_{\sigma(\mu}\vartheta_{,\nu)} - 4h_\rho{}^{[\sigma,\rho]}{}_\sigma\vartheta_{,\mu\nu} \\
&+4\eta_{\mu\nu}\left(2h_\rho{}^{[\sigma,\rho]}{}_{\sigma\delta}\vartheta^{,\delta} + h_\rho{}^{[\sigma,\rho]}{}_\sigma\Box_\eta\vartheta\right)\,,
\end{aligned} \tag{21}$$

$$\begin{aligned}
\tilde{\mathcal{I}}_{\mu\nu}^{(1)} = &-2h_\rho{}^{[\sigma,\rho]}{}_{\sigma(\mu}\vartheta_{,\nu)} - 2\left(h^\delta{}_{[\rho,(\nu]\mu)}{}^\rho - h^\rho{}_{[\rho,(\nu]\mu)}{}^\delta\right)\vartheta_{,\delta} \\
&-2\left(h_{(\nu[\mu),\rho]\delta}{}^\rho + h^\rho{}_{[\rho,(\mu]\nu)\delta}\right)\vartheta^{,\delta} \\
&-2\left(h^\delta{}_{[\rho,(\mu]}{}^\rho\vartheta_{,\nu)\delta} - h^\rho{}_{[\rho,(\mu]}{}^\delta\vartheta_{,\nu)\delta}\right) \\
&+\eta_{\mu\nu}\left\{2h_\rho{}^{[\sigma,\rho]}{}_{\sigma\delta}\vartheta^{,\delta} + \left(h^{\sigma[\rho,\delta]}{}_\rho - h_\rho{}^{[\rho,\delta]\sigma}\right)\vartheta_{,\sigma\delta}\right\} \\
&+\Box_\eta\vartheta\left(h_{(\mu}{}^\delta{}_{,\nu)\delta} - \frac{1}{2}\Box_\eta h_{\mu\nu} - \frac{1}{2}h_{,\mu\nu}\right)\,,
\end{aligned} \tag{22}$$

$$\begin{aligned}
\tilde{\mathcal{J}}_{\mu\nu}^{(1)} = &-8\left(h^\delta{}_{[\rho,(\nu]\mu)}{}^\rho + h^\rho{}_{[\rho,(\nu]\mu)}{}^\delta - \frac{1}{2}h_{,\mu\nu}{}^\delta\right. \\
&\left.+\frac{1}{2}\Box_\eta h_{\mu\nu}{}^{,\delta}\right)\vartheta_{,\delta} + 4\left(h_{\sigma[\mu,\delta]\nu} - h_{\nu[\mu,\delta]\sigma}\right)\vartheta^{,\sigma\delta}\,,
\end{aligned} \tag{23}$$

$$\begin{aligned}
\tilde{\mathcal{K}}_{\mu\nu}^{(1)} = &\,\vartheta^{,\delta}{}_{,\sigma}\eta_{\nu\alpha}\bar{\varepsilon}^{\alpha\sigma\beta\gamma}\left(h_{\mu[\gamma,\beta]\delta} + h_{\delta[\beta,\gamma]\mu}\right) \\
&-2\vartheta^{,\delta}\epsilon_{\delta\sigma\chi\mu}h^\sigma{}_{[\alpha}{}^{,\alpha\chi}{}_{\nu]} + (\mu \leftrightarrow \nu)\,,
\end{aligned} \tag{24}$$

where $\Box_\eta$ is the d'Alembertian of flat spacetime, $h = h_\mu{}^\mu$, and $T_{\mu\nu}^{(\vartheta)}$ is given as

$$T_{\mu\nu}^{(\vartheta)} = \beta\left(\vartheta_{,\mu}\vartheta_{,\nu} - \frac{1}{2}\eta_{\mu\nu}\vartheta_{,\delta}\vartheta^{,\delta}\right)\,. \tag{25}$$

The quantity $\delta T_{\mu\nu}^{\mathrm{mat}}$ stands for the perturbation to the energy-momentum tensor for matter. Even when dealing with BHs, $\delta T_{\mu\nu}^{\mathrm{mat}} \neq 0$ because we treat BHs as distributional point particles and their trajectories are generically modified at $\mathcal{O}(\varsigma)$. However, in this paper we concentrate on the dissipative sector of the theory only, and not on modifications to the shape of the orbits (conservative dynamics). The latter does modify the GW phase evolution [6, 28], as we discuss in Sec. VIII.

The evolution equation for the metric perturbation takes on the same form (a sourced wave equation) as that for the scalar field. The source terms in both of these equations depend on the GR metric perturbation, which we here assume to be that of a compact binary quasi-circular inspiral in the PN approximation, i.e. moving at small velocities relative to the speed of light and producing weak gravitational fields. We provide explicit expressions for the GR metric perturbation in the subsequent subsection.

C. Post-Newtonian metric and trajectories

In this subsection, we provide explicit expressions for the linear metric perturbation in GR that we use to evaluate all source terms. We are here interested in a binary system, composed of two compact objects with masses m_1 and m_2 and initially separated by a distance $r_{12} \equiv b$. The objects' trajectories can be parameterized via

$$\mathbf{x}_1 \equiv x_1^i = +\frac{m_2}{m} b \left[\cos \omega t, \sin \omega t, 0\right], \qquad (26)$$

$$\mathbf{x}_2 \equiv x_2^i = -\frac{m_1}{m} b \left[\cos \omega t, \sin \omega t, 0\right], \qquad (27)$$

where $m \equiv m_1 + m_2$ is the total mass and where we have assumed they are located on the x–y plane. Throughout this paper, vectors are sometimes denoted with a boldface. We also define

$$\mathbf{x}_{12} \equiv x_{12}^i = x_1^i - x_2^i, \qquad (28)$$

$$\mathbf{n}_{12} \equiv n_{12}^i = (x_1^i - x_2^i)/b, \qquad (29)$$

$$\mathbf{n}_A \equiv n_A^i = (x^i - x_A^i)/r_A, \qquad (30)$$

where we follow the conventions of [27], with

$$r_A \equiv |x^i - x_A^i|. \qquad (31)$$

We further assume these objects are on a quasi-circular orbit with leading-order angular velocity $\omega = (1/b)(m/b)^{1/2}$ and orbital velocity $v = (m/b)^{1/2}$. The orbital separation b is assumed constant, as its time-evolution is driven by GW emission at high-order in v/c.

The GR spacetime metric for such a binary is expanded as in Eq. (13). In the near zone, the metric perturbation is given by

$$h_{00} = 2U_1 + (1 \leftrightarrow 2) + \mathcal{O}(v^4), \qquad (32)$$

$$h_{0i} = -4V_{1i} + (1 \leftrightarrow 2) + \mathcal{O}(v^5), \qquad (33)$$

$$h_{ij} = 2U_1 \delta_{ij} + (1 \leftrightarrow 2) + \mathcal{O}(v^4), \qquad (34)$$

where $\mathcal{O}(v^A)$ stands for an $(A/2)$PN remainder, i.e. a term of $\mathcal{O}((v/c)^A)$, and the notation $+(1 \leftrightarrow 2)$ means that one should add the same terms with the labels 1 and 2 interchanged. The potentials U_A and V_{Ai} with $A = (1, 2)$ are defined as

$$U_A = \int \frac{\rho'_A}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3 x', \qquad V_{Ai} = \int \frac{\rho'_A v'_{Ai}}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3 x', \qquad (35)$$

where ρ_A and $v_A^i \equiv \dot{x}_A^i$ are the density and the center of mass velocities of the respective objects, with the overhead dot standing for time differentiation. Field variables associated with a prime, e.g. ρ'_A, are to be evaluated at $\boldsymbol{x}'$. In the point-particle limit, the metric becomes

$$h_{00} = \frac{2m_1}{r_1} + (1 \leftrightarrow 2), \qquad (36)$$

$$h_{0i} = -\frac{4m_1}{r_1} v_1^i + (1 \leftrightarrow 2), \qquad (37)$$

$$h_{ij} = \frac{2m_1}{r_1} \delta_{ij} + (1 \leftrightarrow 2), \qquad (38)$$

with remainders of relative $\mathcal{O}(v^2)$. We have kept the PN leading terms in the metric that are proportional to m_A only, but higher-order terms can be found in [52], while terms proportional to the spin of each BH can be found in [53].

IV. SCALAR FIELD EVOLUTION

In this section, we solve the evolution equation for the scalar field both for field points in the far and near-zones, as defined in Sec. IV A. The former will allow us to evaluate the energy flux carried by the scalar field at infinity, while the latter will be essential to find effective source terms that reproduce the known strong field solutions and to solve the evolution equations for the metric deformation.

A. Zones

As shown in Fig. 2, let us decompose the geometry into three regions: an inner zone (IZ), a near zone (NZ) and a far zone (FZ); see e.g. [54–56] for further details. The IZs are centered at each object with radii $\mathcal{R}_{\mathrm{IZ}}$. These radii are defined as the boundary inside which either $T_{\mu\nu}^{\mathrm{mat}} \neq 0$ or the usual PN approximation breaks down due to strong-gravity effects. We here take them to be sufficiently larger than m_A and much less than b. The NZ is centered at the binary's center of mass with radius $\mathcal{R}_{\mathrm{NZ}}$ and excluding the IZs. This radius is defined as the boundary outside which time-derivatives cannot be assumed to be small compared with spatial derivatives due to the wave-like nature of the metric perturbation. We here take this boundary to be roughly equal to λ_{GW}, where λ_{GW} denotes the GW wavelength. The FZ is also centered at the binary's center of mass, but it extends outside $\mathcal{R}_{\mathrm{NZ}}$.

One can only apply the PN formalism when the gravitational field is weak and velocities are small. When we deal with strong field sources like BHs and NSs, therefore, one can use the PN scheme in the NZ and FZ only. In the IZs, one may not be able to use PN theory, since the gravitational field may be too strong. In this case, we have to asymptotically match our PN solution in the
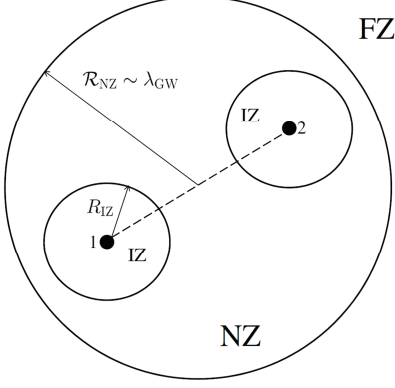
FIG. 2: We consider three zones, inner zone (IZ), near zone (NZ) and far zone (FZ). The IZs are centered at each object and their radii $\mathcal{R}_{\rm IZ}$ satisfy $\mathcal{R}_{\rm IZ} \ll b$. The NZ is centered at the center of mass of the two bodies and the radius $\mathcal{R}_{\rm NZ}$ satisfies $\mathcal{R}_{\rm NZ} \sim \lambda_{\rm GW}$, where $\lambda_{\rm GW}$ is the GW wavelength.

NZ with the strong field solutions valid in the IZs, inside some buffer regions that overlap both NZ and each IZ (see Refs. [21, 57, 58] for a description of how to carry this out in GR). The strong field solution for BHs was found in Refs. [28] and [6] in the class of theories considered here.

B. Near zone solutions

Since the NZ is in the weak field regime, we can apply the PN formalism to compact binary systems. Let us consider the even and odd-parity sectors separately.

1. Even-parity sector

The evolution equation for the even-parity sector is

$$\Box_\eta \vartheta = -64\pi^2 \frac{\alpha_1}{\beta}\rho^2 - 64\pi^2 \frac{\alpha_2}{\beta}\rho^2 - \frac{2\alpha_3}{\beta}\left(h_{\alpha\beta,\mu\nu}h^{\alpha[\beta,\mu]\nu} + h_{\alpha\beta,\mu\nu}h^{\mu[\nu,\alpha]\beta}\right), \quad (39)$$

with $\rho \equiv \rho_1 + \rho_2$ and remainders of $\mathcal{O}(h^3)$.

First, let us consider weakly-gravitating objects, i.e. not BHs or NSs, in which case the PN expansion is valid also in the IZ. By substituting the GR PN metric of Eqs. (32)-(34), the NZ solution to the above wave equation at leading PN order becomes

$$\vartheta = 16\pi\frac{\alpha_1}{\beta}\int_\mathcal{M}\rho'^2\frac{d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|} + 16\pi\frac{\alpha_2}{\beta}\int_\mathcal{M}\rho'^2\frac{d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|} + \frac{1}{\pi}\frac{\alpha_3}{\beta}\int_\mathcal{M}\left(2U'_{,ij}U'_{,ij} + \Box_\eta U'\Box_\eta U'\right)\frac{d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|}, \quad (40)$$

again with remainders of $\mathcal{O}(h^3)$, with $U \equiv U_1 + U_2$ and $\mathcal{M}$ denoting the constant-time, NZ+IZ hypersurface. We

can safely neglect the contribution from the FZ, since the fall-off of the source term is sufficiently fast.

The solution in Eq. (40) can be simplified by integrating by parts several times and using that $\Box U = -4\pi\rho$ and $\Box|\boldsymbol{x}-\boldsymbol{x}'|^{-1} = -4\pi\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')$ to obtain

$$\vartheta = 16\pi\frac{\alpha_1}{\beta}\int_\mathcal{M}\rho'^2\frac{d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|} + 16\pi\frac{\alpha_2}{\beta}\int_\mathcal{M}\rho'^2\frac{d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|}$$
$$+48\pi\frac{\alpha_3}{\beta}\int_\mathcal{M}\rho'^2\frac{d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|}$$
$$-8\frac{\alpha_3}{\beta}\int_\mathcal{M}\rho'U'_{,i}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)_{,i}d^3x'$$
$$-4\frac{\alpha_3}{\beta}\int_\mathcal{M}U'_{,i}U'_{,i}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')d^3x'. \quad (41)$$

Expanding this solution in terms of particles 1 and 2, we arrive at

$$\vartheta = \vartheta_{\rm self} + \vartheta_{\rm cross}, \quad (42)$$

with

$$\vartheta_{\rm self} = \frac{16\pi}{\beta}(\alpha_1 + \alpha_2 + 3\alpha_3)\int_\mathcal{M}\rho'^2_1\frac{d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|}$$
$$-8\frac{\alpha_3}{\beta}\int_\mathcal{M}\rho'_1U'_{1,i}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)_{,i}d^3x'$$
$$-4\frac{\alpha_3}{\beta}U_{1,i}U_{1,i} + (1\leftrightarrow 2), \quad (43)$$

and

$$\vartheta_{\rm cross} = -8\frac{\alpha_3}{\beta}\left[\int_\mathcal{M}\left(\rho'_1U'_{2,i} + \rho'_2U'_{1,i}\right)\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)_{,i}d^3x' + U_{1,i}U_{2,i}\right]. \quad (44)$$

$\vartheta_{\rm self}$ is the part of ϑ that can be evaluated by considering a single object only, while $\vartheta_{\rm cross}$ is the part that depends on the fields of both bodies.

The integrals that define both $\vartheta_{\rm self}$ and $\vartheta_{\rm cross}$ have support in the IZs only, and thus, the NZ integral operator is homogeneous (source-free). When we discuss the NZ behavior of fields associated with compact objects, such as BHs or NSs, we cannot directly evaluate such IZ integrals. These are derived under the assumption that the PN expansion is valid everywhere, which fails for compact objects in the IZs. Instead, we need to determine these homogeneous solutions through asymptotic matching. Before doing so, it is helpful to study the meaning of each term for weakly-gravitating objects.

Neglecting the size of the weakly-gravitating objects, the first term in Eq. (43) in the NZ is evaluated as

$$\int_\mathcal{M}\rho'^2_1\frac{d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|} \approx \frac{1}{r_1}\int_\mathcal{M}\rho'^2_1d^3x', \quad (45)$$

with remainders of relative $\mathcal{O}(m/r)$, while the second term becomes

$$
\begin{aligned}
\int_{\mathcal{M}} \rho_1' \, U_{1,i}' \left(\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} \right)_{,i} d^3x' &\approx \frac{n_1^i}{r_1^2} \int_{\mathcal{M}} \rho_1' U_{1,i}' d^3x' \\
&= -\frac{n_1^i}{r_1^2} \int_{\mathcal{M}} \rho_1(\boldsymbol{x}') \left(\int_{\mathcal{M}} \rho_1(\boldsymbol{y}) \frac{x'^i - y^i}{|\boldsymbol{x}' - \boldsymbol{y}|^2} d^3y \right) d^3x' \\
&= 0 \,.
\end{aligned}
\tag{46}
$$

The last equality can be shown by exchanging the integration variables[3]. Thus, one can approximate ϑ_{self} as

$$
\vartheta_{\text{self}} = \frac{q_1}{r_1} - 4\frac{\alpha_3 m_1^2}{\beta r_1^4} + (1 \leftrightarrow 2) \,,
\tag{47}
$$

with the scalar monopole charge defined by

$$
q_A \equiv \frac{16\pi}{\beta} (\alpha_1 + \alpha_2 + 3\alpha_3) \int_{\text{IZ}} \rho_A'^2 d^3x' \,,
\tag{48}
$$

with $A = (1, 2)$. Here we put "IZ" to the integral to emphasize that the integration can be restricted to both IZs because the integrand is localized.

The first term in Eq. (47) represents the monopole field around object 1. These monopole fields give the leading PN contribution in the NZ unless both monopole charges q_1 and q_2 vanish. This is indeed the case in EDGB theory, where $(\alpha_1, \alpha_2, \alpha_3) = (1, -4, 1)\alpha_{\text{EDGB}}$. We will later show that this cancellation does really survive even if we consider NSs. If this cancellation occurs, the higher order terms of $\mathcal{O}(m^2/r^2)$ in the expansion of Eq. (45) become the dominant contribution to ϑ. The second term in Eq. (47) is much higher PN order compared with the first term and hence sub-dominant in the NZ.

Let us now consider ϑ_{self} for compact objects, where the IZ integrals must be treated carefully. Since the PN expansion is no longer valid in the IZ, one cannot use the simple extrapolation of the above result. In Sec. IV C, we match the NZ solution to the one obtained for isolated BHs in the strong-field [6, 28]. We will not discuss the matching for NSs in this paper, but the order of magnitude estimate

$$
q_A = \sum_{i=1}^{3} q_{i,A} = \sum_{i=1}^{3} \frac{\alpha_i}{\beta} \mathcal{O}\left(\frac{m_A^2}{R_A^3} \right)
\tag{49}
$$

should still be valid, where R_A is the radius of the Ath NS. When $\alpha_1 + \alpha_2 + 3\alpha_3 = 0$, the cancellation observed in the weakly gravitating objects may still persist even for NSs. However, the cancellation will not in general be exact, except for the EDGB subcase. In EDGB theory,

———

[3] In fact, this integral vanishes to all orders in x. This is because $(\rho_1 U_{1,i})_{,i}$ is spherically symmetric, and thus, when it acts as a source to a wave equation, the solution should either scale as $1/r$ or it should vanish identically. We have here shown that there is no $1/r$ part.

the NS scalar monopole charge vanishes independently of the equation of state. Mathematically speaking, this is because the monopole charge is given by the integral of the Gauss-Bonnet invariant $R_{\text{GB}} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, which vanishes for any simply-connected, asymptotically flat geometry. A more explicit proof is given in Appendix A.

Let us now return to the ϑ_{cross} contribution and consider first weakly-gravitating objects. To evaluate Eq. (44), one can use point-particle expansions of the potentials and the density, i.e. $\rho_A = m_A \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_A)$ and $U_A = m_A/r_A$. Simple substitution leads to

$$
\begin{aligned}
\vartheta_{\text{cross}} &\approx 8\frac{\alpha_3 m_1 m_2}{\beta m^4} \left[m^4 \left(\frac{n_1^j}{r_1^2} \frac{n_2^j}{r_2^2} + \frac{n_{12}^j}{b^2} \frac{n_2^j}{r_2^2} \right. \right. \\
&\left. \left. - \frac{n_{12}^j}{b^2} \frac{n_1^j}{r_1^2} \right) + \mathcal{O}\left(\frac{m^5}{r^5} \right) \right] \,.
\end{aligned}
\tag{50}
$$

The first term in parentheses comes from the term $U_{1,i}U_{2,i}$ in Eq. (44). The remaining two terms come from the integral in Eq. (44). The second and third terms in parentheses look like scalar dipole moments for bodies 2 and 1 respectively. However, a Taylor expansion about the center of mass of each body, shows that the $1/r_A^2$ piece of ϑ_{cross} cancels, which implies that there is no scalar dipole.

Let us now consider ϑ_{cross} for compact objects. As discussed in the previous paragraph, one might expect a scalar dipole charge induced by the acceleration of object 1 due to the gravitational field of object 2 ($\propto U_{2,i}(\boldsymbol{x}_1)$). In GR, however, acceleration is understood as geodesic motion in a perturbed geometry. The deviation of the local geometry from the unperturbed isolated geometry originates due to tides, and this is a relative 4PN effect. This is much smaller than the scalar monopole charge contribution from ϑ_{self}.

To summarize, the dominant contribution to ϑ comes from the monopole charge associated with each object, which depends on its internal structure.

2. Odd-parity sector

In the odd-parity case, the scalar field evolution equation is

$$
\Box_\eta \vartheta = -\frac{2\alpha_4}{\beta} \epsilon^{\alpha\beta\mu\nu} h_{\alpha\delta,\gamma\beta} h_\nu{}^{[\gamma,\delta]}{}_\mu \,,
\tag{51}
$$

plus terms of $\mathcal{O}(h^3)$. Again, we first consider weakly gravitating objects. At leading PN order, the above equation becomes

$$
\begin{aligned}
\Box_\eta \vartheta &= \frac{2\alpha_4}{\beta} \epsilon_{ijk} (h_{00,mi} h_{k0,jm} + h_{0l,jm} h_{kl,im}) \\
&= -32\frac{\alpha_4}{\beta} \epsilon_{ijk} U_{,im} V_{k,jm} \,,
\end{aligned}
\tag{52}
$$

with remainders of relative $\mathcal{O}(v^2)$. As in the even-parity case, we write the solution to this wave equation as

$$\vartheta = \vartheta_{\text{self}} + \vartheta_{\text{cross}} \, , \qquad (53)$$

where

$$\vartheta_{\text{self}} = \frac{8}{\pi}\frac{\alpha_4}{\beta}\epsilon_{ijk}\int_{\mathcal{M}} U'_{1,im}V'_{1k,jm}\frac{d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|} + (1 \leftrightarrow 2) \, , \qquad (54)$$

and

$$\vartheta_{\text{cross}} = \frac{8}{\pi}\frac{\alpha_4}{\beta}\epsilon_{ijk}\int_{\mathcal{M}} U'_{1,im}V'_{2k,jm}\frac{d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|} + (1 \leftrightarrow 2) \, . \qquad (55)$$

Let us first consider self-interaction terms ϑ_{self}. Integrating by parts several times, we find

$$\vartheta_{\text{self}} = -16\frac{\alpha_4}{\beta}\epsilon_{ijk}\left[\int_{\mathcal{M}} \rho'_1 V'_{1k,j}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)_{,i} d^3x' \right.$$
$$+ \int_{\mathcal{M}} U'_{1,i}\rho'_1 v'_{1k}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)_{,j} d^3x'$$
$$\left. + \int_{\mathcal{M}} U'_{1,i}V'_{1k,j}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')d^3x' + (1 \leftrightarrow 2)\right] , (56)$$

where we have used the relations $\Box U_1 = -4\pi\rho_1$, $\Box V_1^k = -4\pi\rho_1 v_1^k$ and $\Box|\boldsymbol{x}-\boldsymbol{x}'|^{-1} = -4\pi\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')$. The third term vanishes when we take the point-particle limit[4], i.e. $\rho_A = m_A\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_A)$, $U_A = m_A/r_A$ and $V_{Ai} = m_A v_{Ai}/r_A$.

Let us evaluate the first and the second terms in the NZ. Keeping only the leading PN term in the NZ, we find

$$\vartheta_{\text{self}} = 16\frac{\alpha_4}{\beta}\epsilon_{ijk}\frac{n_{1,i}}{r_1^2}\int_{\mathcal{M}} \rho'_1(V'_{1k,j} - U'_{1,j}v'_{1k})d^3x' + (1 \leftrightarrow 2)$$
$$= \frac{n_{1,i}}{r_1^2}\mu_i^{(1)} + (1 \leftrightarrow 2) \, , \qquad (57)$$

where we have defined

$$\mu_i^{(A)} \equiv 32\frac{\alpha_4}{\beta}\epsilon_{ijk}\int_{\text{IZ}} \rho'_A V'_{Ak,j}d^3x' \, . \qquad (58)$$

This leading-order PN term in ϑ_{self} represents a magnetic-type dipole.

As in the even-parity case, to extend this result to compact objects we have to determine the value of $\mu_i^{(A)}$ by matching the NZ solution in Eq. (57) to a strong field solution. This will be carried out in Sec. IV C for the BH case. For NSs, we just present an order of magnitude estimate based on a simple extrapolation of weakly-gravitating results:

$$\mu_{(A)}^i = \frac{\alpha_4}{\beta}\mathcal{O}\left(\frac{m_A S_A^i}{R_A^3}\right) \, , \qquad (59)$$

where S_A^i is the spin angular momentum of the object. Following the procedure in Appendix A, we can show that NSs cannot have scalar monopole charge in the dynamical CS case.

Next, we consider the cross term ϑ_{cross} in the weakly-gravitating case. Integrating by parts several times, we find

$$\vartheta_{\text{cross}} = -16\frac{\alpha_4}{\beta}\epsilon_{ijk}\left[\int_{\mathcal{M}} \rho'_1 V'_{2k,j}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)_{,i} d^3x' \right.$$
$$+ \int_{\mathcal{M}} U'_{1,i}\rho'_2 v'_{2k}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)_{,j} d^3x'$$
$$\left. + \int_{\mathcal{M}} U'_{1,i}V'_{2k,j}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')d^3x' + (1 \leftrightarrow 2)\right] . \qquad (60)$$

One can take the point-particle limit of this expression without any trouble to obtain

$$\vartheta_{\text{cross}} = -16\frac{\alpha_4 m_1 m_2}{\beta m^4}\,\epsilon_{ijk}v_{12k}\left[m^4\left(\frac{n_{12}^i n_1^j}{r_1^2 b^2} + \frac{n_{12}^i n_2^j}{r_2^2 b^2}\right.\right.$$
$$\left.\left. + \frac{n_1^i n_2^j}{r_1^2 r_2^2}\right) + \mathcal{O}\left(\frac{m^5}{r^5}\right)\right] . \qquad (61)$$

Since $\mu_i^{(A)}$ in Eq. (57) do not depend on v, these terms are of relative $\mathcal{O}(v^5)$ compared to the leading-order term of ϑ_{self}.

As for compact objects, the results found in the even-parity case also apply here. Terms proportional to $1/r_A^2$ in the above expression suggest that each object has a dipole component induced by the companion. When we expand this expression around $r_A \ll b$, however, the terms proportional to $1/(r_A^2 b^2)$ cancel each other, as in the even-parity case, leading to no induced dipole moment. Even if this were not the case, however, the corrections to the dipole moment would be higher order than the contributions from ϑ_{self}.

To conclude, the dominant contribution to ϑ is clearly that of ϑ_{self} given in Eq. (57), which again depends on the structure of the source and thus violates the effacement principle.

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[4] The vanishing of this term is a general consequence of the symmetry of the system. The source term contains an ϵ^{ijk} symbol, which must be contracted with other vectors to produce a scalar. We here have only two possible vectors to contract with, i.e. the velocity v_1^i and the unit vector n_1^i from object 1. Hence, any contraction with the Levi-Civita symbol should vanish.

C. Matching near zone and strong-field solutions and finding the effective source terms

In alternative theories of gravity, the point-particle limit is not always valid and the multipole moments of compact objects may depend on the internal structure of

the source. In the previous subsections, we found that the dominant contributions to the scalar field come from self-interaction terms, which in turn depend on certain structure constants. In this subsection, we determine these constants by matching the ϑ solution to that of an isolated BH.

1. Even-parity sector

In the even-parity case, the monopole charges q_1 and q_2 in Eq. (47) must be determined by matching to a BH solution. An isolated BH sources a scalar field [28], whose leading PN behavior is

$$\vartheta_{\mathrm{YS}} = \frac{2\alpha_3}{\beta m_A^2} \frac{m_A}{r_A}. \qquad (62)$$

Matching this solution to the NZ solution of Eq. (47) we obtain

$$q_A = \frac{2\alpha_3}{\beta m_A}. \qquad (63)$$

Notice that this monopole charge does not depend on (α_1, α_2), as for pure BH spacetimes, these coupling constants appear in combination with the Ricci scalar and tensor, which vanishes. This is to be contrasted with the NS case, in which q_A depends on α_1 and α_2 as well as α_3 and vanishes in EDGB theory. Interestingly, BHs do not have scalar hair in more traditional (Brans-Dicke type) scalar-tensor theories, while NSs do possess them. This situation is reversed in EDGB theory.

The matching carried out above dealt with the monopole part of ϑ. That is, we have ignored any tidal deformation of either BH induced by its binary companion. In BH perturbation theory, one can calculate the deformation of the isolated BH metric to find that it depends on the sum of electric and magnetic tidal tensors, leading to a metric deformation that scales as $(r_1/b)^2(m_2/b)$ for $r_1 \ll b$ [54–56, 59–61]. Thus, in the IZ of object 1, tidal deformations lead to corrections of $\mathcal{O}(m^3/b^3)$, which are much smaller than the effects considered here. Therefore, it suffices in this section to consider an isolated BH when matching the scalar fields.

With this at hand, we can now treat BHs in even-parity, quadratic modified gravity as delta function sources of matter energy density, and with effective scalar charge density

$$\rho_\vartheta = q_A \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_A). \qquad (64)$$

In the PN expansion such sources reproduce the BH solution found by Yunes and Stein [28] at leading order.

Let us make a few observations about the effective source term approach. First, notice that the scalar field diverges as $m_A \to 0$, which violates the small-coupling approximation. This is related to the fact that as one shrinks a BH, the radius of curvature at the horizon also goes to zero, probing increasingly shorter length scales.

When the small-coupling approximation is violated, one can no longer neglect the scalar field's stress-energy tensor and the $(\mathcal{H}_{\mu\nu}, \mathcal{I}_{\mu\nu}, \mathcal{J}_{\mu\nu}, \mathcal{K}_{\mu\nu})$ tensors that would dominate over the Einstein tensor. Of course, one cannot take this limit seriously, as we are considering here a low-energy effective theory, which is missing higher-curvature terms that would need to be included. Notice also that this is different from the behavior of scalar fields in traditional scalar-tensor theories, where the scalar field vanishes in the $m_A \to 0$ limit.

2. Odd-parity sector

In the odd-parity case, the dipole charges of the respective objects in Eq. (57) are to be determined by matching against the appropriate BH solutions. An isolated non-spinning BH in the odd-parity case does not support a scalar field. By contrast, a spinning BH does, and in the slow-rotation limit, neglecting higher order PN corrections, it is given by [6]

$$\vartheta_{\mathrm{YP}} = -\frac{5}{2} \frac{\alpha_4}{\beta r_A^2} n_A^i \chi_A^i, \qquad (65)$$

where $\chi_A \equiv S_A^i/m_A^2$ is the normalized spin angular momentum vector of the Ath BH. Matching this solution to the NZ $\vartheta_{\mathrm{self}}$ in Eq. (57), we obtain

$$\mu_A^i = \frac{5}{2} \frac{\alpha_4}{\beta} \chi_A^i. \qquad (66)$$

With this at hand, we can now treat BHs in odd-parity, quadratic modified gravity as distributional sources of matter energy density and effective scalar charge density

$$\rho_\vartheta = -\mu_A^i \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_A)_{,i}. \qquad (67)$$

In the PN expansion, such sources reproduce the BH solution found by Yunes and Pretorius [6] at leading order.

Let us make a few observations about this solution. First, notice that the pseudo-scalar dipole charge is well behaved in the limit $m_A \to 0$, because there is a maximum BH spin $|\chi_A^i| < 1$. Second, notice that in the $|\chi_A^i| \to 0$ limit, this dipole charge vanishes, which is a consequence of Birkhoff's theorem holding in CS gravity [38, 48, 49]. Namely, non-spinning BHs in CS theory are the same as BHs in GR (i.e. Schwarzschild BHs). Therefore, in this case the point-particle limit is well-justified and the metric deformation or the scalar field does not depend on the internal structures of non-spinning sources.

D. Far-zone field point solutions

Let us assume that we have the wave equation

$$\Box_\eta \vartheta = \tau(t, x), \qquad (68)$$

where τ denotes the source term. The far-zone field point solution to this wave equation is given as [62, 63]

$$\vartheta^{\mathrm{FZ}} = -\frac{1}{4\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \partial_M \left[\frac{1}{r} \int_{\mathcal{M}} \tau(u, x'^i) x'^M \right], \quad (69)$$

with $u \equiv t-r$. By using $u_{,i} = -n_i$ and by keeping only terms proportional to $1/r$, the above solution reduces to

$$\vartheta^{\mathrm{FZ}} = -\frac{1}{4\pi} \frac{1}{r} \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\partial^m}{\partial t^m} \int_{\mathcal{M}} \tau(u, x'^i) \left(n_j x'^j \right)^m d^3 x'. \quad (70)$$

Here, the region $\mathcal{M}$ denotes the hypersurface of $t - r =$ const. In the following, we apply these formulas to the even and odd parity cases separately.

1. Even-parity sector

Following the discussion in Sec. IV B, the evolution equation for the scalar field is dominantly

$$\Box_\eta \vartheta = -4\pi q_1 \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_1) + (1 \leftrightarrow 2). \quad (71)$$

From Eq. (70), this wave equation can be solved as

$$\vartheta^{\mathrm{FZ}} = \frac{1}{r} \sum_m \frac{1}{m!} \frac{\partial^m}{\partial t^m} \int_{\mathcal{M}} q_1 \delta^{(3)}(\boldsymbol{x}' - \boldsymbol{x}_1)(n_j\, x'^j)^m d^3 x'$$
$$+ (1 \leftrightarrow 2). \quad (72)$$

The $m = 0$ term gives

$$\vartheta^{\mathrm{FZ}} = \frac{q}{r}, \quad (73)$$

where we have defined the total scalar monopole charge $q \equiv q_1 + q_2$. Recall that this monopole charge q refers to the scalar field, and not to an electromagnetic one. For a BH binary or a NS binary in a quasi circular orbit, q only changes during merger, as mass is carried away in radiation. Thus, monopole radiation is inefficient and suppressed.

For the $m = 1$ case, we find

$$\vartheta^{\mathrm{FZ}} = \frac{\dot{D}_i n^i}{r}, \quad (74)$$

where we have defined the total scalar dipole moment as

$$D^i \equiv q_1 x_1^i + q_2 x_2^i. \quad (75)$$

When we evaluate this for circular orbits, we find

$$\vartheta^{\mathrm{FZ}} = \frac{1}{r} \left(q_1 \frac{m_2}{m} - q_2 \frac{m_1}{m} \right) v_{12i} n^i, \quad (76)$$

where we have defined the relative velocity $v_{12}^k \equiv v_1^k - v_2^k$.

The $m = 1$ term clearly leads to dipole radiation in the FZ, which is less relativistic than GR quadrupole radiation, becoming stronger at smaller velocities. Of course,

this term is proportional to the coupling constants of the theory, which are assumed much smaller than one. . Reference [28] failed to recognize such dipolar emission because they considered the motion of test particles that had no scalar charge. We cannot think of any mechanism that would suppress such dipolar radiation.

2. Odd-parity sector: spinning bodies

As in the previous Section, the evolution equation for the scalar field is dominantly

$$\Box_\eta \vartheta = 4\pi \mu_1^i \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_1)_{,i} + (1 \leftrightarrow 2). \quad (77)$$

By using Eq. (69), the far-zone field point solution is obtained as

$$\vartheta^{\mathrm{FZ}} = -\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \partial_M \left[\frac{1}{r} \int_{\mathcal{M}} \mu_1^i \delta^{(3)}(\boldsymbol{x}' - \boldsymbol{x}_1)_{,i} x'^M d^3 x' \right.$$
$$\left. + (1 \leftrightarrow 2) \right]. \quad (78)$$

When $m = 0$ there is obviously no contribution to the scalar field. When $m = 1$,

$$\int_{\mathcal{M}} \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_1)_{,i} x^j d^3 x = -\delta_{ij}, \quad (79)$$

and thus

$$\vartheta^{\mathrm{FZ}} = \frac{\mu_i n^i}{r^2} + \frac{\dot{\mu}_i n^i}{r}, \quad (80)$$

with $\mu_i \equiv \mu_{1i} + \mu_{2i}$. Notice that we recover the solution of Yunes and Pretorius [6] for the first term of the above equation with μ_A^i given as in Eq. (66). These terms will not strongly radiate because $\dot{\mu}_i$ is non-vanishing only for spin-precessing systems. Even then, such radiation would be suppressed by the ratio of the orbital timescale to the precession timescale.

The $m = 2$ contribution, by contrast, depends on the much shorter orbital timescale. We look for terms of $\mathcal{O}(r^{-1})$ since they are the only ones that contribute to the energy flux at infinity. Keeping in mind that the function being differentiated depends on retarded time, we can rewrite Eq. (78) as

$$\vartheta^{\mathrm{FZ}} = -\frac{1}{r} \sum_m \frac{1}{m!} \frac{\partial^m}{\partial t^m} \int_{\mathcal{M}} \mu_1^i \delta^{(3)}(\boldsymbol{x}' - \boldsymbol{x}_1)_{,i} (n_k\, x'^k)^m d^3 x'$$
$$+ (1 \leftrightarrow 2). \quad (81)$$

When $m = 2$, we have that

$$\mu_1^i \int_{\mathcal{M}} \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_1)_{,i} x^p x^q d^3 x + (1 \leftrightarrow 2) = -2\mu^{pq}, \quad (82)$$

where the pseudo-tensor quadrupole moment (not to be confused with $\mu^i \mu^j$) is defined as

$$\mu^{ij} \equiv x_1^{(i} \mu_1^{j)} + x_2^{(i} \mu_2^{j)}. \quad (83)$$

The $m = 2$ contribution becomes

$$\vartheta^{\text{FZ}} = \frac{1}{r}\ddot{\mu}_{ij}n^{ij} = -\frac{1}{r}\omega^2\mu_{ij}n^{ij}\,, \qquad (84)$$

where the final equality is evaluated on a circular orbit. Notice that such a scalar field will strongly radiate because μ^{ij} depends on the orbital timescale.

3. Odd-parity sector: non-spinning bodies

When both objects are non-spinning, the self-interaction terms produced by the effective source identically vanish. One is then left with the source term constructed from the product of the gravitational fields of objects 1 and 2. These terms will be proportional to $m_1 m_2$. As we will see, there are many contributions that turn out to vanish upon NZ integration. For pedagogical reasons, we will show here explicitly how this happens and eventually arrive at contributions that do not vanish.

The evolution equation for the scalar field to leading PN order is

$$\Box_\eta \vartheta_{\text{FZ}} = -32\frac{\alpha_4}{\beta}\epsilon_{ijk}m_1 m_2 v_{12k}\left(\frac{1}{r_1}\right)_{,im}\left(\frac{1}{r_2}\right)_{,jm}\,, \quad (85)$$

where we substituted the NZ metric components in the point-particle approximation. The leading order term of the solution to this differential equation, i.e. the $m = 0$ term in the sum of Eq. (70), is evaluated as

$$\begin{aligned}
\vartheta_{\text{FZ}} &= \frac{8}{\pi}\frac{\alpha_4}{\beta}m_1 m_2\epsilon_{ijk}\frac{v_{12}^k}{r}\int_{\mathcal{M}}\left(\frac{1}{r_1}\right)_{,im}\left(\frac{1}{r_2}\right)_{,jm}d^3x\\
&= -16\frac{\alpha_4}{\beta}m_1 m_2\epsilon_{ijk}\frac{v_{12}^k}{r}\partial_i^{(1)}\partial_j^{(2)}\partial_m^{(1)}\partial_m^{(2)}Y = 0\,.(86)
\end{aligned}$$

Here we integrated over the NZ+IZ hypersurface $\mathcal{M}$ without taking any care of the strong gravity region in the IZs. One can easily show that the contribution from the IZs is not large in the present case. In the second line, we replaced partial derivatives with respect to x^i acting on $1/r_A$ with (minus the) particle derivatives with respect to x_A^i:

$$\frac{\partial}{\partial x^i} \to -\frac{\partial}{\partial x_A^i} \equiv -\partial_i^{(A)}\,, \qquad (87)$$

with $A = (1, 2)$. We commuted these particle derivatives with the integral, and finally obtained a typical NZ integral, discussed in Appendix B. From Eq. (B4), we know that $Y = b$, and by taking all particle derivatives, the last equality is established.

We could have inferred that the $m = 0$ term in the sum does not contribute for non-spinning BHs without any explicit calculations. The argument here is similar to that in footnote 4. Possible vectors to contract with the Levi-Civita symbol include the velocities v_A^i and the unit vectors n_A^i, but not spin vectors S_A^i, as we here consider non-spinning BHs. In particular, for the $m = 0$

case, there cannot be any FZ vectors n^i present. Thus, all vectors that can be contracted onto the Levi-Civita symbol must lie in the same orbital plane and this obviously vanishes. This argument should be true at all PN orders[5].

Let us then consider the next-order term. This will arise from the leading-order source term [right-hand side of Eq. (85)] with $m = 1$ in the NZ sum:

$$\begin{aligned}
\vartheta_{\text{FZ}} &= \frac{8}{\pi}\frac{\alpha_4}{\beta}\frac{m_1 m_2}{r}n^p\epsilon_{ijk}v_{12}^k\\
&\quad\times\frac{\partial}{\partial t}\int_{\mathcal{M}}\left(\frac{1}{r_1}\right)_{,im}\left(\frac{1}{r_2}\right)_{,jm}x^p d^3x\\
&= -16\frac{\alpha_4}{\beta}\frac{m_1 m_2}{r}n^p\epsilon_{ijk}v_{12}^k\frac{\partial}{\partial t}\partial_i^{(1)}\partial_j^{(2)}\partial_m^{(1)}\partial_m^{(2)}Y_p\,,
\end{aligned}$$
$$(88)$$

where we have used Eq. (B3), which defines Y_p. By direct evaluation, one can show that this term also identically vanishes. The first non-vanishing contribution coming from an $m = 1$ term must then be $\mathcal{O}(v^3)$ smaller than the ordering of the $m = 0$ term.

Finally, let us consider the $(\text{next})^2$-order term. This can arise only from the leading-order source term with $m = 2$ in the NZ sum:

$$\begin{aligned}
\vartheta_{\text{FZ}} &= \frac{4}{\pi}\frac{\alpha_4}{\beta}\frac{m_1 m_2}{r}n^{pq}\epsilon_{ijk}\frac{\partial^2}{\partial t^2}v_{12}^k\\
&\quad\times\int_{\mathcal{M}}\left(\frac{1}{r_1}\right)_{,im}\left(\frac{1}{r_2}\right)_{,jm}x^p x^q d^3x\\
&= -8\frac{\alpha_4}{\beta}\frac{m_1 m_2}{r}n^{pq}\epsilon_{ijk}\frac{\partial^2}{\partial t^2}v_{12}^k\partial_i^{(1)}\partial_j^{(2)}\partial_m^{(1)}\partial_m^{(2)}\\
&\quad\times\left(Y_{\langle pq\rangle} + \frac{1}{3}\delta_{pq}S\right)\,,
\end{aligned}$$
$$(89)$$

which simplifies to

$$\vartheta_{\text{FZ}} = 16\frac{\alpha_4}{\beta}\frac{1}{r}\frac{\eta m\delta m}{b}\epsilon_{ijk}n_{ip}\omega^2 v_{12}^k n_{12}^{jp}\,, \qquad (90)$$

where we have defined the mass difference $\delta m \equiv m_1 - m_2$ and the symmetric mass ratio $\eta \equiv m_1 m_2/m^2$. We have here used Kepler's law and expanded the STF tensors. This is the dominant FZ behavior of the scalar field, which as we see is much suppressed relative to the odd-parity solution we found for spinning BHs.

E. Summary of this section

Let us summarize the results found so far for later use. In the even-parity case, generically at least one of the

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[5] One may think that one can construct a vector that does not lie in the orbital plane by taking the cross product of two vectors that lie on this plane, e.g. $\boldsymbol{n}_{12} \times \boldsymbol{v}_{12}$. However, since GR is parity even, such a vector cannot be present in the PN metric.

	A	B	C	a	b	c	d
Even-P	q_1	0	$\dot{D}_i n^i$	1	0	$-$	$-$
Odd-P, Spins	$\mu_1^i n_1^i$	0	$\ddot{\mu}_i n^{ij}$	2	0	$-$	$-$
Odd-P, No Spins	$\sigma_{\mathrm{NZ}}^{pq} n_{12}^p n_1^q$	$\frac{1}{2}\sigma_{\mathrm{NZ}}^{pq} n_1^p n_2^q$	$\sigma_{\mathrm{FZ}}^{pq} n^{pq}$	2	2	2	2

TABLE I: Scalar field parameters, as defined in Eqs. (91) and (93). The quantities q_1 and μ_1^i are defined in Eqs. (63) and (66), while $\sigma_{\mathrm{NZ}}^{pq}$ is defined in Eq. (92). The quantities D_i and μ_i are defined in Eqs. (75) and (78), while $\sigma_{\mathrm{FZ}}^{pq}$ is given in Eq. (94).

binary component objects will have a scalar monopole charge. Since the scalar field excitation due to the induced monopole is dominant, we neglect all the other less important contributions. Weakly gravitating objects need not have a scalar monopole charge if $\alpha_1 + \alpha_2 + 3\alpha_3 = 0$, and BHs have no scalar monopole charge if $\alpha_3 = 0$. In EDGB theory, NSs have no scalar monopole charge. In the odd-parity case, the dominant contribution is the magnetic-type scalar dipole moment induced by spins. Generically, astrophysical objects will possess spin, but we will continue to include non-spinning results to compare with previous work.

In the NZ, we can parametrize the leading PN terms of the scalar field as

$$\vartheta_{\mathrm{NZ}} = \frac{A}{r_1^a b^b} + \frac{B}{r_1^c r_2^d} + (1 \leftrightarrow 2), \qquad (91)$$

where (A, B, a, b, c, d) are given in Table I and for compactness of the Table we define

$$\sigma_{\mathrm{NZ}}^{pq} \equiv -16\frac{\alpha_4}{\beta}\eta m^2 \epsilon_{pqs} v_{12}^s. \qquad (92)$$

In the FZ, we can parametrize the scalar field as

$$\vartheta_{\mathrm{FZ}} = \frac{C}{r}, \qquad (93)$$

where C is also given in Table I and we define

$$\sigma_{\mathrm{FZ}}^{pq} \equiv 16\frac{\alpha_4}{\beta}\eta\, m\, \delta m \frac{\omega^2}{b}\epsilon_{qjk} v_{12}^k n_{12}^{jp}, \qquad (94)$$

V. METRIC EVOLUTION

In this section, we solve the evolution equations for the metric deformation in the FZ, so that we can calculate the gravitational energy flux at infinity. Note that throughout, we use the Newtonian relationship $v^2 = m/b$ (and similarly for the acceleration). This relationship must be corrected at higher PN order or at $\mathcal{O}(\varsigma)$. As we mentioned earlier, here we do not take into account the corrections to the orbital motion due to the conservative force at $\mathcal{O}(\varsigma)$. These conservative effects do not interfere at $\mathcal{O}(\varsigma)$ with the radiative effects that we are concerned

with in this paper. Therefore the corrections to the GW waveform become a simple summation of these two different types of effects.

For the FZ field points, the solution to the metric deformation equation of motion [Eq. (17)] can be read from Eq. (70):

$$\mathfrak{h}_{ij} = -\frac{8}{r}\sum_{m=0}^{\infty} \frac{1}{m!}\frac{\partial^m}{\partial t^m}\int_{\mathcal{M}} \tilde{\mathcal{C}}_{ij}\,(n^k x'^k)^m d^3 x' + O\left(r^{-2}\right), \qquad (95)$$

where we have defined the source term as

$$\tilde{\mathcal{C}}_{ij} = \alpha_1 \left(\vartheta\tilde{\mathcal{H}}_{ij}^{(0)} + \tilde{\mathcal{H}}_{ij}^{(1)} \right) + \alpha_2 \left(\vartheta\tilde{\mathcal{I}}_{ij}^{(0)} + \tilde{\mathcal{I}}_{ij}^{(1)} \right) \\ + \alpha_3 \left(\vartheta\tilde{\mathcal{J}}_{ij}^{(0)} + \tilde{\mathcal{J}}_{ij}^{(1)} \right) + \alpha_4\tilde{\mathcal{K}}_{ij}^{(1)} - \frac{1}{2}T_{ij}^{(\vartheta)}. \qquad (96)$$

Notice that this corresponds to an IZ+NZ integration for FZ field points, where we have neglected the FZ integration because it is subdominant.

The integrals presented above have to be carried out also in the IZ, where the PN expansion is not valid anymore. In GR, however, such divergences can be ignored, using a regularization scheme. Since both the true solution and an appropriately regularized solution satisfy the field equations in the NZ, their difference due to the IZ contribution is only through a homogeneous solution. Such homogeneous solutions are regular in the NZ and FZ, but can be divergent in the IZ. They are characterized by the multipole moments of the respective objects, which can be determined by studying tidal perturbations around a strongly gravitating object. One can then perform matching of the metric solution, as for the scalar solution, but the metric matching is beyond the scope of this paper. In what follows, we only consider the regularized contribution, following Hadamard *partie finie* (FP) regularization [64]. We comment more on the divergent contribution at the end of this Section.

A. Even-parity sector

Let us focus on the metric perturbation in the even-parity sector first. The leading order term both in the PN and $1/r$ expansion at infinity is formally given by

$$\mathfrak{h}_{ij} = \mathfrak{h}_{ij}^T + \mathfrak{h}_{ij}^{\mathcal{J}}, \qquad (97)$$

$$\mathfrak{h}_{ij}^T \equiv \frac{4}{r}\int_{\mathcal{M}} T_{ij}^{(\vartheta)} d^3 x, \qquad (98)$$

$$\mathfrak{h}_{ij}^{\mathcal{J}} \equiv -\frac{8\alpha_3}{r}\int_{\mathcal{M}} \tilde{\mathcal{J}}_{ij} d^3 x, \qquad (99)$$

where we have defined $\tilde{\mathcal{J}}_{ij} \equiv \vartheta\tilde{\mathcal{J}}_{ij}^{(0)} + \tilde{\mathcal{J}}_{ij}^{(1)}$. The source terms $\tilde{\mathcal{H}}_{\mu\nu}$ and $\tilde{\mathcal{I}}_{\mu\nu}$ do not contribute to this expression since they identically vanish in the NZ where $R_{\mu\nu} = 0$.

We can estimate the order of magnitude of both $\mathfrak{h}_{ij}^{\mathcal{J}}$ and $\mathfrak{h}_{ij}^{T}$ as follows:

$$\mathfrak{h}_{ij}^{T} \sim \mathcal{O}\left(\beta \frac{m}{r} v^{-2} \vartheta^2\right) = \zeta_3 \frac{m}{r} v^2 \times \mathcal{O}\left(1\right), \qquad (100)$$

$$\mathfrak{h}_{ij}^{\mathcal{J}} \sim \mathcal{O}\left(\frac{\alpha_3}{m^2} \frac{m}{r} v^4 \vartheta\right) = \zeta_3 \frac{m}{r} v^2 \times \mathcal{O}\left(v^4\right). \qquad (101)$$

Here we factored out v^2 in the final expressions, since the GR leading quadrupolar field is also proportional to v^2. Clearly, the dominant contribution comes from Eq. (100).

Let us now make this computation more precise. The stress-energy tensor will contain self-interactions of the form $\vartheta_{A,i} \vartheta_{A,j}$ and cross terms of the form $\vartheta_{1,i} \vartheta_{2,j}$. The former case leads to divergent integrals, which must be determined by strong-field matching, so we do not consider them here. Let us concentrate on the latter, which take the form

$$T_{ij}^{(\vartheta)} = \beta \left(\vartheta_{,i} \vartheta_{,j} - \frac{1}{2} \delta_{ij} \vartheta_{,\mu} \vartheta^{,\mu}\right) \qquad (102)$$

$$\approx \beta q_1 q_2 \left[2 \left(\frac{1}{r_1}\right)_{,(i} \left(\frac{1}{r_2}\right)_{,j)} - \delta_{ij} \left(\frac{1}{r_1}\right)_{,k} \left(\frac{1}{r_2}\right)_{,k}\right], \qquad (103)$$

which sources the metric perturbation

$$\mathfrak{h}_{ij} = \frac{4}{r} \int_{\mathcal{M}} T_{ij}^{(\vartheta)} d^3x,$$
$$= -\frac{4\pi}{r} \beta q_1 q_2 \left(2 \partial_i^{(1)} \partial_j^{(2)} b - \delta_{ij} \partial_k^{(1)} \partial_k^{(2)} b\right) + (1 \leftrightarrow 2),$$
$$= -\frac{16\pi}{r} \beta \frac{q_1 q_2}{b} n_{12}^{ij}, \qquad (104)$$

where we used an integration formula for the triangle potential given in Appendix B. We can see that this correction is 0PN relative to the radiative metric perturbation in GR, just as we predicted in Eq. (100). However, this correction turns out to be still smaller in the energy flux than the dipole scalar radiation, which gives a -1PN correction.

B. Odd-parity sector

We now focus on the odd-parity sector, for which the solution is given by the term proportional to α_4 in Eq. (95), namely

$$\mathfrak{h}_{ij} = \mathfrak{h}_{ij}^{T} + \mathfrak{h}_{ij}^{\mathcal{K}}, \qquad (105)$$

$$\mathfrak{h}_{ij}^{\mathcal{K}} \equiv -\frac{8\alpha_4}{r} \int_{\mathcal{M}} \tilde{\mathcal{K}}_{ij}^{(1)} d^3x. \qquad (106)$$

The stress-energy contribution $\mathfrak{h}_{ij}^{T}$ is the same as in Eq. (102).

The $\mathcal{K}$ contribution to Eq. (105) is more involved. The leading-order behavior of the $\mathcal{K}$ tensor is

$$\tilde{\mathcal{K}}_{ij}^{(1)} = \dot{\vartheta}_{,k} \epsilon_{jkl} h_{00,il} + \dot{\vartheta}_{,k} \epsilon_{jlm} (h_{im,lk} + h_{lk,im})$$
$$+ \vartheta_{,kl} \epsilon_{jlm} (h_{i0,mk} + h_{mk,i0} - h_{k0,im} - h_{im,k0})$$
$$- \vartheta_{,k} \epsilon_{ikl} (2h_{0[m,j]lm} - 2\dot{h}_{l[j,m]m} - \dot{h}_{00,jl})$$
$$- 2\dot{\vartheta} \epsilon_{ikl} h_{k[j,m]lm} + (i \leftrightarrow j). \qquad (107)$$

Other terms are of higher PN order. By applying the Lorenz or harmonic gauge condition $h^{\mu\nu}{}_{,\nu} = 0$, substituting $h_{ij} = h_{00} \delta_{ij}$ into Eq. (107), and using $\epsilon_{jkl} h^{\mu\nu,kl} = \epsilon_{jkl} \vartheta_{,kl} = 0$, we get

$$\tilde{\mathcal{K}}_{ij}^{(1)} = 2\dot{\vartheta}_{,k} \epsilon_{jkl} h_{00,il} - 2\vartheta_{,km} \epsilon_{jkl} h_{0[m,i]l}$$
$$- 2\vartheta_{,k} \epsilon_{jkl} h_{0[m,i]lm} + 2\vartheta_{,k} \epsilon_{jkl} \dot{h}_{00,il} + (i \leftrightarrow j). \qquad (108)$$

The $\tilde{\mathcal{K}}_{ij}$ term in Eq. (105) is then a sum of four terms, namely

$$\mathfrak{h}_{ij}^{\mathcal{K}} = \sum_{n=1}^{4} \mathfrak{h}_{ij}^{(n)}, \qquad (109)$$

where we have defined

$$\mathfrak{h}_{ij}^{(1)} = -\frac{16\alpha_4}{r} \int_{\mathcal{M}} \dot{\vartheta}_{,k} \epsilon_{jkl} h_{00,il} d^3x + (i \leftrightarrow j), \qquad (110)$$

$$\mathfrak{h}_{ij}^{(2)} = +\frac{16\alpha_4}{r} \int_{\mathcal{M}} \vartheta_{,km} \epsilon_{jkl} h_{0[m,i]l} d^3x + (i \leftrightarrow j), \qquad (111)$$

$$\mathfrak{h}_{ij}^{(3)} = +\frac{16\alpha_4}{r} \int_{\mathcal{M}} \vartheta_{,k} \epsilon_{jkl} h_{0[m,i]lm} d^3x + (i \leftrightarrow j), \qquad (112)$$

$$\mathfrak{h}_{ij}^{(4)} = -\frac{16\alpha_4}{r} \int_{\mathcal{M}} \vartheta_{,k} \epsilon_{jkl} \dot{h}_{00,il} d^3x + (i \leftrightarrow j). \qquad (113)$$

When we substitute the PN metric into the above terms, the right-hand sides depend on the velocity vectors v_A^i (which depend on time only). The field ϑ is given in Eq. (91) and its derivative can be computed simply from that equation. Since this field is a NZ one, it depends on time through the positions of the objects, which implies that its time derivative can be converted into a spatial derivative via $\partial_t f(r_1) = -v_1^i \partial_i f(r_1)$.

Let us begin by making a simple order of magnitude estimate of how large the regularized contribution is. For this, it suffices to look at Eqs. (100) and (110):

$$\mathfrak{h}_{ij}^{T} \sim \mathcal{O}\left(\beta \frac{m}{r} v^{-2} \vartheta^2\right), \qquad (114)$$

$$\mathfrak{h}_{ij}^{\mathcal{K}} \sim \mathcal{O}\left(\frac{\alpha_4}{m^2} \frac{m}{r} v^5 \vartheta\right). \qquad (115)$$

The ϑ field here is that of the NZ, and hence

$$\mathfrak{h}_{ij}^{T} \sim \zeta_4 \frac{m}{r} v^2 \times \mathcal{O}\left(\chi^2 v^4 + \eta \chi v^9 + \eta^2 v^{14}\right), \qquad (116)$$

$$\mathfrak{h}_{ij}^{\mathcal{K}} \sim \zeta_4 \frac{m}{r} v^2 \times \mathcal{O}\left(\chi v^7 + \eta v^{12}\right), \qquad (117)$$

where χ stands for the magnitude of χ_1^i and χ_2^i. From this analysis, $\mathfrak{h}_{ij}^T$ is clearly larger for rapidly spinning objects, leading to a 2PN effect.

For the non-spinning case, one might expect the $\mathcal{K}$ contribution to lead to a 6PN effect, but as we explain in Appendix C, these leading-order effects actually vanish. This cancellation can also rather easily be seen by integrating by parts in Eqs. (110)-(113). After discarding boundary terms (taking into account the boundary term is equivalent to adding homogeneous solutions, corresponding to deformed multipole moments of compact objects), we obtain expressions of the form $\epsilon_{jkl}\,\vartheta\,h_{00,kl...}$, which obviously vanishes by the antisymmetry of the Levi-Civita tensor. We carry out a more careful analysis in Appendix C, where we explicitly show that the leading and first sub-leading order terms vanish[6]. The first non-vanishing term is then of $\mathcal{O}(v^2)$ smaller than the order of magnitude estimates in Eqs. (116) and (117), leading to 7PN and 4.5PN contributions at $\mathcal{O}(\chi^0)$ and $\mathcal{O}(\chi^1)$, respectively.

Since the largest contribution seems to arise for spinning BHs from the $\mathfrak{h}_{ij}^T$ term, let us consider this in more detail. Two possible contributions are generated here: one that depends only on self-interaction terms, and one that depends on the cross-interaction. The former leads to divergent integrals, which need to be matched from strong-field solutions, and we do not consider these here. The latter leads to the metric deformation

$$
\begin{aligned}
\mathfrak{h}_{ij}^T &= -\frac{4\pi}{r}\beta\mu_1^k\mu_2^l\left(2\partial_{ik}^{(1)}\partial_{jl}^{(2)}Y - \delta_{ij}\partial_{pk}^{(1)}\partial_{pl}^{(2)}Y\right) + (1\leftrightarrow 2)\\
&= \frac{8\pi\beta}{rb^3}\left\{2\mu_1^{(i}\mu_2^{j)} - 12n_{12}^{(i}\mu_1^{j)}\left(n_{12}^k\mu_{2k}\right)\right.\\
&\quad \left. + 3n_{12}^{ij}\left[5\left(n_{12}^k\mu_{1k}\right)\left(n_{12}^l\mu_{2l}\right) - \mu_{1k}\mu_2^k\right]\right\} + (1\leftrightarrow 2),
\end{aligned}
$$
(118)

which is clearly of the order predicted in Eq. (116), i.e. 2PN order relative to GR. This is of the same order as the energy flux correction carried by the pseudo-scalar radiation.

C. Multipole moments

In this Subsection, we discuss the additional contribution from the IZs, which enter as additional homogeneous solutions in the NZ and FZ, These contributions are homogeneous in the sense that they arise from sources that have support only in the IZs, and thus they vanish in the NZ and FZ (see e.g. the discussion prior to Eq. (45)). The homogeneous solutions are characterized by the mass and current multipole moments of the strong-field bodies, which must be determined by matching to strong gravity

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[6] In Appendix C, we only show this for non-spinning BHs, but a similar calculation can be performed for spinning BHs to $\mathcal{O}(\chi)$.

solutions in the IZ. When we solve the non-linear equations of motion iteratively, the source terms in general can be classified into two pieces: a self-interaction part and a cross-interaction part, as in the case of ϑ in Sec. IV. The cross-interaction part is sourced by the companion, while the self-interaction part is not.

The self-interaction part is rather easy to handle because matching involves only a single isolated object. As described in Sec. IV B, these self-interaction terms can be thought of as homogeneous solutions that have support only in the IZ. As such, in the small-coupling approximation, they satisfy homogenous field equations that take Einstein form. If the spin of the object is neglected, the only possible linear perturbation to such a homogeneous solution that is compatible with asymptotic flatness is a shift of the body's mass (in the $1/r$ piece of the (t,t) and diagonal parts of the metric). In essence, this is a consequence of Birkhoff's theorem, which holds for homogeneous solutions. Such a shift is consistent with the strong-field, non-spinning BH solution in EDGB theory found in [28]. In that case, the mass shift is simply $m_A \to (49/80)\zeta_3 m_A$.

For spinning objects, one expects there to be higher multipole moments in the strong-field solution. However, one should be able to absorb current dipole moment modifications by a redefinition of the spin parameter, while the mass dipole moment will be absorbed by the redefinition of the position of the center of mass. Therefore, the leading-order corrections that survive are the mass quadrupole moment, which produces a metric perturbation in the NZ proportional to $1/r^3$. As we will see, when we consider FZ solution, there is an additional factor of v^2 that enters.

Therefore, contributions to the energy flux from the quadrupole or higher multipole moments are at least 3PN order relative to that from the GR quadrupole formula. We will later find that corrections to the energy flux due to scalar radiation appear at -1PN and 2PN relative order for the even and odd-parity cases, respectively. Hence, the contributions from the multipole moments that we discussed here are definitely smaller than those introduced by scalar radiation in the even-parity case, and at most, the same order in the odd-parity case.

Let us take a look at spinning BHs in the odd-parity sector in more detail. At $\mathcal{O}(\chi)$ there is freedom in adding a homogeneous solution proportional to $1/r^2$ in the h_{0i} component. This corresponds to a freedom in shifting the Kerr parameter measured at infinity. Reference [6] set this homogeneous solution to zero so that there is no shift in the Kerr parameter. At $\mathcal{O}(\chi^2)$, there should be corrections proportional to $1/r^3$ in h_{ij} which shifts the quadrupole moment. Since there is no parameter in the Kerr geometry that can absorb this correction in the quadrupole moment, this $1/r^3$ correction cannot be eliminated.

The effective source term that reproduces this correc-

tion should look like

$$\Box \mathfrak{h}_{ij} = -4\pi Q_1 u_i u_j (\delta_{kl} - 3\hat{S}_{1,k}\hat{S}_{1,l})\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_1)_{,kl}$$
$$+ (1 \leftrightarrow 2), \qquad (119)$$

where $Q_A = \mathcal{O}(\zeta_4 m_A a_A^2)$ and $\hat{S}_{A,k} \equiv S_A^i/m_A^2$ is a unit spin angular momentum vector. The solution of this wave equation at $\mathcal{O}(1/r)$ is given by

$$\mathfrak{h}_{ij} = \frac{1}{r}\sum_{m=0}^{\infty}\frac{1}{m!}\frac{\partial^m}{\partial t^m}u_i u_j(\delta_{kl} - 3\hat{S}_{1,k}\hat{S}_{1,l})Q_1$$
$$\times \int_{\mathcal{M}}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_1)_{,kl}(\boldsymbol{n}\cdot\boldsymbol{x})^m d^3 x + (1 \leftrightarrow 2). \qquad (120)$$

The leading-order contributions at $m = 0$ (2PN) and $m = 1$ (2.5PN) vanish, leading to the first non-zero contribution at $m = 2$

$$\mathfrak{h}_{ij} = \mathcal{O}\left(\frac{1}{r}Q\omega^2 v^2\right) = \zeta_4\frac{m}{r}v^2 \times \mathcal{O}\left(\chi^2 v^6\right), \qquad (121)$$

which is 3PN relative to GR. Therefore, the self-interacting correction in the metric at $\mathcal{O}(\chi^2)$ is smaller compared to the corrections in the energy flux carried by the scalar field and the metric field with regularized modification.

The cross-interaction part is more complicated. In this case, we have to consider the induced multipole moments due to the presence of the secondary object. Thus, even if we consider non-spinning objects, higher multipole moments might be induced. Another important difference is that neither the mass monopole nor the spin dipole can be simply absorbed by a redefinition of the mass and spin of each object. This is because the shifts of these multipole moments depend on the orbital parameters, such as separation b. Notice, however, that the effects of the secondary object propagate only through the scalar field or the gravitational tidal force.

The order of magnitude of the former scalar field effect is more complicated to estimate and it depends on the situation. In the even-parity case, ϑ sourced by the secondary body at the position of the primary body is proportional to $1/b$. In EDGB theory, since ϑ has shift symmetry within the context of the classical theory, the effects are suppressed by the gradient of the field, i.e. they are proportional to $1/b^2$. In the odd-parity case, there is again shift symmetry and the monopole scalar charge is absent. Because of these two reasons, the suppression is proportional to $1/b^3$ in CS theory. These suppressions will be sufficient to conclude that the effects are relatively at least 1PN and 3PN in the even and odd-parity cases respectively, which is smaller than the effects induced by scalar radiation.

In the odd-parity non-spinning case, the latter gravitational tidal force dominates over the scalar propagation effect. To calculate this tidal force properly requires asymptotic matching between the IZ solution and

a strong-field, perturbed Schwarzschild solution in CS gravity. Perturbations of the Schwarzschild spacetime can be decomposed as a sum over electric and magnetic tidal tensors (see e.g. [60]). The former scale as $1/b^3(1 + v + v^2 + \ldots)$, while the latter scales as $v/b^3(1 + v + v^2 + \ldots)$ [56]. Such tidal deformations will induce gravitational waves that will scale as the second-time derivatives of the electric and magnetic quadrupole deformations, i.e. they will scale as $\omega^2/b^3(1 + v + v^2 + \ldots)$ and $\omega^2 v/b^3(1 + v + v^2 + \ldots)$. In GR, the leading order effect is induced by the electric quadrupole moment and it scales as ω^2/b^3, a 5PN order effect. In CS, we expect the magnetic quadrupole moment to provide the leading-order deformation, and at the level of energy flux, this couples to the GR metric perturbation produced by either the mass octupole or current dipole moment, leading to 6PN correction. This interpretation seems to be consistent with the results of Pani et al. [65].

VI. ENERGY FLUX

The inspiral of a compact binary system is controlled by the system's change in binding energy and angular momentum. The binding energy changes according to the dissipation of energy carried by all dynamical fields, which here includes the metric perturbation and the scalar field. The stress-energy tensor (SET) associated with each field quantifies the density and flux of energy and momentum. The energy loss is calculated as the integral of the energy flux through a 2-sphere of radius r in the limit $r \to \infty$ and in the direction of the sphere's outward unit normal n^i. That is, for some field φ (be it h_{ij}, $\mathfrak{h}_{ij}$, or ϑ) with SET $T_{\mu\nu}^{(\varphi)}$,

$$\dot{E}^{(\varphi)} = \lim_{r\to\infty}\int_{S_r^2}\left\langle T_{ti}^{(\varphi)}n^i\right\rangle_\omega r^2 d\Omega, \qquad (122)$$

where the angle brackets with subscript ω stand for orbit averaging.

The total energy flux can be ordered in powers of ς as

$$\dot{E} = \dot{E}_{\rm GR} + \varsigma\,\delta\dot{E} + \mathcal{O}(\varsigma^2). \qquad (123)$$

The GR energy flux $\dot{E}^{\rm GR}$ is given by the GR metric perturbation only, without any contributions from the scalar field at $\mathcal{O}(\varsigma^0)$, as there is no scalar field in GR. For circular orbits, this is

$$\dot{E}_{\rm GR} = -\frac{32}{5}\eta^2 v^{10}. \qquad (124)$$

The $\mathcal{O}(\varsigma)$ correction, $\delta\dot{E}$, can be decomposed into

$$\delta\dot{E} = \delta\dot{E}^{(\vartheta)} + \delta\dot{E}^{(\mathfrak{h})}, \qquad (125)$$

where the first term is the scalar field contribution and the second term is the contribution of the deformed metric perturbation.

The scalar field contribution is calculated with the SET given by Eq. (7):

$$\delta\dot{E}^{(\vartheta)} = \beta \lim_{r\to\infty} \int_{S_r^2} \left\langle \dot{\vartheta}\, n^i\, \partial_i\vartheta \right\rangle_\omega r^2 d\Omega\,. \qquad (126)$$

Since we are taking the $r \to \infty$ limit, ϑ must be that valid in the FZ.

The metric deformation contribution to the energy flux is slightly more subtle. This modification to the GR flux can have three distinct sources: (i) the effective SET in terms of h_{ij} and $\mathfrak{h}_{ij}$ may be functionally different, but as shown in [40], this is not so for the class of theories we consider here[7]; (ii) The orbital equations of motion, and the associated relations $m/b = v^2$ and $\omega = v^3/m$, might be modified at $\mathcal{O}(\varsigma)$, as was partially calculated in [28]; (iii) The generation mechanism of the FZ metric perturbation is modified, i.e. the radiative part of the metric perturbation is deformed. We consider here only the dissipative modifications introduced by (iii), as (ii) would require an analysis of the equations of motion, which is beyond the scope of this paper[8].

Letting $H_{\alpha\beta} = h_{\alpha\beta} + \varsigma\mathfrak{h}_{\alpha\beta} + \mathcal{O}(\varsigma^2)$, the effective SET of GWs is given by [40]

$$T_{\mu\nu}^{(H)} = \frac{1}{32\pi} \left\langle H_{\alpha\beta,(\mu}^{\rm TT} H_{\rm TT}^{\alpha\beta},_{\nu)} \right\rangle_\lambda\,, \qquad (127)$$

where the angle brackets with a subscript λ stand for a quasi-local average over several wavelengths and TT stands for the transverse-traceless projection

$$H_{ij}^{\rm TT} = \Lambda_{ij,kl} H_{kl}\,, \qquad \Lambda_{ij,kl} = P_{ik}P_{jl} - \frac{1}{2} P_{ij}P_{kl}\,, \quad (128)$$

with $P_{ij} = \delta_{ij} - n_{ij}$ the projector onto the plane perpendicular to the line from the source to a FZ field point. Expanding this SET in orders of ς, the $\mathcal{O}(\varsigma^0)$ part leads to $\dot{E}_{\rm GR}$, while the $\mathcal{O}(\varsigma)$ part is

$$T_{\mu\nu}^{(\mathfrak{h})} = \frac{1}{16\pi} \left\langle h_{\alpha\beta,(\mu}^{\rm TT} \mathfrak{h}_{\rm TT}^{\alpha\beta},_{\nu)} \right\rangle_\lambda\,, \qquad (129)$$

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[7] Reference [40] showed that the TT gauge exists in quadratic gravity as $r \to \infty$. Any non-TT propagating mode that is sourced in the NZ vanishes in the FZ at all orders. This is in contrast to scalar-tensor theories in the Jordan frame, where the scalar "breathing" mode is present in the metric. This difference comes from the way the metric deformation and the scalar field couple in the field equations. In the quadratic gravity case, ϑ does not multiply $G_{\mu\nu}$ in the field equations (the Einstein-Hilbert sector of the action is unmodified), while the opposite is true in scalar-tensor theories in the Jordan frame. Therefore, in the former $\mathfrak{h}_{\mu\nu}$ and ϑ decouple in the $r \to \infty$ limit and there is no breathing mode. In contrast, in the latter the coupling between $\mathfrak{h}_{\mu\nu}$ and ϑ remains in the limit $r \to \infty$, leading to a non-vanishing breathing mode and a modification to the effective SET.

[8] The distinction between (ii) and (iii) can be ambiguous at higher PN order, because how the orbital parameters are modified depends on the gauge choice. However, as long as we impose the harmonic gauge condition on both GR and the deformed metric perturbations, we do not have to worry about this gauge issue at least up to next-to-leading PN order.

which leads to

$$\delta\dot{E}^{(\mathfrak{h})} = \frac{1}{16\pi} \lim_{r\to\infty} \int_{S_r^2} \left\langle \left\langle h_{\alpha\beta,(t}^{\rm TT} \mathfrak{h}_{\rm TT}^{\alpha\beta},_{i)} \right\rangle_\lambda n^i \right\rangle_\omega r^2 d\Omega\,. \tag{130}$$

As before, the $h_{\alpha\beta}$ and $\mathfrak{h}_{\alpha\beta}$ are those valid in the FZ.

A. Scalar field correction to the energy flux

1. Even-parity sector

In the even-parity case, $\vartheta^{\rm FZ}$ is dominated by the dipole component [Eq. (74)], which we repeat here for convenience: $\vartheta^{\rm FZ} = \dot{D}_i n^i/r$, where D_i is the NZ dipole given in Eq. (75). This is inserted into the energy loss formula, Eq. (126). Since the FZ scalar field depends on retarded time, both time and spatial derivatives can be written as time derivatives of the NZ moments. This gives

$$\delta\dot{E}^{(\vartheta)} = -\beta \int_{S_\infty^2} \left\langle \ddot{D}_i \ddot{D}_j n^{ij} \right\rangle_\omega d\Omega = -\frac{4\pi}{3}\beta \left\langle \ddot{D}^i \ddot{D}_i \right\rangle_\omega\,, \tag{131}$$

which for circular orbits gives

$$\delta\dot{E}^{(\vartheta)} = -\frac{4\pi}{3}\beta\omega^4|D|^2 = -\frac{4\pi}{3}\frac{\beta}{m^4}(m_2 q_1 - m_1 q_2)^2 v^8\,. \tag{132}$$

Note that here, as before, the $m \to 0$ limit diverges, because the effective theory breaks down on short length scales and $\varsigma \ll 1$ is violated.

When the compact bodies are BHs, their scalar monopole charges are given by Eq. (63), $q_A = 2\alpha_3/(\beta m_A)$, which then leads to

$$\delta\dot{E}^{(\vartheta)} = -\frac{1}{3}\zeta_3 \frac{1}{\eta^2}\frac{\delta m^2}{m^2}v^8\,. \tag{133}$$

Comparing this with the GR energy flux, we find

$$\frac{\delta\dot{E}^{(\vartheta)}}{\dot{E}^{\rm GR}} = \frac{5}{96}\zeta_3 \frac{1}{\eta^4}\frac{\delta m^2}{m^2}v^{-2}\,, \tag{134}$$

a relative -1PN effect. That is, the energy lost to the scalar field due to dipole radiation would enter as a lower-order in v effect than the energy loss in GR. If one takes the limit $m_2 \to \infty$ while keeping (m_1, v) fixed, then the above ratio scales as m_1^{-4}; i.e. the energy flux ratio is sensitive to the smallest horizon scale of the system. The effect is of a similar size for comparable stellar-mass binary and EMRI system. A SMBH-SMBH binary experiences the smallest effect.

2. Odd-parity sector: spinning bodies

The scalar field $\vartheta^{\rm FZ}$ is here dominated by the quadrupole component [Eq. (84)], which we repeat here for convenience $\vartheta^{\rm FZ} = \ddot{\mu}_{ij} n^{ij}/r = -\omega^2 \mu_{ij} n^{ij}/r$, where

the quadrupole tensor μ_{ij} is defined in Eq. (83). Inserting this into the energy loss formula [Eq. (126)] gives

$$\delta\dot{E}^{(\vartheta)} = -\beta \int_{S^2_\infty} \left\langle \dddot{\mu}_{ij}\dddot{\mu}_{kl}n^{ijkl}\right\rangle_\omega d\Omega ,$$

$$= -\frac{4\pi}{15}\beta\left\langle \left[2\dddot{\mu}_{ij}\dddot{\mu}^{ij} + \left(\dddot{\mu}^i_{\ i}\right)^2\right]\right\rangle_\omega . \quad (135)$$

Let us evaluate this for quasi-circular orbits with non-precessing spins. The third time derivative of the quadrupole tensor μ_{ij} becomes

$$\dddot{\mu}^{ij} = b^{-3}\left(m_1 v_{12}^{(i}\mu_2^{j)} - m_2 v_{12}^{(i}\mu_1^{j)}\right) , \quad (136)$$

and the total energy flux is

$$\delta\dot{E}^{(\vartheta)} = -\frac{5}{48}\zeta_4\left[\bar{\Delta}^2 + 2\left\langle(\bar{\Delta}\cdot\hat{v}_{12})^2\right\rangle_\omega\right]v^{14} , \quad (137)$$

where $\hat{v}_{12}$ is the unit vector in the direction of the relative velocity and the dimensionless quantity $\bar{\Delta}$ is defined as

$$\bar{\Delta}^i \equiv \frac{m_2}{m}\chi_1\hat{S}_1^i - \frac{m_1}{m}\chi_2\hat{S}_2^i . \quad (138)$$

Notice that $\delta\dot{E}^{(\vartheta)}$ in Eq. (137) is finite in the EMRI limit. Note also that when both spins are perpendicular to the orbital plane, $\bar{\Delta}$ is as well, and the second term of $\delta\dot{E}^{(\vartheta)}$ vanishes. Comparing Eq. (137) with GR,

$$\frac{\delta\dot{E}^{(\vartheta)}}{\dot{E}^{\mathrm{GR}}} = \frac{25}{1536}\zeta_4\frac{1}{\eta^2}\left[\bar{\Delta}^2 + 2\left\langle(\bar{\Delta}\cdot\hat{v}_{12})^2\right\rangle_\omega\right]v^4 , \quad (139)$$

hence scalar radiation in the odd-parity sector is clearly a relative 2PN effect. This effect was not included in the work of Pani et al. [65], who found a 7PN correction, since their simulations did not include spins. If one takes the limit $m_2 \to \infty$ while keeping (m_1, v) fixed, then the above ratio scales as $m_1^{-2}m_2^{-2}$; i.e. the energy flux ratio is sensitive to the geometric mean of the two horizon scales in the system. This implies that the effect is greatest for comparable stellar-mass binaries.

3. Odd-parity sector: non-spinning bodies

The odd-parity ϑ_{FZ} in Eq. (90) can be used to evaluate the energy loss in Eq. (126):

$$\delta\dot{E}^{(\vartheta)} = -256\kappa\zeta_4\delta m^2\eta^2\left(\frac{m}{b}\right)^8\int_{S^2_\infty}d\Omega\left[\partial_t\left(\epsilon^{ijk}n^{ip}v_{12}^k n_{12}^{jp}\right)\right]^2$$

$$= -256\kappa\zeta_4\eta^2\frac{\delta m^2}{m^2}\left(\frac{m}{b}\right)^{10}\int_{S^2_\infty}d\Omega\left(\epsilon^{ijk}n^{ip}v_{12}^{kp}n_{12}^j\right)^2$$

$$= -\frac{64}{15}\zeta_4\eta^2\frac{\delta m^2}{m^2}\left(\frac{m}{b}\right)^{12} . \quad (140)$$

Compared to the GW radiation in GR [Eq. (124)], this scalar radiation becomes

$$\frac{\delta\dot{E}^{(\vartheta)}}{\dot{E}^{\mathrm{GR}}} = \frac{2}{3}\frac{\delta m^2}{m^2}\zeta_4\,v^{14} , \quad (141)$$
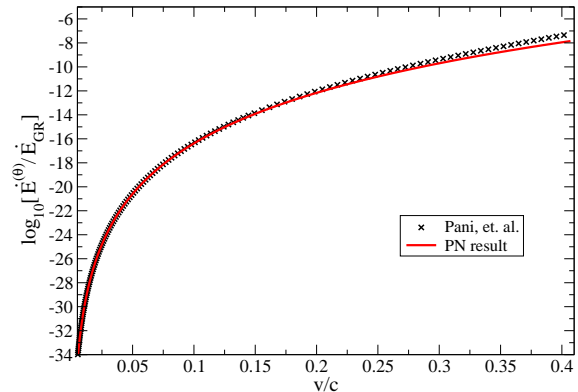

FIG. 3: Comparison of Eq. (141) to the numerical results of Pani et al. [65]. The latter can be mapped to the generic quadratic gravity action of Eq. (4) by letting $\alpha_4 = -\alpha_{\mathrm{CS}}/4$, which then implies that $\zeta_4 = -\zeta_{\mathrm{CS}}/16$. We here used $\zeta_4 = 6.25\times10^{-3}$, which is equivalent to their parameter $\zeta_{\mathrm{CS}} = 0.01$. Observe that at low velocities, in the regime where the PN approximation is valid, the two curves agree.

which shows that this is a relative 7PN effect. In contrast with the cases of even-parity and odd-parity with spins, this effect is dominantly controlled by the total mass, rather than the mass ratio. The effect is greatest for a system of stellar-mass BHs.

The above result can be compared to numerical calculations recently performed by Pani et al. [65]. They estimated the effect of scalar radiation in dynamical CS gravity [38] for non-spinning, circular EMRIs. They numerically solved the master perturbation equations on a Schwarzschild background to obtain the time evolution of the scalar field and the metric perturbation, caused by a non-spinning point particle. Figure 3 compares their results to ours, found in Eq. (141). Observe that the numerical results of Pani et al. are in excellent agreement with our post-Newtonian calculation, which extends it to comparable mass-ratios (notice the factor of $\delta m/m$).

B. Metric deformation correction to the energy flux

For the even-parity case, the correction to the energy flux that arises from the deformation to the gravitational metric perturbation is at least of 0PN order relative to GR. This is higher PN order compared to the scalar dipole radiation found in Sec. VI A 1, and thus, we will not consider it further.

For the odd-parity case with spinning BHs, one of the leading contribution comes from the metric correction sourced by $T_{ij}^{(\vartheta)}$, which is given in Eq. (118). Inserting this metric perturbation into Eq. (130), the energy flux

correction relative to GR becomes

$$\frac{\delta \dot{E}^{(h)}}{\dot{E}^{\rm GR}} = \frac{75}{16}\frac{\zeta_4}{\eta}\chi_1\chi_2 \left\langle \hat{S}_1^i \hat{S}_2^j \left(2\hat{v}_{ij}^{12} - 3n_{<ij>}^{12}\right)\right\rangle_\omega v^4 ,$$
(142)

which is of relative 2PN order, just as the contribution due to scalar radiation in Eq. (139). Notice that both the metric deformation and scalar field corrections to the energy flux are of $\mathcal{O}(\chi^2)$, but the latter is larger by a factor of $\mathcal{O}(\eta^{-1})$.

We expect $\mathcal{O}(\chi)$ corrections to the energy flux due to the metric deformation to be higher PN order. For very slowly spinning binaries, however, they may give larger corrections compared to the $\mathcal{O}(\chi^2)$ 2PN ones presented here.

In the odd-parity sector with non-spinning objects, the regularized contributions to the metric deformation can only provide energy flux corrections of at least 7PN order. However, as explained in Sec. V C, we expect that matching strong-field solutions to the non-regular NZ ones may generate 6PN corrections in the energy flux, similar to those found by Pani et al. [65].

VII. IMPACT ON GRAVITATIONAL WAVE PHASE

How do all these modifications to the energy flux affect the GW observable? To answer this question, we compute the Fourier transform of the phase of the GW response function in the stationary phase approximation (SPA), where we assume the GW phase changes much more rapidly than the GW amplitude [66].

We begin by parameterizing all the corrections to the energy flux that we have studied so far via the following power law:

$$\dot{E} = \dot{E}_{\rm GR}(1 + Av^a) ,$$
(143)

where (A,a) are summarized in Table II for the four different sectors considered.

With the generic energy flux parameterization, the orbital phase for a quasi-circular inspiral becomes

$$\phi(F) = \int^F \frac{dE}{d\omega}\left(\frac{dE}{dt}\right)^{-1}\omega d\omega$$
$$= \phi_{\rm GR}(F)\left[1 + \frac{5}{a-5}A(2\pi mF)^{a/3}\right] , \quad (144)$$

where F and $\omega = 2\pi F$ are the linear and angular orbital frequency, $\phi_{\rm GR} = -1/(32\eta)(2\pi mF)^{-5/3}$ is the GR orbital phase and $E(\omega) = -(\mu/2)(m\omega)^{2/3}$ is the binary's binding energy to Newtonian order. Recall here that $m = m_1 + m_2$ is the total mass of the binary, while $\mu = m_1 m_2/m$ is the reduced mass and $\eta = \mu/m$ is the symmetric mass ratio. Equation (144) is not valid when $a = 5$ (a 2.5PN correction), as then the integrand becomes proportional to ω^{-1}, which leads to a log term.

Before we compute the Fourier phase, we must first define t_0, the time at which the stationary phase condition is satisfied $F(t_0) = f/2$, where f is the GW frequency. This condition can be solved to yield

$$t_0 = t_{0,\rm GR}\left(1 - \frac{8}{8-a}A(\pi mf)^{a/3}\right) , \quad (145)$$

where $t_{0,\rm GR}$ is the GR t_0. Again, this expression is not valid at $a = 8$, because once more the correction to $t_0(f)$ would be a log term.

With this at hand, we can now compute the Fourier phase in the SPA:

$$\Psi_{\rm GW} = 2\phi(t_0) - 2\pi ft_0$$
$$= \Psi_{\rm GR}\left[1 - \frac{40}{(a-5)(a-8)}A\eta^{-a/5}(\pi\mathcal{M}f)^{a/3}\right] , \quad (146)$$

where $\Psi_{\rm GR} \equiv (3/128)(\pi\mathcal{M}f)^{-5/3}$, and where $\mathcal{M} = \eta^{3/5}m$ is the chirp mass. Again, these expressions are not valid when $a = 5$ or $a = 8$, for the reasons described above.

The corrections to the GW phase found here map directly to the parameterized post-Einsteinian (ppE) framework [31]. In that framework, one postulates that modified gravity theories affect the Fourier phase of the GW response function in the SPA via

$$\Psi_{\rm GW}^{\rm ppE} = \Psi_{\rm GR} + \beta_{\rm ppE}\left(\pi\mathcal{M}f\right)^{b_{\rm ppE}} , \quad (147)$$

where $(\beta_{\rm ppE}, b_{\rm ppE})$ are ppE parameters. We see that this is identical to the corrections introduced by a change in the energy flux, with the mapping

$$\beta_{\rm ppE} = -\frac{15}{16}\frac{A}{(a-5)(a-8)}\eta^{-a/5} , \qquad b_{\rm ppE} = \frac{a-5}{3} . \quad (148)$$

This is not surprising, as the ppE framework was in part motivated by studying power-law (in velocity) modifications to the energy flux and the binding energy [31].

We have then found that a large number of energy flux corrections associated with extra gravitational and scalar field emissions can be mapped to the ppE framework. In the even parity case, the leading-order frequency exponent $b_{\rm ppE} = -7/3$, while in the odd-parity case $b_{\rm ppE} = -1/3$, unless the binary is non-spinning in which case $b_{\rm ppE} = +3$.

The results found in this paper could help in the generalization of the ppE framework to more generic quasi-circular inspirals. The original framework considered only non-spinning, equal mass inspirals, while recently Cornish et al. [32] generalized it to non-spinning, unequal mass systems through $A \to A\eta^c$. In this paper we have found that A does not only depend on a simple power law of η, but also on the mass difference $\delta m/m = \sqrt{1 - 4\eta}$ and on combinations of the spins. For single detections, however, such a generalization is not needed as one only

Sector	A	a
Even-Parity	$\frac{5}{96}\zeta_3 \frac{1}{\eta^4} \frac{\delta m^2}{m^2}$	-2
Odd-P, Spins (ϑ),	$\frac{25}{1536}\zeta_4 \frac{1}{\eta^2}\left[\bar{\Delta}^2 + 2\left\langle(\bar{\Delta}\cdot\hat{v}_{12})^2\right\rangle_\omega\right]$	$+4$
Odd-P, Spins (h),	$\frac{75}{16}\zeta_4 \frac{1}{\eta}\langle S_1^i S_2^j \left(2\hat{v}_{ij}^{12} - 3n_{<ij>}^{12}\right)\rangle_\omega$	$+4$
Odd-P, No Spin	$\frac{2}{3}\zeta_4 \frac{\delta m^2}{m^2}$	$+14$

TABLE II: Coefficients of the relative energy flux.

measures a single number, $\beta_{\rm ppE}$, and one cannot extract the dependencies on η, $\delta m/m$, and the spins.

Although we currently lack any GW detections, we can still estimate the projected constraints that such detections would place on quadratic gravity. According to Table II, the even-parity sector leads to the strongest deviations from GR, since a is the most negative. Therefore, we consider EDGB theory, $(\alpha_1, \alpha_2, \alpha_3, \beta) = (1, -4, 1, \alpha_{\rm EDGB}^{-1})\alpha_{\rm EDGB}$, as a simple sub-case of the even-parity sector. Let us first imagine that we have detected a GW with Ad. LIGO and signal-to-noise ratio (SNR) of 20 that is consistent with GR and that originates from a non-spinning BH binary with masses $(m_1, m_2) = (6, 12)M_\odot$. Given such a detection, Cornish *et al.* [32] estimated the projected bound $|\beta_{\rm ppE}| \lesssim 5 \times 10^{-4}$ for $b_{\rm ppE} = -\frac{7}{3}$, which implies $|\alpha_{\rm EDGB}|^{1/2} \lesssim 4 \times 10^5$ cm. Let us now assume that we have detected a GW with LISA classic with and SNR of 879 and still consistent with GR, but that originates from a non-spinning BH binary with masses $(m_1, m_2) = (10^6, 3 \times 10^6)M_\odot$ at $z = 1$. Given such a detection, Cornish *et al.* [32] estimated a bound on $|\beta_{\rm ppE}| \lesssim 10^{-6}$ for the same value of $b_{\rm ppE}$ as before, which leads to $\alpha_{\rm EDGB}^{1/2} \lesssim 10^{10}$ cm. In both cases, notice that these projected bounds are consistent with the small-coupling requirement $\zeta_i \ll 1$; i.e. saturating the projected Ad. LIGO and LISA constraints we have $\zeta_{\rm Ad.\ LIGO} \sim 3 \times 10^{-2}$ and $\zeta_{\rm LISA} \sim 10^{-5}$ for those particular binary systems, which is clearly much less than unity.

Comparing these results with the current constraint obtained by the Cassini satellite, $|\alpha_{\rm EDGB}|^{1/2} < 1.3 \times 10^{12}$cm [35], we see that Ad. LIGO and LISA could constrain $\alpha_{\rm EDGB}$ much more strongly. Unfortunately, it seems difficult to put constraints on EDGB with binary pulsar observations, since NSs have no scalar monopole charge in this theory. We emphasize again that this is opposite to the expectation from scalar-tensor theories, in which NSs have scalar monopole charges while BHs do not. Finally, one cannot estimate the bounds one could place on dynamical CS gravity, since one would have to properly account for modifications to the conservative equations of motion, which we have not calculated here.

VIII. CONCLUSIONS AND DISCUSSIONS

We have studied the binary inspiral problem in a wide class of quadratic gravity theories in the slow-motion, weak-gravity regime. The structure of a compact object in such theories affects the exterior scalar field sourced by the object. Despite this, we can model a compact object by an effective scalar field source characterized by its scalar monopole and dipole moments. The scalar monopole charge is enhanced inversely proportional to the mass of the object, while the dipole charge is independent of the mass for a fixed dimensionless spin parameter. With this effective source, we then derived and solved the modified field equations for the scalar field and metric deformation.

We find that the scalar field generically emits dipole radiation in the even-parity sector, and quadrupole radiation in the odd-parity sector. Such radiation affects the rate of change of the binary energy at relative -1PN order in the even-parity case and relative 2PN order in the odd-parity case. The quadrupole contribution depends quadratically on the BH spins, and thus it is suppressed for non-spinning binaries. In that case, the odd-parity contribution becomes of relative 7PN order, as found numerically in [65]. We have found excellent agreement between their numerical results and our analytical calculations.

We have also calculated the metric perturbation in the FZ and its associated energy flux. In the even-parity sector, the dominant metric contribution leads to a 0PN relative correction in the energy flux, which is smaller than the -1PN correction induced by scalar dipolar radiation. In the odd-parity sector and for spinning BHs, the metric perturbation leads to a 2PN modification to the energy flux, which is of the same order as that induced by quadrupolar scalar radiation. In the odd-parity sector and for non-spinning BHs, we expect the energy flux correction due to the metric deformation is suppressed to at least of 6PN order, as found by Pani *et al.* [65].

Whether these corrections can be measured or constrained depends on whether they are degenerate with GR terms in the physical observable, i.e. the waveform. A -1PN effect cannot be degenerate, as there are no such terms predicted in GR. A 2PN effect, however, could be degenerate with a spin-spin interaction for quasi-circular inspirals with aligned or counter-aligned spin components. That is, a renormalization of the spin magnitudes of both bodies can eliminate this 2PN effect, assuming one truncates the waveform at that order. If higher-order PN waveforms are used, or if the orbit is more generic (i.e. if there is precession or eccentricity), then this degeneracy can be broken.

We also calculated the effects of such energy flux modifications on the gravitational waveform. The waveform phase depends sensitively on the rate of change of the orbital frequency, which in turn is governed by the rate of change of energy. We calculated the corrections that would be induced in the waveform and mapped them

to the ppE framework. We then used a recent ppE study [32] to estimate the constraints that Ad. LIGO and LISA could potentially place on quadratic gravity theories. Given a GW detection, we found that the magnitude of the new length scale introduced by quadratic gravity theories (associated with a ratio of their coupling constants) could constrain at a level controlled by the smallest length-scale probed in the inspirals, i.e. the size of the smallest compact object's event horizon or surface. The best projected bounds achievable with Ad. LIGO will thus come from stellar-mass BH or NS inspirals, while LISA will benefit the most from EMRIs. Since NSs have no scalar monopole charge in EDGB theory, this theory cannot be constrained from binary pulsar observations. This property is diametrically opposite to scalar-tensor theories where BHs have no hair.

There are several possible avenues for future work. Since we here mainly considered corrections due to the dissipative sector of the theory, one possibility is to calculate the non-dissipative corrections that would modify the binding energy (here, we mean both gravitational and scalar binding energy) and the equations of motion. There are two effects that should be accounted for: new scalar-scalar forces and metric deformations. Let us consider the former first. In the even parity case, compact objects have an associated scalar monopole charge, and thus, there is an additional scalar force with a $1/r$ potential that should lead to a relative 0PN non-dissipative correction. Similarly, in the odd-parity case, a spinning compact body possesses a current dipole charge, and hence, dipole-dipole interactions should arise. Since the dipole potential is proportional to $1/r^2$, while the dipole charge couples to the first derivative of the potential, the binding energy and the equations of motion should be corrected at relative 2PN order.

Another non-dissipative modification is induced by deformations of the background metric tensor. In the even-parity sector, such corrections enter at relative 0PN order, as found by Yunes and Stein [28]. In the odd-parity sector, there is no metric deformation for isolated non-spinning BHs, but for spinning ones there is a correction proportional to r^{-4} to the (t, i) components [6], which then leads to a 4.5PN correction in the equations of motion when we consider boosted BHs. This then implies the following: (i) in the even-parity case, the conservative corrections to the equations of motion do not affect the leading-order modification to the waveforms, since this is dominated by the -1PN scalar radiation effect; (ii) in the odd-parity case, the conservative corrections from the metric deformation can be neglected, but those due to the scalar-scalar force will contribute at the same order as the effect calculated here. A complete analysis of the waveform observable would thus require the calculation of such a scalar-scalar, conservative effect.

Another possibility could be to study modified quadratic gravity in the context of BH perturbation theory. This would be a tremendous effort that would have to be split into separate parts. First, one would have to find an analytic, strong-field solution for arbitrarily-fast rotating BHs in quadratic gravity. This has only been found in the slow-rotation limit both in the even-parity [28] and odd-parity sectors [6]. Once this is accomplished, one would have to study the evolution of metric perturbations away from this solution. Such evolution equations would have to be decoupled in terms of some master function to derive Teukolsky-like master equations. Finally, with these equations at hand, one would have to solve them numerically, when the perturbations are sourced by a small object in a tight orbit. Such an analysis would be interesting because one would be able to derive not only the corrections to the energy flux carried out to infinity, but also that which is absorbed by the BH horizons and which we ignored in this paper.

A final follow-up would be to study how NS solutions are modified in quadratic gravity [30, 67] and how the energy flux from NS binaries is modified. This could then lead to direct constraints on quadratic gravity theories from double binary pulsar observations. Such constraints could be stronger , relative to current Solar System constraints, as they could potentially provide constraints of roughly the order of magnitude of the NS radius. Of course, in the case of EDGB theory or dynamical CS gravity, these constraints might not be stronger as NSs have no scalar monopole charge in such theories.

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Appendix A: The Balding of Neutron Stars in EDGB Gravity

In this appendix, we consider the scalar field equation in EDGB gravity for isolated NSs. Integrating the evolution equation, we find

$$\int \sqrt{-g}\,\Box\vartheta\, d^4x \propto \int \sqrt{-g}\,\mathcal{R}^2_{\mathrm{GB}}\, d^4x \,, \qquad (A1)$$

where we have defined the Gauss-Bonnet invariant $\mathcal{R}^2_{\mathrm{GB}} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\delta\sigma}R^{\mu\nu\delta\sigma}$. Since the Gauss-Bonnet combination is a topological invariant, the right-hand side identically vanishes for any simply-connected, asymptotically flat spacetime. Moreover, since we are considering isolated NSs, these must be stationary, and so the time integration can be removed.

With all of this and using Stokes' theorem, Eq. (A1) becomes

$$\int \sqrt{-g}(\partial_i\vartheta) n^i dS = \int \sqrt{-g}(\partial_r\vartheta) dS = 0 \,, \qquad (A2)$$

where n^i is the radial unit vector and the integral is performed over the 2-sphere at spatial infinity. Notice that $\sqrt{-g} \sim r^2$, while the scalar field must decay at infinity for it to have a finite energy.

Equation (A2) does not vanish at spatial infinity for all scalar field solutions, i.e. if we model $\vartheta = \vartheta_n/r$ with ϑ_n a constant, then Eq. (A2) leads to the unique solution $\vartheta_n = 0$. This is a physicists's proof that the EDGB scalar field cannot have scalar monopole charge for a spherically symmetric NS. Similarly, one can show that NSs cannot have scalar monopole charge in dynamical CS gravity; the proof laid out above carries through with the replacement $\mathcal{R}^2_{\mathrm{GB}} \to {}^*R\,R$, since ${}^*R\,R$ is also a topological invariant.

Appendix B: Integration techniques

In this appendix, we provide some useful integration techniques. When computing near-zone integrals, we are faced many times with integrals of the form

$$\int d^3x \frac{x_{\langle L\rangle}}{r_1 r_2} \,. \qquad (B1)$$

When the point-particle approximation is valid, such near-zone integrals can be Hadamard regularized by keeping only the finite part. Let us then define [68]

$$Y_{\langle L\rangle}(\boldsymbol{x}_1, \boldsymbol{x}_2) = -\frac{1}{2\pi}\,\mathrm{FP}_{B=0}\int d^3x\, |\tilde{\boldsymbol{x}}|^B \frac{x_{\langle L\rangle}}{r_1 r_2} \,, \qquad (B2)$$

to be evaluated in the near-zone and where $\mathrm{FP}_{B=0}$ stands for the finite part operator (in the limit $B \to 0$) and $|\tilde{\boldsymbol{x}}|$

is an analytic continuation factor [68]. The solution to this integral is

$$Y_{\langle L\rangle} = \frac{b}{l+1}\sum_{q=0}^{l} x_1^{\langle L-Q} x_2^{Q\rangle} \,. \qquad (B3)$$

The first few $Y_{\langle L\rangle}$ are simply

$$Y_0 = Y = b \,, \qquad Y_i = \frac{b}{2}\left(x_1^i + x_2^i\right) \,, \qquad (B4)$$

$$Y_{\langle ij\rangle} = \frac{b}{3}\left(x_1^{\langle ij\rangle} + x_1^{\langle i}x_2^{j\rangle} + x_2^{\langle ij\rangle}\right) \,, \qquad (B5)$$

$$Y_{\langle ijk\rangle} = \frac{b}{4}\left(x_1^{\langle ijk\rangle} + x_1^{\langle ij}x_2^{k\rangle} + x_1^{\langle i}x_2^{jk\rangle} + x_2^{\langle ijk\rangle}\right) \,. \qquad (B6)$$

The solution to the $Y_{\langle L\rangle}$ integral can also be derived by using certain Poisson integral identities [63]:

$$P\left(f_{,i}g_{,i}\right) = -\frac{1}{2}\left[fg + P\left(fg_{,ii}\right) + P\left(gf_{,ii}\right) - \mathcal{B}_p(fg)\right] \,, \qquad (B7)$$

where we have defined

$$P(f) \equiv \frac{1}{4\pi}\int_{\mathcal{M}} \frac{f(t,x')}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x' \,, \qquad (B8)$$

and the boundary term is

$$\mathcal{B}_p(g) \equiv \frac{1}{4\pi}\oint_{\partial\mathcal{R}} \left[\frac{g(t,x')}{|\boldsymbol{x}-\boldsymbol{x}'|}\partial'_r \ln\left[g(t,x')|\boldsymbol{x}-\boldsymbol{x}'|\right]\right]_{r'=\mathcal{R}} \mathcal{R}^2 d\Omega' \,. \qquad (B9)$$

As usual, we retain only those terms that are independent of the boundary $\mathcal{R}$.

Finally, there is yet another type of integral that commonly appears in near-zone integration:

$$\int_{\mathcal{M}} \frac{d^3x'}{|\boldsymbol{x}'-\boldsymbol{x}_1||\boldsymbol{x}'-\boldsymbol{x}_2||\boldsymbol{x}'-\boldsymbol{x}|} \,. \qquad (B10)$$

Let us then define the so-called *triangle* potential [69]

$$\mathcal{G}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) \equiv \frac{1}{4\pi}\int_{\mathcal{M}} \frac{d^3x'}{|\boldsymbol{x}'-\boldsymbol{x}_1||\boldsymbol{x}'-\boldsymbol{x}_2||\boldsymbol{x}'-\boldsymbol{x}_3|} \,. \qquad (B11)$$

It is a bit of a miracle that the above integral has the closed-form solution $\mathcal{G}(\boldsymbol{x}_A, \boldsymbol{x}_B, \boldsymbol{x}_C) = 1 - \ln\Delta(ABC)$, with $\Delta(ABC) \equiv |\boldsymbol{x}_A-\boldsymbol{x}_B| + |\boldsymbol{x}_B-\boldsymbol{x}_C| + |\boldsymbol{x}_C-\boldsymbol{x}_A|$.

One can show that the triangle potential satisfies a set of relations, including [69]

$$\partial_i^{(1)}\partial_i^{(2)}\mathcal{G}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}) = \frac{1}{2}\left[\frac{1}{b}\left(\frac{1}{r_1}+\frac{1}{r_2}\right) - \frac{1}{r_1 r_2}\right] \,,$$

$$\partial_{il}^{(1)}\partial_{jl}^{(2)}\mathcal{G}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}) = -\frac{1}{2}\left[\frac{n_1^i n_2^j}{r_1^2 r_2^2} + \frac{n_{12}^i n_2^j}{b^2 r_2^2} - \frac{n_{12}^j n_1^i}{b^2 r_1^2}\right.$$

$$\left. + 3\frac{n_{12}^{\langle ij\rangle}}{b^3}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\right] \,, \qquad (B12)$$

and more generally

$$\partial_i^{(B)} \partial_j^{(C)} \mathcal{G}(ABC) = \frac{1}{\Delta(ABC)^2}(n_{AB}^i - n_{BC}^i)(n_{AC}^j + n_{BC}^j) + \frac{1}{r_{BC}\Delta(ABC)}(\delta_{ij} - n_{BC}^i n_{BC}^j),$$
(B13)

where $\mathcal{G}(ABC) \equiv \mathcal{G}(\boldsymbol{x}_A, \boldsymbol{x}_B, \boldsymbol{x}_C)$.

Appendix C: Odd-Parity, Non-Spinning, Regularized Contribution in the Metric Correction

We consider here the odd-parity sector for non-spinning binaries, where, for the scalar field, the magnetic-type dipole moment vanishes, $\mu_A^i = 0$, since $\chi_A = 0$. For the regularized contribution, we only need to consider the cross-interaction terms since the isolated non-spinning BH solution in the odd-parity case is simply the Schwarzschild metric. The $\tilde{K}_{ij}^{(1)}$ source term gives the largest contribution and one is then left only with the pseudo-scalar generated by interaction terms, as given in Eq. (61).

The metric deformation is given by Eq. (95), the $m = 0$ piece of which can be split as in Eqs. (110)-(113). Before tackling each of these terms separately, let us point out that many of them identically vanish. For example, one of the contribution in Eq. (110) is proportional to

$$\begin{aligned}
I_{ijqn} \equiv\; & m_1 \epsilon_{jkl} \int_{\mathcal{M}} \partial_{qnk}^{(1)}\left(\frac{1}{r_1}\right) \partial_{il}^{(1)}\left(\frac{1}{r_1}\right) d^3x \\
& + m_2 \epsilon_{jkl} \int_{\mathcal{M}} \partial_{qnk}^{(1)}\left(\frac{1}{r_1}\right) \partial_{il}^{(2)}\left(\frac{1}{r_2}\right) d^3x \\
=\; & -2\pi m_1 \epsilon_{jkl} \lim_{2\to 1} \partial_{qnk}^{(1)} \partial_{il}^{(2)} Y(\boldsymbol{x}_1, \boldsymbol{x}_2) \\
& -2\pi m_2 \epsilon_{jkl} \partial_{qnk}^{(1)} \partial_{il}^{(2)} Y(\boldsymbol{x}_1, \boldsymbol{x}_2) = 0\,.
\end{aligned}$$
(C1)

It is critical in this calculation and in the calculations that follow to replace the x^i derivatives by particles derivatives, i.e. derivatives with respect to x_1^i and x_2^i.

Let us then tackle the first contribution to the dissipative metric deformation. Equations (110)-(113) can then

be rewritten as

$$\begin{aligned}
\mathfrak{h}_{ij}^{(1)} =\; & 2048\pi \frac{\alpha_4^2}{\beta} \frac{m_1^2 m_2}{r} \Big[b\omega^2 \left(I_{1ij} + I_{2ij}\right) \\
& - v_{1n}(I_{3ijn} + I_{4ijn}) - v_{2n}(I_{5ijn} + I_{6ijn}) \\
& + (i \leftrightarrow j) \Big] + (1 \leftrightarrow 2)\,,
\end{aligned}$$
(C2)

$$\begin{aligned}
\mathfrak{h}_{ij}^{(2)} =\; & -4096\pi \frac{\alpha_4^2}{\beta} \frac{m_1^2 m_2}{r} v_{1[n} \Big[I_{3i]jn} + I_{4i]jn} \\
& + I_{5i]jn} + I_{6i]jn} + (i \leftrightarrow j) \Big] + (1 \leftrightarrow 2)\,,
\end{aligned}$$
(C3)

$$\begin{aligned}
\mathfrak{h}_{ij}^{(3)} =\; & 4096\pi \frac{\alpha_4^2}{\beta} \frac{m_1^2 m_2}{r} v_{1[n} \\
& \times \Big[I_{7i]jn} + I_{8i]jn} + (i \leftrightarrow j) \Big] + (1 \leftrightarrow 2)\,,
\end{aligned}$$
(C4)

$$\begin{aligned}
\mathfrak{h}_{ij}^{(4)} =\; & -2048\pi \frac{\alpha_4^2}{\beta} \frac{m_1^2 m_2}{r} v_{1n} \\
& \times \Big[I_{7ijn} + I_{8ijn} + (i \leftrightarrow j) \Big] + (1 \leftrightarrow 2)\,,
\end{aligned}$$
(C5)

where we have defined

$$\begin{aligned}
I_{1ij} &\equiv \epsilon_{jkl}\epsilon_{pqs} n_{12s} J_{pk,q,il}^{(1)}\,, \\
I_{2ij} &\equiv \epsilon_{jkl}\epsilon_{pqs} n_{12s} J_{p,qk,il}^{(1)}\,, \\
I_{3ijn} &\equiv \epsilon_{jkl}\epsilon_{pqs} v_{12s} J_{pkn,q,il}^{(1)}\,, \\
I_{4ijn} &\equiv \epsilon_{jkl}\epsilon_{pqs} v_{12s} J_{pn,qk,il}^{(1)}\,, \\
I_{5ijn} &\equiv \epsilon_{jkl}\epsilon_{pqs} v_{12s} J_{pk,qn,il}^{(1)}\,, \\
I_{6ijn} &\equiv \epsilon_{jkl}\epsilon_{pqs} v_{12s} J_{p,qkn,il}^{(1)}\,, \\
I_{7ijn} &\equiv \epsilon_{jkl}\epsilon_{pqs} v_{12s} J_{pk,q,iln}^{(1)}\,, \\
I_{8ijn} &\equiv \epsilon_{jkl}\epsilon_{pqs} v_{12s} J_{p,qk,iln}^{(1)}\,,
\end{aligned}$$
(C6)

and

$$J_{A,B,C}^{(p)} = \lim_{3\to p} \partial_A^{(1)} \partial_B^{(2)} \partial_C^{(3)} \mathcal{G}(ABC)\,,$$
(C7)

with A, B, C denoting the multi-index lists. We provide a more detailed discussion of J tensors in Appendix D. One can then show through explicit computation that the two terms combine to give $I_{1ij} + I_{2ij} = 0$, $I_{3ijn} + I_{4ijn} = 0$, $I_{5ijn} + I_{6ijn} = 0$, and $I_{7ijn} + I_{8ijn} = 0$. Therefore $\mathfrak{h}_{ij}^{(1\cdots4)} = 0$ at leading order.

Let us now look at contributions that are smaller by $\mathcal{O}(v)$. Such a correction can arise from two different terms: (i) the $\mathcal{O}(v)$ correction to the source term with $m = 0$ in the sum of Eq. (95), or (ii) the $\mathcal{O}(v^0)$ correction to the source term with $m = 1$ in the sum of Eq. (95). For case (i), the next-order terms consist of two time derivatives and one factor of h_{0i} (or three time derivatives and one factor of h_{ij}), which when combined are $\mathcal{O}(v^2)$ smaller than the $\mathcal{O}(v^0)$ contribution shown to vanish previously. Also, the next-order terms in the PN metric appears at $\mathcal{O}(v^2)$ higher relative to the leading-order terms. Finally, ϑ^{NZ} in Eq. (55) expanded as in Eq. (2.27)

of [62] with $m = 1$ in the sum, gives an $\mathcal{O}(v)$ relative contribution to $\partial_k \vartheta$, but explicit calculation shows that

$$
\begin{aligned}
\vartheta^{\text{NZ}} &= \frac{8}{\pi} \frac{\alpha}{\beta} m_1 m_2 \epsilon_{ijk} \frac{\partial}{\partial t} \left[v_{12k} \int_{\mathcal{M}} \left(\frac{1}{r_1} \right)_{,il} \left(\frac{1}{r_2} \right)_{,jl} d^3 x \right] \\
&= \frac{8}{\pi} \frac{\alpha}{\beta} m_1 m_2 \epsilon_{ijk} \frac{\partial}{\partial t} \left[v_{12k} \partial_{il}^{(1)} \partial_{jl}^{(2)} \int_{\mathcal{M}} \frac{1}{r_1} \frac{1}{r_2} d^3 x \right] \\
&= 16 \frac{\alpha}{\beta} m_1 m_2 \epsilon_{ijk} \frac{\partial}{\partial t} \left[v_{12k} \partial_l^{(1)} \partial_{ijl}^{(2)} b \right] = 0 \, . \qquad \text{(C8)}
\end{aligned}
$$

For case (ii), the resulting $\dot{\mathfrak{h}}_{ij}$ contains one n^i vector. The correction to the energy flux consists of $\dot{\mathfrak{h}}_{ij}$ multiplied by h_{ij}^{TT} and averaged over a 2-sphere. However, since the leading contribution in h_{ij}^{TT} contains even numbers of n^i vectors, the correction only contains angular integrals of odd numbers of n^i's which vanish exactly upon integration.

Since there is no $\mathcal{O}(G^3, v)$ relative contribution to $\partial_t \mathfrak{h}_{ij}$, the first, non-vanishing contribution must be at least $\mathcal{O}(v^2)$ smaller than what we computed in Eqs. (C2)-(C5), which amounts to a 7PN correction to the energy flux carried by the metric deformation, in the odd-parity, non-spinning case.

Appendix D: Evaluating J tensors

Recall that the definition of the J tensors is

$$
J_{A,B,C}^{(p)} = \lim_{3 \to p} \partial_A^{(1)} \partial_B^{(2)} \partial_C^{(3)} \mathcal{G}(ABC) \, . \qquad \text{(D1)}
$$

The limit $3 \to p$ which appears must be taken with care. There may be terms proportional to

$$
\lim_{3 \to p} \frac{1}{r_{p3}} \, , \qquad \text{(D2)}
$$

which have no finite part. In the evaluation of the J tensors, only the finite part of the limit is kept. That is, a function can be expanded as a Laurent series about these points, and the finite part scales as $(r_{p3})^0$ in the limit as $3 \to p$.

Another type of problematic limit is

$$
\lim_{3 \to p} n_{p3}^i \quad \text{or} \quad \lim_{3 \to p} n_{p3}^{ij} \, , \qquad \text{(D3)}
$$

which does not formally exist, since it depends on the path taken as we describe below. Parameterize the path that particle 3 takes to the location of particle p by the continuously differentiable path $\gamma(\lambda)$, with λ a parameter of path length and $\lambda = 0$ the location of particle p. There are an infinite number of paths one could choose, and each can be parameterized in two senses. Taking the limit along this path "from below" (i.e. from smaller values of λ to larger values) yields

$$
\lim_{3 \to p, \, \gamma^-} n_{p3}^i \to -\hat{v}_\gamma^i(0) \, , \qquad \text{(D4)}
$$

where $\hat{v}_\gamma$ is the tangent vector to the curve γ. Taking the limit from above, we find

$$
\lim_{3 \to p, \, \gamma^+} n_{p3}^i \to +\hat{v}_\gamma^i(0) \, . \qquad \text{(D5)}
$$

The limit depends on the path's tangent at the point of particle p, and the direction in which the limit is taken. Clearly, the final answer must be unique, which implies the limit must vanish.

A unique prescription to this problem is formalized as Hadamard regularization [70]. This can be summarized as follows. All possible paths are considered, with tangent vectors $\hat{v}_\gamma$. The average is then taken by integrating, e.g.

$$
\lim_{3 \to p} \cdots n_{p3}^{ij} \cdots = \int \frac{d\Omega(\hat{v}_\gamma)}{4\pi} \cdots \hat{v}_\gamma^{ij} \cdots \, . \qquad \text{(D6)}
$$

The first few such limits, for example, are

$$
\lim_{3 \to p} n_{p3}^i = 0 \, , \qquad \text{(D7)}
$$

$$
\lim_{3 \to p} n_{p3}^{ij} = \frac{1}{3} \delta^{ij} \, . \qquad \text{(D8)}
$$

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