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On Multicarrier Signals Where the PMEPR of a Random Codeword is Asymptotically $\log n$

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***Abstract*—Multicarrier signals exhibit a large peak-to-mean envelope power ratio (PMEPR). In this correspondence, without using a Gaussian assumption, we derive lower and upper probability bounds for the PMEPR distribution when the number of subcarriers n is large. Even though the worst case PMEPR is of the order of n, the main result is that the PMEPR of a random codeword $C = (c_1, \ldots, c_n)$ is $\log n$ with probability approaching one asymptotically, for the following three general cases: i) c_i's are independent and identically distributed (i.i.d.) chosen from a complex quadrature amplitude modulation (QAM) constellation in which the real and imaginary part of c_i each has i.i.d. and even distribution (not necessarily uniform), ii) c_i's are i.i.d. chosen from a phase-shift keying (PSK) constellation where the distribution over the constellation points is invariant under $\pi/2$ rotation, and iii) C is chosen uniformly from a complex sphere of dimension n. Based on this result, it is proved that asymptotically, the Varshamov–Gilbert (VG) bound remains the same for codes with PMEPR of less than $\log n$ chosen from QAM/PSK constellations.**

***Index Terms*—Multicarrier signals, orthogonal frequency-division multiplexing (OFDM), peak-to-mean envelope power ratio (PMEPR), spherical codes, symmetric constellations.**

I. Introduction

Multicarrier modulation has been proposed in different broad-band wireless and wireline applications such as wireless local area networks (WLAN) and digital subscriber line (DSL). Even though multicarrier modulation has a nice performance in a multipath fading environment, it suffers from high amplitude variation which is unfavorable from a practical point of view. Different schemes have been proposed to reduce the peak-to-mean envelope power ratio (PMEPR) such as coding methods, clipping, reserved carriers, and probabilistic methods such as selective mapping and partial transmit sequence [1]–[7].

Unfortunately, the worst case PMEPR of multicarrier signals is rather high and is of the order of n where n is the number of subcarriers. On the other hand, the numerical evaluation of the distribution of PMEPR shows that encountering the worst case n is highly unlikely [8]–[13]. This in fact motivates the problem of finding the PMEPR distribution to quantify how severe that is. In [8], [9], by assuming that the multicarrier signal is a Gaussian process, an expression for the probability distribution of PMEPR is derived. This is a very strong assumption, and when the codewords are chosen from fixed constellations, is mathematically not valid for the joint distribution of n or more samples [14]. Recently, in [12], an upper bound for the PMEPR distribution is shown for quadrature amplitude modulation/phase-shift keying (QAM/PSK) with M^2 points and uniform distribution over the constellation points, and it is shown that the probability of encountering a PMEPR of greater than $(1+\epsilon)\log n$ is going to zero as n increases. On the other hand, in [13], using techniques different from ours, a lower bound for the distribution of PMEPR is obtained when codewords are uniformly distributed over a complex sphere. [13], however, does not perform an asymptotic analysis, which is what we do here. In this correspondence, we generalize the results to a larger class of constellations with even distribution over the constellation points, and we show a stronger result, namely, with high probability the PMEPR *behaves* like $\log n + O(\log\log n)$. In other words, encountering a PMEPR of less than $\log n + O(\log\log n)$ is also highly unlikely.

The results are based on a generalization of the well-known result of Halasz [15] for Littlewood trigonometric polynomials with equiprobable coefficients chosen independently from $\{+1, -1\}$ [10], [6], [12]. In summary, we show that, with probability approaching one, any codeword either with entries chosen independently from the symmetric QAM/PSK constellations or chosen uniformly from a complex sphere has PMEPR of $\log n + O(\log\log n)$ for a large number of subcarriers. We then use this result to determine the achievable rate of codes with given minimum distance and bounded PMEPR.

The rest of the correspondence is outlined as follows. Section II introduces the notation, multicarrier signals, and the PMEPR of a codeword. The lower and upper probability bounds for the PMEPR distribution are derived in Section III. In Section IV, we discuss the consequences of the bounds and we obtain a Varshamov–Gilbert (VG) type bound for the achievable rate of codes with bounded PMEPR and with given minimum Hamming distance.

II. Definition

The complex envelope of a multicarrier signal with n subcarriers may be represented as

$$s_C(t) = \sum_{i=1}^{n} c_i e^{j2\pi i f_0 t}, \qquad 0 \le t \le 1/f_0 \tag{1}$$

where f_0 is the subchannel spacing and $C = (c_1, \ldots, c_n)$ is the complex modulating vector with entries from a given complex constellation. The admissible modulating vectors are called codewords and the ensemble of all possible codewords constitute the code $\mathcal{C}$. For mathematical convenience, we define the normalized complex envelope of a multicarrier signal as

$$s_C(\theta) = \sum_{i=1}^{n} c_i e^{j\theta i}, \qquad 0 \le \theta < 2\pi. \tag{2}$$

Then, the PMEPR of each codeword C in the code $\mathcal{C}$ may be defined as

$$\text{PMEPR}_{\mathcal{C}}(C) = \max_{0\le\theta<2\pi} \frac{|s_C(\theta)|^2}{E\{\|C\|^2\}}. \tag{3}$$

Similarly, the PMEPR of the code $\mathcal{C}$, denoted by $\text{PMEPR}_{\mathcal{C}}$, is defined as the maximum of (3) over all codewords in $\mathcal{C}$. It is clear from the definition of PMEPR that if all the carriers add up coherently, the PMEPR can be of the order of n.

In this correspondence, we will consider two classes of codes, namely, complex symmetric q-ary codes in which each coordi-

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nate is chosen from a complex QAM or PSK constellation with alphabets of cardinality q, and spherical codes in which codewords are points on a complex n-dimensional sphere defined as $\Omega_n = \{(c_1,\ldots,c_n) : \sum_{i=1}^n |c_i|^2 = n\}$.

It is worth noting that for a random q-ary code with independent and identically distributed (i.i.d.) entries chosen from a constellation $E\{\|C\|^2\} = nE_{\rm av}$, where $E_{\rm av}$ is the average energy of the constellation. Also, for spherical codes chosen from Ω_n, $E\{\|C\|^2\} = n$ since all the codewords have constant norm.

Throughout the correspondence we will use the following notations: $\mathcal{C}$ and C represent the code and codeword, c_i denotes the ith coordinate of the modulating vector C, $\log\{\cdot\}$ is the natural logarithm, and $H_q(x) = -x\log_q(x) - (1-x)\log_q(x)$. We use α and β as arbitrarily constants and $f(n) = O(g(n))$ denotes that

$$\lim_{n\to\infty}\left|\frac{f(n)}{g(n)}\right| \le |\alpha|.$$

III. Bounds for Cumulative Distribution Function of PMEPR

It is a commonly assumed in the literature that when the c_i's in (2) are independently chosen, $s_C(\theta)$ can be approximated as a Gaussian process [8], [9]. However, this is not mathematically rigorous for spherical codewords and codewords with entries from a QAM/PSK constellation. In other words, by assuming that the c_i's are i.i.d., even though it is conceivable that any finite samples of $s_C(\theta)$ is jointly Gaussian for large values of n, this statement is not valid for n samples of $s_C(\theta)$.

In this section, without using any Gaussian assumption, we derive upper and lower bounds for the PMEPR distribution for different schemes. The derivation of the bounds are the generalization of a result of Halasz [15] for the asymptotic distribution of the maximum of $|\sum_{i=1}^n a_i \cos i\theta|$ when a_i's are chosen independently from $\{+1,-1\}$ with equal probability. This result is extended in [16] to the maximum of the modulus of polynomials over the unit circle[1] with real independent coefficients c_k and with characteristic function

$$E\{e^{jtc_k}\} = e^{-\alpha_2 t^2 + \alpha_3 t^3 + O(t^4)}$$

for t in some nontrivial interval $[-d, d]$.

Based on our application for orthogonal frequency-division multiplexing (OFDM) signals, we generalize the result of [16] and [15] to polynomials over the unit circle when its coefficients are chosen from the following three general cases: i) c_i's are i.i.d. chosen from a complex QAM constellation in which the real and imaginary parts of c_i each has i.i.d and even distribution, ii) c_i's are i.i.d. chosen from a PSK constellation where the distribution function over the constellation points is invariant under rotation by $\pi/2$, and iii) when the the modulating codeword C is chosen from a complex n-dimensional sphere in which c_k's are no longer independent.

A. Lower Bound for the PMEPR

In this subsection, we obtain a bound for the probability of having a PMEPR slightly less than $\log n$ and we show that asymptotically this probability goes to zero. Theorem 1 derives the bound for QAM constellations and it is later generalized to PSK and spherical codes in Theorems 2 and 3.

Since scaling the constellation does not affect the PMEPR, for mathematical convenience, we assume the maximum energy of the constellation is one and, therefore, the resulting $E_{\rm av}$ is less than one and denotes the normalized average energy of the constellation.

Theorem 1 (Lower Bound: QAM Case): Let $s_C(\theta)$ be as in (2) where $c_i = a_i + jb_i$ and the a_i's and b_i's each has i.i.d. and even distributions. Also, let $\mathcal{C}_q^{\rm QAM}$ be the ensemble of all the admissible codewords C. Then

$$\Pr\left\{{\rm PMEPR}_{\mathcal{C}_q^{\rm QAM}}(C) \le \log n - 6.5\log\log n\right\} \le O\left(\frac{1}{\log^4 n}\right). \quad (4)$$

Proof: Since we are looking for an upper bound for $\Pr\{{\rm PMEPR}_{\mathcal{C}}(C) \le \lambda\}$, hence a lower bound for $\Pr\{{\rm PMEPR}_{\mathcal{C}}(C) > \lambda\}$, instead of considering the maximum of $s_C(\theta)$ over all θ, we may consider the maximum of $s_C^R(\theta) = {\rm Re}\{s_C(\theta)\}$ over its n samples $\theta_m = \pi(2m+1)/n$ for $m = 1,\ldots,n$. Following the proof of [15], we also define $0 \le u(x) \le 1$ as

$$u(x) = \begin{cases} 0, & |x| \le M \\ 1, & |x| \ge M + \Delta \end{cases} \quad (5)$$

where $\Delta = \sqrt{\frac{n}{\log n}}$ and assume

$$M = \sqrt{nE_{\rm av}\log n - 6.5nE_{\rm av}\log\log n} - \sqrt{\frac{n}{\log n}}.$$

We also assume $u(x)$ to be a function which is 10 times differentiable such that $u^{(r)}(x) = O(\Delta^{-r})$ for $1 \le r \le 10$.[2] Based on these assumptions on $u(x)$, in Appendix A, we proved other properties of $u(x)$ that will be used later in the proof. We then define the random variable η as

$$\eta = \sum_{m=1}^n u({\rm Re}\{s_C(\theta_m)\}) = \sum_{m=1}^n u(s_C^R(\theta_m)) = \sum_{m=1}^n \int_{-\infty}^{\infty} e^{jts_C^R(\theta)} v(t)dt \quad (6)$$

where we replaced $u(x)$ by its Fourier transform $v(t)$. To find a lower bound, we use the following inequalities:

$$\begin{aligned}\Pr\left\{\max_{0\le\theta\le2\pi} |s_C(\theta)| \ge M\right\} &\ge \Pr\left\{\max_{0\le m\le n} |s_C^R(\theta_m)| \ge M\right\} \\ &= 1 - \Pr\{\eta = 0\} \\ &\ge 1 - \Pr\{\eta = 0, \eta \ge 2E\{\eta\}\} \\ &= 1 - \Pr\{|\eta - E\{\eta\}| \ge E\{\eta\}\} \\ &\ge 1 - \frac{\sigma_\eta^2}{E^2\{\eta\}}. \quad (7)\end{aligned}$$

The first equality follows from the definition of η which is zero when $|s_C^R(\theta)|$ is less than M. The second equality follows from the fact that η is a nonnegative random variable, and the last inequality is Chebychev's inequality.

Therefore, the evaluation of the lower bound boils down to the asymptotic analysis of the first and second moments of η. In Appendix B, it is shown that $E\{\eta\} \ge O(\log^6 n)$ and

$$\sigma_\eta^2 = E\{\eta^2\} - E^2\{\eta\} \le O(E\{\eta\}\log^2 n + \log^5 n).$$

Therefore, the above results imply that

$$\Pr\left\{\max_{0\le\theta\le2\pi} |s_C(\theta)| \ge \sqrt{nE_{\rm av}\log n - 6.5nE_{\rm av}\log\log n} - \sqrt{\frac{n}{\log n}}\right\} \ge 1 - O\left(\frac{1}{\log^4 n}\right). \quad (8)$$

Using the definition of PMEPR and normalizing both sides of (8) to $P_{\rm av} = nE_{\rm av}$, the theorem follows. □

[1]By polynomials over the unit circle, here we mean polynomials over the complex field evaluated on the unit circle.

[2]Note that $\boldsymbol{u}^{(r)}(\boldsymbol{x}) = \boldsymbol{O}(\boldsymbol{\Delta}^{-r})$ means that for all $\boldsymbol{x}$,

$$\lim_{n\to\infty}\left|\frac{u^{(r)}(x)}{\Delta^{-r}}\right| \le \alpha$$

which implies that the maximum of $|\boldsymbol{u}^{(r)}(\boldsymbol{x})|$ is less than $\boldsymbol{\alpha\Delta}^{-r}$ for large $\boldsymbol{n}$.

As mentioned in Appendix B, the derivation of the lower bound relies on the characteristic function of $s_C^R(\theta)$. For the PSK case, the real and imaginary parts of c_i are not independent; however, we can still use a similar argument to generalize the result to PSK constellations in which the distribution over the constellation points is invariant under $\pi/2$ rotation.

Theorem 2 (Lower Bound: PSK Case): Let $c_i = e^{j\beta_i}$ where the c_i's i.i.d. chosen from a q-ary PSK constellation in which the probability density function (pdf) of c_i is invariant under rotation by $\pi/2$ and $\mathcal{C}_q^{\text{PSK}}$ is the ensemble of all codewords C. Then

$$\Pr\{\text{PMEPR}_{\mathcal{C}_q^{\text{PSK}}}(C) \le \log n - 6.5 \log\log n\} \le O\left(\frac{1}{\log^4 n}\right). \quad (9)$$

Proof: In the PSK case, we can write the characteristic function of $s_C^R(\theta)$ as

$$\Phi_{\text{PSK}}(t) = E\{e^{jts_C^R(\theta)}\} = \prod_{i=1}^{n} E\{e^{jt\text{Re}\{c_i e^{j\theta i}\}}\} = \prod_{i=1}^{n} E\{\cos(t\cos(i\theta+\beta_i))\} = \prod_{i=1}^{n} E\{\cos t\beta_i'\} \quad (10)$$

where $\beta_i' = \cos(i\theta + \beta_i)$ has an even distribution since the β_i's are chosen from a PSK constellation such that β_i, $\beta_i + \pi/2$, and consequently, $\beta_i + \pi$ are equiprobable. Furthermore, for $|t| < 1$ the characteristic function is positive. Therefore, using the result of Appendix C, we can then write $E\{\cos(t\beta_i')\} = e^{-E\{(\beta_i')^2\}t^2/2 + \alpha t^4 + O(t^6)}$ for $|t| < 1$, where the second moment of β_i' can be evaluated as

$$E\{(\beta_i')^2\} = E\{\cos^2(i\theta+\beta_i)\} = \frac{1}{2} + \frac{1}{2}E\{\cos(2i\theta + 2\beta_i)\} = \frac{1}{2} \quad (11)$$

and the second term is zero since $2\beta_i$ has the same probability as $\pi + 2\beta_i$ due to the fact that the pdf of β_i is invariant under rotation by $\pi/2$. Therefore, replacing (11) into $E\{\cos(t\beta_i')\}$ and then into (10), we get, $\Phi_{\text{PSK}}(t) = e^{-nt^2/4 + n\alpha t^4 + O(nt^6)}$ for $|t| \le 1$. Now, we can use the same argument as that of Theorem 1 to find the mean and variance of η as in (B6) and (B18), respectively. The theorem follows similarly by setting $E_{\text{av}} = 1$ for PSK constellations. □

To generalize the result to spherical codes, we initially need to find the characteristic function of $s_C^R(\theta)$ when the codeword C is uniformly distributed over Ω_n. Clearly, all c_i's are dependent, however the following lemma provides the characteristic function of $s_C^R(\theta)$ when C is uniformly distributed over Ω_n.

Lemma 1: Let $C = (c_1, \ldots, c_n)$ be a random complex vector uniformly distributed over Ω_n and $s_C^R(\theta)$. Then

$$E\{e^{jts_C^R(\theta)}\} = \frac{2^n\Gamma(n)}{|t|^n n^{n-1}} J_n(n|t|).$$

Proof: Let $c_i = a_i + jb_i$ for $i = 1, \ldots, n$. As a first step to find the characteristic function of $s_C^R(\theta)$, since

$$\sum_{k=1}^{n} \sin^2 k\theta + \cos^2 k\theta = n$$

we can state that

$$\begin{aligned} p(s_C^R(\theta)) &= p(\langle (a_1,\ldots,a_n,b_1,\ldots,b_n), (\cos\theta,\ldots,\cos n\theta, \sin\theta,\ldots,\sin n\theta)\rangle) \\ &= p(\langle U(a_1,\ldots,a_n,b_1,\ldots,b_n), (\sqrt{n},0,\ldots,0)\rangle) = p(\sqrt{n}a_1') \end{aligned} \quad (12)$$

where $p(x)$ denotes the pdf of the random variable x

$$(a_1',\ldots,a_n',b_1',\ldots,b_n') = U(a_1,\ldots,a_n,b_1,\ldots,b_n)$$

$\langle\cdot,\cdot\rangle$ denotes the inner product of two vectors, and U is any orthogonal matrix such that $U(\cos\theta, \cos 2\theta, \ldots, \sin n\theta) = (\sqrt{n}, 0\ldots,0)$.

Moreover, since the vector $(a_1,\ldots,a_n,b_1,\ldots,b_n)$ has an isotropic distribution [17], the distribution of the vector remains the same under multiplication by orthogonal matrices, and therefore,

$$p(s_C^R(\theta)) = p(\sqrt{n}a_1') = p(\sqrt{n}a_1) = p(\sqrt{n}r_1\phi_1).$$

Now we can use (12) to write

$$\begin{aligned} E\{e^{jts_C^R(\theta)}\} &= E\{e^{jt\sqrt{n}r_1\cos\phi_1}\} = E\left\{\int_0^{2\pi} \frac{1}{2\pi} e^{jt\sqrt{n}r_1\cos\phi_1} d\phi_1\right\} \\ &= E\{J_0(t\sqrt{n}r_1)\} \end{aligned} \quad (13)$$

where we used the definition of the Bessel function and the fact that ϕ_1 has the uniform distribution proved in Appendix D. Since $J_0(x)$ is an even function, the characteristic function is an even function of t and we can, therefore, focus on $t > 0$. Using the distribution of r_1 computed in Appendix D, we can write (13) as

$$\begin{aligned} E\{e^{jts_C^R(\theta)}\} &= \frac{2}{n^{n-1}} \int_0^{\sqrt{n}} rJ_0(t\sqrt{n}r)(n-r^2)^{n-1} dr \\ &= 2n\int_0^1 uJ_0(tnu)(1-u^2)^{n-1}du = \frac{2^n\Gamma(n)}{t^n n^{n-1}} J_n(nt) \end{aligned} \quad (14)$$

for $t > 0$, where we used the identity

$$\int_0^1 x(1-x^2)^{n-1} J_0(bx)dx = \frac{2^{n-1}\Gamma(n)}{b^n} J_n(bx)$$

for $b > 0$ [18]. Since the characteristic function is even, the lemma follows from (14). □

For large values of n and $0 < t < 1$, we can use the asymptotic expansion of the Bessel function of order n as [18]

$$J_n(nt) = \frac{e^{-n(\cosh^{-1}\frac{1}{t} - \sqrt{1-t^2})}}{\sqrt{2\pi n(1-t^2)^{1/2}}} (1 + O(1/n)) \quad (15)$$

for $0 < t < 1$. Therefore, we can use the asymptotic expansion

$$\Gamma(n) = e^{-n} n^{n-1/2}\sqrt{2\pi}(1 + O(1/n))$$

for large n [18] together with (15) and replace them into (14), to obtain

$$\begin{aligned} E\{e^{jts_C^R(\theta)}\} &= E\{J_0(t\sqrt{n}r_1)\} = e^{-nf(t)}(1 + O(1/n)) \\ &= e^{-n(\alpha_1 t^2 + \alpha_2 t^4 + O(t^6))}(1 + O(1/n)) \end{aligned} \quad (16)$$

for $|t| < 1$ and large n. Therefore, $\log E\{e^{jts_C^R(\theta)}\}$ is an even positive function of t for large n and $|t| < 1$. Now using the Taylor expansion of $\log E\{e^{jts_C^R(\theta)}\}$ as in Appendix C, we can write α_1 in (16) as

$$n\alpha_1 = \frac{1}{2}E\left\{\left(s_C^R(\theta)\right)^2\right\} = \frac{1}{2}E\{na_1^2\} = E\{nr_1^2\cos^2\phi_1\} = \frac{n}{4}. \quad (17)$$

Therefore, the characteristic function of $s_C^R(\theta)$ can be written as

$$E\{e^{jts_C^R(\theta)}\} = e^{-nt^2/4 + n\alpha t^4 + O(nt^6)}(1 + O(1/n)) \quad (18)$$

for large n and $|t| < 1$. This, in fact, allows us to generalize the lower bound for random spherical codes in the following theorem.

Theorem 3 (Lower Bound: Spherical Codes): Let $s_C(\theta)$ be as in (2) where C is chosen uniformly from Ω_n. Also, let $\mathcal{C}_s$ be the ensemble of all the admissible codewords. Then

$$\Pr\{\text{PMEPR}_{\mathcal{C}_s}(C) \le \log n - 6.5\log\log n\} \le O\left(\frac{1}{\log^4 n}\right). \quad (19)$$

Proof: Using Lemma 1 in which the characteristic function of $s_C^R(\theta)$ is computed for $|t| < 1$, and using the identity $e^{-a} = e^{-b} + O(|b-a|)$, we may write $E\{e^{jts_C^R(t)}\} = e^{-nt^2/4} + O(nt^4)$, where C

is uniformly distributed over Ω_n. We can now follow the same line as in the proof of Theorem 1 to write the mean of η as in (B6). Similarly, in order to calculate the second moment of η, we have to compute

$$\Phi_s(t,\tau;\theta_m,\theta_l) = E\{e^{j\sum_{k=1}^n a_k(t\cos k\theta_m+\tau\cos k\theta_l)-b_k(t\sin k\theta_m+\tau\sin k\theta_l)}\}. \quad (20)$$

Since

$$\sum_{k=1}^n (t\cos k\theta_m+\tau\cos k\theta_l)^2+(t\sin k\theta_m+\tau\sin k\theta_l)^2 = n(t^2+\tau^2)$$

for $m\neq l$ and $m+l\neq n$, by using a similar argument as in the proof of Lemma 1, we get

$$\Phi_s(t,\tau;\theta_m,\theta_l) = E\left\{e^{j\sqrt{n(t^2+\tau^2)}a_1}\right\} \quad (21)$$

where a_1 is as defined in Lemma 1. Consequently, it can be then concluded that for $m\neq l$ and $m+l\neq n$, we have

$$\Phi_s(t,\tau;\theta_m,\theta_l) = e^{-n(t^2+\tau^2)/4+n\alpha(t^2+\tau^2)^2+nO((t^2+\tau^2)^3)}(1+O(1/n)). \quad (22)$$

We can similarly prove that $E\{\eta\}$ and $E\{\eta^2\}$ are as in (B6) and (B18), respectively. So using Chebychev's inequality as in Theorem 1, we can complete the proof for random spherical codes. □

B. Upper Bound for the PMEPR

In this subsection, Theorem 4 obtains the probability of having PMEPR slightly greater than $\log n$ for the QAM case and shows that this probability goes to zero as n tends to infinity. This will be extended to PSK and spherical codes in Theorem 5 and 6, respectively.

Theorem 4 (Upper Bound: QAM Case): Consider the setting of Theorem 1. Then

$$\Pr\left\{\text{PMEPR}_{\mathcal{C}_q^{\text{QAM}}}(C)\geq \log n+5.5\log\log n\right\}\leq O\left(\frac{1}{\log^4 n}\right). \quad (23)$$

Proof: We first define the real function $s_C(\gamma,\theta)$ as

$$s_C(\gamma,\theta) = \text{Re}\{e^{j\gamma}s_C(\theta)\} = \sum_{i=1}^n a_i\cos(i\theta+\gamma)-b_i\sin(i\theta+\gamma) \quad (24)$$

consequently, we define K as

$$K = \max_{\gamma,\theta}|s_C(\gamma,\theta)| = |s_C(\gamma_0,\theta_0)| = \max_\theta |s_C(\theta)|.$$

As mentioned in [15] and used in [16], the point of introducing γ is that we can now deal with the maximum of a real function $s_C(\gamma,\theta)$ and generalize the result of Halasz to complex polynomials over the unit circle. Let $u(x)$ be as defined in (5) with the only difference that here

$$M = \sqrt{nE_{\text{av}}\log n+5.5nE_{\text{av}}\log\log n}.$$

We also assume that E_{av} is the normalized average energy of the constellation. Consequently, we define the random variable η as

$$\eta = \int_0^{2\pi}\int_0^{2\pi} u(s_C(\gamma,\theta))d\theta d\gamma = \int_0^{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty} e^{jtu(s_C(\gamma,\theta))}v(t)dtd\theta d\gamma$$

where we substitute $u(x)$ by its Fourier transform $v(t)$. As the second step, by using the Taylor expansion of $s_C(\gamma,\theta)$ around its maximum absolute value, it is shown in [16], [15] that if $\eta\leq\frac{1}{n\log^2 n}$ then $K\leq M+2\Delta$. Therefore,

$$\Pr\{K\leq M+2\Delta\}\geq\Pr\left\{\eta\leq\frac{1}{n\log^2 n}\right\}\geq 1-n\log^2 nE\{\eta\} \quad (25)$$

where we used Markov's inequality to deduce (25). Therefore, the derivation of the upper bound boils down to computation of the mean of η. As in the derivation of the lower bound, we start by computing the characteristic function of $s_C(\gamma,\theta)$

$$\begin{aligned}E\{e^{jt\sum_{i=1}^n a_i\cos(i\theta+\gamma)-b_i\sin(i\theta+\gamma)}\} &= e^{-\frac{t^2}{2}E_{\text{av}1}\sum_{i=1}^n\cos^2(i\theta+\gamma)-\frac{t^2}{2}E_{\text{av}2}\sum_{i=1}^n\sin^2(i\theta+\gamma)}\\ &\quad + t^4\left\{\alpha_1\sum_{i=1}^n\cos^4(i\theta+\gamma)+\alpha_2\sum_{i=1}^n\sin^4(i\theta+\gamma)\right\}\\ &\quad\times e^{-\frac{t^2}{2}E_{\text{av}1}\sum_{i=1}^n\cos^2(i\theta+\gamma)-\frac{t^2}{2}E_{\text{av}2}\sum_{i=1}^n\sin^2(i\theta+\gamma)}\\ &\quad + O(nt^6+n^2t^8)\end{aligned} \quad (26)$$

for $|t|\leq 1$ where we used $e^{-a}=e^{-b}+(b-a)e^{-b}+O((b-a)^2)$ as in (B11) and both $E_{\text{av}1}$ and $E_{\text{av}2}$ are as defined in Theorem 1. We can now take the expectation of η as shown in (27) at the bottom of the page. Using the results in Appendix A, we can simplify the expectation of η as follows: the first and second terms follows from (A6) and $p=0,4$, and the last term can be computed by (A3) to get

$$E\{\eta\}=O\left(\frac{\sqrt{n}e^{-\frac{M^2}{nE_{\text{av}}}}}{M}\right)+O\left(\frac{n\sqrt{n}e^{-\frac{M^2}{nE_{\text{av}}}}}{M\Delta^4}\right)+O\left(\frac{n}{\Delta^6}+\frac{n^2}{\Delta^8}\right) \quad (28)$$

Therefore, by setting the value of M and Δ, and using the Markov inequality, we conclude that $E\{\eta\}=O\left(\frac{1}{n\log^6 n}\right)$ to get

$$\begin{aligned}\Pr\{K\leq M+2\Delta\}&\geq\Pr\left\{\eta\leq\frac{1}{n\log^2 n}\right\}\\ &\geq 1-n\log^2 nO\left(\frac{1}{n\log^6 n}\right)=1-O\left(\frac{1}{\log^4 n}\right).\end{aligned} \quad (29)$$

The theorem follows by using the definition of PMEPR for large values of n. □

The next theorem presents the same asymptotic result for the PSK constellations.

$$\begin{aligned}E\{\eta\} &= \int_0^{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty}E\{e^{jt\sum_{i=1}^n a_i\cos^2(i\theta+\gamma)-b_i\sin^2(i\theta+\gamma)}\}v(t)dtd\theta d\gamma\\ &= \int_0^{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty}e^{-\frac{t^2}{2}E_{\text{av}1}\sum_{i=1}^n\cos^2(i\theta+\gamma)-\frac{t^2}{2}E_{\text{av}2}\sum_{i=1}^n\sin^2(i\theta+\gamma)}v(t)dtd\gamma d\theta\\ &\quad + O\left(n\int_0^{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty}t^4e^{-\frac{t^2}{2}E_{\text{av}1}\sum_{i=1}^n\cos^2(i\theta+\gamma)-\frac{t^2}{2}E_{\text{av}2}\sum_{i=1}^n\sin^2(i\theta+\gamma)}|v(t)|dtd\gamma d\theta\right)+O\left(\int_{-\infty}^{\infty}(nt^6+n^2t^8)|v(t)|dt\right).\end{aligned} \quad (27)$$

Theorem 5 (Upper Bound: PSK Case): Consider the setting of Theorem 2. Then

$$\Pr\left\{\text{PMEPR}_{\mathcal{C}_q^{\text{PSK}}}(C) \geq \log n + 5.5 \log\log n\right\} \leq O\left(\frac{1}{\log^4 n}\right). \quad (30)$$

Proof: We first compute the characteristic function of $s_C(\gamma,\theta) = \sum_{i=1}^n \cos(i\theta + \beta_i + \gamma)$

$$E\{e^{jts_C(\gamma,\theta)}\} = \prod_{i=1}^n E\{\cos(t\cos(i\theta+\beta_i+\gamma))\}$$
$$= \prod_{i=1}^n E\{\cos(t\beta_i'')\} \quad (31)$$

where $\beta_i'' = \cos(i\theta + \beta_i + \gamma)$ has an even distribution since β_i is chosen from a symmetric PSK constellation. Since the distribution of c_i is invariant under rotation by $\pi/2$, using the result in Appendix B, and following the same line as in the proof of Theorem 2, we can write the characteristic function as

$$E\left\{e^{jts_C(\gamma,\theta)}\right\} = e^{-nt^2/4+n\alpha t^4+O(nt^6)}$$

for $|t| < 1$. The theorem follows by using the characteristic function of $s_C^R(\theta)$ and following the same line as in the proof of Theorem 4. □

Theorem 6 (Upper Bound: Spherical Codes): Let C be a codeword uniformly chosen from Ω_n and $\mathcal{C}_s$ be the ensemble of all those codewords C. Then

$$\Pr\left\{\text{PMEPR}_{\mathcal{C}_s}(C) \geq \log n + 5.5\log\log n\right\} \leq O\left(\frac{1}{\log^4 n}\right). \quad (32)$$

Proof: First of all, we derive the characteristic function of $s_C(\gamma,\theta)$ when the codeword C is uniformly distributed over Ω_n. We can use the result of Lemma 1 to show that

$$E\left\{e^{jts_C(\gamma,\theta)}\right\} = e^{-nt^2/4+\alpha nt^4+O(nt^6)}(1+O(1/n))$$
$$= e^{-nt^2/4} + \alpha(nt^4+O(nt^6))e^{-nt^2/4} + O(n^2t^8)$$
$$= e^{-nt^2/4} + \alpha nt^4 e^{-nt^2/4} + O(nt^6+n^2t^8) \quad (33)$$

where we used $e^{-a} = e^{-b} + (b-a)e^{-b} + O((b-a)^2)$ for $a, b > 0$. Fortunately, the characteristic function of $s_C(\gamma,\theta)$ allows us to use a similar approach as in Theorem 4 to evaluate the mean of η as

$$E\{\eta\} = \int_0^{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty} E\{e^{jts_C(\gamma,\theta)}\}v(t)dtd\gamma d\theta$$
$$= \int_0^{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty} e^{-nt^2/4}v(t)dtd\gamma d\theta$$
$$+ n\int_0^{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty} t^4 e^{-nt^2/4}v(t)dtd\gamma d\theta$$
$$+ O\left(\int_{-\infty}^{\infty}(nt^6+n^2t^8)|v(t)|dt\right).$$

Using the result of Appendix A and similar to Theorem 4, we can simplify (34) to get $E\{\eta\} = O\left(\frac{1}{n\log^6 n}\right)$. The theorem follows using a similar argument as in Theorem 4 and setting the value of M and Δ. □

IV. Summary and Discussion

To get a better insight into the above results, let $\mathcal{C}_t$ correspond to $\mathcal{C}_q^{\text{PSK}}$, $\mathcal{C}_q^{\text{QAM}}$, or $\mathcal{C}_s$ as random codes over the corresponding constellations. Using the inequality $\Pr(A)+\Pr(B)-1 \leq \Pr(A\cap B) \leq \Pr(A)$, and Theorems 1 to 6, we may write

$$1 > \Pr\{\log n + 5.5\log\log n > \text{PMEPR}_{\mathcal{C}_t}(C) > \log n - 6.5\log\log n\}$$
$$> 1 - O\left(\frac{1}{\log^4 n}\right). \quad (34)$$

Equation (34) shows that with probability approaching unity, the PMEPR of any codewords randomly chosen from symmetric QAM/PSK or Ω_n behaves like $\log n + O(\log\log n)$ asymptotically. This result implies that for a large number of subcarriers, clipping the signal with a threshold value of less than $\log n$ may cause severe distortion in the signal. On the other hand, by using probabilistic methods [2] in which we randomly map the data to different codewords and choose the best one in terms of PMEPR to transmit, we cannot further reduce the PMEPR below $\log n$. Meanwhile these methods performs very well for moderate values of n since $\log n$ is reasonably small.

Another class of methods to reduce the PMEPR is to use coding not only to introduce a large minimum distance but also to reduce PMEPR [6]. It has been shown in [6] that the VG bound for spherical codes with PMEPR less than $8\log n$ remains the same as that of spherical codes without PMEPR restriction. This shows that there exist high rate and large minimum-distance spherical codes with PMEPR of $8\log n$.

In fact, we can use the result of Section III to derive a VG-type bound on the rate of a code with given minimum distance and PMEPR of less than $\log n$. Here, we use the minimum Hamming distance which is defined as the minimum number of coordinates in which any two codewords are different [19]. The rate of $\mathcal{C}$ is also defined as $R = \frac{1}{n}\log_q|\mathcal{C}|$ where $|\mathcal{C}|$ is the cardinality of the set $\mathcal{C}$.

Corollary 1: Let $\mathcal{Q}_q$ be a complex q-ary symmetric PSK or QAM constellation, $R > 0$, and $0 \leq \delta \leq \frac{q-1}{q}$. If

$$R \leq 1 - H_q(\delta) - O\left(\frac{1}{n\log^4 n}\right) \quad (35)$$

then, asymptotically, there exists a code $\mathcal{C}$ of length n with entries chosen from $\mathcal{Q}_q$, rate R, minimum Hamming distance $d_{min} = \lfloor\delta n\rfloor$, and $\text{PMEPR}_{\mathcal{C}} < \log n + 5.5\log\log n$.

Proof: The proof follows by first excluding codewords with PMEPR larger than $\log n + 5.5\log\log n$, and then using VG type argument to construct a code with the minimum Hamming distance $d_{min} = \lfloor\delta n\rfloor$ [19]. □

According to Corollary 1, it follows that not only there exist spherical high rate codes with PMEPR of $8\log n$, but there also exist codes chosen from usual constellations like QAM and PSK with the same asymptotic. On the other hand, this result does not contradict the existence of exponentially many codewords with constant PMEPR. However, the ratio of the number of these codewords to q^n has to tend to zero asymptotically. So there still remains an open problem of what is the rate of codes with constant PMEPR?

Appendix A

We adopt the following lemma from [15] with modifications to the fifth and sixth inequalities that are required for the generalization to polynomials over the unit circle with complex coefficients.

Lemma 2: Let $u(x)$ be a continuous differentiable function as defined in the proof of Theorem 1 and $v(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x)e^{jtx}dx$. Then we have the following properties:

i)

$$|t^r v(t)| = O\left(\frac{1}{\Delta^{r-1}}\right), \qquad 1 \leq r \leq 10 \quad (A1)$$

ii)

$$\int_{-\infty}^{\infty}|v(t)|dt = O\left(\frac{M}{\Delta}\right) \quad (A2)$$

iii)

$$\int_{-\infty}^{\infty}|t^p v(t)|dt = O\left(\frac{1}{\Delta^p}\right), \qquad 1 \leq p \leq 8 \quad (A3)$$

iv)

$$\int_{|t|>l_0} |v(t)|dt = O(1/\Delta^9), \quad \text{for any constant } l_0 > 0 \tag{A4}$$

v)

$$\left|\int_{-\infty}^{\infty} e^{-nE_{\rm av}t^2/4} t^p v(t)dt\right| = O\left(\frac{\sqrt{n}e^{-\frac{M^2}{nE_{\rm av}}}}{M\Delta^p}\right), \qquad 1 \le p \le 8 \tag{A5}$$

vi)

$$\int_0^{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}\left(E_{\rm av1}\sum_{i=1}^n \cos^2(i\theta+\gamma)+E_{\rm av2}\sum_{i=1}^n \sin^2(i\theta+\gamma)\right)} \times t^p v(t)dtd\gamma d\theta = O\left(\frac{\sqrt{n}e^{-\frac{M^2}{nE_{\rm av}}}}{M\Delta^p}\right) \tag{A6}$$

where

$$M = \sqrt{nE_{\rm av}\log n + O(\log\log n)}$$

$\Delta = \sqrt{\frac{n}{\log n}}$, and $E_{\rm av} = E_{\rm av1} + E_{\rm av2}$.

Proof: For the proof of (A1) to (A4) refer to [15]. In order to prove (A5), we use Parseval's theorem and the properties of Fourier transform to obtain

$$\left|\int_{-\infty}^{\infty} e^{-nt^2/4} t^p v(t)dt\right| = \left|\frac{1}{\sqrt{\pi n}}\int_{-\infty}^{\infty} e^{-x^2/n} u^{(p)}(x)dt\right|. \tag{A7}$$

Now we can use the fact that $u^{(p)}(x)$ is zero for $|x| < M$ and equals $O(1/\Delta^r)$ for $|x| > M$ to rewrite the integral as

$$O\left(\left|\int_{-\infty}^{\infty} e^{-nt^2/4} t^p v(t)dt\right|\right) = O\left(\frac{1}{\sqrt{n}\Delta^p}\int_{|x|>M} e^{-x^2/n}dx\right) = O\left(\frac{Q\left(\frac{M}{\sqrt{n/2}}\right)}{\Delta^p}\right) \tag{A8}$$

where

$$Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^{\infty} e^{-x^2/2}dx.$$

Using the asymptotic expansion $Q(x) = \frac{e^{-x^2/2}}{x\sqrt{2\pi}}(1 - O(1/x^2))$ [18], we get

$$O\left(\frac{Q\left(\frac{M}{\sqrt{n/2}}\right)}{\Delta^p}\right) = O\left(\frac{\sqrt{n}e^{-\frac{M^2}{nE_{\rm av}}}}{M\Delta^p}\right). \tag{A9}$$

Equation (A5) follows from (A8) and (A9). To prove (A6), we first use (A5) to write the inner integral in (A6) as

$$\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}(E_{\rm av1}\sum_{i=1}^n \cos^2(i\theta+\gamma)+E_{\rm av2}\sum_{i=1}^n \sin^2(i\theta+\gamma))} t^p v(t)dt = O\left(\frac{\sqrt{n}}{M\Delta^p}e^{-\frac{M^2}{2(E_{\rm av1}\sum_{i=1}^n \cos^2(i\theta+\gamma)+E_{\rm av2}\sum_{i=1}^n \sin^2(i\theta+\gamma))}}\right). \tag{A10}$$

We then use the following inequality for large values of n similar to [15],

$$\sum_{i=1}^n E_{\rm av1}\cos^2(i\theta+\gamma) + E_{\rm av2}\sin^2(i\theta+\gamma) = \frac{(E_{\rm av1}+E_{\rm av2})n}{2} + \frac{(E_{\rm av1}-E_{\rm av2})}{2}\sum_{i=1}^n \cos 2(i\theta+\gamma) \le \begin{cases} \frac{E_{\rm av}n}{2} + \frac{n|E_{\rm av1}-E_{\rm av2}|}{2\log n}, & \frac{\log n}{n} \le |\theta| \le \pi/2 \\ nE_{\rm av}, & \text{everywhere} \end{cases} \tag{A11}$$

where we used the inequality $\sum_{i=1}^n \cos 2(i\theta+\alpha) \le \frac{1}{2|\sin\theta|}$ and considering that $|\sin\theta| > \theta/2$ for $\theta < 0.1$, the inequality follows for large n. Therefore, we may write

$$\int_0^{2\pi}\int_0^{2\pi} e^{-\frac{M^2}{2(E_{\rm av1}\sum_{i=1}^n \cos^2(i\theta+\gamma)+E_{\rm av2}\sum_{i=1}^n \sin^2(i\theta+\gamma))}} d\theta d\gamma \le \frac{4\pi\log n}{n}e^{-\frac{M^2}{2nE_{\rm av}}} + 4\pi^2 e^{-\frac{M^2}{nE_{\rm av}+\frac{n|E_{\rm av1}-E_{\rm av2}|}{\log n}}}. \tag{A12}$$

We can now use the fact that $\frac{\log n}{n} \le e^{-\frac{M^2}{2nE_{\rm av}}}$ for

$$M = \sqrt{nE_{\rm av}\log n + O(\log\log n)}$$

and also using,

$$e^{-\frac{M^2}{nE_{\rm av}+\frac{n|E_{\rm av1}-E_{\rm av2}|}{\log n}}} = e^{-M^2/nE_{\rm av}} \times e^{\frac{M^2}{E_{\rm av}n\log n+|\alpha_1-\alpha_2|n}} = O(e^{-M^2/nE_{\rm av}}) \tag{A13}$$

to bound (A12) by $O(e^{-\frac{M^2}{nE_{\rm av}}})$. Equation (A6) follows from (A10) and (A13). □

Appendix B

In order to calculate the mean and variance of η, we substitute $s_C^R(\theta)$ in (6) to get

$$\eta = \sum_{m=1}^n \int_{-\infty}^{\infty} e^{jt\sum_{k=1}^n a_k\cos k\theta_m - b_k\sin k\theta_m} v(t)dt. \tag{B1}$$

Using the independence of a_k's and b_k's, we obtain

$$E\{e^{jt\sum_{k=1}^n a_k\cos k\theta_m - b_k\sin k\theta_m}\} = \prod_{k=1}^n E\{e^{jta_k\cos k\theta_m}\}E\{e^{jtb_k\sin k\theta_m}\} \triangleq \Phi_{QAM}(t;\theta_m). \tag{B2}$$

It is shown in Appendix C that for $|t| < 1$

$$E\{e^{ja_kt}\} = e^{-E_{\rm av1}t^2/2-\alpha_1t^4+O(t^6)}$$

and similarly

$$E\{e^{jb_kt}\} = e^{-E_{\rm av2}t^2/2-\alpha_2t^4+O(t^6)}$$

where $E_{\rm av1}$ and $E_{\rm av2}$ are the average energy of a_k and b_k, and, therefore, $E_{\rm av} = E_{\rm av1} + E_{\rm av2}$ is the average energy of c_k. Now using $e^{-a} = e^{-b} + O(|b-a|)$ for $a, b \ge 0$, we can write (B2) as[3]

$$\Phi_{QAM}(t;\theta_m) = e^{-\frac{t^2}{2}\left(\sum_{k=1}^n E_{\rm av1}\cos^2 k\theta_m + E_{\rm av2}\sin^2 k\theta_m\right)+nO(t^4)} = e^{-nE_{\rm av}t^2/4} + O(nt^4) \tag{B3}$$

for $|t| < 1$, where we used the identities

$$\sum_{k=1}^n \cos^2 k\theta_m = \sum_{k=1}^n \sin^2 k\theta_m = n/2$$

for $\theta_m = \pi(2m+1)/n$. To evaluate the expectation of η in (B1), we replace (B3) in (B1) for $|t| < 1$, and use one as the upper bound for the absolute value of the characteristic function for $|t| > 1$ to get

$$E\{\eta\} = n\int_{-1}^{1} e^{-nE_{\rm av}t^2/4}v(t)dt + O\left(n^2\int_{-1}^{1} t^4|v(t)|dt\right) + O\left(n\int_{|t|>1}|v(t)|dt\right). \tag{B4}$$

We may then extend the first integral to infinity and include the resulting error in the third term, also by extending the second integral to

[3]Note that since the characteristic function is less than **1** as shown in Appendix C, $\boldsymbol{a}$ has to be nonnegative.

infinity the third term can be included in the second integral. Finally, (B4) simplifies to

$$E\{\eta\} = n\int_{-\infty}^{\infty} e^{-nE_{\rm av}t^2/4}v(t)dt + O\left(n^2\int_{-\infty}^{\infty} t^4|v(t)|dt\right). \tag{B5}$$

Using the property (A3) of $u(x)$ shown in Appendix A, we may substitute the second term by $O\left(\frac{n^2}{\Delta^4}\right)$ and using $\Delta = \sqrt{\frac{n}{\log n}}$, we get

$$E\{\eta\} = n\int_{-\infty}^{\infty} e^{-nE_{\rm av}t^2/4}v(t)dt + O\left(\log^2 n\right). \tag{B6}$$

In order to find the second moment of η, we may write η^2 as

$$\eta^2 = \sum_{m=1}^{n}\sum_{l=1}^{n} u(s_C^R(\theta_m))u(s_C^R(\theta_l)). \tag{B7}$$

Therefore, after substituting the Fourier transform of $u(x)$ in (B7), to evaluate each term of the double summation of (B7), we should compute (B8) at the bottom of the page.The inner expectation in (B8) can be split using the independence of a_k's and b_k's as

$$\begin{aligned} &E\{e^{j\sum_{k=1}^{n} a_k(t\cos k\theta_m + \tau\cos k\theta_l) - b_k(t\sin k\theta_m + \tau \sin k\theta_l)}\} \\ &= \prod_{k=1}^{n} E\{e^{ja_k(t\cos k\theta_m + \tau\cos k\theta_l)}\} \\ &\quad \times \prod_{k=1}^{n} E\{e^{-jb_k(t\sin k\theta_m + \tau\sin k\theta_l)}\} \\ &\triangleq \Phi'_{QAM}(t,\tau;\theta_m,\theta_l). \end{aligned} \tag{B9}$$

As we stated for the calculation of $E\{\eta\}$, for $|t| \le 1$, we have

$$E\{e^{ja_kt}\} = e^{-E_{\rm av1}t^2/2 - \alpha_1 t^4 + O(t^6)}$$

and

$$E\{e^{jb_kt}\} = e^{-E_{\rm av2}t^2/2 - \alpha_2 t^4 + O(t^6)}.$$

Therefore, for $|t|, |\tau| < 1/2$, each expectation in (B9) can be written (B10) (also shown at the bottom of the page), where we used

$$O((t\cos k\theta_m + \tau\cos k\theta_l)^6) = O((|t| + |\tau|)^6)$$

for the last term in the exponent. We can also write a similar equation for b_k. After substituting (B10) into (B9), we can use the second-order approximation

$$e^{-a} = e^{-b} + (b-a)e^{-b} + O((b-a)^2)$$

for $a, b > 0$, to write (B9) as (B11) at the bottom of the page, for $|t|, |\tau| \le 1/2$. We can further simplify (B11) by using the identities

$$\sum_{k=1}^{n} (t\cos k\theta_m + \tau\cos k\theta_l)^2 = \sum_{k=1}^{n} (t\sin k\theta_m + \tau\sin k\theta_l)^2 = n(t^2+\tau^2)/2$$

for $m \ne l$ and $m + l \ne n$, and $E_{\rm av} = E_{\rm av1} + E_{\rm av2}$, to get

$$\begin{aligned} &\Phi'_{\rm QAM}(t,\tau;\theta_m,\theta_l) \\ &\quad = e^{-\frac{1}{2}nE_{\rm av}(t^2+\tau^2)} + O(n(|t|+|\tau|)^4)e^{-\frac{1}{2}nE_{\rm av}(t^2+\tau^2)} \\ &\qquad + O(n(|t|+|\tau|)^6) + O(n^2(|t|+|\tau|)^8) \end{aligned} \tag{B12}$$

for $|t|, |\tau| < 1/2$, $m \ne l$, and $m + l \ne n$. For the other $2n$ terms (i.e., $m = l$ or $m + l = n$) in (B1), we can use the following inequality:

$$2\sum_{m=1}^{n} u(s_C^R(\theta_m))u(s_C^R(\theta_l)) \le 2\sum_{m=1}^{n} u(s_C^R(\theta_m)) = 2\eta \tag{B13}$$

since $0 \le u(x) \le 1$. Now placing (B12) into (B8) and then into (B7) for $|t|, |\tau| < 1/2$, using one as an upper bound for $|t|, |\tau| > 1/2$, and using (B13) for $2n$ terms with $m = l$ or $m + l = n$, we obtain

$$\begin{aligned} &E\{\eta^2\} \\ &\le (n^2 - 2n)\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} e^{-nE_{\rm av}(t^2+\tau^2)/4}v(t)v(\tau)dtd\tau \\ &\quad + O\left(n^3\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}(|t|+|\tau|)^4 e^{-n(t^2+\tau^2)/4}|v(t)||v(\tau)|dtd\tau\right) \\ &\quad + O\left(n^3\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}(|t|+|\tau|)^6|v(t)||v(\tau)|dtd\tau\right) \\ &\quad + O\left(n^4\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}(|t|+|\tau|)^8|v(t)||v(\tau)|dtd\tau\right) \\ &\quad + O\left(n^2\int_{|t|>1/2}\int_{|\tau|>1/2}|v(t)||v(\tau)|dtd\tau\right) + 2E\{\eta\}. \end{aligned} \tag{B14}$$

$$u(s_C^R(\theta_m))u(s_C^R(\theta_l)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left\{e^{j\sum_{k=1}^{n} a_k(t\cos k\theta_m+\tau\cos k\theta_l) - b_k(t\sin k\theta_m + \tau\sin k\theta_l)}\right\} v(t)v(\tau)dtd\tau. \tag{B8}$$

$$E\{e^{ja_k(t\cos k\theta_m+\tau\cos k\theta_l)}\} = e^{-1/2\sum_{k=1}^{n}\left\{E_{\rm av1}(t\cos k\theta_m+\tau\cos k\theta_l)^2 + \alpha_1(t\cos k\theta_m + \tau\cos k\theta_l)^4\right\} + O(n(|t|+|\tau|)^6)} \tag{B10}$$

$$\begin{aligned} \Phi'_{QAM}(t,\tau;\theta_m,\theta_l) = &\, e^{-\frac{1}{2}\sum_{k=1}^{n} E_{\rm av1}(t\cos k\theta_m + \tau\cos k\theta_l)^2 + E_{\rm av2}(t\sin k\theta_m + \tau\sin k\theta_l)^2} \\ &+ \sum_{k=1}^{n}\{\alpha_1(t\cos k\theta_m + \tau\cos k\theta_l)^4 + \alpha_2(t\sin k\theta_m + \tau\sin k\theta_l)^4 + O(n(|t|+|\tau|)^6)\} \\ &\times e^{-\frac{1}{2}\sum_{k=1}^{n} E_{\rm av1}(t\cos k\theta_m + \tau\cos k\theta_l)^2 + E_{\rm av2}(t\sin k\theta_m + \tau\sin k\theta_l)^2} \\ &+ O(n^2(|t|+|\tau|)^8) \end{aligned} \tag{B11}$$

To evaluate (B14), we may extend the integrals in the first four terms from $-\infty$ to ∞ to find an upper bound for $E\{\eta^2\}$. So we may write (B14) as

$$\begin{aligned} E\{\eta^2\} &\le (n^2-2n)\left(\int_{-\infty}^{\infty} e^{-nE_{\rm av}(t^2+\tau^2)/4} v(t)v(\tau)dtd\tau\right) \\ &+ O\left(n^3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(|t|+|\tau|)^4 e^{-nE_{\rm av}(t^2+\tau^2)/4}|v(t)||v(\tau)|dtd\tau\right) \\ &+ O\left(n^3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(|t|+|\tau|)^6|v(t)||v(\tau)|dtd\tau\right) \\ &+ O\left(n^4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(|t|+|\tau|)^8|v(t)||v(\tau)|dtd\tau\right) \\ &+ O\left(n\int_{|t|>1/2}|v(t)|dt\right)^2 + 2E\{\eta\}. \end{aligned} \tag{B15}$$

Now we can use (A4) in Appendix A to write the fourth term in (B15) as $O\left(\frac{n^2}{\Delta^{18}}\right)$. The second term in (B15) will be also simplified to

$$\begin{aligned} &O\left(n^3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(|t|+|\tau|)^4 e^{-nE_{\rm av}(t^2+\tau^2)/4}|v(t)||v(\tau)|dtd\tau\right) \\ &= O\left(n^3\sum_{p=0}^{4}\int_{-\infty}^{\infty} t^p e^{-nE_{\rm av}t^2/4}dt\int_{-\infty}^{\infty}\tau^{4-p}e^{-nE_{\rm av}\tau^2/4}d\tau\right) \\ &= O\left(\frac{n^4 e^{-2M^2/nE_{\rm av}}}{\Delta^4 M^2}\right) \end{aligned} \tag{B16}$$

where we used the identities (A5) with $p=k$ and $p=4-k$. The third term similarly can be evaluated as

$$\begin{aligned} &O\left(n^3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(|t|+|\tau|)^6|v(t)||v(\tau)|dtd\tau\right) \\ &= O\left(n^3\left\{\sum_{k=1}^{5}\int_{-\infty}^{\infty}|t^k v(t)|dt\int_{-\infty}^{\infty}|t^{6-k}v(t)|dt \right.\right.\\ &\qquad\left.\left. +\ 2\int_{-\infty}^{\infty}|v(\tau)|d\tau\int_{-\infty}^{\infty}|t^6 v(t)|dt\right\}\right) \\ &= O\left(\frac{n^3}{\Delta^6}\right) + O\left(n^3\times\frac{1}{\Delta^6}\times\frac{M}{\Delta}\right) \\ &= O\left(\frac{n^3}{\Delta^6}\right) + O\left(\frac{n^3 M}{\Delta^7}\right) \end{aligned} \tag{B.17}$$

where we again used (A2) and (A3) to evaluate both terms in (B17). Along the same lines as in the evaluation of the third term, the fourth term can also be shown to be $O\left(\frac{n^4 M}{\Delta^9}\right)$. Therefore, setting the value of M and Δ, we may write

$$\begin{aligned} E\{\eta^2\} &\le 2E\{\eta\} + n^2\left(\int_{-\infty}^{\infty} e^{-nE_{\rm av}t^2/4}v(t)dt\right)^2 \\ &\quad + O\left(\frac{n^4 e^{-2M^2/nE_{\rm av}}}{\Delta^4 M^2}\right) + O\left(\frac{n^3 M}{\Delta^7}\right) + O\left(\frac{n^4 M}{\Delta^9}\right) \\ &= 2E\{\eta\} + n^2\left(\int_{-\infty}^{\infty} e^{-nE_{\rm av}t^2/4}v(t)dt\right)^2 \\ &\quad + O(\log^5 n) + O(\log n) + O(\log^2 n). \end{aligned} \tag{B18}$$

On the other hand, it is easy to find a lower bound for $E\{\eta\}$ by using the definition of $u(x)$ and Parseval's theorem and show that $E\{\eta\} \ge O(\log^6 n)$ [15]. Equivalently, this implies that

$$\sigma_\eta^2 = E\{\eta^2\} - E^2\{\eta\} = O\left(E\{\eta\}\log^2 n + \log^5 n\right).$$

APPENDIX C

In this appendix, we calculate the characteristic function of a bounded random variable with even probability distribution function.

Lemma 3: Let c be a real random variable with even pdf, variance $E_{\rm av}$, and maximum energy 1, i.e., $|c|^2 \le 1$. Then for $|t| < 1$, we have

$$\log \Phi_c(t) = -E_{\rm av}t^2/2 + a_4 t^4 + O(t^6) \tag{C1}$$

where $\Phi_c(t) = E\{e^{jtc}\}$.

Proof: It is clear that when the pdf of c is even then the characteristic function is real, so $\Phi_c(t) = E\{\cos(tc)\}$. Since the pdf of c is nonnegative and the maximum energy of c is one, $\Phi_c(t)$ is a real positive function. $\Phi_c(t)$ is also inifnitely differentiable for $|t| < 1$ since $E\{c^k\cos(tc)\}$ is bounded for any k. Now we can write the Taylor expansion

$$\log \Phi_c(t) = \sum_{i=0}^{\infty} a_i t^i.$$

Since $\Phi_c(0) = 1$, a_0 will be zero. Furthermore, since the pdf of c is even, $\Phi_c(t)$ and $\log\Phi_c(t)$ will be even and, therefore, $a_{2k+1} = 0$, for $k = 0, 1, \ldots$. The values of a_2 can be computed as

$$a_2 = \frac{-1}{2}\frac{\Phi''(0)}{\Phi(0)} = -\frac{1}{2}E\{c^2\}.$$

Therefore, for $|t| < 1$, we can write the Taylor's expansion

$$\log(\Phi(t)) = -E\{c^2\}t^2/2 + a_4 t^4 + O(t^6). \qquad \square$$

.

APPENDIX D

Lemma 4: Let $C = (c_1, \ldots, c_n)$ be a random complex vector uniformly distributed over Ω_n. Let $c_i = r_i e^{j\phi_i}$. Then r_1 and ϕ_1 are independent with the following distribution:

$$p(r_1) = \frac{2}{n^{n-1}} r_1 (n - r_1^2)^{n-1} \tag{D1}$$

$$p(\phi_1) = \frac{1}{2\pi}. \tag{D2}$$

Proof: Since the vector C is uniformly distributed over Ω_n, $p(C) = \frac{1}{V}\delta(CC^* - n)$ where $V = \frac{\pi^n n^n}{\Gamma(n)}$. Now let us define $C' = (c_2, \ldots, c_n)$. Then, we can write $p(c_1)$ as

$$p(c_1) = \frac{1}{V}\int \delta(CC^* - n)dC' = \frac{1}{2\pi V}\int\int e^{j\omega(CC^*-n)}dC'd\omega \tag{D3}$$

where we used the definition of $\delta(x)$. In order to make the integral converging, we multiply the integral by $1 = e^{n\beta - CC^*\beta}$ in which $\beta > 0$. Therefore,

$$p(c_1) = \frac{1}{2\pi V}\int\int e^{n(\beta - j\omega)} e^{-r_1^2(\beta - j\omega)} e^{-C'C'^*(\beta - j\omega)}dC'd\omega. \tag{D4}$$

It is shown in [17] that if P and Q are Hermition $M \times M$ matrices and $P > 0$

$$\int e^{-y^*(P+jQ)y}dy = \frac{\pi^M}{\det(P+jQ)}.$$

So setting $z = \beta - j\omega$, we get

$$p(c_1) = \frac{\pi^{n-1}}{2\pi V j}\int \frac{e^{(n-r_1^2)z}}{z^{n-1}}dz = \frac{\pi^{n-1}}{V\Gamma(n-1)}(n - r_1^2)^{n-1}. \tag{D5}$$

Therefore, we can now compute the probability distribution of r_1 and ϕ_1 as follows:

$$p(r_1) = \int r_1 p(r_1, \phi_1) d\phi_1 = \frac{2\pi^n}{V\Gamma(n-1)} r_1 (n - r_1)^{n-1}$$

$$= \frac{2}{n^{n-1}} r_1 (n - r_1^2)^{n-1}$$

$$p(\phi_1) = \int r_1 p(r_1, \phi_1) dr_1 = \frac{1}{2\pi}.$$

Also, since $p(r_1, \phi_1) = p(r_1)p(\phi_1)$, r_1 and ϕ_1 are independent. □

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User Capacity of Asynchronous CDMA Systems With Matched Filter Receivers and Optimum Signature Sequences

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***Abstract*—For a symbol-asynchronous (but chip-synchronous) single-cell code-division multiple-access (CDMA) system, we define a system-wide quantity called the total squared asynchronous correlation (TSAC) which, for arbitrary signature sets, depends on the users' delay profile. We develop a lower bound for TSAC that is independent of the users' delays. We show that if the signature set achieves this TSAC lower bound, then the user capacity of the asynchronous CDMA system using matched filters becomes the same as that of a single-cell synchronous CDMA system; in this case, there is no loss in user capacity due to asynchronism. We present iterative signature adaptation algorithms, which, when executed sequentially by the users, appear to converge to these optimum signature sequences; however, the existence, for all user delay profiles, of signature sequences achieving this lower bound remains a significant open problem.**

***Index Terms*—Asynchronous code-division multiple access (CDMA), CDMA user capacity, interference avoidance, minimum mean-square error (MMSE) filters, optimum signature sequence sets, Welch bound equality (WBE) sequences, Welch bound.**

I. INTRODUCTION

For code-division multiple-access (CDMA) systems, there has been recent progress in understanding the influence of signature sequences on the overall system capacity [3]–[5]. In particular, for a single-cell synchronous CDMA system with equal received powers, [3] showed that one can always choose the signature sequences to be Welch bound equality (WBE) sequences [6]–[8] and that WBE sequences maximize the user capacity, i.e., the maximum number of supportable users at a common signal-to-interference ratio (SIR) target level for a fixed processing gain. A generalized version of this problem where users have arbitrary (unequal) received powers was solved in [4].

In this correspondence, we investigate the user capacity of an *asynchronous* single-cell CDMA system with matched filters, under the assumption that the users' signature sequences can be optimized. Even though the system is symbol-asynchronous, we assume that it is chip-synchronous (e.g., as in [9]) in order to make the analysis tractable. We also assume that short sequences are used; that is, the length of the signature sequences is equal to one symbol duration, and that the signature sequences are repeated at every symbol interval. We define a quantity called the total squared asynchronous correlation (TSAC) of a signature sequence set. For arbitrary signature sequences, the TSAC depends on the users' delay profile. We identify a lower bound on the TSAC that is independent of the users' delay profile. For those delay profiles for which there exist signature sets that achieve the TSAC lower bound, we show that an asynchronous system in which each user employs a matched filter receiver over a single-symbol interval has the same user

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