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Gravitational-wave measurements of the mass and angular momentum of a black hole

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A deformed black hole produced in a cataclysmic astrophysical event should undergo damped vibrations which emit gravitational radiation. By fitting the observed gravitational waveform $h(t)$ to the waveform predicted for black-hole vibrations, it should be possible to deduce the hole's mass M and dimensionless rotation parameter $a=(c/G)(\text{angular momentum})/M^2$. This paper estimates the accuracy with which M and a can be determined by optimal signal processing of data from laser-interferometer gravitational-wave detectors. It is assumed that the detector noise has a white spectrum and has been made Gaussian by cross correlation of detectors at different sites. Assuming, also, that only the most slowly damped mode (which has spheroidal harmonic indices $l=m=2$) is significantly excited—as probably will be the case for a hole formed by the coalescence of a neutron-star binary or a black-hole binary—it is found that the one-sigma uncertainties in M and a are $\Delta M/M \simeq 2.2\rho^{-1}(1-a)^{0.45}$, $\Delta a \simeq 5.9\rho^{-1}(1-a)^{1.06}$, where $\rho \simeq h_s(\pi f S_h)^{-1/2}(1-a)^{-0.22}$. Here ρ is the amplitude signal-to-noise ratio at the output of the optimal filter, h_s is the wave's amplitude at the beginning of the vibrations, f is the wave's frequency (the angular frequency ω divided by 2π), and S_h is the frequency-independent spectral density of the detectors' noise. These formulas for ΔM and Δa are valid only for $\rho \gtrsim 10$. Corrections to these approximate formulas are given in Table II.

I. INTRODUCTION AND SUMMARY

In 1971 Press showed[1] that black holes can vibrate, and in fact have normal modes of vibration; and in 1974 Teukolsky and Press[2] showed that the gravitational waves emitted by a black hole will always be dominated, after an initial transient period, by a superposition of the outputs of a set of discrete normal (or quasinormal) modes. Since then the vibration frequencies ω of the normal modes and their radiation-reaction-induced damping times τ have been computed as functions of the hole's mass M and dimensionless rotation parameter $a=(c/G)(\text{angular momentum})/M^2$ by Chandrasekhar and Detweiler,[3] Detweiler,[4] Leaver,[5] and others.

In 1977, when Detweiler's[4] calculations revealed that for the most slowly damped mode of a rotating hole the waves' parameters $\{\omega,\tau\}$ are a unique and invertible function of the hole's parameters $\{M, a\}$, the possibility arose of being able to infer a hole's M and a from the waves it emits. This possibility is enhanced whenever, among all the hole's modes, the most slowly damped one is preferentially excited. Detweiler[6] has argued that this will be the case if the hole is rapidly rotating (if a is very near unity). Moreover, it will likely be the case for the most interesting and strongest emitting of all black-hole events: the formation of a deformed hole by the coalescence of a neutron-star binary or a black-hole binary. The reason is that during the coalescence the binary will have a rotating shape corresponding to spheroidal harmonic indices $l=m=2$, and the most slowly damped mode has precisely these indices.[4,5]

Although the idea of determining a hole's M and a from measurements of its gravitational waves has been around since 1977, nobody has yet estimated the accuracy with which this can be done, i.e., the rms errors ΔM and Δa due to the noise in the detectors to be expected in such a determination. This paper is devoted to an estimate of ΔM and Δa and their correlation.

Our estimate will rely on a number of assumptions.

(i) Which normal modes are present in the ringdown waves and in what mixture? Motivated by the above discussion, we shall restrict attention to the case where only the most slowly damped (fundamental), $l=m=2$ mode is present.

(ii) What is the transient waveform that precedes the ringdown? This transient, for a coalescing compact binary system, should consist initially of periodic waves whose frequency increases due to the spiraling orbital motion that brings the two bodies together, and then a burst due to the start of the coalescence itself. We suspect, but have not tried to prove, that ΔM and Δa will be rather insensitive to that transient, provided we express them in terms of the signal-to-noise ratio ρ for the ringdown waves and leave the initial transient out of ρ. Furthermore, the signal-to-noise ratio for the transient waveform may be small in comparison with that of the ringdown: in some model simulations this is true,[7] and for rapidly rotating holes (a near unity) the high Q factor of the ringdown enhances its signal-to-noise ratio. We shall presume for simplicity that there is no transient; and, more specifically, that the waves' waveform is

$$h_{jk}^{\mathrm{TT}}(t)=\begin{cases} e_{jk}\,A e^{-(t-t_s)/\tau_s}\sin\omega_s(t-t_s) & \text{for } t\ge t_s\,, \\ 0 & \text{for } t<t_s\,. \end{cases} \tag{1.1}$$

Here e_{jk} is the polarization tensor, A is the amplitude, ω_s and τ_s are the normal-mode frequency and damping time, and t_s is the waves' arrival time. (The subscript s, standing for "signal," is used to distinguish this ω_s, τ_s, and t_s from the values ω_0, τ_0, t_0 that are estimated by the experimenters and the values ω_k, τ_k, t_k that the experimenters use in their optimal filters; see Sec. II below.)

(iii) What is the spectral density $S_h(f)$ of the noise in the detector? [For a detailed discussion of $S_h(f)$ see Ref. 8.] The most promising of all gravitational-wave detectors are the multikilometer laser-interferometer detectors (also called "interferometric detectors" or "beam detectors") that are being planned in the United States, Britain, Germany, France, Italy, Japan, and Russia; for a review see Ref. 8. These are broadband detectors; and in the frequency ranges of optimal sensitivity their noise is likely to be white, $S_h(f)$ independent of f. Accordingly, we shall assume white noise. Since black-hole waves have, for the mode we have chosen, a quality factor [4,5]

$$Q_s \equiv \tfrac{1}{2}\omega_s\tau_s \simeq 2(1-a)^{-0.45} \gtrsim 2 \ , \tag{1.2}$$

the band of frequencies involved in the signal (1.1) is relatively narrow, $\Delta f < f$. This narrowness means that our results should not be very sensitive to the white-noise assumption. *Note*: the signal $h(t)$ which is to be compared with $S_h(f)$ is the projection of h_{jk}^{TT} on the unit vectors l_j and m_k which point along the beam detector's legs,

$$h(t) = \begin{cases} h_s e^{-(t-t_s)/\tau_s} \sin\omega_s(t-t_s) & \text{for } t \geq t_s \ , \\ 0 \ \text{ for } t < t_s \ , \end{cases} \tag{1.3}$$

$$h_s \equiv A e_{jk}(l^j l^k - m^j m^k) \ ;$$

see Ref. 8.

(iv) What are the statistical properties of the detector noise? Individual detectors exhibit some excitations due to local, non-Gaussian noise—e.g., due to sudden strain releases in the wires suspending the detector's mirrors or to inadequately shielded voltage fluctuations in the electric power lines. In order to have any likelihood at all of successful detection of waves it is essential to remove such excitations from the detectors' output data. Fortunately, the non-Gaussian noise comes in short spikes, separated by long intervals of purely Gaussian noise. Those short spikes are uncorrelated between two widely separated detectors and thus are easily removed by cross correlation. Thus we shall assume, in accord with the experimenters' past experience, that the remaining noise is Gaussian.

(v) What method is used to analyze the data? Wiener optimal filtering: more specifically, we shall assume that the data are run through a set of filters, each of which is optimized for detecting a signal of the form (1.3) but with values of the frequency ω_k, damping time τ_k, and arrival time t_k which differ from those of the signal. (The experimenters, of course, do not know in advance what ω_s, τ_s, and t_s are.) The experimenters choose as their best estimates of ω_s, τ_s, and t_s those filter values ω_k, τ_k and t_k which give the output with the largest signal-to-noise ratio. We shall denote these best estimates by ω_0, τ_0, and t_0.

In a large number of different measurements with identical input signals, but with Gaussianly fluctuating noise, this procedure will give different values of ω_0, τ_0, and t_0. These values will be Gaussianly distributed with means ω_s, τ_s, and t_s, if the signal-to-noise ratio is high enough. Correspondingly, the values M_0 and a_0 of the hole's mass and angular-momentum parameter inferred from ω_0 and τ_0 will be Gaussianly distributed, with means M and a, respectively. The bottom-line result of this paper is the Gaussian probability distribution (integrated over start times) for the inferred M_0 and a_0:

$$P(M_0,a_0) = \frac{Ma}{2\pi\Delta M\Delta a(1-C_{Ma}^2)^{1/2}} \exp\left[\frac{-1}{2(1-C_{Ma}^2)}\left[\frac{(M_0-M)^2}{\Delta M^2} - \frac{2C_{Ma}(M_0-M)(a_0-a)}{\Delta M\Delta a} + \frac{(a_0-a)^2}{\Delta a^2}\right]\right]. \tag{1.4}$$

The variances ΔM and Δa of the inferred mass M_0 and angular momentum a_0 turn out to be (Sec. IV)

$$\begin{aligned} \Delta M/M &= 2.2\rho^{-1}(1-a)^{0.45} f_M(a) \ , \\ \Delta a &= 5.9\rho^{-1}(1-a)^{1.06} f_a(a) \ , \end{aligned} \tag{1.5a}$$

where f_M and f_a are functions that are nearly equal to unity and are tabulated in Table II, ρ is the amplitude signal-to-noise ratio at the output of the filter, and these formulas are valid only for $\rho \gtrsim 10$. Because the best information about M and a comes from the waves' frequency (their ringdown time is less well determined), the fluctuations of M_0 and a_0 away from the true values, $\delta M \equiv M_0 - M$ and $\delta a \equiv a_0 - a$, are strongly correlated; the correlation coefficient appearing in (1.4) is

$$C_{Ma} = 0.976 f_{Ma}(a) \ , \tag{1.5b}$$

where f_{Ma} (tabulated in Table II) is very nearly equal to unity throughout the range $0 \leq a \leq 1$.

It is important to note that the signal-to-noise ratio ρ at the output of the filter depends not only on the waves' amplitude h_s and the detector's noise S_h; it also depends on how long the waves last, i.e., on their quality factor Q_s [Eq. (1.2)]:

$$\begin{aligned} \rho &= h_s(\omega_s S_h)^{-1/2} 2Q_s^{3/2}(1+4Q_s^2)^{-1/2} \\ &= h_s[2/(\omega_s S_h)]^{1/2}(1-a)^{-0.22} f_\rho(a) \\ &= h_s S_h^{-1/2}(GM/c^3)^{1/2} \\ &\quad \times 2.26(1-a)^{-0.15} f'_\rho(a) \ . \end{aligned} \tag{1.6}$$

Here f_ρ and f'_ρ are correction functions close to unity that are tabulated in Table II. The faster the hole rotates, the larger is its quality factor, and thus for fixed initial wave amplitude h_s, the larger is the signal-to-noise ratio ρ and the better determined are the hole's mass and angular momentum. The determination improves not only due to the increase in ρ. Expressions (1.5a) also show a direct and larger improvement with increasing a in addition to that produced by ρ. They also show that for slowly rotating holes, $a \lesssim 0.8$, the rotation parameter is less accurately determined than the mass, $\Delta a > \Delta M/M$; but for $a \gtrsim 0.8$ it is better determined, $\Delta a < \Delta M/M$.

The body of this paper, in which these and other results are derived, is organized as follows: Section II outlines, briefly, the theory of optimal filtering of signals that are contaminated by noise. Section III uses that theory to determine, for $\rho \gtrsim 5$, the accuracies $\Delta\omega$, $\Delta\tau$ with which the parameters ω_s, τ_s of the waveform (1.3) can be deduced in the presence of the white, Gaussian noise S_h. Section IV translates those $\Delta\omega$ and $\Delta\tau$ into corresponding accuracies (and Gaussian probability distributions) for the inferred mass M_0 and rotation parameter a_0 of the hole. Finally, Sec. V points the direction toward future, followup research.

II. GENERAL APPROACH

In this section we sketch, briefly, the application of Wiener's theory of optimal filtering to our problem.

The experimenters' initial task is to estimate the signal parameters t_s, ω_s, and τ_s from their experimental data—data consisting of the signal (1.3) corrupted by detector noise.

The simplest variant of the Wiener optimal filter deals with a slightly different task: The parameters t_s, ω_s, and τ_s [and thence the full signal $h(t)$] are presumed known in advance, and it is desired merely to determine whether or not the signal is present. For this task the optimal filter $K(t)$ is the one which, when integrated against the noisy signal, gives the largest integrated signal-to-noise ratio.[9] More specifically, let the uncontaminated signal be $h(t)$ [Eq. (1.3)] and let the noise (a Gaussian random process) be $n(t)$. Then the value obtained as output of the optimal filter is

$$W=\int_{-\infty}^{\infty} K(t)[h(t)+n(t)]dt=S+\nu \ , \qquad (2.1)$$

where

$$S=\int_{-\infty}^{\infty} K(t)h(t)dt \ , \qquad \nu=\int_{-\infty}^{\infty} K(t)n(t)dt \ , \qquad (2.2)$$

and $K(t)$ (the optimal filter) is defined by

$$\tilde{K}(f) \propto \tilde{h}(f)/S_h(f) \ . \qquad (2.3)$$

Here the tildes denote Fourier transforms, $S_h(f)$ represents the spectral density of the noise $n(t)$, and the constant of proportionality is arbitrary. Note that, while S is a constant, independent of the moment of detection [because if $h(t)$ is shifted in time $K(t)$ is shifted too], ν, in a given experiment, is just an instance of a random variable, and will be different if detected at a different time or even by another identical detector at the same time, since $n(t)$ is a random process. The filter (2.3) is optimal in that it gives the maximum possible value for the output signal-to-noise ratio S/N, where N is the standard deviation of ν considered as a random variable, i.e., $N \equiv \sigma_\nu$.

In our case, the signal's parameters ω_s, τ_s, t_s, and amplitude are unknown; and thus the experimenters cannot know in advance the exact form for the optimal filter function (2.3). However, since the general shape of $h(t)$ is known, and since the optimal filter gives the maximum signal-to-noise ratio S/N on output for the signal it is tailored to, we can (and shall) assume some arbitrary initial parameters (ω_k, τ_k, t_k) for the filter function and then perform a fine-tuning, changing these parameters in order to maximize the output S/N. The maximum value of S/N will occur when (ω_k, τ_k, t_k) are equal to (ω_s, τ_s, t_s).

This maximization of S/N cannot be accomplished exactly in practice, since at the output of the filter we do not know the output signal S and the output noise ν separately, but only their sum W. The best we can do is take the total output W as an estimate of S, apply the above-described procedure to maximize W/N, and thereby obtain estimates (ω_0, τ_0, t_0) of the exact signal parameters. Obviously, the weaker the noise (the higher S/N), the closer these estimates will be to (ω_s, τ_s, t_s).

In this paper we shall compute the uncertainties in (ω_0, τ_0)—i.e., the amounts by which they are expected to deviate from (ω_s, τ_s). Our computation will be based on the statistical properties of the background noise and the effect of the filtering and optimization processes on those statistical properties and on the total (corrupted) signal. We shall carry out this analysis analytically with appropriate approximations for the weak-noise case (large S/N). The same procedure, implemented numerically, could give details of the uncertainties in ω_s, τ_s for the strong-noise case; but we shall not attempt such calculations.

Once the uncertainties in ω_0, τ_0 are known, these can be (and will be) translated into corresponding uncertainties for the mass and angular momentum of the black hole. This can be readily performed using the known numerical results that relate these two sets of parameters.[5,10] We will also find the correlation between these uncertainties.

III. SIGNAL PARAMETERS

The signal function we will use is the damped sinusoid described by Eq. (1.3), which starts at $t=t_s$. For ease of calculation, we will choose $t_s=0$, so that the estimate t_0 will be distributed around zero, and $h(t)$ will have the form

$$h(t)=\begin{cases} h_s e^{-t/\tau_s}\sin\omega_s t & \text{if } t \geq 0 \ , \\ 0 \quad \text{if } t<0 \ . \end{cases} \qquad (3.1)$$

We will follow the process described in Sec. II to obtain the estimates ω_0, τ_0, t_0 for ω_s, τ_s, and $t_s=0$ and to determine how much uncertainty is introduced in the process.

Now, since we assumed that the detector introduces white noise, its spectral density will be constant: $S_h(f)=S_h$ for all f. Then, according to (2.3) the optimal filter would be proportional to the signal $h(t)$. However, since in practice we do not know the values of ω_s, τ_s, or t_s, we are forced to use, as our filter,

$$K(t)=\begin{cases} e^{-(t-t_k)/\tau_k}\sin\omega_k(t-t_k) & \text{if } t\geq t_k\ , \\ 0 \quad \text{if } t<t_k\ , \end{cases} \tag{3.2}$$

where ω_k, τ_k, and t_k are the parameters that have arbitrary initial values, and are fine-tuned to maximize W/N.

By inserting Eqs. (3.1) and (3.2) into (2.2), we obtain, for the filtered signal,

$$S=\int_{-\infty}^{\infty}K(t)h(t)dt$$
$$=\begin{cases} \frac{1}{2}h_s e^{t_k/\tau_k}(I_- - I_+) & \text{if } t_k<0\ , \\ \frac{1}{2}h_s e^{-t_k/\tau_s}(J_- - J_+) & \text{if } t_k\geq 0\ , \end{cases} \tag{3.3}$$

with

$$I_{\pm}=(\tau\cos\omega_k t_k+\tau^2\omega_{\pm}\sin\omega_k t_k)/(1+\omega_{\pm}^2\tau^2)\ ,$$
$$J_{\pm}=(\tau\cos\omega_s t_k\mp\tau^2\omega_{\pm}\sin\omega_s t_k)/(1+\omega_{\pm}^2\tau^2)\ , \tag{3.4}$$

where $\tau\equiv\tau_k\tau_s/(\tau_k+\tau_s)$ and $\omega_{\pm}\equiv\omega_k\pm\omega_s$.

We will now analyze ν, the output of the filter when the input is $n(t)$. We can look at ν in two very different ways. First, we can consider it as just the real number obtained, added to the useful output S, in one specific experiment. Second, we can view it as the random variable corresponding to the different results that an ensemble of identical detectors would give for the same experiment. We must use the first viewpoint when we try to reproduce the steps that would be followed in the analysis of the data from an actual experiment, e.g., the fine-tuning of parameters by maximization of the output signal-to-noise ratio. We need, however, to use the second point of view when we want to study the statistical variations that are to be expected in actual experiments, given the fact that $n(t)$ is unpredictable.

From (2.2) and (3.2), ν is given by

$$\nu=\int_{-\infty}^{\infty}K(t)n(t)dt$$
$$=\int_0^{\infty}e^{-t/\tau_k}\sin\omega_k t\, n(t+t_k)dt\ . \tag{3.5}$$

Viewing this ν as a random variable, we see that its distribution is Gaussian, since it is a linear combination of the Gaussian random variables $n(t+t_k)$ (Ref. 11). As an aid in evaluating the variance $\sigma_\nu{}^2$ of ν we introduce the random process

$$\hat{\nu}(t')=\int_{-\infty}^{\infty}K(t-t')n(t)dt\ , \tag{3.6}$$

so that $\nu=\hat{\nu}(0)$. The spectral density of this $\hat{\nu}(t')$ will then be[12]

$$S_{\hat{\nu}}(f)=|\tilde{K}(f)|^2 S_h(f)=S_h|\tilde{K}(f)|^2\ . \tag{3.7}$$

Consequently, the variance of the random process $\hat{\nu}(t')$ will be

$$\sigma_{\hat{\nu}}{}^2=\int_0^{\infty}S_{\hat{\nu}}(f)df=S_h\int_0^{\infty}|\tilde{K}(f)|^2 df$$
$$=(S_h/2)\int_{-\infty}^{\infty}|\tilde{K}(f)|^2 df$$
$$=(S_h/2)\int_{-\infty}^{\infty}K^2(t)dt\ , \tag{3.8}$$

where the first equality follows from the definition of spectral density,[13] and the last one from Parseval's theorem. However, the random variable $\nu=\hat{\nu}(0)$ is just the value of the random process $\hat{\nu}$ at one specific time, so their variances are the same: $\sigma_\nu^2=\sigma_{\hat{\nu}}^2$. Hence,

$$N^2\equiv\sigma_\nu^2=\langle\nu^2\rangle=(S_h/2)\int_{-\infty}^{\infty}K^2(t)dt$$
$$=(S_h/8)\omega_k^2\tau_k^3/(1+\omega_k^2\tau_k^2)\ . \tag{3.9}$$

We now want to perform the maximization of our "best estimate" of the signal-to-noise ratio, $W/N=(S+\nu)/N$. It is impossible to do this analytically to obtain an expression for the optimal values of ω_k, τ_k, and t_k since the expressions (3.3) and (3.9) are not simple enough. If, however, we restrict ourselves to the case in which the noise is sufficiently low, i.e., S/N is big enough, we can assume that the optimal values for the filter parameters will be relatively close to the exact signal parameters. In this case we can write

$$\omega_k=\omega_s(1+\varepsilon)\ ,\quad \tau_k=\tau_s(1+\eta)\ ,$$
$$t_k=\zeta/\omega_s\ , \tag{3.10}$$

where we need only consider values of ε, η, and ζ much smaller than 1. We can then show that, in this approximation, we get the simple expression

$$S/N=\rho(1-\alpha_1\varepsilon^2-\alpha_2\eta^2-\alpha_3\zeta^2$$
$$+\beta_1\varepsilon\eta+\beta_2\varepsilon\zeta+\beta_3\eta\zeta)\ , \tag{3.11}$$

where the coefficients can be expressed in terms of the quantities

$$Q_s\equiv\tfrac{1}{2}\omega_s\tau_s\ ,$$
$$q_s\equiv 2Q_s(1+4Q_s^2)^{-1/2}=\omega_s\tau_s(1+\omega_s^2\tau_s^2)^{-1/2}\ , \tag{3.12}$$

which are dependent only on the product of the signal's frequency and damping time, and not on each separately. The parameter Q_s is the resonance factor or quality factor for the damped wave. Note that q_s can only take values between 0 and 1, but for the values of ω_s and τ_s corresponding to the fundamental normal mode with $l=m=2$ of Kerr black holes, Q_s ranges from about 2 to ∞ [see Eq. (4.3)], so q_s is always very close to unity. In terms of these parameters the coefficients in (3.11) are

$$\rho=q_s h_s(\tau_s/2S_h)^{1/2}\ ,$$
$$\alpha_1=\tfrac{1}{2}q_s^4-\tfrac{1}{4}q_s^2+Q_s^2\ ,$$
$$\alpha_2=\tfrac{1}{2}q_s^4-\tfrac{3}{4}q_s^2+\tfrac{3}{8}\ ,$$
$$\alpha_3=\tfrac{1}{2}q_s^{-2}\ , \tag{3.13}$$
$$\beta_1=-\ q_s^4+q_s^2\ ,$$
$$\beta_2=Q_s\ ,$$
$$\beta_3=-\tfrac{1}{4}Q_s^{-1}\ .$$

We can do something similar with the noise contribution to W/N: we can construct a series expansion for ν in powers of ε, η, and ζ and keep the leading terms. In order to do this, we note that from (3.5) and (3.10) the filter function depends not only on t, but also on ε, η, and ζ; i.e., it can be expressed as $K(\varepsilon,\eta,\zeta,t)$. Therefore, it can be expanded in powers of ε, η, and ζ. Keeping terms up to first order, we get

$$K(\varepsilon,\eta,\zeta,t)=\begin{cases} f_0(t)+f_1(t)\varepsilon+f_2(t)\eta+f_3(t)\zeta & \text{if } t\geq 0 , \\ 0 \quad \text{if } t<0 , \end{cases} \tag{3.14}$$

where

$$\begin{aligned} f_0(t)&=e^{-t/\tau_s}\sin\omega_s t , \\ f_1(t)&=\omega_s t e^{-t/\tau_s}\cos\omega_s t , \\ f_2(t)&=(t/\tau_s)e^{-t/\tau_s}\sin\omega_s t , \\ f_3(t)&=e^{-t/\tau_s}(\tfrac{1}{2}Q_s^{-1}\sin\omega_s t-\cos\omega_s t) . \end{aligned} \tag{3.15}$$

Therefore, the expansion for ν is

$$\nu=\nu_0+\nu_1\varepsilon+\nu_2\eta+\nu_3\zeta , \tag{3.16}$$

where

$$\nu_i=\int_0^\infty f_i(t)n(t)dt \qquad \text{for} \quad i=0,\dots,3 \tag{3.17}$$

can be viewed as four Gaussian random variables independent of the filter. Next, taking into account the dependence of N on ε, η, and ζ, we get

$$\nu/N=b(c_0+c_1\varepsilon+c_2\eta+c_3\zeta) , \tag{3.18}$$

where

$$\begin{aligned} &b=q_s^{-1}(8/S_h\tau_s)^{1/2} , \\ &c_0=\nu_0 , \quad c_1=\nu_1-(1-q_s^2)\nu_0 , \\ &c_2=\nu_2-(\tfrac{3}{2}-q_s^2)\nu_0 , \quad c_3=\nu_3 . \end{aligned} \tag{3.19}$$

Note that the c_i, like the ν_i, are Gaussian random variables. By adding expressions (3.11) and (3.18) we obtain $W/N=(S+\nu)/N$.

Now, we are ready to maximize $(S+\nu)/N$. The values $(\varepsilon,\eta,\zeta)=(\varepsilon_0,\eta_0,\zeta_0)$ for which

$$\nabla_{\varepsilon,\eta,\zeta}[(S+\nu)/N]=0 \tag{3.20}$$

are our best estimate of $(\varepsilon_s,\eta_s,\zeta_s)$ for a given experiment. By imposing (3.20) with the help of (3.11) and (3.18) we obtain the linear equations

$$\begin{bmatrix} 2\alpha_1 & -\beta_1 & -\beta_2 \\ -\beta_1 & 2\alpha_2 & -\beta_3 \\ -\beta_2 & -\beta_3 & 2\alpha_3 \end{bmatrix} \begin{bmatrix} \varepsilon_0 \\ \eta_0 \\ \zeta_0 \end{bmatrix} = \rho^{-1} b \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} . \tag{3.21}$$

The solution of these linear equations, rewritten using (3.13) and (3.19), is

$$\begin{aligned} \delta\omega_s/\omega_s&\equiv\varepsilon_0=k(d_0\nu_0+d_1\nu_1+d_2\nu_2+d_3\nu_3) , \\ \delta\tau_s/\tau_s&\equiv\eta_0=k(e_0\nu_0+e_1\nu_1+e_2\nu_2+e_3\nu_3) , \end{aligned} \tag{3.22}$$

with

$$\begin{aligned} d_0&=-(\tfrac{1}{4}+\tfrac{1}{16}Q_s^{-2})q_s^2+(\tfrac{9}{8}+\tfrac{1}{16}Q_s^{-2})-\tfrac{3}{4}q_s^{-2} , \\ d_1&=q_s^2-(\tfrac{3}{2}+\tfrac{1}{16}Q_s^{-2})+\tfrac{3}{4}q_s^{-2} , \\ d_2&=-q_s^2+\tfrac{3}{4} , \\ d_3&=(Q_s+\tfrac{1}{4}Q_s^{-1})q_s^4-(\tfrac{3}{2}Q_s+\tfrac{1}{4}Q_s^{-1})q_s^2+\tfrac{3}{4}Q_s , \\ e_0&=-(Q_s^2+\tfrac{1}{4})q_s^2+\tfrac{7}{2}Q_s^2-3Q_s^2q_s^{-2} , \\ e_1&=-q_s^2+\tfrac{3}{4} , \\ e_2&=q_s^2-(Q_s^2+\tfrac{1}{2})+2Q_s^2q_s^{-2} , \\ e_3&=-(Q_s+\tfrac{1}{4}Q_s^{-1})q_s^4+(Q_s+\tfrac{1}{8}Q_s^{-1})q_s^2-\tfrac{1}{2}Q_s , \\ k&=4\Delta^{-1}/(h_s q_s^2\tau_s) , \end{aligned} \tag{3.22a}$$

where Δ, the determinant of the 3×3 matrix in (3.21), is given by

$$\begin{aligned} \Delta=&-(Q_s^2+\tfrac{1}{2}+\tfrac{1}{16}Q_s^{-2})q_s^4 \\ &+(\tfrac{7}{2}Q_s^2+1+\tfrac{1}{32}Q_s^{-2})q_s^2 \\ &-(\tfrac{15}{4}Q_s^2+\tfrac{1}{2})+\tfrac{3}{2}Q_s^2q_s^{-2} . \end{aligned} \tag{3.22b}$$

Analogous expressions can be obtained for ζ_0, but we do not give them, since in this paper we are not interested in the accuracy of the start time. (Our only reason for including $\zeta=\omega_s t_k$ in the analysis was to take account of its impact on the accuracy of read-out of ω_s and τ_s.)

We should notice that ε_0 and η_0 will, in general, be correlated, since they are ultimately dependent on the same random process $n(t)$. Furthermore, since this process is Gaussian and the dependencies are linear, they will have a joint normal distribution, with probability density

$$f(\varepsilon_0,\eta_0)=\frac{1}{2\pi\sigma_{\varepsilon_0}\sigma_{\eta_0}(1-r_{\varepsilon_0\eta_0}^2)^{1/2}}\exp\left[-\frac{1}{2(1-r_{\varepsilon_0\eta_0}^2)}\left[\frac{\varepsilon_0^2}{\sigma_{\varepsilon_0}^2}-\frac{2r_{\varepsilon_0\eta_0}\varepsilon_0\eta_0}{\sigma_{\varepsilon_0}\sigma_{\eta_0}}+\frac{\eta_0^2}{\sigma_{\eta_0}^2}\right]\right] , \tag{3.23}$$

where $r_{\varepsilon_0\eta_0}$ is their *correlation coefficient*, and σ_{ε_0} and σ_{η_0} are the standard deviations of ε_0 and η_0, respectively (after integrating over the other variable). Once we have computed σ_{ε_0}, σ_{η_0}, and $r_{\varepsilon_0\eta_0}$ we will know from (3.23) all the statistical properties of ε_0 and η_0.

In order to compute σ_{ε_0}, σ_{η_0}, and $r_{\varepsilon_0\eta_0}$, we must express ε_0 and η_0 as linear combinations of the values of $n(t)$ [i.e., as integrals over $n(t)$]. By combining equations (3.17) and (3.22) we obtain the explicit expressions

$$\varepsilon_0=\int_0^\infty g(t)n(t)\ , \qquad \eta_0=\int_0^\infty h(t)n(t)\ , \tag{3.24}$$

where

$$g(t)=k\sum_{i=0}^{3} d_i f_i(t)\ ,$$
$$h(t)=k\sum_{i=0}^{3} e_i f_i(t)\ . \tag{3.24a}$$

Then by analogy with (3.5) and (3.8) we obtain, for the variances,

$$\begin{aligned}(\Delta\omega_s/\omega_s)^2 &\equiv \sigma_{\varepsilon_0}^2=\langle \varepsilon_0^2\rangle \\ &=(S_h/2)\int_0^\infty g^2(t)dt \\ &=(k^2S_h/2)\sum_{i,j=0}^{3} d_id_jI_{ij}\ , \\ (\Delta\tau_s/\tau_s)^2 &\equiv \sigma_{\eta_0}^2=\langle \eta_0^2\rangle \\ &=(S_h/2)\int_0^\infty h^2(t)dt \\ &=(k^2S_h/2)\sum_{i,j=0}^{3} e_ie_jI_{ij}\ ,\end{aligned} \tag{3.25}$$

where

$$I_{ij}\equiv\int_0^\infty f_i(t)f_j(t)dt\ . \tag{3.26}$$

Here we have defined the "typical fractional errors" $\Delta\omega_s/\omega_s$ and $\Delta\tau_s/\tau_s$ in the signal parameters to be their standard deviations. [These standard deviations should not be confused with the actual—but unknown—fractional errors $\delta\omega_s/\omega_s\equiv(\omega_0-\omega_s)/\omega_s$ and $\delta\tau_s/\tau_s=(\tau_0-\tau_s)/\tau_s$.] Explicit calculation of the integrals I_{ij} yields

$$\begin{aligned}&I_{00}=\tfrac{1}{4}\tau_sq_s^2\ ,\\ &I_{11}=\tfrac{1}{2}\tau_sQ_s^2(-4q_s^6+9q_s^4-6q_s^2+2)\ ,\\ &I_{22}=\tfrac{1}{8}\tau_sq_s^2(4q_s^4-9q_s^2+6)\ ,\\ &I_{33}=\tfrac{1}{4}\tau_s\ ,\\ &I_{01}=I_{10}=\tfrac{1}{4}\tau_sq_s^2(1-q_s^2)\ ,\\ &I_{02}=I_{20}=\tfrac{1}{8}\tau_sq_s^2(3-2q_s^2)\ ,\\ &I_{03}=I_{30}=0\ ,\\ &I_{12}=I_{21}=\tfrac{1}{2}\tau_sQ_s^2(-4q_s^6+11q_s^4-10q_s^2+3)\ ,\\ &I_{13}=I_{31}=-\tfrac{1}{4}\tau_sQ_sq_s^2\ ,\\ &I_{23}=I_{32}=\tfrac{1}{4}\tau_sQ_s(1-q_s^2)\ .\end{aligned} \tag{3.27}$$

To compute the correlation coefficient $r_{\varepsilon_0\eta_0}$ we begin by evaluating

$$\begin{aligned}r_{\varepsilon_0\eta_0}\sigma_{\varepsilon_0}\sigma_{\eta_0}&=\langle\varepsilon_0\eta_0\rangle\\ &=\left\langle\int_0^\infty g(t)n(t)dt\int_0^\infty h(t')n(t')dt'\right\rangle\\ &=\int_0^\infty dt\int_0^\infty dt'g(t)h(t')\langle n(t)n(t')\rangle\\ &=\int_0^\infty dt\int_0^\infty dt'g(t)h(t')(S_h/2)\delta(t-t')\\ &=(S_h/2)\int_0^\infty g(t)h(t)dt\ .\end{aligned} \tag{3.28}$$

Here we have used the equality $\langle n(t)n(t')\rangle=(S_h/2)\delta(t-t')$, which follows from the fact that the noise's correlation function $C_n(\tau)\equiv\langle n(t)n(t+\tau)\rangle$ is the cosine transform of its spectral density $S_h(f)$ (Wiener-Khintchine theorem). Next, using (3.24a) and (3.26) in (3.28) we obtain

$$\begin{aligned}\langle\varepsilon_0\eta_0\rangle&=r_{\varepsilon_0\eta_0}\sigma_{\varepsilon_0}\sigma_{\eta_0}\\ &=(k^2S_h/2)\sum_{i,j=0}^{3} e_id_jI_{ij}\ .\end{aligned} \tag{3.29}$$

From (3.25) and (3.29) we can draw the conclusion that the correlation coefficient $r_{\varepsilon_0\eta_0}$ is a function of Q_s only; i.e., it does not depend on the frequency or damping time of the signal separately. Moreover, it does not depend on the noise level that is present in the detector or on the initial amplitude of the signal.

Using the fact that $Q_s>2$ and $q_s\approx1$, we can see that only one or a few of the terms in the sums (3.25) and (3.29) make significant contributions to σ_{ε_0}, σ_{η_0}, and $r_{\varepsilon_0\eta_0}$. By identifying and evaluating the dominant terms, we can find the following analytical expressions for the uncertainties in the signal parameters and their correlation:

$$\begin{aligned}&\Delta\omega_s/\omega_s\equiv\sigma_{\varepsilon_0}=\rho^{-1}Q_s^{-1}f_\omega(Q_s)\ ,\\ &\Delta\tau_s/\tau_s\equiv\sigma_{\eta_0}=2\rho^{-1}f_\tau(Q_s)\ ,\\ &C_{\omega_s\tau_s}\equiv\mathrm{Corr}(\delta\omega_s/\omega_s;\delta\tau_s/\tau_s)\\ &\qquad\equiv r_{\varepsilon_0\eta_0}=-\tfrac{1}{2}Q_s^{-1}f_{\omega\tau}(Q_s)\ .\end{aligned} \tag{3.30}$$

Here ρ [Eq. (3.13)] is the signal-to-noise ratio, aside from small corrections that are shown in Eq. (3.11); and f_ω, f_τ, $f_{\omega\tau}$ are "correction functions" which depend on the quality factor Q_s and are very close to unity. These correction functions are tabulated in Table I.

These results can be summarized as follows: The fractional uncertainties in the frequency and damping time are inversely proportional to the signal-to-noise-ratio ρ. The uncertainty in the frequency is very nearly inversely proportional to the signal's quality factor, while the uncertainty in the damping time is essentially independent of the quality factor. Finally, the correlation coefficient, which is independent of the signal-to-noise ratio, is nearly inversely proportional to Q_s; and, given the numerical values of Q_s for the fundamental normal mode with

TABLE I. Corrections for the uncertainties and correlation of the signal's frequency and damping time as functions of Q_s. These corrections are defined in Eqs. (3.30).

Q_s	f_ω	f_τ	$f_{\omega\tau}$
2.1	0.9480	1.1021	0.7517
5.0	0.9902	1.0196	0.9513
10.0	0.9975	1.0050	0.9876
15.0	0.9989	1.0022	0.9945
20.0	0.9994	1.0012	0.9969
25.0	0.9996	1.0008	0.9980
30.0	0.9997	1.0006	0.9986
35.0	0.9998	1.0004	0.9990
40.0	0.9998	1.0003	0.9992
45.0	0.9999	1.0002	0.9994
50.0	0.9999	1.0002	0.9995
55.0	0.9999	1.0002	0.9996
60.0	1.0000	1.0001	0.9996
65.0	1.0000	1.0001	0.9997
70.0	1.0000	1.0001	0.9997
75.0	1.0000	1.0001	0.9998
80.0	1.0000	1.0001	0.9998

$l=m=2$ of Kerr black holes, the correlation coefficient turns out to be small for all the range of interest, and absolutely negligible for rapidly rotating holes (high Q_s). This means that the errors in the values of ω_s and τ_s obtained in one specific experiment will be essentially independent of each other.

From these results we can see that the condition $\varepsilon, \eta \ll 1$, on which our analysis relies, is satisfied if and only if

$$\rho \gg 1 \ . \tag{3.31}$$

By using the results obtained to check the errors in the approximate expansions (3.11) and (3.14) for typical values of ε_0 and η_0, it is found that with $\rho=5$ these are $\lesssim 10\%$ for all Q_s, so the results can be considered valid for $\rho \gtrsim 5$.

Let us now briefly look at the initial amplitude h_s of the signal. We will show how it can be determined from the experiment, and why the lack of knowledge of its precise value does not significantly affect our knowledge of the uncertainties and correlation coefficient of the frequency and damping time.

From the experimental data, we have to take $W=S+\nu$ as our best estimate for S. Then, from this best estimate, using (3.3) and using ω_0, τ_0, t_0 in place of ω_k, τ_k, t_k, we obtain a best estimate of h_s. Because of the presence of ν in $W=S+\nu$ and the deviations of ω_0, τ_0, t_0 from ω_s, τ_s, t_s, this procedure produces fractional errors of order ρ^{-1} in our estimated h_s. Similarly, there are fractional errors of order ρ^{-1} in our estimate W/N for the value of $\rho=S/N$—and these errors produce fractional errors of order ρ^{-1} in our knowledge of the values of $\Delta\omega_s/\omega_s$, $\Delta\tau_s/\tau_s$, and $C_{\omega_s\tau_s}$ [Eqs. (3.30)]. For $\rho \gtrsim 5$ we can regard these errors as negligible.

IV. BLACK-HOLE PARAMETERS

We now describe the final step of our analysis: the translation from the values and uncertainties for the signal parameters ω_s and τ_s into those of the mass M and angular momentum parameter a of the black hole. In order to make this translation, we need the functional relationship between the two pairs of variables (ω_s, τ_s) and (M, a). That relationship has been computed numerically by Leaver[5] using the theory of small perturbations of Kerr black holes. We shall write that relationship in the form[4]

$$\omega_s = f(a)/M \ , \qquad \tau_s = g(a) M \ , \tag{4.1}$$

where the functions $f(a)$ and $g(a)$ are plotted in Fig. 3(c) of Leaver (Ref. 5); and we shall write the inverse relation as

$$a=\phi(Q_s) \ , \qquad M=\psi(Q_s)/\omega_s \ . \tag{4.2}$$

The functions $\phi(Q_s)$ and $\psi(Q_s)$ can be determined from tables of $f(a)$ and $g(a)$ (obtained by private communication from Leaver, since they are not tabulated in Ref. 5 for the case $l=m=2$).

It is interesting to notice that the angular momentum parameter a depends only on the resonance factor Q_s and, furthermore, that this dependence is monotonic and thus invertible. An analytical expression for the inverse function is

$$\begin{aligned} Q_s &= \tfrac{1}{2}\omega_s\tau_s = f(a)g(a) \\ &= 2(1-a)^{-0.45} f_Q(a) \ , \end{aligned} \tag{4.3}$$

where the approximate expression is corrected by f_Q, which is close to unity, and is tabulated in Table II. For the sake of completeness we also give an analytical expression for the function $f(a)$, which determines the frequency ω_s:

$$f(a) = [1-0.63(1-a)^{0.3}] f_f(a) \ , \tag{4.4}$$

where the correction is also given in Table II.

We now want to know how the uncertainties in the signal parameters are translated into the corresponding uncertainties of the hole's parameters. In order to determine this we replace ω_s and τ_s in (4.2) with the values ω_0 and τ_0 obtained by the procedure described in Sec. III, thus obtaining approximate values a_0 and M_0 (the experimenter's best estimates) for the hole's angular momentum parameter a and mass M. These can be written as

$$a_0 = a + \xi \ , \qquad M_0 = M(1+\mu) \ , \tag{4.5}$$

where ξ and μ are the errors the experimenter makes. Then, using expressions (3.10) and the approximations $\varepsilon_0, \eta_0 \ll 1$, we obtain linearized expressions for these errors:

$$\begin{aligned} \xi &= \delta a = A\varepsilon_0 + B\eta_0 \ , \\ \mu &= \delta M/M = C\varepsilon_0 + D\eta_0 \ , \end{aligned} \tag{4.6}$$

with A, B, C and D given by

$$\begin{aligned} A &= B = Q_s \phi'(Q_s) \ , \\ D &= C+1 = Q_s \psi'(Q_s)/\psi(Q_s) \ . \end{aligned} \tag{4.7}$$

TABLE II. Corrections for quality factor, signal-to-noise ratio, and hole's mass and angular momentum parameter as functions of a. These corrections are defined in Eqs. (4.3), (4.4), (4.10), (4.11), and (4.15).

a	f_Q	f_f	f_a	f_M	f_{Ma}	f_ρ	f'_a	f'_M	f'_ρ
0.0001	1.0501	1.0100	1.0237	0.9417	1.0000	0.9969	1.0044	0.9239	1.0192
0.1000	1.0402	0.9934	1.0165	0.9541	1.0000	0.9947	1.0097	0.9478	1.0067
0.2000	1.0297	0.9789	1.0094	0.9685	1.0000	0.9922	1.0162	0.9750	0.9933
0.3000	1.0182	0.9668	1.0026	0.9846	1.0001	0.9895	1.0242	1.0058	0.9789
0.4000	1.0057	0.9572	0.9954	1.0025	1.0001	0.9864	1.0332	1.0406	0.9634
0.5000	0.9918	0.9505	0.9870	1.0215	1.0002	0.9828	1.0429	1.0794	0.9464
0.6000	0.9763	0.9475	0.9762	1.0409	1.0001	0.9786	1.0521	1.1219	0.9278
0.7000	0.9587	0.9494	0.9605	1.0589	0.9997	0.9739	1.0585	1.1670	0.9074
0.8000	0.9389	0.9587	0.9442	1.0724	0.9990	0.9688	1.0666	1.2114	0.8853
0.9000	0.9184	0.9815	0.9233	1.0764	0.9977	0.9649	1.0679	1.2450	0.8646
0.9200	0.9149	0.9891	0.9215	1.0755	0.9973	0.9650	1.0689	1.2475	0.8621
0.9400	0.9124	0.9983	0.9171	1.0722	0.9969	0.9659	1.0644	1.2445	0.8616
0.9600	0.9121	1.0099	0.9036	1.0636	0.9963	0.9685	1.0441	1.2289	0.8655
0.9800	0.9187	1.0252	0.9246	1.0512	0.9959	0.9763	1.0468	1.1902	0.8832
0.9850	0.9236	1.0297	0.9285	1.0427	0.9958	0.9806	1.0383	1.1661	0.8942
0.9900	0.9326	1.0346	0.9408	1.0307	0.9956	0.9876	1.0304	1.1289	0.9131
0.9950	0.9523	1.0392	0.9695	1.0083	0.9955	1.0018	1.0169	1.0577	0.9533

Given the shapes of the functions $\phi(Q_s)$ and $\psi(Q_s)$, it is found empirically that the linear expansion (4.6) is a good approximation as long as the condition $\varepsilon_0, \eta_0 \ll 1$ holds. More precisely: using σ_{ε_0}, σ_{η_0} from (3.30) as typical values for ε_0, η_0, it turns out that the differences between the exact ξ and μ [defined by (4.5)] and their approximate values [defined by (4.6)] are $\lesssim 30\%$ for $\rho \gtrsim 10$. So we will consider the results obtained below to be valid only for $\rho \gtrsim 10$.

Finally, we need to determine the probability distribution of ξ and μ. Since ε_0 and η_0 have a joint Gaussian probability distribution, ξ and μ, which are the linear combinations (4.6) of ε_0 and η_0, also have a joint Gaussian distribution[11] with variances and correlation given by

$$\sigma_\xi^2 = A^2\sigma_{\varepsilon_0}^2 + B^2\sigma_{\eta_0}^2 + 2ABr_{\varepsilon_0\eta_0}\sigma_{\varepsilon_0}\sigma_{\eta_0} ,$$
$$\sigma_\mu^2 = C^2\sigma_{\varepsilon_0}^2 + D^2\sigma_{\eta_0}^2 + 2CDr_{\varepsilon_0\eta_0}\sigma_{\varepsilon_0}\sigma_{\eta_0} , \tag{4.8}$$
$$r_{\xi\mu}\sigma_\xi\sigma_\mu = AC\sigma_{\varepsilon_0}^2 + BD\sigma_{\eta_0}^2 + (AD+BC)r_{\varepsilon_0\eta_0}\sigma_{\varepsilon_0}\sigma_{\eta_0} .$$

Correspondingly, the typical errors in the estimated black-hole mass M and angular momentum parameter a, and the correlation of those errors, have the general form

$$\Delta a \equiv \sigma_\xi = \rho^{-1}F(Q_s) = \rho^{-1}\hat{F}(a) ,$$
$$\Delta M/M \equiv \sigma_\mu = \rho^{-1}G(Q_s) = \rho^{-1}\hat{G}(a) , \tag{4.9}$$
$$C_{Ma} \equiv \mathrm{Corr}(\delta a ; \delta M/M)$$
$$\equiv r_{\xi\mu} = H(Q_s) = \hat{H}(a) ,$$

where the functions F, G, H and $\hat{F}$, $\hat{G}$, $\hat{H}$ are computable from Eqs. (3.30), (4.2), (4.7), (4.8), and Leaver's numerical results for $\phi(Q_s)$ and $\psi(Q_s)$. It is important to note that both the uncertainties Δa and $\Delta M/M$ are inversely proportional to the output signal-to-noise ratio ρ, and that their correlation depends only on a, and not on M. The author has evaluated the functions $\hat{F}$, $\hat{G}$, and $\hat{H}$ by the above prescription. The numerical results can be expressed in the form

$$\Delta M/M = 2.2\rho^{-1}(1-a)^{0.45}f_M(a) , \tag{4.10a}$$
$$\Delta a = 5.9\rho^{-1}(1-a)^{1.06}f_a(a) , \tag{4.10b}$$
$$C_{Ma} = 0.976 f_{Ma}(a) , \tag{4.10c}$$

where f_M, f_a, and f_{Ma} are corrections that are close to unity and are given in Table II.

Expression (4.10b) shows that, for a given ρ, the uncertainty in the angular momentum parameter a decreases in a nearly linear way as a increases, vanishing for $a=1$. If we now take into consideration the multiplicative factor, it turns out that to get a reasonable precision in an estimate of a black hole's angular momentum, the hole would have to rotate very rapidly, or else we would need a very high signal-to-noise ratio.

By contrast with a, the fractional error $\Delta M/M$ in the mass does not decrease so rapidly with increasing rotation of the black hole. However, the multiplicative factor in (4.10a) is small enough that $\Delta M/M$ can be small for all the range of a (including the Schwarzschild case, $a=0$) with just a moderate signal-to-noise ratio ($\rho \gtrsim 10$). There is a crossover at $a \simeq 0.8$; above the crossover $\Delta a < \Delta M/M$, i.e., a is better determined than M.

We should note the fact that ρ is not independent of a, since it is defined in terms of τ_s (or ω_s) and Q_s [Eq. (3.13)]. For fixed signal amplitude h_s and detector noise S_h, ρ increases with increasing a as given by

$$\rho = h_s[2/(\omega_s S_h)]^{1/2}(1-a)^{-0.22}f_\rho(a)$$
$$= h_s S_h^{-1/2}(GM/c^3)^{1/2}2.26(1-a)^{-0.15}f'_\rho(a) , \tag{4.11}$$

where in the last step we replaced ω_s with its dependency on a and M as given by Eq. (4.1), and where f_ρ and f'_ρ are corrections given in Table II.

The correlation between the two uncertainties [Eq. (4.10c)] turns out to be remarkably high: it is essentially independent of the rotation rate and very close to unity. This is due to the fact that δa and $\delta M/M$ are a linear combination of the $\delta\omega_s/\omega_s$ and $\delta\tau_s/\tau_s$ [Eq. (4.6)], and since the damping time is much less well determined than the frequency ($\delta\omega_s/\omega_s \ll \delta\tau_s/\tau_s$), the uncertainties in the mass and angular momentum are produced almost entirely by the uncertainty in the damping time. Thus, the errors δa and $\delta M/M$ must be highly correlated. This high correlation means that if the error is big (or small) in one of the parameters a or M, it is highly likely that the error in the other is also big (or small) and of the same sign.

We can quantify this statement a little better by studying the pair of uncorrelated variables

$$z_{\pm} \equiv \frac{\xi}{\sigma_\xi} \pm \frac{\mu}{\sigma_\mu} = \frac{\delta a}{\Delta a} \pm \frac{\delta M}{\Delta M} , \tag{4.12}$$

which are linear combinations of $\delta a \equiv \xi$ and $\delta M/M \equiv \mu$. Using the analog of Eq. (4.8), it can be shown that the correlation of z_+ and z_- vanishes and that their standard deviations are given by

$$\sigma_{z_\pm} = \sqrt{2(1 \pm r_{\xi\mu})} . \tag{4.13}$$

We see that, as we should have expected, one of these variables, z_+, is very poorly determined relative to the other, since its standard deviation is nearly equal to 2 for all a, while the other, z_-, is very well determined, since its standard deviation is very small, $\sigma_{z_-} \simeq 0.22$ independent of a. This means that we can expect to have

$$|\delta a/\Delta a - \delta M/\Delta M| \lesssim 0.22 . \tag{4.14}$$

That is, for one specific experiment, the ratio of the actual error δa that we make in our estimate of a to the typical error Δa does not differ from the ratio of the actual error in the mass δM to the typical error ΔM by more than 0.22 typically. Thus, the errors in a and M have almost the same relative magnitude and the same sign. This might be of importance in case there is some independent and more precise determination of either the mass or the angular momentum but not both. Then we could readily obtain an almost equally better estimate of the other parameter with very high certainty.

Finally, we reexpress $\Delta M/M$ and Δa of Eqs. (4.10) in a form that depends solely on the signal's amplitude h_s, the detectors' spectral density of noise S_h, and the hole's M and a. This form is obtained by inserting expression (4.11) for $\rho(M,a,S_h,h_s)$ into (4.10). The result is

$$\Delta M/M = 0.14 \left[\frac{10^{-20}}{h_s}\right] \left[\frac{S_h^{1/2}}{10^{-23}\ \mathrm{Hz}^{-1/2}}\right] \times \left[\frac{10 M_\odot}{M}\right] (1-a)^{0.60} f'_M(a) , \tag{4.15a}$$

$$\Delta a = 0.37 \left[\frac{10^{-20}}{h_s}\right] \left[\frac{S_h^{1/2}}{10^{-23}\ \mathrm{Hz}^{-1/2}}\right] \times \left[\frac{10 M_\odot}{M}\right] (1-a)^{1.21} f'_a(a) , \tag{4.15b}$$

where f'_M and f'_a are correction functions tabulated in Table II, and where one should keep in mind that the waves' frequency ω_s and damping time τ_s are given in terms of M and a by Eqs. (4.1), (4.3), and (4.4).

V. ISSUES FOR FUTURE RESEARCH

This paper constitutes a first, approximate, study of the problem of extracting black-hole parameters from broadband gravitational-wave data. Several issues not treated here deserve future study.

This paper's analysis is valid only for rather large signal-to-noise ratios, $\rho \gtrsim 10$. However, most gravitational-wave bursts observed by future Earth-based detectors are likely to have $\rho \simeq 5$ [see Eq. (34) of Ref. 8]. Our analysis could be extended to such bursts (or even smaller ρ's) by using a numerical implementation of our optimal filter algorithm, together with a numerical (Monte Carlo) simulation of the detector noise. This could also be achieved by evaluating exactly the uncertainties in the black hole's mass and angular momentum, instead of using a linear approximation [Eq. (4.6)], since we found that it is this step that introduces the largest errors, reducing the range of validity of the results from $\rho \gtrsim 5$ to $\rho \gtrsim 10$. We should also note that the Monte Carlo approach would be worthwhile in itself, since it would also make it possible to analyze the effect on the accuracy of the estimates due to arbitrary transient waveforms and other changes in our initial assumptions.

In this paper attention was restricted to black-hole events in which only the most slowly damped, $l = m = 2$ mode is excited. While many black-hole events should satisfy this restriction (see the abstract and Introduction), others will not. For example, axisymmetric collapse will excite only $m = 0$ modes and is likely to excite several such modes significantly.[14] It would be useful to extend this paper's analysis to such multimode situations.

This paper ignored the gravitational-wave transient that precedes the ringdown waves. It would be useful to redo the analysis with waveforms that include the transients. One especially important case would be the gravitational waves from the spiraling orbital motion and the coalescence of a two-hole binary system to form a single, larger hole. In this case the full waveform would consist of a Keplerian, spiraling portion (periodic with increasing frequency) [Eqs. (42) of Ref. 8], followed by a several-cycle coalescence wave, followed by the ringdown wave. Although the precise form of the coalescence wave is not yet known (future supercomputer simulations will tell it to us), a reasonable guess at it could be made for exploratory purposes. It would be interesting to see how much can be learned about the two initial holes and the final hole, in the presence of detector noise, from the combination of the three pieces of the waveform: spiraling, coalescence, and ringdown. Such a study would constitute a marriage and extension of this paper's results and methods, and those of Smith,[15] who has studied the extraction of information from the spiraling portion of the waveform.

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