Time-dependent collision cascades

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We use the linearized Boltzmann equation to discuss the time dependence of atomic collision cascades. The atoms are presumed to interact via a power-law potential, the power being denoted $(-2/a)$. We wish to ascertain whether the equation, and/or certain special solutions, show striking changes in behavior at certain special values of a. We find a value of a which distinguishes energy-conserving cascades from anomalous cascades, and another which signals the failure of the linear approximation. Scaling, and also the emergence of similarity solutions, are discussed.

I. INTRODUCTION

We discuss two aspects of the linear theory of atomic collision cascades—their connection with fractal geometry, and the fact that cascades produced by certain interatomic potentials do not conserve energy. We begin by noting that the theory, a mature theory, has been enlivened recently[1,2] by the notion of the fractal nature of the cascade. A fractal dimension D may be ascribed to a cascade in several ways.[1,2] All arguments are simple, and lead to the same dimension, which depends upon the nature of the scattering law. The discussion is based upon the "power-law" model of the scattering, a model containing a single parameter which we denote by a. ($2m$ in Refs. 1 and 2) and which implies scattering by a potential $r^{-2/a}$. One considers the range $0<a<2$ or $0<m<1$. The simple arguments depend upon the energy dependence of what is, in effect, a mean free path for collision. For the power model the length depends upon E as E^a, and the fractal dimension D turns out to be $1/a$, for a certain interval.

Assigning an exotic dimension to a cascade may have a certain appeal but one would hope that the connection would improve our insight into physical phenomena as well. In this regard, both sets of authors argue that when the fractal is "space filling" ($D=3$) the associated cascade is, in fact, a "spike"—a significant fraction of the atoms in a small volume have been set into motion. Thus there is something special about the point $a=\frac{1}{3}$ in the interval $0<a<2$. One suspects that a linear picture is unjustified when $a<\frac{1}{3}$; in any case one would be reluctant to ascribe to a physical process a dimension greater than 3. But there is something special about the point $a=1$ also, for the two-branched "tree" that one draws to represent the cascade has finite, nonzero measure for $1<a<2$. In that half-interval $D=1$ and the cascade does not deserve the adjective "fractal." Thus the fractal region appears to be limited to $\frac{1}{3}<a<1$.

One finds other special points in the half-interval $0<a<1$. Winterbon *et al.*[2] suggest that the onset of nonlinearity occurs "earlier," at $a=\frac{2}{3}$, when the overlap of subcascades becomes important. Finally, it has been noted[3] (in the context of solutions to the associated transport equation) that the point $a=\frac{1}{2}$ (Maxwell molecules) separates one type of mathematical behavior from another. Thus, simple physical arguments suggest that the points $a=\frac{1}{3},\frac{1}{2},\frac{2}{3},1$ are special. Indeed, if the temporal evolution of a real cascade is best represented by slowly varying $a(t)$, the nature of the cascade may change as $a(t)$ passes through the special points.

In this paper we consider solutions to appropriate transport (Boltzmann) equations which depend upon the parameter a. [We should like to know whether $a=\frac{1}{3},\frac{2}{3},1$ distinguishes one type of solution (distribution) from another.] The case $a=\frac{1}{2}$ is so interesting that it forms the second "aspect" of our study. It seems not to be generally appreciated that, in a certain sense, cascades with $a>\frac{1}{2}$ do not conserve energy, while those with $a\leq\frac{1}{2}$ do. This surprising fact has been noted by Williams.[4] Here, we embellish his observations somewhat. Failure of an "obvious" conservation law (mass) has been noted recently by scientists studying the phenomenon of "fragmentation."[5,6] One finds it also in the study of ion-implanation (neutron-transport theory).[7] But for Ref. 4 all of these observations have been quite recent. It is surprising, then, to find a clear description of the nonconservation of measure in a class of Markov processes (of which the cited papers are an example) in a paper written several decades ago.[8] The author, A. F. Filippov, acknowledges that the problem was suggested to him by the mathematician A. N. Kol'mogorov.

Finally, it is not obvious that the Boltzmann equation is able to subsume all of the physical ideas which lead to the special values of a. Solutions to the equation express the properties of ensembles of cascades, and the ensemble averaging can wash out fine distinctions. On the other hand, the solutions properly express the stochastic nature of collision chains, and provide a more accurate picture than that of the idealized tree,[1] whose branches are given a fixed length appropriate to the mean energy loss. In any case, one wants to see what the Boltzmann equation offers, and that is the purpose of this paper.

II. ANALYSIS

We consider a one-component system of infinite extent. In the linear picture the majority of the atoms are at rest; the Boltzmann equation describes the "hot" component through its distribution function, $n(\mathbf{r},E,\hat{\Omega},t)$. The equation is[9,10]

$$\frac{\partial}{\partial t}n(\mathbf{r},E,\hat{\Omega},t)+v\cdot\frac{\partial}{\partial r}n(\mathbf{r},E,\hat{\Omega},t)=\int dE'd\hat{\Omega}'\{v'n(\mathbf{r},E',\hat{\Omega}',t)[\Sigma(E',E,\hat{\Omega}'\cdot\hat{\Omega})+\Sigma(E',E'',\hat{\Omega}'\cdot\hat{\Omega}'')]$$
$$-vn(\mathbf{r},E,\hat{\Omega},t)\Sigma(E,E',\hat{\Omega}\cdot\hat{\Omega}')\}+S(\mathbf{r},E,\hat{\Omega},t)\ . \qquad (1)$$

It is familar except, perhaps, for the two in-scattering terms. The first counts "incident" particles which are scattered into $(E,\hat{\Omega})$ from $(E',\hat{\Omega}')$; the second counts "recoil" particles scattered into $(E,\hat{\Omega})$ from rest at a rate corresponding to the scattering of incident particles from $(E',\hat{\Omega}')$ into $(E'',\hat{\Omega}'')$ where $E''=E'-E$ and the angles $\cos^{-1}(\hat{\Omega}'\cdot\hat{\Omega})$ and $\cos^{-1}(\hat{\Omega}'\cdot\hat{\Omega}'')$ sum to $\pi/2$.

The cascade is generated by an atom of energy E_0 introduced at $r=0$. The appropriate equation for the distribution function is quite complicated. We begin by considering only the integrated quantity $n(E,t)$, obtained by integrating Eq. (1) over $\mathbf{r}$ and $\hat{\Omega}$. We obtain the simpler equation

$$\frac{\partial}{\partial t}n(E,t)=\int dE'\{v'n(E',t)[\Sigma(E',E)+\Sigma(E',E'-E)]$$
$$-vn(E,t)\Sigma(E,E')\}$$
$$+\delta(t)\delta(E-E_0)\ .$$

The crucial quantity $\Sigma(E,E')$ (the "P_0-scattering kernel," in the language of neutron physics) is deduced from the atom-atom potential. If the two-body potential is not cut off, but is permitted to extend to infinity, $\Sigma(E,E')$ diverges as $E'\to E$, and the total scattering cross-section is undefined. In the power-law model the precise form of $\Sigma(E,E')$ is

$$\Sigma(E,E')=\frac{\Sigma(E)}{E}f(T/E),\quad T=E-E'$$
$$\Sigma(E)=\Sigma_*\left[\frac{E_*}{E}\right]^{2m}=\frac{A}{E^{2m}}=\frac{A}{E^a}\ ,$$
$$f(x)=\begin{cases}\dfrac{1}{x^{1+m}}, & 0<x\le 1,\quad 0<m<1\\ 0, & \text{otherwise.}\end{cases}$$

Thus

$$\tau(E)\frac{\partial}{\partial t}F(E,t)+F(E,t)\int_0^E\frac{dE'}{E}f(1-E'/E)$$
$$=\int_E^\infty\frac{dE'}{E'}F(E',t)[f(1-E/E')+f(E/E')]$$
$$+\delta(t)\delta(E-E_0)\ , \qquad (2)$$

where $v\tau(E)=[\Sigma(E)]^{-1}$ and $F=v\Sigma(E)n$. Though, strictly speaking, collisions neither begin nor end in this picture, it is appropriate to think of $[\Sigma(E)]^{-1}$, whose dimension is length, as a flight length between (significant) collisions, for an atom of energy E, and $\tau(E)$ as a corresponding flight time. Also, $l(E)\equiv[\Sigma]^{-1}$.

One can do a certain amount of analysis of Eq. (2). But the model has a limiting case that is irresistable—the "hard-sphere" (HS) approximation, in which $f(x)$ is set equal to unity. The singular, forward-focussed scattering in the center-of-mass system, chracteristic of the E^{-a} cross section, is replaced by isotropic scattering. Doubtless this model, which is treated less than enthusiastically by workers in the field, can lead to serious error in situations where only a few collisions are involved, and will lead to quantitative error in general. But it should show qualitatively features quite accurately. We begin by studying the hard-sphere model.

A. Hard-sphere model

The equation is now

$$\tau(E)\frac{\partial}{\partial t}F(E,t)+F(E,t)=2\int_E^{E_0}\frac{dE'}{E'}F(E',t)$$
$$+\delta(t)\delta(E-E_0)\ . \qquad (3)$$

Since we will be interested in the (mean) energy present in the cascade, we multiply Eq. (3) by E and integrate from E to E_0. Thus

$$\frac{\partial}{\partial t}W(E,t)+E^2\int_E^{E_0}\frac{dE'}{E'}F(E',t)=E_0\delta(t)\ , \qquad (4)$$

where

$$W(E,t)=\int_E^{E_0}dE'E'\tau(E')F(E',t)\ .$$

The change in (mean) energy of atoms in the interval (E,E_0) is expressed as an increase due to the introduction of "hot" atoms, and a decrease caused by the slowing down of atoms below E. This term may be thought of as a current density of energy, evaluated at the lower boundary of the interval.

If we remove the time variation by Laplace transform,

$$\tilde{F}(E,S)=\int_0^\infty dt\ e^{-st}F(E,t)\ ,$$

the equation in E may be solved, by differentiation, to obtain[11]

$$\tilde{F}(E,s)=\frac{\delta(E-E_0)}{1+s\tau_0}+\frac{2}{E_0}\frac{1}{1+s\tau_0}\frac{1}{1+s\tau(E)}$$
$$\times\exp\left[2\int_E^{E_0}\frac{dE'}{E'}\frac{1}{1+s\tau(E')}\right]\ ,$$
$$\tau_0=\tau(E_0)\ . \qquad (5)$$

The transform of the energy contained in the cascade is

$$s\tilde{W}(E=0,s)=E_0-\frac{E_0}{1+s\tau_0}\exp\left[-\int_0^{E_0}\frac{dE}{E}\frac{2s\tau(E)}{1+s\tau(E)}\right]\ , \qquad (6)$$

and since we believe in the conservation of energy we expect to find $s\tilde{W}(E=0,s)=E_0$. That is, the (transform of the) time derivative of the total energy in the cascade vanishes. But this result requires that the exponential

vanishes—that the integral diverges. The divergence, in turn, depends upon the behavior of the mean-free time between collisions as the energy falls to zero. For example, if $\tau(E)$ decreases as any positive power of E, it would appear that energy is not being conserved. In terms of our parameters a and m, the critical values are $a=\frac{1}{2}$ and $m=\frac{1}{4}$. Thus, Maxwell molecules are the "marginal" case. We shall call the energy-conserving case "normal," the other, "anomalous."

Equation (4) gives some insight into the apparent paradox. First inspect Eq. (5) for $\tilde{F}(E,s)$, as $E\to 0$, to obtain

$$\tilde{F}(E,s)\sim f(s)/\tau(E),\quad \tilde{F}(E,s)\sim g(s)/E^2\ ,$$

for the normal and anomalous cases, respectively. Next, evaluate the term corresponding to the energy current density as $E\to 0$. One finds that the limiting value of the current is zero in the normal case, and is some nonzero constant in the anomalous case. In the anomalous case, there is a flow of energy "into" $E=0$, which behaves as a sink.

The following argument shows another aspect of the anomalous case. Neglect fluctuation and regard the cascade as an atom having energy E_0 and colliding after an interval $\tau(E_0)$, creating two atoms with energies $E_1=\frac{1}{2}E_0$. These collide after an interval $\tau(E_1)$, creating four atoms with energies $E_2=\frac{1}{2}E_1$, and so on. The time for n collisions, bringing one to E_n, is

$$\sum_{k=0}^{n-1}\tau(E_k)\sim\int_0^{n-1}dk\ \tau(E_k)\propto\int_{E_{n-1}}^{E_0}\frac{dE'}{E'}\tau(E')\ .$$

But the integral is essentially the one we encountered earlier; if it converges as $E_n\to 0$ we have the anomalous case, Here that case shows itself via the feature that the cascades reaches *any* energy, no matter how small, in less than some finite time. On the other hand, a divergent integral indicates that the smaller the final energy, the longer the time required. One can get more explicit results by taking power-law behavior for $\tau(E)$. Thus $\tau(E)\sim E^{a'}$ gives a geometric series that is summable in the anomalous case.

Since we believe that the atoms taking part in the cascade share a fixed amount of energy, which is conserved, we must regard this curious effect as a failure of the Boltzmann equation as a description of nature. The physical basis for the failure is clear enough. The equation is based upon the notion of a sequence of collisions—a collision followed by a free flight, then another collision, etc. Implicit here is the assumption that the duration of a collision is negligible—that the ratio "duration of encounter" to "duration of flight between encounters" is negligible. Thus, one should be cautious when working with a Boltzmann equation which permits the (mean) duration of flight $\tau(E)$ to vanish. It is not necessarily fatal, for the "analytic continuation" of a model beyond its *a priori* limits sometimes gives reasonable results. But not this time.

1. *Solution for power-law cross sections*

When $\tau(E)=cE^{a'}=\tau_0(E/E_0)^{a'}$ $(-\frac{1}{2}<a'<\frac{3}{2}$, where $a'=a-\frac{1}{2})$ the Laplace inversion may be carried out to obtain[12]

$$F(E,t)=\frac{\delta(E-E_0)}{\tau_0}e^{-t/\tau_0}+\frac{2}{E_0\tau_0}\left[\frac{t}{\tau(E)}\right]\times\begin{cases}e^{-t/\tau(E)}{}_1F_1\left[1+\frac{2}{a'};2;\left[\frac{1}{\tau(E)}-\frac{1}{\tau_0}\right]t\right]\\ e^{-t/\tau_0}{}_1F_1\left[1-\frac{2}{a'};2;\left[\frac{1}{\tau_0}-\frac{1}{\tau(E)}\right]t\right]\end{cases}\tag{7}$$

The two forms are useful for $a'>0$ and $a'<0$, respectively, the function ${}_1F_1$ being positive for positive argument when its parameters are positive. In our terminology, the case $a'>0$ is anomalous, while $a'<0$ (as well as $a'=0$) is normal. In terms of the parameter m, the value $\frac{1}{4}$ separates the two cases.

The solution in terms of special functions is convenient. One can verify the behavior of the collision density as $E\to 0$, noted earlier, for the two cases. Then one can note the special case $a'=1$, which leads to an explicit expression for the decay of the energy,

$$E(t)=E_0e^{-t_1}(1+t_1+\tfrac{1}{2}t_1^2),\quad t_1=t/\tau_0\ .$$

The solutions show the development of similarity solutions quite clearly. In the anomalous case one sees that for energies such that $\tau(E)\ll\tau_0$ the collision density is a function of $x=t/\tau(E)$ alone. That function behaves as $x^{2-2/a'}$ for x large (E small). It does not possess all positive moments. A self-similar solution is obtained in the normal case for all E,t such that $(1/\tau_0-1/\tau(E))t\gg 1$. That function is the civilized xe^{-x} and possesses all positive moments.

2. *From cascade to spike*

Consider now the argument given by Cheng, Nicolet, and Johnson[1] (CNJ) identifying $a=\frac{1}{3}$ with the formation of a spike. Two lengths are introduced, $l_0=l(E_0)$, and $l_*=n_0^{-1/3}$. The first length, the free path associated with source energy, is (roughly) the spatial size of the cascade; the second, inferred from n_0, the density of undisturbed atoms, is a measure of the atomic "graininess" of the medium. The ratio l_0/l_* is much greater than unity. Then, describing the cascade by a simple tree, whose branches are shrunk by a factor of $1/2^a$ in each generation, one calculates the number of atoms taking part in the cascade (2^N) if N collisions are required to reach scale l_*. A comparison of densities leads one at once to $2(l_0/l_*)^{1/a-3}$ as a measure of the fraction of atoms disturbed, a measure of the hypothesis of linearity. Since $l_0/l_*\gg 1$, the critical nature of $a=\frac{1}{3}$ $(a'=-\frac{1}{6})$ is obvious.

Does the argument hold in kinetic theory? We want to count all of the collisions occurring in the interval (E,E_0) for all time subsequent to the insertion of the energetic particle. That is, we want

$$\int_E^{E_0} dE' \int_0^\infty dt\, F(E',t) = \int_E^{E_0} dE' \tilde{F}(E',s=0)$$

Inspection of Eq. (5) shows this quantity to be $2(E_0/E)$ per source particle, when $E_0 \gg E$. But this leads at once to the CNJ criterion, either by direct computation, or by recognizing that the main point in their argument is that 2^N, the number disturbed, is nothing more than E_0/E, E being the energy reached after N branches. The stochastic nature of the collision process has no effect upon the criterion—the extra factor of 2 being related, no doubt to an issue of counting, and is of no consequence.

We might summarize this section by noting that solving the transport equation for a simple model of the time-energy cascade suggests that the parameter values $a=\frac{1}{2},\frac{1}{3}$ $(a'=0,-\frac{1}{6})$ are indeed special. However, the solution lends no special distinction to the values $a=\frac{2}{3}$ or 1.

B. Refinements

Are the time-energy cascades markedly different when a more accurate model of scattering is used? Let us examine the Laplace-Mellin–transform versions of Eq. (2). It is convenient to introduce dimensionless variables $E=E_0\epsilon$, $\tau=\tau_0\epsilon^{a'}$, $t=\tau_0 t_1$, $F(E,t)=(\tau_0 E_0)^{-1}H(\epsilon,t_1)$, and to recall that the collision density vanishes unless $0\le\epsilon\le 1$. Then, with

$$\hat{H}(p,s)=\int_0^\infty d\epsilon\, \epsilon^{p-1}\tilde{H}(\epsilon,s)\ ,$$

$$\tilde{H}(\epsilon,s)=\int_c \frac{dp}{2\pi i}\epsilon^{-p}\hat{H}(p,s)\ ,$$

and the convolution transform

$$f(p)H(p)=\int_0^\infty dE\, E^{p-1}\int_0^\infty dE'/E' f(E/E')H(E')\ ,$$

the transport equation becomes the functional equation,

$$s\hat{H}(p+a',s)+D(p)\hat{H}(p,s)=1\ , \tag{8}$$

with

$$D(p)=\int_0^1 dx\, f(x)[(1-x^{p-1})-(1-x)^{p-1}]\ .$$

$D(p)$, whose properties rule the analysis, may be expressed in several ways. For example,

$$D(p)=\frac{1}{m}\left[\Gamma(1-m)\frac{\Gamma(p)}{\Gamma(p-m)}-\frac{p-1}{p-(1+m)}\right]$$

demonstrates that the function is meromorphic in p, having poles at $p=(1+m),0,-1,-2,\ldots$. When $p\to\infty$ in the right half-plane, $D(p)\sim(1/m)\Gamma(1-m)p^m$. The zeroes of $D(p)$ are particularly important. The zero at $p=2$ is apparent; it is easy to show that there are no zeroes with $\mathrm{Re}(p)>2$. Other zeros, with $\mathrm{Re}(p)<2$ occur as solutions to a transcendental equation. One conjectures that there are infinitely many. (They are needed to represent the "Placzek wiggles" in the slowing down density.[13] Finally, one notes that in the hard-sphere model $[f(x)=1]$, $D(p)=(p-2)/p$.

All this is canonical material. Hidden in the analysis is the fact that the angle-integrated cross section is unbounded. The divergence is hidden by cancellations in the transport equation. The cancellation enables one to calculate via Mellin transform, without obtaining results that are obviously unphysical. But strictly speaking, if we do not cut off the interatom potential, we may never speak in terms of (completed) collisions, and the validity of the Boltzmann-transport picture is questionable.

The flaw becomes visible when one seeks to extract the behavior of the uncollided beam by considering Eq. (8) as $p\to\infty$. We expect $F(E,t)=\delta(E-E_0)e^{-t_1}$ in the limit, which requires that $\hat{H}(p,s)\to 1/(1+s)$. The latter requires $D(p)\to 1$, true for the hard-sphere model but not for the general case. The flaw is remedied by the introduction of a "cutoff". Imagine $f(x)$ to be altered very slightly near $x=0$, $f(x)\to\theta(x)f(x)$, with $\theta(x)=O(x^\alpha)$ so that the integral $\int_0^1 dx\, \theta f(x)$ exists. Then $D(p)$ will approach a constant, the value of the integral, as $p\to\infty$, and proper physical behavior will be obtained. The constant may be absorbed into the definition of dimensionless time, so that $D(p)\to 1$. This procedure is not innocuous, for the cutoff determines the time scale for the cascade. One expectes that in particular application a narrow range of acceptable cutoffs will exist. Otherwise the analysis loses physical relevance and becomes an exercise in computation, merely.

1. *Nature of $\hat{H}(p,s)$*

Let us turn to the questions of energy conservation, and the formation of spikes. Since we do not have explicit solutions when we leave the hard-sphere case, the discussion must be somewhat abstract. What behavior do we expect for $\hat{H}(p,s)$? Being with $F(E,t)$. Since this is an exercise in slowing down, we expect that for fixed E, the collision density will fall to zero as $t\to\infty$ and that its time integral will be convergent. Thus the half-plane of regularity of its time transform, $\tilde{H}(\epsilon,s)$ is at least as large as $\mathrm{Re}s>0$. In fact it is larger, for the cascade is described by a spectrum of collision times $\tau(E_0)$ to $\tau(E)$ in the interval E_0 to E. In the normal case $(a'<0,\ a'=-\kappa)$ these cause the s plane to be cut from $-\tau_0/\tau(E)$ to -1, in the anomalous case from -1 to $-\tau_0/\tau(E)$. We conclude that the half-plane of regularity for $\tilde{H}(\epsilon,s)$ is $\mathrm{Re}s>-\tau_0/\tau(E)$ in the normal case, and $\mathrm{Re}s>-1$ in the anomalous case.

Our real interest is in the Mellin transform, $\hat{H}(p,s)$. The s-plane behavior for fixed p stems, roughly from the time variation of energy moments of the collision density. Since all energies are involved, one suspects that a superposition of the results for $\tilde{H}(\epsilon,s)$ will be obtained. One expects the s plane to be cut along the real axis from 0 to -1 in the normal case, and from -1 to $-\infty$ in the anomalous case.

Now consider the p dependence of $\hat{H}(p,s)$ for fixed s. $F(E,t)$ is nonzero in a finite interval of E, $(0,E_0)$, and we demand that the distribution in energy, $w(E,t)=-dW(E,t)/dE=E\tau(E)F(E,t)$, is integrable. If $\tau(E)$ is power law, the Mellin transform is analytic in some right-half-plane of p. Analytical continuation "leftward" is effected by Eq. (8), which we write as

$$H(p,s)=D^{-1}(p)[1-sH(p+a',s)],\quad a'>0$$

and

$$sH(p,s)=1-D(p+\kappa)H(p+\kappa,s), \quad a'=-\kappa<0$$

in the two cases. In the anomalous case p_*, the first singularity encountered via contribution, is a pole at the leading zero of $D(p)$, $p_x=2$, in the normal case it is at the leading singularity of $D(p+\kappa)$ when $s\neq 0$. If $s=0$ a special discussion is required. Note that p_* is model dependent in the normal case, but "universal" ($p_*=2$) in the anomalous case. Some examples of the model dependence are hard-sphere (HS), $p_*=-\kappa$; power law (PL), $p_*=1+m$; and power law, cut off (PLC), $p_*=1+m-\alpha$.

2. Solutions

We may construct formal solutions to Eq. (8) by iteration, finding

$$\hat{H}(p,s)=\sum_{n=0}^{\infty}(-s)^n D^{-1}(p)D^{-1}(p+a')D^{-1}(p+2a')\cdots \times D^{-1}(p+na'), \quad a'>0 \tag{9}$$

and

$$s\hat{H}(p,s)=\sum_{n=0}^{\infty}(-s)^n D(p+\kappa) \times D(p+2\kappa)\cdots D(p+n\kappa)\,, \quad a'=-\kappa<0\,. \tag{10}$$

The first series converges for $|s|<1$ for models HS and PLC. That is as the general discussion indicates; $s=-1$ is a singular point. The series converges for all s for model PL, quite contrary to the general discussion, which was based on the notion of (complete) collisions, an idea that it absent from PL.

In the HS model, the sum may be done explicitly, giving

$$\hat{H}(p,s)=\left[\frac{p}{p-2}\right]{}_2F_1\left[1,1+p/a';1+\frac{p-2}{a'};-s\right]. \tag{11}$$

The rich literature dealing with the hypergeometric function[14] describes the analytic continuation of ${}_2F_1$ beyond the unit circle; $F(\ldots;-s)$ is analytic in the entire s plane, cut along the real axis from -1 to $-\infty$. Thus, $\hat{H}(p,s)$ has the predicted behavior. One of several integral representations displays $H(p,s)$ explicitly,

$$\hat{H}(p,s)=\frac{e^{-2\pi i/a'}}{D(p)}\frac{\Gamma\left[1+\frac{2}{a'}\right]\Gamma\left[1+\frac{p-2}{a'}\right]}{\Gamma\left[1+\frac{p}{a'}\right]} \times\frac{1}{2\pi i}\int_0^{(1-)}\frac{dt}{1+ts}\frac{t^{p/a'}}{(1-t)^{1+2/a'}}$$

Equation (11) also exhibits the "universal" singularity $p_*=2$, and other singularities at $1+(p-2)/a'=-n$ ($n=0,1,\ldots$).

The second series converges for $|s|>1$ for HS and PLC, following the general discussion for the "normal" case. The series diverges for all s in PL being, at best, asymptotic, and suggesting that $\hat{H}(p,s)$ is singular at infinity. It is likely that $\hat{H}(p,s)$ is analytic in the finite s plane, as it is in the anomalous case. Again, the sum may be done explicitly in HS. We find

$$s\hat{H}(p,s)={}_2F_1(1;1+(p-2)/\kappa,1+p/\kappa;-1/s)\,. \tag{12}$$

Continuation leads to $\hat{H}(p,s)$ analytic in the cut s plane, cut from 0 to -1, as predicted. A particular representation is

$$sH(p,s)=-\frac{i}{2}\frac{e^{-i\pi B}}{\sin(\pi B)}\frac{\Gamma\left[1+\frac{p}{\kappa}\right]}{\Gamma(B)\Gamma\left[\frac{2}{\kappa}\right]} \times\int_1^{(0^+)}\frac{dt}{1+t/s}\frac{t^{(p-2)/\kappa}}{(1-t)^{1-2/\kappa}}\,, \quad B=1+(p-2)/\kappa, \quad B\neq 1,2,\ldots\,.$$

If we turn to the p-plane behavior we note that Eq. (12) displays the expected singularities at $p+\kappa=-n\kappa$ ($n=0,1,\ldots$). Finally, one may entertain the notion of continuation in the parameter a'. The special solutions obtained in HS suggest that the point $a'=0$ is singular. It is likely to be an essential singularity—hardly a surprise.

3. Conservation of energy

Denote the Laplace time transform of $E(t)$, the energy in the cascade, by $\tilde{W}(s)$ [$E(t)\equiv W(E=0,t)$]. In terms of the function $\hat{H}(p,s)$ it is

$$\tilde{W}(s)=E_0\hat{H}(a'+2,s)\,.$$

In terms of the functional equation satisfied by $\hat{H}(p,s)$, we have

$$s\tilde{W}(s)=E_0[1-\lim_{p\to 2}D(p)\hat{H}(p,s)]\,,$$

and normal behavior would require that the limiting value of the product $D\hat{H}$ be zero. But the convergent series, Eq. (9), shows that this is not so in the anomalous case. We find for all models

$$s\tilde{W}(s)=E_0\left[1-\sum_{n=0}^{\infty}(-s)^n D^{-1}(2+a')D^{-1}(2+2a')\cdots \times D^{-1}(2+na')\right]. \tag{13}$$

But the product *is* zero for the convergent series of the normal case. (Doubtless it is true for normal PL too, but we are unable to prove it.) Thus our earlier conclusions about "nonconservation of energy" are borne out in a more general context. Note that the series in Eq. (13) may be summed, in HS, to give

$$s\tilde{W}(s)=E_0\left[1-\frac{1}{(1+s)^{1+2/a'}}\right],$$

which leads to

$$E(t_1)/E=1-\frac{1}{\Gamma\left[1+\frac{2}{a'}\right]}\int_0^{t_1}dy\,y^{2/a'}e^{-y}.$$

This expression of the loss of energy to the "sink" at $E=0$ contains, as a special case, the example ($a'=1$) noted earlier.

4. *From cascade to spike (CNJ)*

As before, we want all of the collisions occurring in the interval (E,E_0) for all time subsequent to the insertion of the energetic atom. That quantity is

$$\int_E^{E_0}dE'\int_0^{\infty}dt\,F(E',t)=\int_{\epsilon}^{\prime}e\epsilon'\tilde{H}(\epsilon',s=0),$$

which may be extracted from $\hat{H}(p,s=0)$. The behavior of the integral as $E\to 0$ is controlled by p_*, the dominant p-plane singularity of $\hat{H}(p,s=0)$. We noted earier that $p_*=2$ in the anomalous case, for all s. There, $H(p,0)=D^{-1}(p)$.

With this result, calculation gives $1/D'(2)(E_0/E)$ for the double integral, as $E\to 0$. The coefficient depends upon m; it is 2 in the hard-sphere limit. Thus, in the anomalous case, the argument about the critical nature of $a=\frac{1}{3}$ ($m=\frac{1}{6}$) goes through as before.

Earlier, we deferred the discussion of the normal case, when $s=0$. Consider that situation; at issue is the limiting value of $sH(p-\kappa,s)$. [Or, in (E,t) variables, the infinite-time integral of the time derivative of the collision density.] In the normal case, the collision frequency, $1/\tau(E)$, vanishes as $E\to 0$, and there is the possibility of an accumulation and consequent singularity, at $E=0$. This is reflected in the behavior of $H(p,s)$ in the s plane; the origin is a singular point and the limiting value of $sH(p-\kappa,s)$ is uncertain. We have only the explicit results of HS to guide us. Earlier, we noted that setting $s=0$ in Eq. (5) for $\tilde{F}(E,s)$ produced E^{-2} behavior quite independent of the nature of $\tau(E)$. We can arrive at the same result via the (p,s) representation by using Eq. (12) in conjunction with the asymptotic representation

$$F(1,B;C;z)\sim\left[\frac{C-1}{B-1}\right](-z)^{-1}+\frac{\Gamma(C)\Gamma(1-B)}{\Gamma(C-B)}(-z)^{-B},$$

valid when $z\to\infty$, $|\arg(-z)|<\pi$. One sees that $sH(p-\kappa,s)$ vanishes in the limit, provided $\mathrm{Re}(p)>2$. Analytical continuation then gives $D(p)H(p,0)=1$ everywhere, and the E^{-1} behavior for the double integral. The analysis of HS is the best that we can do in the "normal" case at this time. It is very likely that the behavior is universal, and with it the critical nature of $a=\frac{1}{3}$.

III. SUMMARY

We have discussed time-dependent cascades from the point of view of the linearized Boltzmann equation. The atoms are presumed to interact via a potential of the form $r^{-2/a}$ with and without cutoff. The parameter a lies in the range $0<a<2$. The purpose of our investigation has been to ascertain whether the equation, and/or certain special solutions, show striking changes in behavior at certain special values of a. These values have been sugested by elementary arguments based on scaling, and on the geometry of collision "trees." We find the most obvious special value to be $a=\frac{1}{2}$, which distinguishes energy-conserving cascades from anomalous cascades. For the latter, the point $E=0$ appears to be an energy sink. A little analysis then substantiates the special nature of $a=\frac{1}{3}$, CNJ's (Ref. 1) criterion for both a "spike" and a space-filling cascade. Other special points, $a=\frac{2}{3},1$ (Ref. 2) do not manifest themselves. Here, the linearized Boltzmann equation is too crude a tool for the analysis. But the equation does exhibit "scaling" in that we can extract special solutions to the initial-value problem which evolve into similarity solutions, functions of $t/\tau(E)$.

For simplicity, we have limited our discussion to distributions in time and energy. Of course, the same issues arise in the general case, described by Eq. (1). There, multiplication by E and integration over velocity $(E,\hat{\Omega})$ should produce a differential equation for a conserved (energy) density, a continuity equation. Again, the question of the vanishing of the integrated scattering terms arises. In the anomalous case the integration leaves a residium—a space- and time-dependent "sink"—a flow of energy into the origin of velocity space. One must keep this term in mind in doing one's bookkeeping. Of course the entire issure may be avoided by altering the physical model, but that has its price. In any case, caveat emptor.

Note added in proof. Dr. Williams has informed me, recently, of yet another paper[15] giving an example of nonconservation of measure, in the kinetics of coagulation.

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