Anchoring games for parallel repetition

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Abstract

Two major open problems regarding the parallel repetition of games are whether an analogue of Raz's parallel-repetition theorem holds for (a) games with more than two players, and (b) games with quantum players using entanglement. We make progress on these problems: we introduce a class of games we call *anchored*, and then prove exponential-decay parallel repetition theorems for anchored games in the multiplayer and entangled players settings. We then introduce a simple transformation on games called *anchoring*, inspired in part by the Feige-Kilian transformation [SICOMP '00], and show that this transformation turns any (multiplayer) game into an anchored game. Together, our parallel repetition theorem and our anchoring transformation provide a simple and efficient hardness-amplification technique for general games in the multiplayer and quantum settings.

1 Introduction

Multiplayer games are central objects of study in complexity theory and quantum information. For simplicity we focus our exposition on the two-player case. A *two-player one-round* game is specified by finite question sets $\mathcal{X}, \mathcal{Y}$, finite answer sets $\mathcal{A}, \mathcal{B}$, a probability distribution μ over $\mathcal{X} \times \mathcal{Y}$, and a verification predicate $V : \mathcal{X} \times \mathcal{Y} \times \mathcal{A} \times \mathcal{B} \to \{0, 1\}$ that determines the acceptable question and answer combinations. The game is played as follows: a referee samples questions $(x, y) \in \mathcal{X} \times \mathcal{Y}$ according to μ and sends x to the first player and y to the second. Each player replies with an answer, $a \in \mathcal{A}$ and $b \in \mathcal{B}$ respectively. The referee accepts if and only if $V(x, y, a, b) = 1$, in which case we say that the players win the game. Multiplayer games arise naturally in settings ranging from hardness of approximation [Hås01, Vaz13], interactive proof systems [BOGKW88, FRS88], and the study of Bell inequalities and non-locality in quantum physics [Bel64, CHSH69].

The main quantity associated with a multiplayer game G is its *value*: the maximum acceptance probability achievable by the players, where the probability is taken over the questions, as chosen by the referee, and the players' answers. Different notions of value arise from different restrictions on allowed strategies for the players. The most important for us are the *classical value* (denoted

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by $\text{val}(G)$) and the *entangled value* (denoted by $\text{val}^*(G)$). The former is obtained by restricting the players to classical strategies, where each player's answer is a function of its question only (both private and shared randomness are in principle allowed, but easily seen not to help). The latter allows for quantum strategies, in which each player's answer is obtained as the outcome of a local measurement performed on a quantum state shared by the players. The use of quantum states *does not* allow communication between the players, but it does allow for correlations between their questions and answers that cannot be reproduced by any classical strategy.

We study the behavior of $\text{val}(G)$ and $\text{val}^*(G)$ under parallel repetition. In the n-fold parallel repetition G^n of a game G the referee samples $(x_1, y_1), \ldots, (x_n, y_n)$ independently from μ, and sends $(x_1, \ldots, x_n)$ to the first player and $(y_1, \ldots, y_n)$ to the second. The players respond with answer tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ respectively, and they win if and only if their answers satisfy $V(x_i, y_i, a_i, b_i) = 1$ for all i.

Clearly, if the players play each instance of G in G^n independently of each other (i.e. according to a *product strategy*), their success probability is the n-th power of their success probability in G. The main obstacle to proving a parallel repetition theorem is that players need not employ product strategies – their answers for the i-th instance of G may depend on their questions in the j-th instance for $j \neq i$. Indeed, it is known that there are games G for which non-product strategies enable the players to win G^n with probability significantly greater than $\text{val}(G)^n$ [FV02, Raz11].

Nevertheless, the parallel repetition theorem of Raz [Raz98] establishes that if G is a two-player game such that $\text{val}(G) < 1$ the value $\text{val}(G^n)$ decays exponentially with n. The two following decades have seen a substantial amount of research on this question, connecting the problem of parallel repetition to topics such as the Unique Games conjecture, hardness of approximation, communication complexity, and more [BHH$^+$08, Rao11, BRWY13]. The most important application of parallel repetition is its use as a generic and efficient method for performing *gap amplification*, or *hardness amplification*. Suppose a certain problem – deciding membership in a language L, or breaking a given cryptosystem – has been reduced to the task of distinguishing between $\text{val}(G) = 1$ and $\text{val}(G) < 1 - \varepsilon$ for a certain G. Parallel repetition can be employed to claim that the presumedly easier task of distinguishing between $\text{val}(H) = 1$ and $\text{val}(H) < \delta$ for all games H is at least as hard as the original problem, by letting $H = G^n$ for some $n = \text{poly}(\log \delta^{-1}, \varepsilon^{-1})$.

In spite of much research—and partial results, as surveyed below—two major questions about parallel repetition remain open: what effect does parallel repetition have on (a) games with an arbitrary number of players, and (b) games with quantum players using entanglement? In this paper, we make progress on both questions. Our main results can be summarized as follows; see Theorems 11 and 17 for precise statements.

Theorem 1 (Main theorem, informal). *There exists a polynomial-time transformation (called anchoring) that takes the description of an arbitrary k-player game G and returns a game $G_\perp$ with the following properties:*

1. $\text{val}(G_\perp) = \frac{1}{4} + \frac{3}{4}\text{val}(G)$.

2. $\text{val}^*(G_\perp) = \frac{1}{4} + \frac{3}{4}\text{val}^*(G)$.

3. If $\text{val}(G) = 1 - \varepsilon$, then $\text{val}(G_\perp^n) \leq \exp(-\Omega(\varepsilon^3 \cdot n))$.

4. If the number of players $k = 2$ and $\text{val}^*(G) = 1 - \delta$, then $\text{val}^*(G_\perp^n) \leq \exp(-\Omega(\delta^8 \cdot n))$,

where the implied constants in the $\Omega(\cdot)$ only depend on the number of players k and the cardinality of the answer sets.

Remark 2. We expect item 4. from the theorem to extend to the case of the entangled value of k-player anchored games for $k > 2$ using the same techniques, but leave the details for a future version of this paper

The idea of modifying the game to facilitate its analysis under parallel repetition originates in work of Feige and Kilian [FK00] which predates Raz's parallel repetition theorem. Feige and Kilian introduce a transformation that converts an arbitrary game G to a so-called *miss-match* game G^{FK}. The transformation is *value-preserving* in the sense that there is a precise affine relationship $\text{val}(G^{FK}) = (2 + \text{val}(G))/3$. Furthermore Feige and Kilian are able to show that the value of the n-fold repetition of G^{FK} decays *polynomially* in n whenever $\text{val}(G) < 1$. This enables them to establish a general gap amplification result without having to prove a parallel repetition theorem for arbitrary games. This is sufficient for many applications, including to hardness of approximation, for which it is enough that the gap amplification procedure be efficient and value-preserving.

Theorem 1 adopts a similar approach to that of Feige and Kilian by providing an arguably even simpler transformation, *anchoring*, which preserves both the classical and entangled value of a game and for which we are able to prove an exponential decay under parallel repetition. In contrast, the transformation considered by Feige and Kilian does not in general preserve the entangled value. We proceed to describe our transformation and then discuss the role it plays in facilitating the proof of our parallel repetition theorem.

The anchoring transformation. Our parallel repetition results apply to a class of games we call *anchored*. The anchoring transformation of Theorem 1 produces games of this type; however, anchored games can be more general. We give a full definition of anchored games in Section 2. First we describe the anchoring transformation.

Definition 3 (Basic anchoring). *Let G be a two player game with question distribution μ on $\mathcal{X} \times \mathcal{Y}$, and verification predicate V. In the α-anchored game $G_\perp$ the referee chooses a question pair $(x, y) \in \mathcal{X} \times \mathcal{Y}$ according to μ, and independently and with probability α replaces each of x and y with an auxiliary "anchor" symbol $\perp$ to obtain the pair $(x', y') \in (\mathcal{X} \cup \{\perp\}) \times (\mathcal{Y} \cup \{\perp\})$ which is sent to the players as their respective questions. If any of x', y' is $\perp$ the referee accepts regardless of the players' answers; otherwise, the referee checks the players' answers according to the predicate V.*

For a choice of $\alpha = 1 - \frac{\sqrt{3}}{2}$ it holds that both $\text{val}(G_\perp) = \frac{3}{4}\text{val}(G) + \frac{1}{4}$ and $\text{val}^*(G_\perp) = \frac{3}{4}\text{val}^*(G) + \frac{1}{4}$. One can think of $G_\perp$ as playing the original game G with probability 3/4, and a trivial game with probability 1/4. The term "anchored" refers to the fact that question pairs chosen according to μ are all "anchored" by a common question $(\perp, \perp)$. Though the existence of this anchor question

makes the game $G_\perp$ *easier* to play than the game G, it facilitates showing that the repeated game $G_\perp^n$ is *hard*. At a high level, the anchor questions provide a convenient way to handle the complicated correlations that may arise when the players use non-product strategies in the repeated game, as we explain next.

Proving parallel repetition by breaking correlations. In virtually all known (information theoretic) proofs of parallel repetition theorems, the key step consists in arguing that the players' success probability in most instances of G individually cannot be substantially larger than the value of G itself, *even when conditioned on the player winning a significant fraction of the instances.* Coupled with the possibility of using non-product strategies this conditioning introduces correlations between the player's questions which make the task of bounding their success probability in the remaining instances of G non-trivial.

In the proof of his parallel repetition theorem, Raz [Raz98] introduced a technique, further refined in subsequent work of Holenstein [Hol09], to *break* such correlations. The idea consists of introducing a *dependency-breaking random variable* Ω satisfying two properties: (1) Ω can be sampled jointly, using shared randomness, by all players, and (2) conditioned on Ω, the players are able to locally generate questions and answers from the same distribution as they would in the repeated game, *conditioned on winning* a (not too large) subset of instances. These two requirements are at odds with each other, and the main difficulty is to design an Ω that satisfies both simultaneously. In Raz's proof this heavily relies both on the fact that the game involves only two players and that the players' strategies can be assumed to be deterministic.

Extending this approach to more players, or quantum strategies, remains a challenge. Rather than solving the general problem directly, we sidestep it and instead analyze the parallel repetition of anchored games, for which designing an appropriate dependency-breaking variable (or, in the case of entangled players, a dependency-breaking quantum state) is easier, though by no means trivial. Combined with the anchoring operation this yields a simple and efficient method to achieve hardness amplification for arbitrary games in the multiplayer and entangled-player settings. We give a more detailed explanation of how this is achieved in Section 2 below.

1.1 Related work

We refer to the surveys by Feige and Raz [Fei95, Raz10] for an extensive historical account of the classical parallel repetition theorem and its connections to the hardness of approximation and multiprover interactive proof systems, and instead focus on more recent results, specifically those pertaining to the quantum or multiplayer parallel repetition.

The first result on the parallel repetition of entangled-player games was obtained by Cleve et al. [CSUU08] for XOR games. This was extended to the case of unique games by Kempe, Regev and Toner [KRT08]. Kempe and Vidick [KV11] studied a Feige-Kilian type repetition for the entangled value of two-player games, and obtained a polynomial rate of decay. The Feige-Kilian transformation does not in general preserve the entangled value, and their result does not pro-

vide a hardness amplification technique for arbitrary entangled games.

Dinur et al. [DSV14] extend the analytical framework of Dinur and Steurer [DS14] to obtain an exponential-decay parallel repetition theorem for the entangled value of two-player projection games. Their analysis is notable because it provides a quantum extension of the *correlated sampling lemma*, a key component of Holenstein's solution to the correlation-breaking problem. However their use of the quantum correlated sampling lemma appears to heavily rely on symmetries of projection games, and it is unclear how to extend the argument to general games. Chailloux and Scarpa [CS14a] and Jain et al. [JPY14] prove exponential-decay parallel repetition for *free two-player games*, i.e. games with a product question distribution. Their analysis, as well as the follow-up work Chung et al. [CWY15], is based on extending the information-theoretic approach of Raz and Holenstein.

Turning to the multiplayer setting, very little is known. It is folklore that free games with any number of players satisfy a parallel repetition theorem, and this was explicitly proved for both the classical and quantum case in [CWY15]. Multiplayer parallel repetition has been studied in the setting of *non-signaling strategies*, a superset of entangled strategies which allows the players to generate any correlations that do not imply communication. Buhrman et al. [BFS13] show that the non-signaling value of a game G with any number of players decays exponentially under parallel repetition, with a rate of decay that depends on the entire description of the game G. Arnon-Friedman et al. [AFRV14] and Lancien and Winter [LW15] achieve similar results using a different technique based on "de Finetti reductions".

The transformation from general games into anchored games that we introduce is inspired by the work of Feige and Kilian [FK00]. This alternative approach to achieving gap amplification is also used by Moshkovitz [Mos14], who shows how projection games can be "fortified", and gives a simple and elegant proof that the classical value of fortified games decays exponentially under parallel repetition (see also the follow-up work by Bhangle et al. [BSVV15]). In an upcoming paper we prove a parallel repetition theorem for the entangled value of fortified games.

1.2 Organization

In Section 2 we give an overview of the techniques underlying our main results, mainly focusing on the general ideas and leaving the specifics to each subsequent section. Section 3 introduces some preliminaries, including the definition of anchored games. In Section 4 we present the result on the parallel repetition of multiplayer classical anchored games; for the benefit of the readers more interested in the classical aspects of the paper, this section is mostly self-contained. Section 5 contains the proof of the result on the parallel repetition of two-player entangled games. The issues arising in the quantum setting are significantly more subtle than in the classical case, and hence the argument in this section is more involved, requiring a more careful analysis as well as several new technical ideas. We conclude in Section 6 by recounting a few open problems related to parallel repetition.

2 Technical overview

We give a technical overview of anchored games and their parallel repetition. For concreteness we focus on the case of two-player games. For the full definition of k-player anchored games, see Section 3.3.

Definition 4 (Two-player anchored games). *Let G be a two-player game with question alphabet $\mathcal{X} \times \mathcal{Y}$ and distribution μ. For any $0 < \alpha \le 1$ we say that G is α-anchored if there exists subsets $\mathcal{X}_\perp \subseteq \mathcal{X}$ and $\mathcal{Y}_\perp \subseteq \mathcal{Y}$ such that, denoting by μ the respective marginals of μ on both coordinates,*

1. *Both $\mu(\mathcal{X}_\perp), \mu(Y_\perp) \ge \alpha$,*

2. *Whenever $x \in \mathcal{X}_\perp$ or $y \in \mathcal{Y}_\perp$ it holds that $\mu(x, y) = \mu(x) \cdot \mu(y)$.*

Informally, a game is *anchored* if each player independently has a significant probability of receiving a question from the set of "anchor questions" $\mathcal{X}_\perp$ and $\mathcal{Y}_\perp$. An alternative way of thinking about the class of anchored games is to consider the case where μ is uniform over a set of edges in a bipartite graph on vertex set $\mathcal{X} \times \mathcal{Y}$; then the condition is that the induced subgraph on $\mathcal{X}_\perp \times \mathcal{Y}_\perp$ is a complete bipartite graph that is connected to the rest of $\mathcal{X} \times \mathcal{Y}$ and has weight at least α. In other words, a game G is anchored if it contains a free game that is connected to the entire game.

It is easy to see that the games $G_\perp$ output by the anchoring transformation given in Definition 3 are α-anchored. Free games are automatically 1-anchored (set $\mathcal{X}_\perp = \mathcal{X}$ and $\mathcal{Y}_\perp = \mathcal{Y}$), but the class of anchored games is much broader; indeed assuming the Exponential Time Hypothesis it is unlikely that there exists a similar (efficient) reduction from general games to free games [AIM14]. Additionally, since free games are anchored games, our parallel repetition theorems automatically reproduce the quantum and multiplayer parallel repetition of free games results of [JPY14, CS14a, CWY15], albeit with worse parameters.

Dependency-breaking variables and states. Essentially all known proofs of parallel repetition proceed via reduction, showing how a "too good" strategy for the repeated game G^n can be "rounded" into a strategy for G with success probability strictly greater than $\mathrm{val}(G)$, yielding a contradiction.

Let S^n be a strategy for G^n that has a high success probability. By an inductive argument one can identify a set of coordinates C and an index i such that $\Pr(\text{Players win round } i|W) > \mathrm{val}(G) + \delta$, where W is the event that the players' answers satisfy the predicate V in all instances of G indexed by C. Given a pair of questions (x, y) in G the strategy S embeds them in the i-th coordinate of a n-tuple of questions

$$x_{[n]}y_{[n]} = \begin{pmatrix} x_1, x_2, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n \\ y_1, y_2, \ldots, y_{i-1}, y, y_{i+1}, \ldots, y_n \end{pmatrix}$$

that is distributed according to $\mathsf{P}_{X_{[n]}Y_{[n]}|X_i=x, Y_i=y, W}$. The players then simulate S^n on $x_{[n]}$ and $y_{[n]}$ respectively to obtain answers $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$, and return (a_i, b_i) as their answers

in G. The strategy S succeeds with probability precisely $\Pr(\text{Win } i|W)$ in G, yielding the desired contradiction.

As S^n need not be a product strategy, conditioning on W may introduce correlations that make $P_{X_{[n]}Y_{[n]}|X_i=x,Y_i=y,W}$ impossible to sample exactly. A key insight in Raz' proof of parallel repetition is that it is still possible for the players to *approximately* sample from this distribution. Drawing on the work of Razborov [Raz92], Raz introduced a *dependency-breaking variable* Ω with the following properties:

(a) Given $\omega \sim P_\Omega$ the players can locally sample $x_{[n]}$ and $y_{[n]}$ according to $P_{X_{[n]}Y_{[n]}|X_i=x,Y_i=y,W}$,

(b) The players can jointly sample from P_Ω using shared randomness.

In [Hol09] Ω is defined so that a sample ω fixes at least one of $\{x_{i'}, y_{i'}\}$ for each $i' \neq i$. It can then be shown that conditioned on x, Ω is nearly (though not exactly) independent of y, and vice-versa. In other words,

$$P_{\Omega|X_i=x,W} \approx P_{\Omega|X_i=x,Y_i=y,W} \approx P_{\Omega|Y_i=y,W} \tag{1}$$

where "$\approx$" denotes closeness in statistical distance. Eq. (1) suffices to guarantee that the players can *approximately* sample the same ω from $P_{\Omega|X_i=x,Y_i=y,W}$ with high probability, achieving point (b) above. This sampling is accomplished through a technique called *correlated sampling*.

This argument relies heavily on the assumption that there are only two players who employ a deterministic strategy. With more than two players, it is not known how to design an appropriate dependency-breaking variable Ω that satisfies requirements (a) and (b) above: in order to be jointly sampleable, Ω needs to fix as few inputs as possible; in order to allow players to locally sample their inputs conditioned on Ω, the variable needs to fix as many inputs as possible. These two requirements are in direct conflict as soon as there are more than two players.

In the quantum case the rounding argument seems to require that Alice and Bob jointly sample a *dependency-breaking state* $|\Omega_{x,y}\rangle$, which again depends on both their inputs. Although it is technically more complicated, as a first approximation $|\Omega_{x,y}\rangle$ can be thought of as the players' post-measurement state, conditioned on W. Designing a state that simultaneously allows Alice and Bob to (a) simulate the execution of the i-th game in G^n conditioned on W, and (b) locally generate $|\Omega_{x,y}\rangle$ without communication is the main obstacle to proving a fully general parallel repetition theorem for entangled games.

It has long been known that in the free games case (i.e. games with product question distributions) these troubles with the dependency-breaking variable disappear, and consequently we have parallel repetition theorems for free games for the multiplayer and quantum settings [CWY15]. With free games involving more than two players, it can be shown that

$$P_{\Omega|X_i=x,Y_i=y,Z_i=z,...,W} \approx P_{\Omega|W}, \tag{2}$$

on average over question tuples $(x,y,z,\ldots)$. In the quantum case, [JPY14, CS14a, CWY15] showed how to construct dependency-breaking states $|\Omega_{X_i=x,Y_i=y,W}\rangle$ and local unitaries U_x and V_y such that

$$(U_x \otimes V_y)|\Omega\rangle \approx |\Omega_{X_i=x,Y_i=y,W}\rangle \tag{3}$$

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for some fixed quantum state $|\Omega\rangle$. This eliminates the need for the players to use correlated sampling, as they can simply share a sample from $\mathsf{P}_{\Omega|W}$ or the quantum state $|\Omega\rangle$ from the outset.

Breaking correlations in repeated anchored games. Rather than providing a complete extension of the framework of Raz and Holenstein to the multiplayer and quantum settings, we interpolate between the case of free games and the general setting by showing how the same framework of dependency-breaking variables and states can be extended to anchored games – without using correlated sampling. We introduce dependency-breaking variables Ω and states $|\Phi_{x,y}\rangle$ so that we can prove analogous statements to (2) and (3) in the anchored games setting.

The analysis for anchored games is more intricate than for free games. Proofs of the analogous statements for free games in [JPY13, CS14b, CWY15] make crucial use of the fact that all possible question tuples are possible. An anchored game can be far from having this property. Instead, we use the anchors as a "home base" that is connected to all questions. Intuitively, no matter what question tuple $(x, y, z, \ldots)$ we are considering, it is only a few replacements away from the set of anchor questions. Thus the dependency of the variable Ω or state $|\Phi_{x,y}\rangle$ on the questions can be iteratively removed by "switching" each players' question to an anchor as

$$\mathsf{P}_{\Omega|X_i=x,Y_i=y,Z_i=z,W} \approx \mathsf{P}_{\Omega|X_i=x,Y_i=y,Z_i\in\perp,W} \approx \mathsf{P}_{\Omega|X_i=x,Y_i\in\perp,Z_i\in\perp,W} \approx \mathsf{P}_{\Omega|X_i\in\perp,Y_i\in\perp,Z_i\in\perp,W},$$

where "$X_i \in \perp$" is shorthand for the event that $X_i \in \mathcal{X}_\perp$.

Dealing with quantum strategies adds another layer of complexity to the argument. The local unitaries U_x and V_y involved in (3) are quite important in the arguments of [JPY14, CS14a, CWY15]. The difficulty in extending the argument for free games to the case of general games is to show that these local unitaries each only depend on the input to a single player. In fact with the definition of $|\Omega_{x,y}\rangle$ used in these works it appears likely that this statement does not hold, thus a different approach must be found.

When the game is anchored, however, we are able to use the anchor question in order to show the existence of unitaries U_x and V_y that achieve (3) and depend only on a single player's question each. Achieving this requires us to introduce dependency-breaking states $|\Omega_{x,y}\rangle$ that are more complicated than those used in the free games case; in particular they include information about the *classical* dependency-breaking variables of Raz and Holenstein.

We prove (3) for anchored games by proving a sequence of approximate equalities: first we show that for most x there exists U_x such that $(U_x \otimes \mathbb{I})|\Omega_{\perp,\perp}\rangle \approx |\Omega_{x,\perp}\rangle$, where $|\Omega_{\perp,\perp}\rangle$ denotes the dependency-breaking state in the case that both Alice and Bob receive the anchor question "$\perp$", and $|\Omega_{x,\perp}\rangle$ denotes the state when Alice receives x and Bob receives "$\perp$". Then we show that for all y such that $\mu(y|x) > 0$ there exists a unitary V_y such that $(\mathbb{I} \otimes V_y)|\Omega_{x,\perp}\rangle \approx |\Omega_{x,y}\rangle$. Accomplishing this step requires ideas and techniques going beyond those in the free games case. Interestingly, a crucial component of our proof is to argue the existence of a local unitary $R_{x,y}$ that depends on *both* inputs x and y. The unitary $R_{x,y}$ is not implemented by Alice or Bob in the simulation, but it is needed to show that V_y maps $|\Omega_{x,\perp}\rangle$ onto $|\Omega_{x,y}\rangle$.

One can view our work as pushing the limits of arguments for parallel repetition that do not require some form of correlated sampling, a procedure that seems inherently necessary to analyze the general case. Our results demonstrate that such procedure is not needed for the purpose of achieving strong gap amplification theorems for multiplayer and quantum games.

3 Preliminaries

3.1 Probability distributions

We largely adopt the notational conventions from [Hol09] for probability distributions. We let capital letters denote random variables and lower case letters denote specific samples. We will use subscripted sets to denote tuples, e.g., $X_{[n]} := (X_1, \ldots, X_n)$, $x_{[n]} = (x_1, \ldots, x_n)$, and if $C \subset [n]$ is some subset then X_C will denote the sub-tuple of $X_{[n]}$ indexed by C. We use P_X to denote the probability distribution of random variable X, and $\mathsf{P}_X(x)$ to denote the probability that $X = x$ for some value x. For multiple random variables, e.g., X, Y, Z, $\mathsf{P}_{XYZ}(x, y, z)$ denotes their joint distribution with respect to some probability space understood from context.

We use $\mathsf{P}_{Y|X=x}(y)$ to denote the conditional distribution $\mathsf{P}_{YX}(y, x)/\mathsf{P}_X(x)$, which is defined when $\mathsf{P}_X(x) > 0$. When conditioning on many variables, we usually use the shorthand $\mathsf{P}_{X|y,z}$ to denote the distribution $\mathsf{P}_{X|Y=y,Z=z}$. For example, we write $\mathsf{P}_{V|\omega_{-i},x_i,y_i}$ to denote $\mathsf{P}_{V|\Omega_{-i}=\omega_{-i},X_i=x_i,Y_i=y_i}$. For an event W we let $\mathsf{P}_{XY|W}$ denote the distribution conditioned on W. We use the notation $\mathbb{E}_X f(x)$ and $\mathbb{E}_{\mathsf{P}_X} f(x)$ to denote the expectation $\sum_x \mathsf{P}_X(x) f(x)$.

Let P_{X_0} be a distribution of $\mathcal{X}$, and for every x in the support of P_{X_0}, let $\mathsf{P}_{Y|X_1=x}$ be a conditional distribution defined over $\mathcal{Y}$. We define the distribution $\mathsf{P}_{X_0}\mathsf{P}_{Y|X_1}$ over $\mathcal{X} \times \mathcal{Y}$ as

$$(\mathsf{P}_{X_0}\mathsf{P}_{Y|X_1})(x, y) := \mathsf{P}_{X_0}(x) \cdot \mathsf{P}_{Y|X_1=x}(y).$$

Additionally, we write $\mathsf{P}_{X_0Z}\mathsf{P}_{Y|X_1}$ to denote the distribution $(\mathsf{P}_{X_0Z}\mathsf{P}_{Y|X_1})(x, z, y) := \mathsf{P}_{X_0Z}(x, z) \cdot \mathsf{P}_{Y|X_1=x}(y)$.

For two random variables X_0 and X_1 over the same set $\mathcal{X}$, $\mathsf{P}_{X_0} \approx_\varepsilon \mathsf{P}_{X_1}$ indicates that the total variation distance between P_{X_0} and P_{X_1},

$$\|\mathsf{P}_{X_0} - \mathsf{P}_{X_1}\| := \frac{1}{2} \sum_{x \in \mathcal{X}} |\mathsf{P}_{X_0}(x) - \mathsf{P}_{X_1}(x)|,$$

is at most ε.

The following simple lemma will be used repeatedly.

Lemma 5. *Let Q_F and S_F be two probability distributions of some random variable F, and let $\mathsf{R}_{G|F}$ be a conditional probability distribution for some random variable G, conditioned on F. Then*

$$\|\mathsf{Q}_F\mathsf{R}_{G|F} - \mathsf{S}_F\mathsf{R}_{G|F}\| = \|\mathsf{Q}_F - \mathsf{S}_F\|.$$

Proof. Note that $\|Q_F R_{G|F} - S_F R_{G|F}\|$ is equal to

$$
\frac{1}{2} \sum_{f,g} |Q(f)R(g|f) - S(f)R(g|f)| = \frac{1}{2} \sum_f |Q(f) - S(f)| \cdot \left(\sum_g R(g|f) \right)
$$

$$
= \frac{1}{2} \sum_f |Q(f) - S(f)|
$$

$$
= \|Q_F - S_F\|.
$$

$\square$

3.2   Quantum information theory

For comprehensive references on quantum information we refer the reader to [NC10, Wil13].

For a vector $|\psi\rangle$, we use $\||\psi\rangle\|$ to denote its Euclidean length. For a matrix A, we will use $\|A\|_1$ to denote its *trace norm* $\mathrm{Tr}(\sqrt{AA^\dagger})$. A density matrix is a positive semidefinite matrix with trace 1. The *fidelity* between two density matrices ρ and σ is defined as $F(\rho, \sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1$. The Fuchs-van de Graaf inequalities relate fidelity and trace norm as

$$
1 - F(\rho, \sigma) \leq \frac{1}{2} \|\rho - \sigma\|_1 \leq \sqrt{1 - F(\rho, \sigma)^2}. \tag{4}
$$

For Hermitian matrices A, B we write $A \preceq B$ to indicate that $A - B$ is positive semidefinite. We use $\mathbb{I}$ to denote the identity matrix. For an operator X and a density matrix ρ, we write $X[\rho]$ for $X\rho X^\dagger$. A *positive operator valued measurement* (POVM) with outcome set $\mathcal{A}$ is a set of positive semidefinite matrices $\{E^a\}$ labeled by $a \in \mathcal{A}$ that sum to the identity.

We will use the convention that, when $|\psi\rangle$ is a pure state, ψ refers to the rank-1 density matrix $|\psi\rangle\langle\psi|$. We use subscripts to denote system labels; so ρ_{AB} will denote the density matrix on the systems A and B. A *classical-quantum state* ρ_{XE} is classical on X and quantum on E if it can be written as $\rho_{XE} = \sum_x p(x)|x\rangle\langle x|_X \otimes \rho_{E|X=x}$ for some probability measure $p(\cdot)$. The state $\rho_{E|X=x}$ is by definition the E part of the state ρ_{XE}, conditioned on the classical register $X = x$. We write $\rho_{XE|X=x}$ to denote the state $|x\rangle\langle x|_X \otimes \rho_{E|X=x}$. We often write expressions such as $\rho_{E|x}$ as shorthand for $\rho_{E|X=x}$ when it is clear from context which registers are being conditioned on. This will be useful when there are many classical variables to be conditioned on.

Entropic quantities.   For two positive semidefinite operators ρ, σ, the *relative entropy* $S(\rho\|\sigma)$ is defined to be $\mathrm{Tr}(\rho(\log\rho - \log\sigma))$. The *relative min-entropy* $S_\infty(\rho\|\sigma)$ is defined as $\min\{\lambda : \rho \preceq 2^\lambda \sigma\}$.

Let ρ_{AB} be a bipartite state. The mutual information $I(A:B)_\rho$ is defined as $S(\rho^{AB}\|\rho^A \otimes \rho^B)$. For a classical-quantum state ρ_{XAB} that is classical on X and quantum on AB, we write $I(A;B|x)_\rho$ to indicate $I(A;B)_{\rho_x}$.

The following technical lemmas will be used in Section 5.

Proposition 6 (Pinsker's inequality). *For all density matrices ρ, σ, $\frac{1}{2}\|\rho - \sigma\|_1^2 \leq S(\rho\|\sigma)$.*

Lemma 7 ([JPY14], Fact II.8). *Let $\rho = \sum_z \mathsf{P}_Z(z)|z\rangle\langle z| \otimes \rho_z$, and $\rho' = \sum_z \mathsf{P}_{Z'}(z)|z\rangle\langle z| \otimes \rho'_z$. Then $S(\rho'\|\rho) = S(\mathsf{P}_{Z'}\|\mathsf{P}_Z) + \mathbb{E}_{Z'}\left[S(\rho'_z\|\rho_z)\right]$. In particular, $S(\rho'\|\rho) \geq \mathbb{E}_{Z'}\left[S(\rho'_z\|\rho_z)\right]$.*

We will also use the following Lemma from [CWY15].[1] Here we present an argument that obtains better parameters ([CWY15] proved that $\sum_{i=1}^n I(X_i : A)_\rho \leq 2S(\rho_{XA}\|\sigma_{XA})$.)

Lemma 8 ([CWY15], Quantum Raz's Lemma). *Let ρ and σ be two CQ states with $\rho_{XA} = \rho_{X_1 X_2 \ldots X_n A}$ and $\sigma = \sigma_{XA} = \sigma_{X_1} \otimes \sigma_{X_2} \otimes \ldots \otimes \sigma_{X_n} \otimes \sigma_A$ with $X = X_1 X_2 \ldots X_n$ classical in both states. Then*

$$\sum_{i=1}^n I(X_i : A)_\rho \leq S(\rho_{XA}\|\sigma_{XA}). \tag{5}$$

The conditions on ρ and σ stated in the lemma are equivalent to them satisfying the following form

$$\rho_{XA} = \sum_x \mathsf{P}_X(x)|x\rangle\langle x| \otimes \rho_{A|X=x}, \qquad \sigma_{XA} = \sum_x \mathsf{P}'_X(x)|x\rangle\langle x| \otimes \sigma_A,$$

where $x = (x_1, x_2, \ldots, x_n)$ is an n-tuple, P_X an arbitrary distribution, and $\mathsf{P}'_X(x) = \prod_{i=1}^n \mathsf{P}'_{X_i}(x_i)$ a product distribution.

Proof of Lemma 8. By the chain rule (Lemma 7) we have

$$S(\rho_{XA}\|\sigma_{XA}) = S(\rho_{X_1}\|\sigma_{X_1}) + \underset{x_1 \leftarrow \rho_{X_1}}{\mathbb{E}} S(\rho_{X_2|X_1=x_1}\|\sigma_{X_2}) + \ldots + \underset{x \leftarrow \rho_{X_1 \cdots X_n}}{\mathbb{E}} S(\rho_{A|X=x}\|\sigma_A), \tag{6}$$

where $x_1 \leftarrow \rho_{X_1}$ means sampling x_1 according to the classical distribution ρ_{X_1}, and similarly for $x \leftarrow \rho_{X_1 \cdots X_n}$. Consider any of the first n terms in (6). We have

$$\underset{x_{<i} \leftarrow \rho_{X_1 X_2 \ldots X_{i-1}}}{\mathbb{E}} S(\rho_{X_i|x_{<i}}\|\sigma_{X_i}) \geq \underset{x_{<i} \leftarrow \rho_{X_1 X_2 \ldots X_{i-1}}}{\mathbb{E}} S(\rho_{X_i|x_{<i}}\|\rho_{X_i}) = I(X_1 \ldots X_{i-1} : X_i)_\rho,$$

where $\rho_{X_i|x_{<i}}$ stands for $\rho_{X_i|X_{<i}=x_{<i}}$. Now consider the last term in (6):

$$\underset{x \leftarrow \rho_X}{\mathbb{E}} S(\rho_{A|X=x}\|\sigma_A) \geq \underset{x \leftarrow \rho_X}{\mathbb{E}} S(\rho_{A|X=x}\|\rho_A) = S(\rho_{XA}\|\rho_X \otimes \rho_A)$$

$$= I(X : A)_\rho = \sum_{i=1}^n I(X_i : A|X_1 X_2 \ldots X_{i-1})_\rho.$$

Summing up the last two equations and using $I(X_i : AX_1 \ldots X_i) = I(X_i : X_1 \ldots X_{i-1}) + I(X_i : A|X_1 \ldots X_{i-1})$ implies

$$S(\rho_{XA}\|\sigma_{XA}) \geq \sum_{i=1}^n I(X_i : AX_1 \ldots X_{i-1})_\rho \geq \sum_{i=1}^n I(X_i : A)_\rho,$$

where the last inequality follows from strong subadditivity, i.e., $I(X_i : X_1 \ldots X_{i-1}|A)_\rho \geq 0$. □

3.3 Games, parallel repetition, and anchoring

We formally define k-player one-round games, their parallel repetition, and anchored games.

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[1]Some versions of this lemma, though in a less compact form, also appear in [JPY14, CS14a].

Multiplayer games. A k-player game $G = (\mathcal{X}, \mathcal{A}, \mu, V)$ is specified by a question set $\mathcal{X} = \mathcal{X}^1 \times \mathcal{X}^2 \times \cdots \times \mathcal{X}^k$, answer set $\mathcal{A} = \mathcal{A}^1 \times \mathcal{A}^2 \times \cdots \times \mathcal{A}^k$, a probability measure μ on $\mathcal{X}$, and a verification predicate $V : \mathcal{X} \times \mathcal{A} \to \{0, 1\}$. Throughout this paper, we use superscripts in order to denote which player an input/output symbol is associated with. For example, we write x^1 to denote the input to the first player, and a^t to denote the output of the t-th player. Finally, to denote the tuple of questions/answers to all k players we write $x = (x^1, \ldots, x^k)$ and $a = (a^1, \ldots, a^k)$ respectively.

The *classical value* of a game G is denoted by $\mathrm{val}(G)$ and defined as

$$\mathrm{val}(G) := \sup_{f^1, \ldots, f^k} \mathbb{E}_{(x^1, \ldots, x^k) \sim \mu} \left[V \left((x^1, \ldots, x^k), (f^1(x^1), \ldots, f^k(x^k)) \right) \right]$$

where the supremum is over all functions $f_i : \mathcal{X}_i \to \mathcal{A}_i$; these correspond to deterministic strategies used by the players. It is easy to see that the classical value of a game is unchanged if we allow the strategies to take advantage of public or private randomness.

The *entangled value* of G is denoted by $\mathrm{val}^*(G)$ and defined as

$$\mathrm{val}^*(G) := \sup_{\substack{|\psi\rangle \in (\mathbb{C}^d)^{\otimes k} \\ M^1, \ldots, M^k}} \mathbb{E}_{(x^1, \ldots, x^k) \sim \mu} \sum_{\substack{(a^1, \ldots, a^k): \\ V\left((x^1, \ldots, x^k), (a^1, \ldots, a^k)\right) = 1}} \langle \psi | M^1(x^1, a^1) \otimes \cdots \otimes M^k(x^k, a^k) | \psi \rangle$$

where the supremum is over all integer $d \geq 2$, k-partite pure states $|\psi\rangle$ in $(\mathbb{C}^d)^{\otimes k}$, and $M^1, \ldots, M^k$ for each player. Each M^t is a set of POVM measurements $\{M(x^t, a^t)\}_{a^t \in \mathcal{A}^t}$ acting on $\mathbb{C}^d$, one for each question $x^t \in \mathcal{X}^t$.

Repeated games. Let $G = (\mathcal{X}, \mathcal{A}, \mu, V)$ be a k-player game, with $\mathcal{X} = \mathcal{X}^1 \times \cdots \times \mathcal{X}^k$ and $\mathcal{A} = \mathcal{A}^1 \times \cdots \times \mathcal{A}^k$. Let $\mu^{\otimes n}$ denote the product probability distribution over $\mathcal{X}^{\otimes n} = \bigotimes_{i=1}^n \mathcal{X}_i$, where each $\mathcal{X}_i$ is a copy of $\mathcal{X}$. Similarly let $\mathcal{A}^{\otimes n} = \bigotimes_{i=1}^n \mathcal{A}_i$ where each $\mathcal{A}_i$ is a copy of $\mathcal{A}$. [2] Let $V^{\otimes n} : \mathcal{X}^{\otimes n} \times \mathcal{A}^{\otimes n} \to \{0, 1\}$ denote the verification predicate that is 1 on question tuple $(x_1, \ldots, x_n) \in \mathcal{X}^{\otimes n}$ and answer tuple $(a_1, \ldots, a_n) \in \mathcal{A}^{\otimes n}$ iff for all i, $V(x_i, a_i) = 1$. We define the n-fold parallel repetition of G to be the k-player game $G^n = (\mathcal{X}^{\otimes n}, \mathcal{A}^{\otimes n}, \mu^{\otimes n}, V^{\otimes n})$.

When working with games with more than 2 players, we use subscripts to denote which game round/coordinate a question/answer symbol is associated with. For example, by x_i^t we mean the question to the t-th player in the i-th round. While this is overloading notation slightly (because superscripts are meant to indicate tuples), we use this convention for the sake of readability. When x^n refers to a tuple $(x_1, \ldots, x_n)$ and when x_i^t refers to the t-th player's question in the i-th coordinate should be clear from context.

Anchored games. We give the general definition of an anchored game.

Definition 9 (Multiplayer Anchored Games). *A game $G = (\mathcal{X}, \mathcal{A}, \mu, V)$ is called α-anchored if there exists $\mathcal{X}_\perp^t \subseteq \mathcal{X}^t$ for all $t \in [k]$ where*

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[2] We will use the tensor product notation ("$\otimes$") to denote product across coordinates in a repeated game, and the traditional product notation ("$\times$") to denote product across players.

1. $\mu(\mathcal{X}_\perp^t) \geq \alpha$ for all $t \in [k]$, and

2. for all $x \in \mathcal{X}$,

$$\mu(x) = \mu(x|_{\overline{F}_x}) \cdot \prod_{t \in F_x} \mu(x^t) \tag{7}$$

where for all question tuples $x = (x^1, x^2, \ldots, x^k) \in \mathcal{X}$, $F_x \subseteq [n]$ denotes the set of coordinates of x that lie in the anchor, i.e.

$$F_x = \{t \in [k] : x^t \in \mathcal{X}_\perp^t\}$$

and $\overline{F}_x$ denotes the complement, i.e., $[n] - F_x$.

Here for a set $S \subseteq [n]$, $\mu(x|_S)$ denotes the marginal probability of the question tuple x restricted to the coordinates in S, i.e.

$$\mu(x|_S) = \sum_{x'|_S = x|_S} \mu(x').$$

When $k = 2$ this definition coincides with the definition of two-player anchored games in Definition 4. Additionally, just like the two-player case, one can easily extend the anchoring transformation given in Definition 3 to arbitrary k-player games:

Proposition 10. *Let $G = (\mathcal{X}, \mathcal{A}, \mu, V)$ be a k-player game. Let $G_\perp$ be the k-player game where the referee samples $(x^1, x^2, \ldots, x^k)$ according to μ, replaces each x^t with an auxiliary symbol $\perp$ independently with probability α, and checks the players' answers according to V if all $x^t \neq \perp$, and otherwise the referee accepts. Then $G_\perp$ is an α-anchored game satisfying*

$$\mathrm{val}(G_\perp) = 1 - (1-\alpha)^k \cdot (1 - \mathrm{val}(G)), \qquad \mathrm{val}^*(G_\perp) = 1 - (1-\alpha)^k \cdot (1 - \mathrm{val}^*(G)). \tag{8}$$

Proof. We give the proof for the classical value; the same argument carries over to the entangled value. First, it is clear that $\mathrm{val}(G_\perp) \geq (1 - (1-\alpha^k)) + (1-\alpha)^k \cdot \mathrm{val}(G)$. For the other direction, consider an optimal strategy for $G_\perp$. Under this strategy, we can express the entangled value as

$$\mathrm{val}(G_\perp) = (1-\alpha)^k \cdot \Pr(W|\forall t, \ x^t \neq \perp) + (1 - (1-\alpha^k)) \cdot \Pr(W|\exists t \text{ s.t. } x^t = \perp)$$

where W is the event that the players win. The optimal strategy for $G_\perp$ yields a strategy for G that wins with probability $\Pr(W|\forall t, \ x^t \neq \perp)$, which can be at most $\mathrm{val}(G)$. Since $\Pr(W|\exists t \text{ s.t. } x^t = \perp) = 1$, we obtain the desired equality. $\qquad\square$

4   Classical multiplayer games

Perhaps the most well-known open problem about the classical parallel repetition of games is whether an analogue of Raz's theorem holds for games with more than two players. While the two-player case already presented a number of non-trivial difficulties, proving a parallel repetition theorem for three or more players is believed to require substantially new ideas.[3]

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[3]This is mainly because the Raz/Holenstein framework, if extended to a multiplayer parallel repetition theorem in full generality, would likely also yield new lower bound techniques for multiparty communication complexity, an area that has long resisted progress (especially for the important multiparty direct sum/product problems).

In this section we make some progress on the multiplayer parallel repetition question: we prove a parallel repetition theorem for anchored games involving any number of players.

Theorem 11. *Let $G = (\mathcal{X}, \mathcal{A}, \mu, V)$ be a k-player α-anchored game such that $\mathrm{val}(G) \le 1 - \varepsilon$. Then*

$$\mathrm{val}(G^n) \le \exp\left(-\frac{\alpha^{2k} \cdot \varepsilon^3 \cdot n}{384 \cdot s \cdot k^2}\right), \tag{9}$$

where $s = \log|\mathcal{A}|$.

Combined with the anchoring operation described in Proposition 10, we obtain a gap amplification transformation that can be applied to any k-player game, yielding a decay of the value that matches, at least qualitatively, what one would expect from a general parallel repetition theorem.

From a more quantitative point of view, even in the two-player setting the optimal exponent of ε in (9) remains unknown. Perhaps more importantly, it is unclear whether the exponential dependence in k, due to the term α^k, in the bound is necessary; known lower bounds [Fei95, CWY15] only show the need for a polynomial dependence on k in the exponent.

For the remainder of this section we fix a k-player α-anchored game $G = (\mathcal{X}, \mathcal{A}, \mu, V)$, an integer n, and a deterministic strategy for the k players in the repeated game G^n that achieves success probability $\mathrm{val}(G^n)$. In Section 4.1 we introduce the notation, random variables and basic lemmas for the proof. The proof of Theorem 11 itself is given in Section 4.2.

4.1 Breaking classical multipartite correlations

We refer to Section 3.3 for basic notation related to multiplayer games.

Let $C \subseteq [n]$ a fixed set of coordinates for the repeated game G^n of size $|C| = n - m$. It will be convenient to fix $C = \{m + 1, m + 2, \ldots, n\}$; the symmetry of the problem will make it clear that this is without loss of generality. Let $Z = A_C = (A_C^1, A_C^2, \ldots, A_C^k)$ denote the players' answers associated with the coordinates indexed by C.

For $t \in [k]$ let $\mathcal{Y}^t = (\mathcal{X}^t \setminus \mathcal{X}_\perp^t) \cup \{\perp\}$, and define a random variable

$$Y^t = \begin{cases} X^t, & X^t \in \mathcal{X}^t \setminus \mathcal{X}_\perp^t \\ \perp, & X^t \in \mathcal{X}_\perp^t \end{cases}. \tag{10}$$

Let $\mathcal{Y} = \mathcal{Y}^1 \times \mathcal{Y}^2 \times \ldots \times \mathcal{Y}^k$ and $Y = (Y^1, Y^2, \ldots, Y^k)$. For G^n we write

$$Y^{\otimes n} = (Y_1, Y_2, \ldots, Y_n) = \left(\left(Y_1^1, \ldots, Y_1^k\right), \left(Y_2^1, \ldots, Y_2^k\right), \ldots, \left(Y_n^1, \ldots, Y_n^k\right)\right).$$

Note that each k-tuple Y_i is a deterministic function of X_i. Furthermore, we will write Y_i^{-t} to denote Y_i with the t-th coordinate Y_i^t omitted.

For $i \in [n]$ let D_i be a subset of $[k]$ of size $k - 1$ chosen uniformly at random, and $\overline{D}_i \in [k]$ its complement in $[k]$. Let $M_i = Y_i^{D_i}$ denote the coordinates of Y associated to indices in D_i. Define the *dependence-breaking random variable* Ω_i as

$$\Omega_i = \begin{cases} (D_i, M_i) & i \in \overline{C} \\ X_i & i \in C \end{cases}. \tag{11}$$

The importance of Ω is captured in the following lemma.

Lemma 12. *(Local Sampling) Let X, Z, Ω be as above. Then $\mathsf{P}_{X_{-i}|X_i\Omega_{-i}Z}$ is a product distribution across the players:*

$$\mathsf{P}_{X_{-i}|X_i\Omega_{-i}Z} = \prod_{t=1}^{k} \mathsf{P}_{X_{-i}^t|\Omega_{-i}^t Z^t X_i^t}.$$

Proof. Conditioned on $M_i = Y_i^{D_i}$ each $X_i = (X_i^1, X_i^2, \ldots, X_i^k)$ is a product distribution, hence $\mathsf{P}_{X_{-i}|\Omega_{-i}X_i}$ is product. Since for $t \in [k]$ Z^t is a deterministic function of X^t the same holds of $\mathsf{P}_{X_{-i}|\Omega_{-i}ZX_i}$. $\square$

Lemma 12 crucially relies on the sets D_j being of size $k - 1$: if two or more of the players' questions are unconstrained in a coordinate it is no longer necessarily true that $\mathsf{P}_{X_{-i}|\Omega_{-i}ZX_i}$ is product across all players.

Let $W = W_C = \bigwedge_{i=1}^{C} W_i$ denote the event that the players' answers Z to questions in the coordinates indexed by C satisfy the predicate V. Let

$$\delta = \frac{|C|\log|\mathcal{A}| + \log \frac{1}{\Pr(W_C)}}{m}. \tag{12}$$

The following lemma and its corollary are direct consequences of analogous lemmas used in the analysis of repeated two-player games, as stated in e.g. [Hol09, Lem. 5] and [Hol09, Cor. 6]. They do not depend on the structure of the game, and only rely on W being an event defined only on (X_C, Z).

Lemma 13. *We have*

$$(i) \qquad \mathbb{E}_{i\in[m]} \|\mathsf{P}_{X_iY_i\Omega_i|W} - \mathsf{P}_{X_iY_i\Omega_i}\| \leq \sqrt{\delta}.$$

$$(ii) \qquad \mathbb{E}_{i\in[m]} \|\mathsf{P}_{X_iY_iZ\Omega_{-i}|W} - \mathsf{P}_{X_i|Y_i}\mathsf{P}_{Y_iZ\Omega_{-i}|W}\| \leq \sqrt{\delta}$$

$$(iii) \qquad \mathbb{E}_{i\in[m]} \|\mathsf{P}_{Y_iZ\Omega|W} - \mathsf{P}_{Y_i|\Omega_i}\mathsf{P}_{Z\Omega|W}\| \leq \sqrt{\delta}.$$

Proof. Item (i) follows directly from [Hol09, Lem. 5] by taking $U_i = X_iY_i\Omega_i$. For (ii) apply [Hol09, Cor. 6] with $U_i = X_i$ and $T = (Y_1, Y_2, \ldots, Y_m, X_C)$ to get

$$\mathbb{E}_{i\in[m]} \|\mathsf{P}_{X_iZY_{[m]}X_C|W} - \mathsf{P}_{X_i|Y_i}\mathsf{P}_{Y_iZY_{[m]\backslash\{i\}}X_C|W}\| \leq \sqrt{\delta}, \tag{13}$$

which is stronger than (ii); (ii) follows by marginalizing $Y_i^{\overline{D_i}}$ in each term. Finally, the same corollary applied with $U_i = Y_i$ and $T = \Omega$ shows (iii). $\square$

Corollary 14.

$$\mathbb{E}_{i\in[m]} \sum_{t=1}^{k} \|\mathsf{P}_{Y_i}\mathsf{P}_{Z\Omega_{-i}|WY_i} - \mathsf{P}_{Y_i}\mathsf{P}_{Z\Omega_{-i}|WY_i^{-t}}\| \leq 3k \cdot \sqrt{\delta}.$$

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Proof. We have $P_{Y_i|\Omega_i}P_{Z\Omega|W} = P_{Y_i|\Omega_i}P_{\Omega_i|W}P_{Z\Omega_{-i}|W\Omega_i}$. Applying Lemma 5 with $Q_F = P_{\Omega_i|W}$, $S_F = P_{\Omega_i}$, and $R_{G|F} = P_{Y_i|\Omega_i}P_{Z\Omega_{-i}|W\Omega_i}$, we see that

$$\mathop{\mathbb{E}}_{i\in[m]} \|P_{Y_i|\Omega_i}P_{Z\Omega|W} - P_{Y_i\Omega_i}P_{Z\Omega_{-i}|W\Omega_i}\| = \mathop{\mathbb{E}}_{i\in[m]} \|P_{\Omega_i|W} - P_{\Omega_i}\| \le \sqrt{\delta},$$

where the last inequality follows from Lemma 13, item (i). Combining the above with item (iii) of the same Lemma, we have

$$\mathop{\mathbb{E}}_{i\in[m]} \|P_{Y_iZ\Omega|W} - P_{Y_i\Omega_i}P_{Z\Omega_{-i}|W\Omega_i}\| \le 2\sqrt{\delta}. \tag{14}$$

Noting that Ω_i is determined by Y_i (the D_i are completely independent of everything else), (14) implies

$$\mathop{\mathbb{E}}_{i\in[m]}\mathop{\mathbb{E}}_{t\in[k]} \|P_{Y_iZ\Omega_{-i}|W} - P_{Y_i}P_{Z\Omega_{-i}|WY_i^{-t}}\| = \mathop{\mathbb{E}}_{i\in[m]} \|P_{Y_iZ\Omega_{-i}|W} - P_{Y_i}P_{Z\Omega_{-i}|W\Omega_i}\|$$
$$\le 2\sqrt{\delta}.$$

Finally, notice that Lemmas 5 and 13 imply $\mathop{\mathbb{E}}_{i\in[m]} \|P_{Y_iZ\Omega_{-i}|W} - P_{Y_i}P_{Z\Omega_{-i}|WY_i}\| = \mathop{\mathbb{E}}_{i\in[m]} \|P_{Y_i} - P_{Y_i|W}\| \le \sqrt{\delta}$; the desired result follows. $\qquad\square$

4.2   Proof of the parallel repetition theorem

This section is devoted to the proof of Theorem 11. The main ingredient of the proof is given in the next proposition.

Proposition 15. *Let $C \subseteq [n]$ and X, Z, Ω_{-i} be defined as in Section 4.1. Then*

$$\mathop{\mathbb{E}}_{i\in[m]} \|P_{X_i\Omega_{-i}Z|W} - P_{X_i}P_{\Omega_{-i}Z|W,Y_i=\perp^k}\| \le (6k\alpha^{-k} + 1)\sqrt{\delta}, \tag{15}$$

where δ is defined in (12).

Theorem 11 follows from this proposition in a relatively standard fashion; this is done at the end of this section. Let us now prove Proposition 15 assuming a certain technical statement, Lemma 16. This lemma is proved immediately after.

Proof of Proposition 15. First observe that

$$\|P_{X_i\Omega_{-i}Z|W} - P_{X_i}P_{\Omega_{-i}Z|W,Y_i=\perp^k}\| = \|P_{X_iY_i\Omega_{-i}Z|W} - P_{X_iY_i}P_{\Omega_{-i}Z|W,Y_i=\perp^k}\|$$

as Y_i is a deterministic function of X_i. Applying Lemma 13, item (ii) we get

$$\mathop{\mathbb{E}}_{i\in[m]} \|P_{X_iY_i\Omega_{-i}Z|W} - P_{X_i|Y_i}P_{Y_i\Omega_{-i}Z|W}\| \le \sqrt{\delta}.$$

The latter distribution can be written as $P_{Y_i|W}P_{X_i|Y_i}P_{\Omega_{-i}Z|WY_i}$. Applying Lemma 5 with $Q_F = P_{Y_i|W}$ and $S_F = P_{Y_i}$ we see that

$$\|P_{X_i|Y_i}P_{Y_i\Omega_{-i}Z|W} - P_{X_iY_i}P_{\Omega_{-i}Z|WY_i}\| = \|P_{Y_i|W} - P_{Y_i}\|,$$

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which is bounded by $\sqrt{\delta}$ on average over i by Lemma 13, item (i). Hence

$$\underset{i\in[m]}{\mathbb{E}}\left\|\mathsf{P}_{X_i\Omega_{-i}Z|W}-\mathsf{P}_{X_i}\mathsf{P}_{\Omega_{-i}Z|W,Y_i=\perp^k}\right\|\leq 2\sqrt{\delta}+\underset{i\in[m]}{\mathbb{E}}\left\|\mathsf{P}_{X_iY_i}\mathsf{P}_{\Omega_{-i}Z|WY_i}-\mathsf{P}_{X_iY_i}\mathsf{P}_{\Omega_{-i}Z|W,Y_i=\perp^k}\right\|$$

$$=2\sqrt{\delta}+\underset{i\in[m]}{\mathbb{E}}\left\|\mathsf{P}_{Y_i}\mathsf{P}_{\Omega_{-i}Z|WY_i}-\mathsf{P}_{Y_i}\mathsf{P}_{\Omega_{-i}Z|W,Y_i=\perp^k}\right\|,$$

where the equality follows from Lemma 5 applied with $\mathsf{R}_{G|F}=\mathsf{P}_{X_i|Y_i}$. Applying the triangle inequality,

$$\underset{i\in[m]}{\mathbb{E}}\left\|\mathsf{P}_{X_iY_i}\mathsf{P}_{\Omega_{-i}Z|WY_i}-\mathsf{P}_{X_iY_i}\mathsf{P}_{\Omega_{-i}Z|W,Y_i=\perp^k}\right\|$$

$$=\underset{i\in[m]}{\mathbb{E}}\left\|\mathsf{P}_{Y_i}\mathsf{P}_{\Omega_{-i}Z|WY_i}-\mathsf{P}_{Y_i}\mathsf{P}_{\Omega_{-i}Z|W,Y_i=\perp^k}\right\|$$

$$\leq\underset{i\in[m]}{\mathbb{E}}\sum_{t=1}^{k}\left\|\mathsf{P}_{Y_i}\mathsf{P}_{\Omega_{-i}Z|WY_i^{<t}=\perp^{t-1},Y_i^{\geq t}}-\mathsf{P}_{Y_i}\mathsf{P}_{\Omega_{-i}Z|WY_i^{\leq t}=\perp^{t},Y_i^{>t}}\right\| \tag{16}$$

$$\leq 6k\alpha^{-k}\cdot\sqrt{\delta}, \tag{17}$$

where (16) is proved by Lemma 16 below and (17) follows from Corollary 14. $\qquad\square$

Lemma 16. *Let $S\subset[k]$ and $t\in\overline{S}$. Then*

$$\left\|\mathsf{P}_{Y_i}\mathsf{P}_{\Omega_{-i}Z|WY_i^{S}=\perp^{S},Y_i^{\overline{S}}}-\mathsf{P}_{Y_i}\mathsf{P}_{\Omega_{-i}Z|WY_i^{S\cup\{t\}}=\perp^{S\cup\{t\}},Y_i^{\overline{S}\setminus\{t\}}}\right\|$$

$$\leq 2\alpha^{-(|S|+1)}\cdot\left\|\mathsf{P}_{Y_i}\mathsf{P}_{Z\Omega_{-i}|WY_i}-\mathsf{P}_{Y_i}\mathsf{P}_{Z\Omega_{-i}|WY_i^{-t}}\right\|. \tag{18}$$

Proof. In the proof for ease of notation we omit the subscript i and write Y instead of Y_i. After relabeling we may assume $S=\{1,2,\ldots,r-1\}$ and $t=r$ where $1\leq r<k$. Expanding the expectation over Y explicitly we can rewrite the left-hand side of (18) as

$$\left\|\mathsf{P}_Y\cdot\left(\mathsf{P}_{\Omega_{-i}Z|W,y^{\geq r},y^{<r}=\perp^{r-1}}-\mathsf{P}_{\Omega_{-i}Z|W,y^{>r},y^{\leq r}=\perp^r}\right)\right\|. \tag{19}$$

Next we use a symmetrization argument to bound the above expression. Consider a random variable $\hat{Y}$ that is a copy of Y, and is coupled to Y in the following way: $\hat{Y}^{-r}=Y^{-r}$, and conditioned on any setting of $Y^r=y^r$, $\hat{Y}^r$ and Y^r are independent. Using the fact that $\Pr[\hat{Y}^r=\perp]\geq\alpha$ conditioned on any value of $Y^{-r}=U^{-r}=y^{-r}$, we get that the expression in (19) is at most

$$\alpha^{-1}\left\|\mathsf{P}_{Y^{-r}}\mathsf{P}_{Y^r|Y^{-r}}\mathsf{P}_{\hat{Y}^r|Y^{-r}}\cdot\left(\mathsf{P}_{\Omega_{-i}Z|W,y^{>r},y^r,y^{<r}=\perp^{r-1}}-\mathsf{P}_{\Omega_{-i}Z|W,y^{>r},\hat{y}^r,y^{<r}=\perp^{r-1}}\right)\right\|.$$

Using the triangle inequality and symmetry of Y and $\hat{Y}$, this expression can be bounded by

$$2\alpha^{-1}\cdot\left\|\mathsf{P}_Y\cdot\left(\mathsf{P}_{\Omega_{-i}Z|W,y^{>r},y^r,y^{<r}=\perp^{r-1}}-\mathsf{P}_{\Omega_{-i}Z|W,y^{>r},y^{<r}=\perp^{r-1}}\right)\right\|,$$

which after noting that the quantity $\left\|\mathsf{P}_{\Omega_{-i}Z|W,y^{>r},y^r,y^{<r}=\perp^{r-1}}-\mathsf{P}_{\Omega_{-i}Z|W,y^{>r},y^{\leq r}=\perp^r}\right\|$ is independent of the variable $Y^{<r}$, can be rewritten as

$$2\alpha^{-1}\cdot\left\|\mathsf{P}_{Y^{\geq r}}\cdot\left(\mathsf{P}_{\Omega_{-i}Z|W,y^{>r},y^r,y^{<r}=\perp^{r-1}}-\mathsf{P}_{\Omega_{-i}Z|W,y^{>r},y^{<r}=\perp^{r-1}}\right)\right\|.$$

Using that the event that $Y^{<r} = \perp^{r-1}$ occurs with probability at least α^{r-1} and $P_{Y \geq r | Y^{<r} = \perp^{r-1}} = P_{Y \geq r}$ by the anchor property, we can finally bound (19) by

$$2\alpha^{-r} \cdot \left\| P_Y P_{Z\Omega_{-i}|WY} - P_Y P_{Z\Omega_{-i}|WY^{-r}} \right\|,$$

which is the desired result. $\qquad\square$

We prove Theorem 11 by iteratively applying Proposition 15 as follows.

Proof of Theorem 11. Let $C_0 = \emptyset$ and $\delta_0 = 0$. While $(6k\alpha^{-k} + 1)\sqrt{\delta_s} \leq \varepsilon/2$, by Proposition 15, we can choose $i \in \overline{C}_s$ with $\left\| P_{X_i \Omega_{-i} Z | W} - P_{X_i} P_{\Omega_{-i} Z | W, Y_i = \perp^k} \right\| \leq \varepsilon/2$. Set $C_{s+1} = C_s \cup \{i\}$ and $\delta_{s+1} = (|C_{s+1}| \log |\mathcal{A}| + \log 1 / \Pr(W_{C_{s+1}}))/m$. First we show that throughout this process the bound

$$\Pr[W_{C_s}] \leq (1 - \varepsilon/2)^{|C_s|} \tag{20}$$

holds. Since by the choice of i one has $\left\| P_{X_i \Omega_{-i} Z | W_C} - P_{X_i} P_{\Omega_{-i} Z | W_C, Y_i = \perp^k} \right\| \leq \varepsilon/2$, to establish (20) it will suffice to show that

$$\Pr(W_i | W_C) \leq \mathrm{val}(G) + \left\| P_{X_i \Omega_{-i} Z | W_C} - P_{X_i} P_{\Omega_{-i} Z | W_C, Y_i = \perp^k} \right\|. \tag{21}$$

The proof of (21) is based on a rounding argument. Consider the following strategy for G: First, the players use shared randomness to obtain a common sample from $P_{\Omega_{-i} Z | W_C, Y_i = \perp^k}$. After receiving her question x_i^*, player $t \in [k]$ samples questions for the remaining coordinates according to $P_{X_{-i}^t | \Omega_{-i}^t Z^t X_i^t}$, forming the tuple $X^t = (X_{-i}^t, x_i^*)$. She determines her answer $a_i^t \in \mathcal{A}_i^t$ according to the strategy for G^n. The distribution over questions X implemented by players following this strategy is

$$P_{X_i} P_{\Omega_{-i} Z | W_C Y_i = \perp^k} \prod_{t=1}^{k} P_{X_{-i}^t | \Omega_{-i}^t Z^t X_i^t},$$

which by Lemma 12 is equal to

$$P_{X_i} P_{\Omega_{-i} Z | W_C Y_i = \perp^k} P_{X_{-i} | \Omega_{-i} Z}.$$

On the other hand from the definition of Ω_{-i} we have

$$P_{X\Omega_{-i} Z | W_C} = P_{X_i \Omega_{-i} Z | W_C} P_{X_{-i} | \Omega_{-i} Z W_C} = P_{X_i \Omega_{-i} Z | W_C} P_{X_{-i} | \Omega_{-i} Z}.$$

Applying Lemma 5 with $R = P_{X_{-i} | \Omega_{-i} Z}$ it follows that

$$\left\| P_{XZ\Omega_{-i} | W_C} - P_{X_i} P_{\Omega_{-i} Z | W_C Y_i = \perp^k} P_{X_{-i} | \Omega_{-i} Z} \right\| = \left\| P_{X_i \Omega_{-i} Z | W_C} - P_{X_i} P_{\Omega_{-i} Z | W_C, Y_i = \perp^k} \right\|.$$

Now by definition the winning probability of the extracted strategy for G is at most $\mathrm{val}(G)$, and (21) follows.

Let now C be the final set of coordinates when the above-described process stops; at this point we must have

$$\delta = \frac{|C| \log |\mathcal{A}| + \log \frac{1}{\Pr(W_C)}}{n - |C|} > \frac{\alpha^{2k} \varepsilon^2}{48 \cdot k^2}.$$

18

If $|C| \geq n/2$ we are already done by (20). Suppose $\frac{|C|\log|\mathcal{A}|+\log(\frac{1}{\Pr[W_C]})}{n} > \frac{\alpha^{2k}\epsilon^2}{96 \cdot k^2}$. If $\log(\frac{1}{\Pr(W_C)}) \geq \frac{n\cdot\alpha^{2k}\epsilon^2}{192\cdot k^2}$ we are again done; hence, we can assume

$$\frac{|C|\log|\mathcal{A}|}{n} > \frac{\alpha^{2k}\varepsilon^2}{192 \cdot k^2}.$$

Now plugging the lower bound on the size of C in (20) we get

$$\mathrm{val}(G^n) \leq \Pr(W_C) \leq \exp\left(-\frac{\alpha^{2k} \cdot \varepsilon^3 \cdot n}{384 \cdot k^2 \cdot s}\right)$$

where $s = \log|\mathcal{A}|$, which completes the proof. $\qquad\square$

Some remarks on multiplayer parallel repetition for general games. We conclude this section with some remarks about Theorem 11 and the more general problem of multiplayer parallel repetition. Our analysis of repeated anchored games follows the information-theoretic approach of Raz and Holenstein. It is a natural question, predating this work by many years, whether one can extend this framework to prove parallel repetition for general multiplayer games?

At first sight the Raz/Holenstein framework may seem quite suitable for multiplayer parallel repetition. For instance, it is folklore that classically the approach extends to the case of free games with any number of players, and furthermore, many of the other technical components of the proof readily carry over in much generality. Despite these positive signs, attempts to extend Raz's original argument to the general multiplayer setting have so far failed for different and rather interesting *technical reasons*. Embarrassingly, to our knowledge, it is not even known how to extend the information-theoretic approach to prove that the value of a repeated k-player game decays at all![4]

We give an example of one of the difficulties in proving a multiplayer parallel repetition theorem for general games. Consider the problem of defining an appropriate dependency-breaking variable Ω in the multiplayer setting. There are two competing demands on Ω: on one hand the breaking of dependencies between the players' respective questions seems to require it to contain as many of the players' questions as possible for each coordinate $i \in \overline{C}$. In fact, if the correlations between the players inputs' are generic, it seems hard to avoid the need to keep at least $k-1$ inputs in each Ω_i, as we do in Lemma 12. On the other hand, for correlated sampling to be possible, it seems necessary for Ω to specify very few of the questions per coordinate, or in fact in the generic case, at most 1; as soon as $k \geq 3$ both requirements are in direct contradiction.

An insight behind our result is that it is sometimes possible to decouple the above two competing demands on Ω (i.e. the *dependency-breaking* and the *correlated sampling* components). More precisely, when the base game is anchored, we show how to define a useful dependency-breaking variable (or quantum state, in the entangled players setting) that can be sampled *without* correlated sampling. With correlated sampling out of the way, the aforementioned conflict between

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[4]One can modify a Ramsey-theoretic argument of Verbitsky to show that if $\mathrm{val}(G) < 1$, then $\mathrm{val}(G^n)$ must go to 0 eventually as n grows [Ver96], but the bound on the rate of decay is extremely poor.

correlated sampling and dependency-breaking disappears, allowing us to proceed with the argument.

5 Parallel repetition of anchored games with entangled players

This section is devoted to the analysis of the entangled value of repeated anchored games. While we expect the arguments to carry over to the multiplayer case without any additional difficulty we leave this for future work and focus on two-player entangled games. We use $\mathcal{X}$ and $\mathcal{Y}$ (resp. $\mathcal{A}$ and $\mathcal{B}$) to denote the two players' respective question (resp. answer) sets in G, and following a well-established convention we name the first player Alice and the second Bob. Our main result is the following.

Theorem 17. *Let* $G = (\mathcal{X} \times \mathcal{Y}, \mathcal{A} \times \mathcal{B}, \mu, V)$ *be a two-player* α*-anchored game satisfying* $\mathrm{val}^*(G) = 1 - \varepsilon$. *Then*

$$\mathrm{val}^*(G^n) \leq \exp(-\Omega(\alpha^{16} \cdot \varepsilon^8 \cdot n/s)),$$

where $s = \log |\mathcal{A}||\mathcal{B}|$.

Thus as in the classical multiplayer case, the anchoring operation described in Proposition 10 provides a general gap amplification transformation for the entangled value of any two-player game.

For the remainder of the section we fix an α-anchored two-player game $G = (\mathcal{X} \times \mathcal{Y}, \mathcal{A} \times \mathcal{B}, \mu, V)$ with entangled value $\mathrm{val}^*(G) = 1 - \varepsilon$ and anchor sets $\mathcal{X}_\perp \subseteq \mathcal{X}$, $\mathcal{Y}_\perp \subseteq \mathcal{Y}$ for Alice and Bob, respectively. Without loss of generality we will assume that $\alpha \leq 1/2$, because if a game is α-anchored for $\alpha > 1/2$, then it is also $1/2$-anchored. We also fix an optimal strategy for G^n, consisting of a shared entangled state $|\psi\rangle^{E_A E_B}$ and POVMs $\{A(x_{[n]}, a_{[n]})\}$ and $\{B(y_{[n]}, b_{[n]})\}$ for Alice and Bob respectively. Without loss of generality we assume that $|\psi\rangle$ is invariant under permutation of the two registers, i.e. there exist basis vectors $\{|v_j\rangle\}_j$ such that $|\psi\rangle = \sum_j \sqrt{\lambda_j}|v_j\rangle|v_j\rangle$.

5.1 Setup

We introduce the random variables, entangled states and operators that play an important role in the proof of Theorem 17. The section is divided into three parts: first we define the dependency-breaking variable Ω, with a slightly modified definition from the one introduced for the classical multiplayer setting in Section 4. Then we state useful lemmas about conditioned distributions. Finally we describe the states and operators used in the proof.

Dependency-breaking variables. Let $C \subseteq [n]$ a fixed set of coordinates for the repeated game G^n. We fix $C = \{m+1, m+2, \ldots, n\}$, where $m = n - |C|$, as this will easily be seen to hold without loss of generality. Let $(X_{[n]}, Y_{[n]})$ be distributed according to $\mu_{[n]}$ and $(A_{[n]}, B_{[n]})$ be defined from $X_{[n]}$ and $Y_{[n]}$ as follows:

$$\mathsf{P}_{A_{[n]}B_{[n]}|X_{[n]}=x_{[n]},Y_{[n]}=y_{[n]}}(a_{[n]}, b_{[n]}) = \langle\psi|A(x_{[n]}, a_{[n]}) \otimes B(y_{[n]}, b_{[n]})|\psi\rangle.$$

Let (X_C, Y_C) and $Z = (A_C, B_C)$ be random variables that denote the players' questions and answers respectively associated with the coordinates indexed by C. For $i \in [n]$ let W_i denote the event that the players win round i while playing G^n. Let $W_C = \wedge_{i \in C} W_i$. We will often write W in place of W_C when C is clear from context.

We introduce a random variable Ω that is closely related to dependency-breaking variables used in classical proofs of the parallel repetition theorem (such as in Section 4), as well as direct sum theorems in communication complexity. In those works, for all $i \in [n]$, Ω_i fixes at least one of X_i or Y_i (and sometimes both). Thus, conditioned on Ω, $X_{[n]}$ and $Y_{[n]}$ are independent of each other. Our Ω variable is slightly more complicated. It will be important in our analysis to allow, e.g., X_i to be free to take on a value from Alice's anchor $\mathcal{X}_\perp$, regardless of the value of Ω_i. In other words, from Bob's point of view, even when conditioned on Ω_i, Alice's input may randomly choose to take on some anchor value.

Let $D_1, \ldots, D_m$ be independent and uniformly distributed over $\{A, B\}$. Let $M_1, \ldots, M_m$ be independent random variables defined in the following way. For each $i \in [m]$ and $(x, y) \in \mathcal{X} \times \mathcal{Y}$,

$$
M_i = \begin{cases}
\perp & \text{with prob. } 1 - p_A & \text{if } D_i = A \\
\perp/x & \text{with prob. } p_A^2 \cdot \mu(x | x \notin \mathcal{X}_\perp) & \text{if } D_i = A \\
\perp/x & \text{with prob. } p_A \cdot (1 - p_A) \cdot \mu(x | x \in \mathcal{X}_\perp) & \text{if } D_i = A \\
\perp & \text{with prob. } 1 - p_B & \text{if } D_i = B \\
\perp/y & \text{with prob. } p_B^2 \cdot \mu(y | y \notin \mathcal{Y}_\perp) & \text{if } D_i = B \\
\perp/y & \text{with prob. } p_B \cdot (1 - p_B) \cdot \mu(y | y \in \mathcal{Y}_\perp) & \text{if } D_i = B
\end{cases}
$$

where $p_A := (1 - \mu(\mathcal{X}_\perp))^{1/3}$ and $p_B := (1 - \mu(\mathcal{Y}_\perp))^{1/3}$. For $i \in [m]$ let $\Omega_i := (D_i, M_i)$, and $\Omega := (\Omega_1, \ldots, \Omega_m, X_C, Y_C)$. We write Ω_{-i} to denote the random variable $(\Omega_1, \ldots, \Omega_{i-1}, \Omega_{i+1}, \ldots, \Omega_m, X_C, Y_C)$.

We re-introduce random variables $(X_{[n]}, Y_{[n]})$ that depend on Ω. We will show that the distribution of $(X_{[n]}, Y_{[n]})$ is μ^n, as required. For $i \in C$, X_i and Y_i are fixed by Ω. If $i \notin C$, and $D_i = A$, then

$$
X_i = \begin{cases}
x & \text{with prob. } \mu(x | x \in \mathcal{X}_\perp) & \text{if } M_i = \perp \\
\begin{cases} x' & \text{with prob. } (1 - p_A) \cdot \mu(x' | x' \in \mathcal{X}_\perp) \\ x & \text{with prob. } p_A \end{cases} & & \text{if } M_i = \perp/x
\end{cases}
$$

$$
Y_i = \begin{cases}
y & \text{with prob. } \mu(y) & \text{if } M_i = \perp \\
y & \text{with prob. } \mu(y | x) & \text{if } M_i = \perp/x
\end{cases}
$$

The distribution of $M_i X_i Y_i$ conditioned on $D_i = B$ is defined similarly with the roles of X_i and Y_i interchanged. Clearly, $P_{X_{[n]} Y_{[n]}} = P_{X_1 Y_1} \cdots P_{X_n Y_n}$, thus the following claim shows that the marginal distribution $P_{X_{[n]} Y_{[n]}}$ is equal to $\mu_{[n]}$.

Claim 18. *For all $i \in [n]$, for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$, $P_{X_i Y_i}(x, y) = \mu(x, y)$.*

Proof. If $i \in C$ then by definition $X_i Y_i$ are distributed according to μ. Suppose that $i \notin C$. We show that $P_{X_i Y_i | D_i = A}$ is identical to μ. An analogous calculation shows that $P_{X_i Y_i | D_i = B} = \mu$, proving the claim.

Let $(x, y) \in \mathcal{X} \times \mathcal{Y}$, and consider two cases. In the first case, $x \in \mathcal{X} - \mathcal{X}_\perp$. This implies that $M_i = \perp/x$; otherwise, x would have been drawn from $\mathcal{X}_\perp$:

$$
\begin{aligned}
\mathsf{P}_{X_i Y_i | D_i = A}(x, y) &= \mathsf{P}_{M_i | D_i = A}(\perp/x) \cdot \mathsf{P}_{X_i | M_i = \perp/x, D_i = A}(x) \cdot \mathsf{P}_{Y_i | X_i = x, M_i = \perp/x, D_i = A}(y) \\
&= p_A^2 \cdot \mu(x | x \notin \mathcal{X}_\perp) \cdot p_A \cdot \mu(y|x) \\
&= p_A^3 \cdot \mu(x, y) / \mu(x \notin \mathcal{X}_\perp).
\end{aligned}
$$

Using $p_A^3 = 1 - \mu(\mathcal{X}_\perp)$, $\mathsf{P}_{X_i Y_i | D_i = A}(x, y) = \mu(x, y)$. In the second case, $x \in \mathcal{X}_\perp$. Then,

$$
\begin{aligned}
&\mathsf{P}_{X_i Y_i | D_i = A}(x, y) \\
&= \mathsf{P}_{M_i | D_i = A}(\perp) \cdot \mathsf{P}_{X_i | M_i = \perp, D_i = A}(x) \cdot \mathsf{P}_{Y_i | X_i = x, M_i = \perp, D_i = A}(y) \\
&\quad + \sum_{x' \notin \mathcal{X}_\perp} \mathsf{P}_{M_i | D_i = A}(\perp/x') \cdot \mathsf{P}_{X_i | M_i = \perp/x', D_i = A}(x) \cdot \mathsf{P}_{Y_i | X_i = x, M_i = \perp/x', D_i = A}(y) \\
&\quad + \sum_{x' \in \mathcal{X}_\perp} \mathsf{P}_{M_i | D_i = A}(\perp/x') \cdot \mathsf{P}_{X_i | M_i = \perp/x', D_i = A}(x) \cdot \mathsf{P}_{Y_i | X_i = x, M_i = \perp/x', D_i = A}(y) \\
&= (1 - p_A) \cdot \mu(x | x \in \mathcal{X}_\perp) \cdot \mu(y) \tag{22} \\
&\quad + p_A^2 \cdot \sum_{x' \notin \mathcal{X}_\perp} \underbrace{\mu(x' | x' \notin \mathcal{X}_\perp)}_{= \mu(x')/\mu(\mathcal{X} \backslash \mathcal{X}_\perp)} \cdot (1 - p_A) \cdot \underbrace{\mu(x | x \in \mathcal{X}_\perp)}_{= \mu(x)/\mu(\mathcal{X}_\perp)} \cdot \mu(y|x') \tag{23} \\
&\quad + p_A \cdot (1 - p_A) \cdot \sum_{x' \in \mathcal{X}_\perp} \mu(x' | x' \in \mathcal{X}_\perp) \cdot (1 - p_A) \cdot \mu(x | x \in \mathcal{X}_\perp) \cdot \mu(y|x') \tag{24} \\
&\quad + p_A \cdot (1 - p_A) \cdot \mu(x | x \in \mathcal{X}_\perp) \cdot p_A \cdot \mu(y|x) \tag{25} \\
&= \mu(x | x \in \mathcal{X}_\perp) \cdot \mu(y) \Big[\underbrace{1 - p_A}_{(22)} + \underbrace{p_A^2 \cdot (1 - p_A)}_{(23)} + \underbrace{p_A \cdot (1 - p_A) \cdot (1 - p_A)}_{(24)} \\
&\quad\quad + \underbrace{p_A \cdot (1 - p_A) \cdot p_A}_{(25)} \Big] \\
&= \mu(x)\mu(y) \frac{1 - p_A^3}{\mu(\mathcal{X}_\perp)}.
\end{aligned}
$$

But $1 - p_A^3 = \mu(\mathcal{X}_\perp)$, and $\mu(x)\mu(y) = \mu(x, y)$ when $x \in \mathcal{X}_\perp$. So we can conclude in this case too that $\mathsf{P}_{X_i Y_i | D_i = A}(x, y) = \mu(x, y)$. $\qquad\square$

Conditioned distributions. Define $\delta := \frac{1}{m} \left(\log 1 / \Pr(W) + |C| \log |\mathcal{A}||\mathcal{B}| \right)$. For notational convenience we often use the shorthand $X_i \in \perp$ and $Y_i \in \perp$ to stand for $X_i \in \mathcal{X}_\perp$ and $Y_i \in \mathcal{Y}_\perp$, respectively. The following lemma essentially follows from lemmas in [Hol09] and the arguments used in the proof of Lemma 16 in Section 4.

Lemma 19. *The following statements hold on, average over i chosen uniformly in $[m]$:*

1. $\mathbb{E}_i \| \mathsf{P}_{D_i M_i X_i Y_i | W} - \mathsf{P}_{D_i M_i X_i Y_i} \| \le O(\sqrt{\delta})$

2. $\mathbb{E}_i \| \mathsf{P}_{\Omega Z X_i Y_i | W} - \mathsf{P}_{\Omega Z | W} \mathsf{P}_{X_i Y_i | \Omega} \| \le O(\sqrt{\delta})$

3. $\mathbb{E}_i \left\| \mathsf{P}_{X_i Y_i} \mathsf{P}_{\Omega_{-i} Z | X_i \in \perp, Y_i \in \perp, W} - \mathsf{P}_{X_i Y_i} \mathsf{P}_{\Omega_{-i} Z | X_i Y_i W} \right\| \leq O(\sqrt{\delta}/\alpha^2)$

4. $\mathbb{E}_i \left\| \mathsf{P}_{X_i Y_i} \mathsf{P}_{\Omega_{-i} Z | X_i Y_i W} - \mathsf{P}_{X_i Y_i \Omega_{-i} Z | W} \right\| \leq O(\sqrt{\delta}/\alpha^2)$

Quantum states and operators. Recall that we have fixed an optimal strategy for Alice and Bob in the game G^n. This specifies a shared entangled state $|\psi\rangle$, and measurement operators $\{A(x_{[n]}, a_{[n]})\}$ for Alice and $\{B(y_{[n]}, b_{[n]})\}$ for Bob.

Operators. Define, for all $a_C, b_C, x_{[n]}, y_{[n]}$:

$$A(x_{[n]}, a_C) := \sum_{a_{[n]} | a_C} A(x_{[n]}, a_{[n]}) \qquad\qquad B(y_{[n]}, b_C) := \sum_{b_{[n]} | b_C} B(y_{[n]}, b_{[n]})$$

where $a_{[n]} | a_C$ (resp. $b_{[n]} | b_C$) indicates summing over all tuples $a_{[n]}$ consistent with the suffix a_C (resp. $b_{[n]}$ consistent with suffix b_C). For all i, ω_{-i}, x_i, and y_i define:

$$A_{\omega_{-i}}(x_i, a_C) := \underset{X_{[n]} | \omega_{-i}, x_i}{\mathbb{E}} A(x_{[n]}, a_C) \qquad\qquad B_{\omega_{-i}}(y_i, b_C) := \underset{Y_{[n]} | \omega_{-i}, y_i}{\mathbb{E}} B(y_{[n]}, b_C)$$

where recall that $\mathbb{E}_{X_{[n]} | \omega_{-i}, x_i}$ is shorthand for $\mathbb{E}_{X_{[n]} | \Omega_{-i} = \omega_{-i}, X_i = x_i}$. Intuitively, these operators represent the "average" measurement that Alice and Bob apply, conditioned on $\Omega_{-i} = \omega_{-i}$, and $X_i = x_i$ and $Y_i = y_i$. Next, define

$$A_{\omega_{-i}}(\perp, a_C) := \underset{X_{[n]} | \Omega_{-i} = \omega_{-i}, X_i \in \perp}{\mathbb{E}} A(x_{[n]}, a_C) \qquad\qquad B_{\omega_{-i}}(\perp, b_C) := \underset{Y_{[n]} | \Omega_{-i} = \omega_{-i}, Y_i \in \perp}{\mathbb{E}} B(y_{[n]}, b_C).$$

These operators represent the "average" measurement performed by Alice and Bob, conditioned on $\Omega_{-i} = \omega_{-i}$ and $M_i = \perp$. Finally, for all $x_i \in \mathcal{X}$ and $y_i \in \mathcal{Y}$, define

$$A_{\omega_{-i}}(\perp / x_i, a_C) := (1 - p_A) A_{\omega_{-i}}(\perp, a_C) + p_A A_{\omega_{-i}}(x_i, a_C)$$
$$B_{\omega_{-i}}(\perp / y_i, b_C) := (1 - p_B) B_{\omega_{-i}}(\perp, b_C) + p_B B_{\omega_{-i}}(y_i, b_C)$$

where the weights p_A and p_B were defined in Section 5.1. Intuitively, these operators represent the "average" measurements conditioned on $\Omega_{-i} = \omega_{-i}$ and $M_i = \perp / x_i$ for some x_i (or $M_i = \perp / y_i$ for some y_i).

For notational convenience we often suppress the dependence on $(i, \omega_{-i}, z = (a_C, b_C))$ when it is clear from context. Thus, when we refer to an operator such as $A_{\perp / x}$, we really mean the operator $A_{\omega_{-i}}(\perp / x_i, a_C)$.

States. For all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, define the following (unnormalized) states:

$$|\Phi_{x,y}\rangle := \sqrt{A_x} \otimes \sqrt{B_y} |\psi\rangle \qquad\qquad\qquad |\Phi_{x,\perp}\rangle := \sqrt{A_x} \otimes \sqrt{B_\perp} |\psi\rangle$$
$$|\Phi_{\perp / x, \perp}\rangle := \sqrt{A_{\perp / x}} \otimes \sqrt{B_\perp} |\psi\rangle \qquad\qquad\qquad |\Phi_{\perp / x, y}\rangle := \sqrt{A_{\perp / x}} \otimes \sqrt{B_y} |\psi\rangle \qquad (26)$$
$$|\Phi_{\perp, \perp}\rangle := \sqrt{A_\perp} \otimes \sqrt{B_\perp} |\psi\rangle$$

together with the normalization factors

$$\gamma_{x,y} := \left\| |\Phi_{x,y}\rangle \right\| \qquad\qquad\qquad\qquad \gamma_{x,\perp} := \left\| |\Phi_{x,\perp}\rangle \right\|$$

$$\gamma_{\perp/x,\perp} := \||\Phi_{\perp/x,\perp}\rangle\| \qquad\qquad \gamma_{\perp/x,y} := \||\Phi_{\perp/x,y}\rangle\|$$
$$\gamma_{\perp,\perp} := \||\Phi_{\perp,\perp}\rangle\|$$

Note that these normalization factors are the square-roots of the probabilities that a certain pair of answers $z = (a_C, b_C)$ occurred, given the specified inputs and the dependency-breaking variables. For example, revealing the depencies on ω_{-i} and z, we have

$$\gamma^{\omega_{-i},z}_{x_i,y_i} = \sqrt{P_{Z|\omega_{-i},x_i,y_i}(z)}.$$

We denote the normalized states by $|\widetilde{\Phi}_{x,y}\rangle = |\Phi_{x,y}\rangle/\gamma_{x,y}$, $|\widetilde{\Phi}_{x,\perp}\rangle = |\Phi_{x,\perp}\rangle/\gamma_{x,\perp}$, $|\widetilde{\Phi}_{\perp/x,\perp}\rangle = |\Phi_{\perp,\perp}\rangle/\gamma_{\perp/x,\perp}$, $|\widetilde{\Phi}_{\perp/x,\perp/y}\rangle = |\Phi_{\perp/x,y}\rangle/\gamma_{\perp/x,y}$, and $|\widetilde{\Phi}_{\perp,\perp}\rangle = |\Phi_{\perp,\perp}\rangle/\gamma_{\perp,\perp}$.

5.2 Proof of the parallel repetition theorem

The proof of Theorem 17 follows the same outline as that of Theorem 11 given in Section 4, but the details differ significantly. The proof is based on the following main lemma.

Lemma 20. *For every i, ω_{-i}, $z = (a_C, b_C)$, x_i and y_i there exists unitaries U_{ω_{-i},z,x_i} acting on E_A and V_{ω_{-i},z,y_i} acting on E_B such that*

$$\mathbb{E}_i \mathbb{E}_{X_iY_i} \mathbb{E}_{\Omega_{-i}Z|W} \left\| (U_{\omega_{-i},z,x_i} \otimes V_{\omega_{-i},z,y_i}) \left|\widetilde{\Phi}^{\omega_{-i},z}_{\perp,\perp}\right\rangle - \left|\widetilde{\Phi}^{\omega_{-i},z}_{x_i,y_i}\right\rangle \right\|^2 = O(\delta^{1/4}/\alpha^4).$$

The proof of Lemma 20 is given in Section 5.2.1. Assuming the lemma, we prove Theorem 17.

Proof of Theorem 17. Using the same reasoning that allowed us to derive Theorem 11 from (21), the proof of Theorem 17 will follow once it is established that for any $C \subseteq [n]$,

$$\mathbb{E}_{i \in [n]\backslash C} \Pr(W_i|W_C) \le \mathrm{val}^*(G) + O(\delta^{1/8}/\alpha^2). \tag{27}$$

As for (27), the proof of (27) is based on the rounding argument, but it now involves entangled strategies. Fix $C \subseteq [n]$ and $i \in [n]\backslash C$, and let $W = W_C$. For every ω_{-i}, $z = (a_C, b_C)$, $x_i \in \mathcal{X}$, $y_i \in \mathcal{Y}$, $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$, define

$$\hat{A}_{\omega_{-i}}(x_i, a_i) := (A_{\omega_{-i}}(x_i, a_C))^{-1/2} \left(\sum_{a_{[n]}|a_i,a_C} A_{\omega_{-i}}(x_i, a_{[n]}) \right) (A_{\omega_{-i}}(x_i, a_C))^{-1/2}$$

$$\hat{B}_{\omega_{-i}}(y_i, b_i) := (B_{\omega_{-i}}(y_i, b_C))^{-1/2} \left(\sum_{b_{[n]}|b_i,b_C} B_{\omega_{-i}}(y_i, b_{[n]}) \right) (B_{\omega_{-i}}(y_i, b_C))^{-1/2}$$

where $a_{[n]}|a_i, a_C$ (resp. $b_{[n]}|b_i, b_C$) denotes summing over tuples $a_{[n]}$ that are consistent with a_C and a_i (resp. $b_{[n]}$ that are consistent with b_C and b_i), and $(A_{\omega_{-i}}(x_i, a_C))^{-1/2}$ and $(B_{\omega_{-i}}(y_i, b_C))^{-1/2}$ denote pseudoinverses. Note that the $\{\hat{A}_{\omega_{-i}}(x_i, a_i)\}_{a_i}$ and $\{\hat{B}_{\omega_{-i}}(y_i, b_i)\}_{b_i}$ are positive semidefinite operators that sum to identity, so form valid POVMs.

Consider the following strategy to play game G. Alice and Bob share classical public randomness, and for every setting of i, ω_{-i}, z, the bipartite state $|\widetilde{\Phi}^{\omega_{-i},z}_{\perp,\perp}\rangle$. Upon receiving questions $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ respectively they perform the following:

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1. Alice and Bob use public randomness to sample (i, ω_{-i}, z) conditioned on W.

2. Alice applies $U_{\omega_{-i}, z, x}$ to her register of $|\widetilde{\Phi}_{\omega_{-i}, z}\rangle_{\perp, \perp}$.

3. Bob applies $V_{\omega_{-i}, z, y}$ to his register of $|\widetilde{\Phi}_{\omega_{-i}, z}\rangle_{\perp, \perp}$.

4. Alice measures with POVM operators $\{\hat{A}_{\omega_{-i}}(x, a_i)\}$ and returns the outcome as her answer.

5. Bob measures with POVM operators $\{\hat{B}_{\omega_{-i}}(y, b_i)\}$ and returns the outcome as his answer.

Suppose that, upon receiving questions (x, y) and after jointly picking a uniformly random $i \in [m]$, Alice and Bob could jointly sample ω_{-i}, z from $\mathsf{P}_{\Omega_{-i} Z | W}$ and locally prepare the state $|\widetilde{\Phi}_{\omega_{-i}, z}\rangle_{x, y}$. For a fixed (x, y), ω_{-i} and z, the distribution of outcomes (a_i, b_i) after measuring $\{\hat{A}_{\omega_{-i}}(x, a_i) \otimes \hat{B}_{\omega_{-i}}(y, b_i)\}_{a_i, b_i}$ will be identical to $\mathsf{P}_{A_i B_i | \omega_{-i}, z, x, y}$ (where we mean conditioning on $X_i = x$ and $Y_i = y$). Averaging over $(x, y) \sim \mu$, i, ω_{-i}, and z, the above-defined strategy will win game G with probability at least $\mathbb{E}_i \Pr(W_i | W)$.

Next we show that Alice and Bob are able to *approximately* prepare $|\widetilde{\Phi}_{\omega_{-i}, z}\rangle_{x, y}$ with high probability, and thus produce answers that are approximately distributed according to $\mathsf{P}_{A_i B_i | \omega_{-i}, z, x, y}$, allowing them to win game G with probability greater than $1 - \varepsilon$ — a contradiction.

From Lemma 20, using the fact that for two pure states $|\psi\rangle$ and $|\phi\rangle$, $\|\psi - \phi\|_1 \le \sqrt{2}\| |\psi\rangle - |\phi\rangle\|$, as well as Jensen's inequality,

$$\underset{i}{\mathbb{E}}\ \underset{XY}{\mathbb{E}}\ \underset{\Omega_{-i} Z | W}{\mathbb{E}}\ \left\| (U_{\omega_{-i}, z, x} \otimes V_{\omega_{-i}, z, y}) \left[\widetilde{\Phi}_{\omega_{-i}, z}\right]_{\perp, \perp} - \widetilde{\Phi}_{\omega_{-i}, z} \underset{x, y}{} \right\|_1 = O\left(\frac{\delta^{1/8}}{\alpha^2}\right), \qquad (28)$$

where the second expectation is over (x, y) drawn from μ, and $(U \otimes V)[\widetilde{\Phi}]$ denotes $(U \otimes V)\widetilde{\Phi}(U \otimes V)^\dagger$. Conditioned on a given pair of questions (x, y) and the players sampling (i, ω_{-i}, z) in Step 1., the state that the players prepare after Step 3. in the protocol is precisely $(U_{\omega_{-i}, z, x} \otimes V_{\omega_{-i}, z, y})[\widetilde{\Phi}_{\omega_{-i}, z}]_{\perp, \perp}$. Let $\mathcal{E}_{\omega_{-i}, z \atop x, y}$ denote the quantum-classical channel on density matrices that performs the measurement $\{\hat{A}_{\omega_{-i}}(x, a_i) \otimes \hat{B}_{\omega_{-i}}(y, b_i)\}_{a_i, b_i}$, and outputs a classical register with the measurement outcome (a_i, b_i). Applying $\mathcal{E}_{\omega_{-i}, z \atop x, y}$ to the expression inside the trace norm in (28), using that the trace norm is non-increasing under quantum operations,

$$\underset{i}{\mathbb{E}}\ \underset{XY}{\mathbb{E}}\ \underset{\Omega_{-i} Z | W}{\mathbb{E}}\ \left\| \widetilde{\mathsf{P}}_{A_i B_i | \omega_{-i}, z, x, y} - \mathsf{P}_{A_i B_i | \omega_{-i}, z, x, y} \right\| \le O(\delta^{1/8} / \alpha^2).$$

where $\widetilde{\mathsf{P}}_{A_i B_i | \omega_{-i}, z, x, y}(a_i, b_i)$ deontes the probability of outcome (a_i, b_i) in the above strategy, conditioned on questions (x, y) and the players sampling (i, ω_{-i}, z) in Step 1. Thus

$$\mathsf{P}_I \cdot \mathsf{P}_{\Omega_{-i} Z | W} \cdot \mathsf{P}_{XY} \cdot \widetilde{\mathsf{P}}_{A_i B_i | \Omega_{-i} Z X_i Y_i} \approx_{O(\delta^{1/8}/\alpha^2)} \mathsf{P}_I \cdot \mathsf{P}_{\Omega_{-i} Z | W} \cdot \mathsf{P}_{XY} \cdot \mathsf{P}_{A_i B_i | \Omega_{-i} Z X_i Y_i}$$

$$\approx_{O(\delta^{1/8}/\alpha^2)} \mathsf{P}_I \cdot \mathsf{P}_{\Omega_{-i} Z X_i Y_i | W} \cdot \mathsf{P}_{A_i B_i | \Omega_{-i} Z X_i Y_i}$$

where the $X_i Y_i$ in the conditionals is shorthand for $X_i = x, Y_i = y$. The last approximate equality follows from Lemma 19. Marginalizing $\Omega_{-i} Z$, we get

$$\mathsf{P}_I \cdot \mathsf{P}_{XY} \cdot \widetilde{\mathsf{P}}_{A_i B_i | X_i Y_i} \approx_{O(\delta^{1/8}/\alpha^2)} \mathsf{P}_I \cdot \mathsf{P}_{X_i Y_i A_i B_i | W}. \qquad (29)$$

Under the distribution $P_{X_i Y_i A_i B_i | W}$, the probability that $V(x_i, y_i, a_i, b_i) = 1$ is precisely $\Pr(W_i | W)$. On the other hand, (29) implies that using the protocol described above the players win G with probability at least $\mathbb{E}_i \Pr(W_i | W) - O(\delta^{1/8} / \alpha^2)$. This establishes (27) and concludes the proof of the theorem. $\qquad\square$

5.2.1 Proof of the main lemma

This section is devoted to the proof of Lemma 20. The proof is based on two lemmas. The first defines the required unitaries.

Lemma 21. *For all i, ω_{-i}, z, $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ there exists unitaries $U_{\omega_{-i} z x}$ acting on E_A and $V_{\omega_{-i} z y}$, $V_{\omega_{-i} z}^{x,y}$ acting on E_B such that*

$$\mathop{\mathbb{E}}_{i} \mathop{\mathbb{E}}_{\Omega_{-i} Z | W} \mathop{\mathbb{E}}_{X} \left\| \left| \widetilde{\Phi}_{x,\perp} \right\rangle - U_{\omega_{-i} z x} \left| \widetilde{\Phi}_{\perp,\perp} \right\rangle \right\|^2 = O(\delta^{1/4} / \alpha^2), \tag{30}$$

$$\mathop{\mathbb{E}}_{i} \mathop{\mathbb{E}}_{\Omega_{-i} Z | W} \mathop{\mathbb{E}}_{Y} \left\| V_{\omega_{-i} z y} \left| \widetilde{\Phi}_{\perp,\perp} \right\rangle - \left| \widetilde{\Phi}_{\perp,y} \right\rangle \right\|^2 = O(\delta^{1/4} / \alpha^2), \tag{31}$$

$$\mathop{\mathbb{E}}_{i} \mathop{\mathbb{E}}_{\Omega_{-i} Z | W} \mathop{\mathbb{E}}_{XY} \left\| V_{\omega_{-i} z}^{x,y} \left| \widetilde{\Phi}_{\perp / x, y} \right\rangle - \left| \widetilde{\Phi}_{\perp / x, \perp} \right\rangle \right\|^2 = O(\delta^{1/4} / \alpha^2). \tag{32}$$

where $\mathbb{E}_X$, $\mathbb{E}_Y$, and $\mathbb{E}_{XY}$ denote averaging over $\mu(x)$, $\mu(y)$, and $\mu(x, y)$ respectively.

The proof of Lemma 21 is given in Section 5.2.2. The second lemma relates the normalization factors $\gamma_{x,y}$, $\gamma_{x,\perp}$, $\gamma_{\perp,y}$, $\gamma_{\perp / x, y}$, $\gamma_{\perp / x, \perp}$, $\gamma_{\perp, \perp}$ that appear in the definition of the corresponding normalized states $|\widetilde{\Phi}\rangle$.

Lemma 22. *There exists a set S of triples (i, ω_{-i}, z) that has probability $1 - \delta^{1/4}$ under $P_I \cdot P_{\Omega_{-i} Z | W}$ such that*

$$\frac{1}{m} \sum_{\substack{x,y \\ (i, \omega_{-i}, z) \in S}} P_{XY}(x, y) \cdot P_{\Omega_{-i} Z | W}(\omega_{-i}, z) \left| \gamma^2_{\omega_{-i} z \atop x,y} - \gamma^2_{\omega_{-i} z \atop \perp, \perp} \right| = O(\delta^{1/4} / \alpha^2) \gamma^2, \tag{33}$$

where

$$\gamma = \left(\frac{1}{m} \sum_{i} \sum_{x,y,\omega_{-i},z} P_{XY}(x, y) \cdot P_{\Omega_{-i} Z | W}(\omega_{-i}, z) \cdot \gamma^2_{\omega_{-i} z \atop x,y} \right)^{1/2}.$$

Furthermore, similar bounds as (33) hold where $\gamma_{\omega_{-i} z \atop x,y}$ is replaced by any of $\gamma_{\omega_{-i} z \atop x,\perp}$, $\gamma_{\omega_{-i} z \atop \perp,y}$, $\gamma_{\omega_{-i} z \atop \perp / x,y}$, $\gamma_{\omega_{-i} z \atop \perp / x,\perp}$.

The proof of Lemma 22 uses the following claim.

Claim 23.

$$\frac{1}{m} \sum_{i} \sum_{x,y,(\omega_{-i},z) \in W} P_{XY}(x, y) \left\| P_{\Omega_{-i} Z | X_i = x, Y_i = y}(\omega_{-i}, z) - P_{\Omega_{-i} Z | X_i \in \perp, Y_i \in \perp}(\omega_{-i}, z) \right\| = O\left(\frac{\sqrt{\delta}}{\alpha^2} \right) \Pr(W).$$

Proof. First note that

$$\frac{1}{m}\sum_i \sum_{x,y} \mathsf{P}_{XY}(x,y)\,|\Pr(W|X_i = x, Y_i = y) - \Pr(W)| = \frac{\Pr(W)}{m}\sum_i \|\mathsf{P}_{X_iY_i|W} - \mathsf{P}_{X_iY_i}\|$$

$$= O(\sqrt{\delta})\,\Pr(W), \qquad (34)$$

where the second equality follows from Lemma 19. Using the triangle inequality and $\Pr(X_i \in \perp, Y_i \in \perp) \geq \alpha^2$ we also get

$$\frac{1}{m}\sum_i \sum_{x,y} \mathsf{P}_{XY}(x,y)\,|\Pr(W|X_i = x, Y_i = y) - \Pr(W|X_i \in \perp, Y_i \in \perp)| = O(\sqrt{\delta}/\alpha^2)\,\Pr(W). \qquad (35)$$

Using (34) and letting $\mathsf{P}_{\Omega_{-i}Z|x,y,W}$ denote $\mathsf{P}_{\Omega_{-i}Z|X_i=x,Y_i=y,W}$,

$$\frac{1}{m}\sum_i \sum_{x,y} \mathsf{P}_{XY}(x,y)\sum_{(\omega_{-i},z)\in W}\left|\Pr(W)\cdot\mathsf{P}_{\Omega_{-i}Z|x,y,W}(\omega_{-i},z) - \mathsf{P}_{\Omega_{-i}Z|x,y}(\omega_{-i},z)\right|$$

$$\approx_{O(\sqrt{\delta})\,\Pr(W)} \frac{1}{m}\sum_i \sum_{x,y} \mathsf{P}_{XY}(x,y)\sum_{(\omega_{-i},z)\in W}\left|\mathsf{P}_{\Omega_{-i}Z,W|x,y}(\omega_{-i},z) - \mathsf{P}_{\Omega_{-i}Z|x,y}(\omega_{-i},z)\right|$$

$$= 0.$$

A similar derivation proves

$$\frac{1}{m}\sum_i \sum_{(\omega_{-i},z)\in W}\left|\Pr(W)\cdot\mathsf{P}_{\Omega_{-i}Z|X_i\in\perp,Y_i\in\perp,W}(\omega_{-i},z) - \mathsf{P}_{\Omega_{-i}Z|X_i\in\perp,Y_i\in\perp}(\omega_{-i},z)\right| = O(\sqrt{\delta}/\alpha^2)\,\Pr(W).$$

Combining the previous two bounds with the bound

$$\frac{1}{m}\sum_i \Pr(W)\|\mathsf{P}_{X_iY_i}\mathsf{P}_{\Omega_{-i}Z|X_i\in\perp,Y_i\in\perp,W} - \mathsf{P}_{X_iY_i}\mathsf{P}_{\Omega_{-i}Z|X_iY_i,W}\| \leq O(\sqrt{\delta}/\alpha^2)\,\Pr(W)$$

from Lemma 19 with the triangle inequality proves the claim. $\qquad\square$

Proof of Lemma 22. For any i, x, y and $(\omega_{-i}, z) \in W$ write

$$\mathsf{P}_{XY}(x,y)\cdot\mathsf{P}_{\Omega_{-i}Z|W}(\omega_{-i},z)\cdot\gamma^2_{\omega_{-i},z\atop x,y} = \frac{1}{\Pr(W)}\mathsf{P}_{XY}(x,y)\cdot\mathsf{P}_{\Omega_{-i}Z}(\omega_{-i},z)\cdot\gamma^2_{\omega_{-i},z\atop x,y}$$

$$= \frac{1}{\Pr(W)}\mathsf{P}_{XY}(x,y)\cdot\mathsf{P}_{\Omega_{-i}|x,y}(\omega_{-i})\cdot\mathsf{P}_{Z|\omega_{-i}}(z)\cdot\gamma^2_{\omega_{-i},z\atop x,y},$$

where for the last equality we used $\mathsf{P}_{\Omega_{-i}|X_iY_i} = \mathsf{P}_{\Omega_{-i}}$. From the definition, $\gamma^2_{\omega_{-i},z\atop x,y} = \mathsf{P}_{Z|\omega_{-i},x,y}(z)$,

$$= \frac{1}{\Pr(W)}\mathsf{P}_{XY}(x,y)\cdot\mathsf{P}_{Z|\omega_{-i}}(z)\cdot\mathsf{P}_{\Omega_{-i}Z|x,y}(\omega_{-i},z), \qquad (36)$$

where $\mathsf{P}_{\Omega_{-i}Z|x,y}(\omega_{-i},z)$ denotes $\mathsf{P}_{\Omega_{-i}Z|X_i=x,Y_i=y}(\omega_{-i},z)$. Similarly, we have

$$\mathsf{P}_{XY}(x,y)\cdot\mathsf{P}_{\Omega_{-i}Z|W}(\omega_{-i},z)\cdot\gamma^2_{\omega_{-i},z\atop\perp,\perp} = \frac{1}{\Pr(W)}\mathsf{P}_{XY}(x,y)\cdot\mathsf{P}_{Z|\omega_{-i}}(z)\cdot\mathsf{P}_{\Omega_{-i}Z|\perp,\perp}(\omega_{-i},z). \qquad (37)$$

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By definition

$$\gamma^2 = \frac{1}{m} \sum_{i,\omega_{-i},z} \mathsf{P}_{\Omega_{-i}Z|W}(\omega_{-i},z) \cdot \mathsf{P}_{V|\omega_{-i}}(z),$$

thus for any $\eta > 0$ applying Markov's inequality a fraction at least $1 - \eta$ of (i, ω_{-i}, z) distributed according to $\mathsf{P}_I \cdot \mathsf{P}_{\Omega_{-i}Z|W}$ are such that $\mathsf{P}_{Z|\omega_{-i}}(z) \leq \gamma^2/\eta$. Let S be the set of such triples, and consider summing the difference

$$\mathsf{P}_{XY}(x,y) \cdot \mathsf{P}_{Z|\omega_{-i}}(z) \cdot \left| \mathsf{P}_{\Omega_{-i}Z|x,y}(\omega_{-i},z) - \mathsf{P}_{\Omega_{-i}Z|\perp,\perp}(\omega_{-i},z) \right|$$

over all (x,y) and $(i, \omega_{-i}, z) \in S$. By lines (36) and (37), and applying Claim 23 we obtain

$$\frac{1}{m} \sum_{\substack{x,y \\ (i,\omega_{-i},z) \in S}} \mathsf{P}_{XY}(x,y) \cdot \mathsf{P}_{\Omega_{-i}Z|W}(\omega_{-i},z) \cdot \left| \gamma^2_{\substack{\omega_{-i},z \\ x,y}} - \gamma^2_{\substack{\omega_{-i},z \\ \perp,\perp}} \right| \leq \frac{\gamma^2}{\eta} O\left(\frac{\sqrt{\delta}}{\alpha^2}\right).$$

Choosing $\eta = \delta^{1/4}$ proves the lemma.

□

Proof of Lemma 20. For every (i, ω_{-i}, z), $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ let unitaries $U_{\omega_{-i}zx}$, $V_{\omega_{-i}zy}$ and $V_{\substack{\omega_{-i}z \\ x,y}}$ be as in Lemma 21. For notational convenience we suppress the dependence on (i, ω_{-i}, z) when it is clear from context. Call triples (i, ω_{-i}, z) that satisfy the conclusion of Lemma 22 for $\gamma_{\substack{\omega_{-i}z \\ x,y}}$, $\gamma_{\substack{\omega_{-i}z \\ x,\perp}}$, $\gamma_{\substack{\omega_{-i}z \\ \perp,y}}$, $\gamma_{\substack{\omega_{-i}z \\ \perp/x,y}}$, and $\gamma_{\substack{\omega_{-i}z \\ \perp/x,\perp}}$ simultaneously *good triples*, and let S denote the set of good triples. Using $|a-b|^2 \leq |a^2 - b^2|$ for $a, b \geq 0$,

$$\frac{1}{m} \sum_{\substack{x,y \\ (i,\omega_{-i},z) \in S}} \mathsf{P}_{XY}(x,y) \cdot \mathsf{P}_{\Omega_{-i}Z|W}(\omega_{-i},z) \cdot \left\| |\widetilde{\Phi}_{x,y}\rangle - \gamma^{-1} |\Phi_{x,y}\rangle \right\|^2 \tag{38}$$

$$= \frac{1}{m} \sum_{\substack{x,y \\ (i,\omega_{-i},z) \in S}} \mathsf{P}_{XY}(x,y) \cdot \mathsf{P}_{\Omega_{-i}Z|W}(\omega_{-i},z) \cdot \left| \frac{\gamma - \gamma_{\substack{\omega_{-i}z \\ x,y}}}{\gamma} \right|^2$$

$$\leq \frac{1}{m} \sum_{\substack{x,y \\ (i,\omega_{-i},z) \in S}} \mathsf{P}_{XY}(x,y) \cdot \mathsf{P}_{\Omega_{-i}Z|W}(\omega_{-i},z) \cdot \left| \frac{\gamma^2 - \gamma^2_{\substack{\omega_{-i}z \\ x,y}}}{\gamma^2} \right|$$

$$= O(\delta^{1/4}/\alpha^2), \tag{39}$$

and similar bounds hold for $|\widetilde{\Phi}_{x,\perp}\rangle$, $|\widetilde{\Phi}_{\perp,y}\rangle$ and $|\widetilde{\Phi}_{\perp,\perp}\rangle$. Thus to prove the theorem it will be sufficient to establish the following bound on the distance between *unnormalized* states:

$$\frac{1}{m} \sum_{\substack{x,y \\ (i,\omega_{-i},z) \in S}} \mathsf{P}_{XY}(x,y) \cdot \mathsf{P}_{\Omega_{-i}Z|W}(\omega_{-i},z) \cdot \left\| (U_x \otimes V_y) |\Phi_{\perp,\perp}\rangle - |\Phi_{x,y}\rangle \right\|^2 = O\left(\frac{\delta^{1/4}}{\alpha^2}\right) \gamma^2. \tag{40}$$

To see that (40) is sufficient, observe that by using the lower bound on the probability of S we can bound the desired expression in the statement of Lemma 20 by

$$\frac{1}{m} \sum_{\substack{x,y \\ i,\omega_{-i},v}} \mathsf{P}_{XY}(x,y) \cdot \mathsf{P}_{\Omega_{-i}Z|W}(\omega_{-i},z) \cdot \left\| (U_x \otimes V_y) |\widetilde{\Phi}_{\perp,\perp}\rangle - |\widetilde{\Phi}_{x,y}\rangle \right\|^2$$

$$\leq \frac{1}{m} \sum_{\substack{x,y \\ (i,\omega_{-i},v)\in S}} \mathsf{P}_{XY}(x,y) \cdot \mathsf{P}_{\Omega_{-i}Z|W}(\omega_{-i},z) \cdot \left\| (U_x \otimes V_y) \left| \widetilde{\Phi}_{\perp,\perp} \right\rangle - \left| \widetilde{\Phi}_{x,y} \right\rangle \right\|^2 + O(\delta^{1/4})$$

Then, by the triangle inequality and the bound of (39),

$$\frac{1}{m} \sum_{\substack{x,y \\ (i,\omega_{-i},v)\in S}} \mathsf{P}_{XY}(x,y) \cdot \mathsf{P}_{\Omega_{-i}Z|W}(\omega_{-i},z) \cdot \left\| (U_x \otimes V_y) \left| \widetilde{\Phi}_{\perp,\perp} \right\rangle - \left| \widetilde{\Phi}_{x,y} \right\rangle \right\|^2$$

$$\leq \frac{1}{m} \sum_{\substack{x,y \\ (i,\omega_{-i},v)\in S}} \mathsf{P}_{XY}(x,y) \cdot \mathsf{P}_{\Omega_{-i}Z|W}(\omega_{-i},z) \, 3 \left\| \left| \widetilde{\Phi}_{\perp,\perp} \right\rangle - \gamma^{-1} \left| \Phi_{\perp,\perp} \right\rangle \right\|^2$$

$$+ 3 \left\| \left| \widetilde{\Phi}_{x,y} \right\rangle - \gamma^{-1} \left| \Phi_{x,y} \right\rangle \right\|^2 + 3\gamma^{-2} \left\| (U_x \otimes V_y) \left| \Phi_{\perp,\perp} \right\rangle - \left| \Phi_{x,y} \right\rangle \right\|^2$$

$$\leq \frac{3\gamma^{-2}}{m} \sum_{\substack{x,y \\ (i,\omega_{-i},v)\in S}} \mathsf{P}_{XY}(x,y) \cdot \mathsf{P}_{\Omega_{-i}Z|W}(\omega_{-i},z) \cdot \left\| (U_x \otimes V_y) \left| \Phi_{\perp,\perp} \right\rangle - \left| \Phi_{x,y} \right\rangle \right\|^2 + O(\delta^{1/4}/\alpha^2).$$

Thus we have established the sufficiency of proving (40). Using (39), the bounds stated in Lemma 21 also imply the following bounds on the un-normalized vectors:

$$\frac{1}{m} \sum_{\substack{x \\ (i,\omega_{-i},z)\in S}} \mathsf{P}_X(x) \cdot \mathsf{P}_{\Omega_{-i}Z|W}(\omega_{-i},z) \cdot \left\| \left| \Phi_{x,\perp} \right\rangle - U_{\omega_{-i}zx} \left| \Phi_{\perp,\perp} \right\rangle \right\|^2 = O\left(\frac{\delta^{1/4}}{\alpha^2}\right)\gamma^2, \tag{41}$$

$$\frac{1}{m} \sum_{\substack{y \\ (i,\omega_{-i},z)\in S}} \mathsf{P}_Y(y) \cdot \mathsf{P}_{\Omega_{-i}Z|W}(\omega_{-i},z) \cdot \left\| V_{\omega_{-i}zy} \left| \Phi_{\perp,\perp} \right\rangle - \left| \Phi_{\perp,y} \right\rangle \right\|^2 = O\left(\frac{\delta^{1/4}}{\alpha^2}\right)\gamma^2, \tag{42}$$

$$\frac{1}{m} \sum_{\substack{x,y \\ (i,\omega_{-i},z)\in S}} \mathsf{P}_{XY}(x,y) \cdot \mathsf{P}_{\Omega_{-i}Z|W}(\omega_{-i},z) \cdot \left\| V_{\omega_{-i}z}^{x,y} \left| \Phi_{\perp/x,y} \right\rangle - \left| \Phi_{\perp/x,\perp} \right\rangle \right\|^2 = O\left(\frac{\delta^{1/4}}{\alpha^3}\right)\gamma^2. \tag{43}$$

We show how to combine these bounds to establish (40). We have

$$\left\| U_x \left| \Phi_{\perp,y} \right\rangle - \left| \Phi_{x,y} \right\rangle \right\|^2 = \left\| U_x A_\perp^{1/2} A_{\perp/x}^{-1/2} \left| \Phi_{\perp/x,y} \right\rangle - A_x^{1/2} A_{\perp/x}^{-1/2} \left| \Phi_{\perp/x,y} \right\rangle \right\|^2$$

$$= \left\| U_x A_\perp^{1/2} A_{\perp/x}^{-1/2} \otimes V_{xy} \left| \Phi_{\perp/x,y} \right\rangle - A_x^{1/2} A_{\perp/x}^{-1/2} \otimes V_{xy} \left| \Phi_{\perp/x,y} \right\rangle \right\|^2.$$

Using the triangle inequality again,

$$\leq 3 \left\| \left(U_x A_\perp^{1/2} A_{\perp/x}^{-1/2} \right) \otimes V_{xy} \left| \Phi_{\perp/x,y} \right\rangle - \left(U_x A_\perp^{1/2} A_{\perp/x}^{-1/2} \right) \left| \Phi_{\perp/x,\perp} \right\rangle \right\|^2 \tag{44}$$

$$+ 3 \left\| \left(U_x A_\perp^{1/2} A_{\perp/x}^{-1/2} \right) \left| \Phi_{\perp/x,\perp} \right\rangle - A_x^{1/2} A_{\perp/x}^{-1/2} \left| \Phi_{\perp/x,\perp} \right\rangle \right\|^2 \tag{45}$$

$$+ 3 \left\| A_x^{1/2} A_{\perp/x}^{-1/2} \left| \Phi_{\perp/x,\perp} \right\rangle - A_x^{1/2} A_{\perp/x}^{-1/2} \otimes V_{xy} \left| \Phi_{\perp/x,y} \right\rangle \right\|^2. \tag{46}$$

Using $\| U_x A_\perp^{1/2} A_{\perp/x}^{-1/2} \| \leq 1/\sqrt{1 - p_A}$ the term (44) can be bounded as

$$\left\| \left(U_x A_\perp^{1/2} A_{\perp/x}^{-1/2} \right) \otimes V_{xy} \left| \Phi_{\perp/x,y} \right\rangle - \left(U_x A_\perp^{1/2} A_{\perp/x}^{-1/2} \right) \left| \Phi_{\perp/x,\perp} \right\rangle \right\|^2 \leq \frac{1}{1 - p_A} \left\| V_{xy} \left| \Phi_{\perp/x,y} \right\rangle - \left| \Phi_{\perp/x,\perp} \right\rangle \right\|^2.$$

The term (45) can be re-written as

$$\left\| \left(U_x A_\perp^{1/2} A_{\perp/x}^{-1/2} \right) |\Phi_{\perp/x,\perp}\rangle - A_x^{1/2} A_{\perp/x}^{-1/2} |\Phi_{\perp/x,\perp}\rangle \right\|^2 = \left\| U_x |\Phi_{\perp,\perp}\rangle - |\Phi_{x,\perp}\rangle \right\|^2.$$

Finally, using $\|A_x^{1/2} A_{\perp/x}^{-1/2}\| \le 1/\sqrt{p_A}$ the term (46) can be bounded as

$$\left\| A_x^{1/2} A_{\perp/x}^{-1/2} |\Phi_{\perp/x,\perp}\rangle - A_x^{1/2} A_{\perp/x}^{-1/2} \otimes V_{xy} |\Phi_{\perp/x,y}\rangle \right\|^2 \le \frac{1}{p_A} \left\| |\Phi_{\perp/x,\perp}\rangle - V_{xy} |\Phi_{\perp/x,y}\rangle \right\|^2.$$

Putting the three bounds together, from (46) we get

$$\left\| U_x |\Phi_{\perp,y}\rangle - |\Phi_{x,y}\rangle \right\|^2 \le \frac{6}{\min\{p_A, 1 - p_A\}} \left\| V_{xy} |\Phi_{\perp/x,y}\rangle - |\Phi_{\perp/x,\perp}\rangle \right\|^2 + 3 \left\| U_x |\Phi_{\perp,\perp}\rangle - |\Phi_{x,\perp}\rangle \right\|^2. \tag{47}$$

Recall that we defined $p_A^3 = 1 - \mu(\mathcal{X}_\perp)$, and we are assuming that $\mu(\mathcal{X}_\perp) \le 1/2$. Therefore $\min\{p_A, 1 - p_A\} \ge 3\alpha$. Using that U_x is unitary,

$$\begin{aligned}
\left\| (U_x \otimes V_y) |\Phi_{\perp,\perp}\rangle - |\Phi_{x,y}\rangle \right\|^2 &\le 2 \left\| V_y |\Phi_{\perp,\perp}\rangle - |\Phi_{\perp,y}\rangle \right\|^2 + 2 \left\| U_x |\Phi_{\perp,y}\rangle - |\Phi_{x,y}\rangle \right\|^2 \\
&\le 36\alpha^{-1} \left\| V_{xy} |\Phi_{\perp/x,y}\rangle - |\Phi_{\perp/x,\perp}\rangle \right\|^2 + 6 \left\| U_x |\Phi_{\perp,\perp}\rangle - |\Phi_{x,\perp}\rangle \right\|^2 \\
&\quad + 2 \left\| V_y |\Phi_{\perp,\perp}\rangle - |\Phi_{\perp,y}\rangle \right\|^2,
\end{aligned}$$

where the last inequality is (47). Eqs. (41), (42) and (43) bound the three terms above by $O(\delta^{1/4}/\alpha^4)\gamma^2$ on average over (x, y) weighted by P_{XY}, and $(i, \omega_{-i}, z) \in S$, weighted by $\mathsf{P}_I \cdot \mathsf{P}_{\Omega_{-i}Z|W}$. This proves (40), and the theorem follows. $\qquad\square$

5.2.2 Obtaining local unitaries

In this section we give the proof of Lemma 21, which states the existence of the local unitary transformations needed for the proof of Theorem 17.

Proof of Lemma 21. Recall that we let the entangled state $|\psi\rangle$ and POVMs $\{A(x_{[n]}, a_{[n]})\}$ and $\{B(y_{[n]}, b_{[n]})\}$ constitute an optimal strategy for G^n. Define the operators

$$A_\omega(a_C) := \mathop{\mathbb{E}}_{X_{[n]}|\Omega=\omega} A(x_{[n]}, a_C) \qquad\qquad B_\omega(b_C) := \mathop{\mathbb{E}}_{Y_{[n]}|\Omega=\omega} B(y_{[n]}, b_C).$$

Let ρ denote the reduced density matrix of $|\psi\rangle$ on either system (this is well-defined because we've assumed $|\psi\rangle$ is symmetric).

We first prove (30). Recall the notation $\psi = |\psi\rangle\langle\psi|$ and $X[\rho] = X\rho X^\dagger$. We introduce the following state:

$$\Xi_{\Omega X_{[n]} E_A E_B Z} = \sum_{\omega, x_{[n]}, a_C, b_C} \mathsf{P}_{\Omega X_{[n]}}(\omega, x_{[n]}) \, |\omega \, x_{[n]}\rangle\langle\omega \, x_{[n]}| \otimes \left(A(x_{[n]}, a_C)^{1/2} \otimes B_\omega(b_C)^{1/2} \right) [\psi] \otimes |a_C b_C\rangle\langle a_C b_C|.$$

The state Ξ is defined so that tracing out the entanglement registers E_A and E_B the resulting state $\Xi_{\Omega X_{[n]} A_C B_C}$ is a classical state that can be identified with $\mathsf{P}_{\Omega X_{[n]} A_C B_C}$. We can condition Ξ on the

event W (which is well-defined since the event is determined by the classical random variables Ω and Z) by defining the state

$$\zeta_{\Omega X_{[n]} E_A E_B Z} = \Xi_{\Omega X_{[n]} E_A E_B Z | W'} .\tag{48}$$

In what follows, we will frequently refer to states such as

$$\zeta_{E_B | \omega_{-i}, z, x_i, \perp'}$$

which is shorthand for the state $\zeta_{E_B | \Omega_{-i} = \omega_{-i}, Z = z, X_i = x_i, \Omega_i = (B, \perp)}$, the reduced density matrix of ζ on the system E_B, conditioned on the indicated settings of the classical random variables. The following claim provides the main step of the proof by relating the reduced densities on Bob's registers of states (48) associated with different choices for x_i.

Claim 24.

$$\mathop{\mathbb{E}}_{i} \mathop{\mathbb{E}}_{\Omega_{-i} Z | W} \mathop{\mathbb{E}}_{X_i} \left\| \zeta_{E_B | \omega_{-i}, z, x_i, \perp} - \zeta_{E_B | \omega_{-i}, z, \perp, \perp} \right\|_1^2 = O(\sqrt{\delta}/\alpha^2)\tag{49}$$

Proof. First we observe that $\Pr(W)\xi \preceq \Xi$, thus by definition $S(\xi\|\Xi) \leq S_\infty(\xi\|\Xi) \leq \log 1/\Pr(W)$. Using the chain rule for the relative entropy (Lemma 7),

$$\mathop{\mathbb{E}}_{\Omega V | W} S(\zeta_{X_{[n]} E_B | \omega, z} \| \Xi_{X_{[n]} E_B | \omega, z}) \leq \log \frac{1}{\Pr(W)} .\tag{50}$$

Next we note that for any ω, using Ando's identity

$$\langle \psi | X \otimes Y | \psi \rangle = \text{Tr}(X \sqrt{\rho} Y^\top \sqrt{\rho}),$$

where $|\psi\rangle = \sum \sqrt{\lambda_j} |v_j\rangle |v_j\rangle$, $\rho = \sum \lambda_j |v_j\rangle\langle v_j|$, X, Y are any linear operators and the transpose is taken with respect to the orthonormal basis $\{|v_j\rangle\}$,

$$
\begin{aligned}
\Xi_{X_{[n]} E_B Z | \omega} &= \sum_{x_{[n]}, a_C, b_C} \mathsf{P}_{X_{[n]} | \omega}(x_{[n]}) \, |x_{[n]}\rangle\langle x_{[n]}| \otimes \sqrt{B_\omega(b_C)} \sqrt{\rho} \overline{A(x_{[n]}, a_C)} \sqrt{\rho} \sqrt{B_\omega(b_C)} \otimes |a_C b_C\rangle\langle a_C b_C| \\
&\preceq \sum_{x_{[n]}, a_C, b_C} \mathsf{P}_{X_{[n]} | \omega}(x_{[n]}) \, |x_{[n]}\rangle\langle x_{[n]}| \otimes \sqrt{B_\omega(b_C)} \sqrt{\rho} \overline{A(x_{[n]}, a_C)} \sqrt{\rho} \sqrt{B_\omega(b_C)} \otimes \mathbb{I} \\
&= \sum_{x_{[n]}, b_C} \mathsf{P}_{X_{[n]} | \omega}(x_{[n]}) \, |x_{[n]}\rangle\langle x_{[n]}| \otimes \sqrt{B_\omega(b_C)} \rho \sqrt{B_\omega(b_C)} \otimes \mathbb{I},
\end{aligned}\tag{51}
$$

where the last equality uses $\sum_{a_C} A(x_{[n]}, a_C) = \mathbb{I}$. From (51) and the definition of S_∞ it follows that for all z, $S_\infty(\Xi_{X_{[n]} E_B | \omega, z} \| \Xi_{X_{[n]} | \omega} \otimes \Xi_{E_B | \omega}) \leq |C| \cdot \log |\mathcal{A}||\mathcal{B}|$. Applying Lemma 8,

$$
\begin{aligned}
\mathop{\mathbb{E}}_{i} \mathop{\mathbb{E}}_{\Omega Z | W} I(X_i; E_B | \omega, z)_\xi &\leq \frac{1}{m} \mathop{\mathbb{E}}_{\Omega Z | W} S(\zeta_{X_{[n]} E_B | \omega, z} \| \Xi_{X_{[n]} | \omega, z} \otimes \Xi_{E_B | \omega, z}) \\
&\leq \frac{1}{m} \left(\mathop{\mathbb{E}}_{\Omega Z | W} S(\zeta_{X_{[n]} E_B | \omega, z} \| \Xi_{X_{[n]} E_B | \omega, z}) + \mathop{\mathbb{E}}_{\Omega Z | W} S_\infty(\Xi_{X_{[n]} E_B | \omega, z} \| \Xi_{X_{[n]} | \omega} \otimes \Xi_{E_B | \omega}) \right) \\
&\leq \frac{1}{m} \left(\log \frac{1}{\Pr(W)} + |C| \cdot \log |\mathcal{A}||\mathcal{B}| \right)
\end{aligned}\tag{52}
$$

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$$= \delta \tag{53}$$

where in the third inequality the first term is bounded using (50) and the second using (51). Applying Lemma 19,

$$\mathbb{E}_i \, \mathsf{P}_{D_i M_i | W}(B, \perp) \approx_{O(\sqrt{\delta})} \mathbb{E}_i \, \mathsf{P}_{D_i M_i}(B, \perp) = \frac{1 - p_B}{2} \geq \frac{\alpha}{6},$$

thus from (53) by conditioning on $\Omega_i = (B, \perp)$ we deduce

$$\mathbb{E}_i \, \mathbb{E}_{\Omega Z | \Omega_i = (B, \perp), W} \, I\big(X_i; E_B | \omega, z\big)_{\xi} = O\big(\delta / \alpha\big), \tag{54}$$

as long as $\alpha = \Omega(\sqrt{\delta})$. Next we apply Pinsker's inequality (Lemma 6) and use that X_i is classical in ξ to write

$$\mathbb{E}_i \, \mathbb{E}_{\Omega Z | \Omega_i = (B, \perp), W} \, \mathbb{E}_{X_i | \omega, z} \, \big\| \xi_{E_B | \omega_{-i}, z, x_i, \perp} - \xi_{E_B | \omega, z} \big\|_1^2 \leq \mathbb{E}_i \, \mathbb{E}_{\Omega Z | \Omega_i = (B, \perp), W} \, \mathbb{E}_{X_i | \omega, z} \, S\big(\xi_{E_B | \omega_{-i}, z, x_i, \perp} \, \big\| \, \xi_{E_B | \omega, z}\big)$$

$$= \mathbb{E}_i \, \mathbb{E}_{\Omega Z | \Omega_i = (B, \perp), W} \, I(X_i; E_B | \omega, z)_{\xi}$$

$$= O\big(\delta / \alpha\big)$$

by (54). To conclude Claim 24, we use Lemma 19 to obtain

$$\mathsf{P}_I \cdot \mathsf{P}_{\Omega_{-i} Z X_i | \Omega_i = (B, \perp), W} \approx_{O(\sqrt{\delta}/\alpha^2)} \mathsf{P}_I \cdot \mathsf{P}_{\Omega_{-i} Z | W} \cdot \mathsf{P}_{X_i}.$$

$\qquad\square$

The proof of (30) essentially follows from Claim 24 and Uhlmann's theorem. We give the details. First write $\xi_{E_B | \omega_{-i}, z, x_i, \perp}$ and $\xi_{E_B | \omega_{-i}, z, \perp, \perp}$ explicitly as

$$\xi_{E_B | \omega_{-i}, z, x_i, \perp} \propto \sqrt{B_{\omega_{-i}}(\perp, b_C)} \sqrt{\rho} \, \overline{A_{\omega_{-i}}(x_i, a_C)} \, \sqrt{\rho} \, \sqrt{B_{\omega_{-i}}(\perp, b_C)},$$

$$\xi_{E_B | \omega_{-i}, z, \perp, \perp} \propto \sqrt{B_{\omega_{-i}}(\perp, b_C)} \sqrt{\rho} \, \overline{A_{\omega_{-i}}(\perp, a_C)} \, \sqrt{\rho} \, \sqrt{B_{\omega_{-i}}(\perp, b_C)},$$

which makes it apparent that the states $|\widetilde{\Phi}_{x_i, \perp}\rangle$ and $|\widetilde{\Phi}_{\perp, \perp}\rangle$ introduced in (26) purify $\xi_{E_B | \omega_{-i}, z, x_i, \perp}$ and $\xi_{E_B | \omega_{-i}, z, \perp, \perp}$ respectively. Applying Uhlmann's Theorem, there exists a unitary U_{ω_{-i}, z, x_i} acting on E_A such that

$$\mathbb{E}_i \, \mathbb{E}_{\Omega_{-i} Z | W} \, \mathbb{E}_{X_i} \, \big| \big\langle \widetilde{\Phi}_{x_i, \perp} \big| U_{\omega_{-i}, z, x_i} \big| \widetilde{\Phi}_{\perp, \perp} \big\rangle \big| \geq 1 - \mathbb{E}_i \, \mathbb{E}_{\Omega_{-i} Z | W} \, \mathbb{E}_{X_i} \, \big\| \xi_{E_B | \omega_{-i}, z, x_i, \perp} - \xi_{E_B | \omega_{-i}, z, \perp, \perp} \big\|_1$$

$$\geq 1 - O(\delta^{1/4} / \alpha), \tag{55}$$

where the first inequality follows from the Fuchs-van de Graaf inequality (4) and the second uses Jensen's inequality and (49) from Claim 24. Expanding out the squared Euclidean norm and making sure that U_{ω_{-i}, z, x_i} is chosen so as to ensure that the inner product $\langle \widetilde{\Phi}_{x_i, \perp} | U_{x_i} | \widetilde{\Phi}_{\perp, \perp} \rangle$ is positive real, (55) proves (30).

A nearly identical argument yields (31). It remains to show (32). To do so, conside the state

$$\Pi_{\Omega Y_{[n]} Z} := \sum_{\omega, y_{[n]}, a_C, b_C} P_{\Omega Y_{[n]}}(\omega, y_{[n]}) \, |\omega \, y_{[n]}\rangle\langle \omega \, y_{[n]}| \otimes \left(\sqrt{A_\omega(a_C)} \otimes \sqrt{B(y_{[n]}, b_C)} \right) [\psi] \otimes |a_C b_C\rangle\langle a_C b_C|.$$

and let π denote Π conditioned on the event W. Note that Π is such that the classical state $\Pi_{\Omega Y_{[n]} A_C B_C}$ matches the distribution $P_{\Omega Y_{[n]} A_C B_C}$. The required analogue of Claim 24 can be stated as follows.

Claim 25.

$$\mathbb{E}_i \underset{\Omega_{-i} V | W}{\mathbb{E}} \underset{X_i Y_i}{\mathbb{E}} \left\| \pi_{E_A} \Big|_{\omega_i = (A, \perp / x_i)}^{\omega_{-i}, z, y_i} - \pi_{E_A} \Big|_{\omega_i = (A, \perp / x_i)}^{\omega_{-i}, z, y_i \in \perp} \right\|_1^2 \le O(\sqrt{\delta}/\alpha^4). \tag{56}$$

Proof. The proof follows very closely that of Claim 24. The conditioning on $\Omega_i = (B, \perp)$ that led to (54) is replaced here by conditioning on $D_i = A$ and $M_i \neq \perp$. The main observation needed is that the distribution $P_{M_i Y_i | D_i = A, M_i \neq \perp}$ is a "reshaped" version of $P_{X_i Y_i}$ in the sense that

$$P_{M_i Y_i | D_i = A, M_i \neq \perp} \cong p_A \cdot P_{X_i Y_i | X_i \in \mathcal{X}_\perp} + (1 - p_A) \cdot P_{X_i Y_i | X_i \notin \mathcal{X}_\perp}$$

where "$\cong$" denotes a relabeling of M_i on the left-hand side by X_i on the right-hand side. Using the assumption that $\alpha \le 1/2$, p_A is bounded away from zero and 1 by $\Theta(\alpha)$, and the proof concludes as for Claim 24 (with the extra dependence on α^{-1} coming from the possible imbalance in p_A). $\quad\square$

The densities

$$\pi_{E_A} \Big|_{\omega_i = (A, \perp / x_i)}^{\omega_{-i}, z, y_i} \qquad \text{and} \qquad \pi_{E_A} \Big|_{\omega_i = (A, \perp / x_i)}^{\omega_{-i}, z, y_i \in \perp}$$

are purified by $|\widetilde{\Phi}_{\perp / x_i, y_i}\rangle$ and $|\widetilde{\Phi}_{\perp / x_i, \perp}\rangle$ respectively. Applying Uhlmann's Theorem, there exist unitaries $V_{\omega_{-i}, z, x_i, y_i}$ acting on E_B such that

$$\mathbb{E}_i \underset{\Omega_{-i} V | W}{\mathbb{E}} \underset{X_i Y_i}{\mathbb{E}} \left\| \left|\widetilde{\Phi}_{\perp / x_i, \perp}\right\rangle - V_{\omega_{-i}, z, x_i, y_i} \left|\widetilde{\Phi}_{\perp / x_i, y_i}\right\rangle \right\|^2 \le O(\delta^{1/4}/\alpha^2) \tag{57}$$

by Claim 25, proving (32). $\quad\square$

6 Open problems

Many interesting problems about the parallel repetition of multiplayer and entangled games remain open. Perhaps the most obvious and pressing is the problem of obtaining a complete extension of Raz's theorem for general entangled two-player games. For example, obtaining a fully quantum analogue of Raz's theorem, as was the case for Raz's theorem itself, is likely to have important implications in the setting of communication complexity. One promising candidate approach could be to leverage the recent ideas related to quantum information complexity [Tou14, BGK+15].

Similarly, the problem of obtaining a parallel repetition with exponential decay for *general multiplayer* games remain a fascinating challenge. In our view, however, this problem (even classically) seems more challenging than the two-player entangled case, as its difficulties are related to communication complexity and circuit complexity lower bounds.

One limitation of our result is that it is essentially most suitable in the case of games with value close to 1, in the sense that if $\mathrm{val}(G)$, $\mathrm{val}^*(G)$ are already *subconstant*, our bounds do not take advantage of this fact. Indeed, even if G originally has a value close to 0, the anchoring operation itself pushes the value up to $\Omega(1)$. So it remains open to find a hardness amplification result that replicates the strength of similar theorems obtained recently in the classical setting [DS14, BG15].

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