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TESTABLE IMPLICATIONS OF EXPONENTIAL DISCOUNTING

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Abstract

We develop a behavioral axiomatic characterization of exponentially discounted utility (EDU) over consumption streams. Given is an individual agent's behavior in the market: assume a finite collection of purchases across periods. We show that such behavior satisfies a "revealed preference axiom" if and only if there exists a EDU model (a discount rate per period and a concave utility function over money) that accounts for the given intertemporal consumption.

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1 Introduction

Many areas of economics involve intertemporal decision making. For example, many ideas in macroeconomics, finance and dynamic game theory often hinge on certain intertemporal tradeoffs. The model of exponentially discounted utility (EDU) is by far the most common assumption placed on individual agents in all these areas. The EDU model is an essential tool in the study of intertemporal decisions, at least since Samuelson (1937).

In macro or finance, EDU is used as a way of generating individual's behavior in response to market conditions: prices and wealth. Given prices (or interest rates) and wealth, individuals maximize discounted utility. The underlying justification is that individuals' *behavior* is *as if* they were maximizing an EDU. It is therefore important to understand the behaviors that can be rationalized as if they arose from individuals maximizing discounted utility.

There are different behavioral axiomatizations of EDU in the literature, starting with Koopmans (1960), and followed by Fishburn and Rubinstein (1982), Fishburn and Edwards (1997), and Bleichrodt et al. (2008). All of them take preferences as primitive, or in some cases they take utility over consumption streams as the primitive. The idea is that an analyst can observe all pairwise comparisons of consumption streams, or that the relevant behavior consists of all pairwise comparisons of consumption streams. Note that this assumes knowledge of an infinite number of pairwise comparisons: so the given "dataset" is infinite.

*We thank Kim Border and Chris Chambers for inspiration and advice.

For macroeconomics and finance, however, it is possible that the most relevant behavior is different. Theories in macro and finance use EDU to model market behavior. They predict what agents choose given prices and wealth, which is less demanding than predicting all pairwise comparisons of consumption streams. The purpose of our paper is to characterize the set of market behaviors that is consistent with the EDU model.

Given that EDU is so often assumed as a model of behavior in the market, it seems very important to understand the nature of EDU behavior. What is the class of behaviors, in a macro or finance market setting, that are consistent with EDU? We show that intertemporal consumption (a finite consumption stream) and prices satisfy one revealed preference axiom if and only if there exists a EDU model (i.e., a discount rate and a concave utility function over money) that accounts for intertemporal consumption given the prices. In addition, we assume given a finite number of observed choices, so our datasets will be finite.

This note is a companion paper to our recent paper Echenique and Saito (2013), in which we develop a similar result for subjective expected utility. The argument used to prove our results is very similar in both papers, and the form of the axiom required to characterize EDU is very similar to the one that characterized subjective expected utility.

We proceed to discuss the aforementioned papers that axiomatize the EDU model. All of them use either preferences over consumption streams or a utility function over consumption streams as their primitive. Another important difference with our setup is that they assume infinitely many periods. In contrast, we suppose that we observe finite consumption streams.

Koopmans (1960) proposes the well-known stationarity axiom, which says a preference is not affected if a common first consumption is dropped and the timing of all other consumptions is advanced by one period. The stationarity axiom is used by many followers and the axiom is used together with the assumption that the set of periods is infinite. In Fishburn and Rubinstein (1982) preferences are defined on one-time consumptions in continuous time. In Fishburn and Edwards (1997), preferences are defined on infinite consumption streams that differ in at most finitely many periods. More recently, Bleichrodt et al. (2008) show that Koopmans (1960)'s axioms imply the boundedness of utility function. Then, Bleichrodt et al. (2008) axiomatize the EDU model possibly with unbounded utility function by using preferences defined on infinite consumption streams.

There are several axiomatizations of quasi-hyperbolic disuniting utility model, which is more general than EDU. See Attema et al. (2010) and Olea and Strzalecki (2013) for example. All of them use preferences as their primitives and require the set of periods is infinite.

2 Model and Results

We use the following notational conventions: For vectors $x, y \in \mathbf{R}^n$, $x \leq y$ means that $x_i \leq y_i$ for all $i = 1, \ldots, n$; $x < y$ means that $x \leq y$ and $x \neq y$; and $x \ll y$ means that $x_i < y_i$ for all $i = 1, \ldots, n$. The inner product of two vectors is $x \cdot y = \sum_{i=1}^n x_i y_i$.

2.1 Model

The model to be tested is that of exponentially discounted utility. Suppose T time periods, and index time by $t = 1, \ldots, T$. A *consumption stream* is a vector in $\mathbf{R}^T_+$.

Consider a decision-maker, a consumer, that chooses a consumption stream $x \in \mathbf{R}^T_+$. The consumer's choice solves the following problem:

$$\begin{array}{ll} \max_{x \in \mathbf{R}^T_+} & \sum_{t=1}^T \beta^{t-1} u(x_t) \\ \text{s.t.} & \sum_{t=1}^T p_t x_t \leq I, \end{array} \tag{1}$$

in which $p \in \mathbf{R}^T_{++}$ is a vector of prices, these can be thought of as interest rates; I is the agent's (present-value) wealth; $\beta \in (0, 1]$ is the agent's discount factor, and $u : \mathbf{R}_+ \to \mathbf{R}$ is her utility function. We suppose that u is strictly increasing and concave.

One cannot observe u or β; one can only observe the consumer's behavior.

2.2 Data

We are given a collection of decisions made by our consumer. We observe the choices of a consumption plan made at various prices and income levels.

Definition 1. *A* dataset *is a collection* $(x^k, p^k)_{k=1}^K$*, where* $x^k, p^k \in \mathbf{R}^T_{++}$ *for all* k *and* $x_t^k \neq x_{t'}^{k'}$ *if* $(k, t) \neq (k', t')$*.*

The interpretation of a dataset is as follows. There are K observations, indexed by $k = 1, \ldots, K$. Each observation consists of a consumption stream x^k purchased at some vector of strictly positive prices p^k across periods. Given the assumption that utility is monotone increasing, we take the level of income at observation k to be $p^k \cdot x^k$ (a standard procedure in all studies on revealed preference using consumption data).

A data set can be thought of in two different ways. One the one hand, it could be the plan made by a consumer for consumption over time. On the other hand, it can be his actual choice made in each period. Both interpretations are equivalent because of the dynamic consistency implied by the EDU model. For other models, such as quasi-hyperbolic discounting, the distinction between these two kinds of data can be very important.

The assumption that $x_t^k \neq x_{t'}^{k'}$ if $(k,t) \neq (k',t')$ is for simplicity of the analysis. The essence of our results is true without the assumption: see Section 3.1.

The datasets that are consistent with the theory of exponential discounted utility are those that can be explained by some specification of the unobservable components of the model. Formally,

Definition 2. *A dataset* $(x^k, p^k)_{k=1}^K$ *is* exponential discounted utility rational *(EDU rational) if there is a number* $\beta \in (0,1]$ *and a concave and strictly increasing function* $u : \mathbf{R}_+ \to \mathbf{R}$ *such that, for all* k*,*

$$p^k \cdot y \leq p^k \cdot x^k \Rightarrow \sum_{t \in T} \beta^{t-1} u(y_t) \leq \sum_{t \in T} \beta^{t-1} u(x_t^k).$$

2.3 Theorem

Consider the maximization problem (1). Suppose that the function u is continuously differentiable (an assumption that turns out to be without loss of generality). The first-order condition for an interior solution is

$$\beta^{t-1} u'(x_t) = \lambda^k p_t,$$

where λ^k is a Lagrange multiplier. So if a dataset $(x^k, p^k)_{k=1}^K$ is EDU rational, the discount factor β and utility u must satisfy the above first order condition for each x_t^k and p_t^k.

Suppose that one tries to derive the implications on quantities x of some property of the observed prices. From the first-order conditions, one can obtain that

$$\frac{u'(x_{t'}^{k'})}{u'(x_t^k)} = \frac{\beta^t}{\beta^{t'}} \frac{\lambda^{k'} p_{t'}^{k'}}{\lambda^k p_t^k}.$$

Suppose that $x_t^k > x_{t'}^{k'}$. The concavity of u and $x_t^k > x_{t'}^{k'}$ implies that

$$\frac{\beta^t}{\beta^{t'}} \frac{\lambda^{k'} p_{t'}^{k'}}{\lambda^k p_t^k} \leq 1,$$

but the discount rate β and the Lagrange multipliers $\lambda^{k'}$ and λ^k are unobservable so we cannot conclude anything about the observable $\frac{p_{t'}^{k'}}{p_t^k}$.

There is, however, one implication of EDU and the concavity of u that can unambiguously be obtained, despite the role of unobservables. We can consider a sequence of pairs $(x_t^k, x_{t'}^{k'})$ chosen such that when we divide first-order conditions as above, all Lagrange multipliers cancel out, and the effect of the discount factors is predicted (even though we do not know the value of the discount factor). For example, consider

$$x_{t_1}^{k_1} > x_{t_2}^{k_2} \text{ and } x_{t_3}^{k_2} > x_{t_4}^{k_1}.$$

such that

$$t_1 + t_3 \geq t_2 + t_4.$$

By manipulating first-order conditions we obtain that:

$$\frac{u'(x_{t_1}^{k_1})}{u'(x_{t_2}^{k_2})} \cdot \frac{u'(x_{t_3}^{k_2})}{u'(x_{t_4}^{k_1})} = \left(\frac{\beta^{t_2-1}}{\beta^{t_1-1}} \frac{\lambda^{k_1} p_{t_1}^{k_1}}{\lambda^{k_2} p_{t_2}^{k_2}} \right) \cdot \left(\frac{\beta^{t_4-1}}{\beta^{t_3-1}} \frac{\lambda^{k_2} p_{t_3}^{k_2}}{\lambda^{k_1} p_{t_4}^{k_1}} \right) = \beta^{(t_2+t_4)-(t_1+t_3)} \frac{p_{t_1}^{k_1}}{p_{t_2}^{k_2}} \frac{p_{t_3}^{k_2}}{p_{t_4}^{k_1}}$$

Notice that the pairs $(x_{t_1}^{k_1}, x_{t_2}^{k_2})$ and $(x_{t_3}^{k_2}, x_{t_4}^{k_1})$ have been chosen so that the Lagrange multipliers would cancel out and the discount factors unambiguously increase the value on the left hand side (i.e., $\beta^{(t_2+t_4)-(t_1+t_3)} \geq 1$ for any $\beta \in (0, 1]$).

Now the concavity of u and the assumption that $x_{t_1}^{k_1} > x_{t_2}^{k_2}$ and $x_{t_3}^{k_2} > x_{s_4}^{k_1}$ imply that the product $\beta^{(t_2+t_4)-(t_1+t_3)} \frac{p_{t_1}^{k_1}}{p_{t_2}^{k_2}} \frac{p_{t_3}^{k_2}}{p_{t_4}^{k_1}}$ cannot exceed 1. Since $\beta^{(t_2+t_4)-(t_1+t_3)} \geq 1$ for any $\beta \in (0, 1]$, then $\frac{p_{t_1}^{k_1}}{p_{t_2}^{k_2}} \frac{p_{t_3}^{k_2}}{p_{t_4}^{k_1}}$ cannot exceed 1. Thus, we obtain an implication of EDU for prices, an observable entity. *No matter what the values of the unobservable β and u, we find that the ratio of prices cannot be more than* 1.

In general, the assumption of EDU rationality will require that, for any collection of sequences as above (appropriately chosen so that Lagrange multipliers will cancel out and the discount factors unambiguously increase the product of the ratio of prices) the product of the ratio of prices cannot exceed 1. Formally,

Axiom 1. *For any sequence of pairs* $(x^{k_i}_{t_i}, x^{k'_i}_{t'_i})^n_{i=1}$ *in which*

1. $x^{k_i}_{t_i} \geq x^{k'_i}_{t'_i}$ *for all* i;
2. $\sum_{i=1}^n t_i \geq \sum_{i=1}^n t'_i$;
3. *each* k *appears as* k_i *(on the left of the pair) the same number of times it appears as* k'_i *(on the right):*

The product of prices satisfies that

$$\prod_{i=1}^n \frac{p^{k_i}_{t_i}}{p^{k'_i}_{t'_i}} \leq 1.$$

Our result is that this necessary condition turns out be sufficient as well.

Theorem 1. $(x^k, p^k)^K_{k=1}$ *is EDU rational if and only if it satisfies Axiom 1.*

Note that Axiom 1 is different from our axiom in Echenique and Saito (2013) only in the second requirement for the sequence.

3 Extension

In this section, we extend the results into possibly constant consumption streams and risky consumption streams.

3.1 Constant Consumption Stream

We have assumed that $x^k_t \neq x^{k'}_{t'}$ if $(k, t) \neq (k', t')$. We now relax this assumption. In this section, a *dataset* is a collection $(x^k, p^k)^K_{k=1}$ where for all k $x^k, p^k \in \mathbf{R}^T_{++}$.

When we allow for $x^k_t \neq x^{k'}_{t'}$, then there is a gap in our result: Axiom 1 is still sufficient for strict EDU rationality, but only necessary for EDU rationality with a differentiable utility function. A concave utility function is almost everywhere differentiable, so the gap is "small."

Definition 3. *A dataset* $(x^k, p^k)_{k=1}^K$ *is* smooth EDU rational *if there is a number* $\beta \in (0,1]$ *and a differentiable, concave and strictly increasing function* $u : \mathbf{R}_+ \to \mathbf{R}$ *such that, for all* k*,*

$$p^k \cdot y \leq p^k \cdot x^k \Rightarrow \sum_{t \in T} \beta^{t-1} u(y_t) \leq \sum_{t \in T} \beta^{t-1} u(x_t^k).$$

Theorem 2. *If a dataset satisfies Axiom 1 then it is EDU rational. If a dataset is smooth EDU rational, then it satisfies Axiom 1.*

3.2 Risky Consumption Stream

In many economic applications, decision maker's consumptions often involve uncertainty. In this section, we extend the results to risky consumption streams.

The model to be tested is that of exponentially discounted utility with subjective expected utility. We introduce a new primitive, a finite set S of states. A *risky consumption stream* is a vector in $\mathbf{R}_{++}^{T \times S}$.

Consider a decision-maker, a consumer, that chooses a risky consumption stream $x \in \mathbf{R}_{++}^{T \times S}$. The consumer's choice solves the following problem:

$$\begin{array}{ll} \max_{x \in \mathbf{R}_{++}^{T \times S}} & \sum_{t \in T} \beta^{t-1} \sum_{s \in S} \mu_s u(x_{(t,s)}) \\ \text{s.t.} & \sum_{t \in T} \sum_{s \in S} p_{(t,s)} x_{(t,s)} \leq I. \end{array} \tag{2}$$

Note that (2) is differ from (1) in that the decision maker faces an uncertainty over a realization of a state; and he has a subjective probability over the set of states. Note also that Theorem 1 characterizes (2) with $S = \{1\}$; Echenique and Saito (2013) (Theorem 1) characterize (2) with $T = \{1\}$.

In this section, we observe the choices of a consumption plan across states made at various prices and income levels.

Definition 4. *A* dataset *is a collection* $(x^k, p^k)_{k=1}^K$*, where* $x^k \in \mathbf{R}_{++}^S$ *and* $p^k \in \mathbf{R}_{++}^T$ *for all* k *and* $x_{(t,s)}^k \neq x_{(t',s')}^{k'}$ *if* $(k,t,s) \neq (k',t',s')$*.*

The interpretation of a dataset is as follows. There are K observations, indexed by $k = 1, \ldots, K$. In each observation k, for each period t, the data consists of a consumption $x_t^k \in \mathbf{R}_{++}^S$ across states, purchased at strictly positive prices $p_t^k \in \mathbf{R}_{++}^S$ across states.

The assumption that $x^k_{(t,s)} \neq x^{k'}_{(t',s')}$ if $(k,t,s) \neq (k',t',s')$ is for simplicity of the analysis. The essence of our results is true without the assumption as in Section 3.1.

Definition 5. *A dataset* $(x^k, p^k)_{k=1}^K$ *is* exponential discounting-subjective expected utility rational *(ED-SEU rational) if there is a number* $\beta \in (0,1]$, *a vector* $\mu \in \mathbf{R}^S_{++}$ *such that* $\sum_{s=1}^S \mu_s = 1$ *and a concave and strictly increasing function* $u : \mathbf{R}_+ \to \mathbf{R}$ *such that, for all* k,

$$p^k \cdot y \leq p^k \cdot x^k \Rightarrow \sum_{t \in T} \beta^{t-1} \sum_{s \in S} \mu_s u(y_{(t,s)}) \leq \sum_{t \in T} \beta^{t-1} \sum_{s \in S} \mu_s u(x^k_{(t,s)}).$$

The ED-SEU rationality can be characterized by the following axiom:

Axiom 2. *For any sequence of pairs* $(x^{k_i}_{(t_i,s_i)}, x^{k'_i}_{(t'_i,s'_i)})_{i=1}^n$ *in which*

1. $x^{k_i}_{(t_i,s_i)} \geq x^{k'_i}_{(t'_i,s'_i)}$ *for all* i;
2. $\sum_{i=1}^n t_i \geq \sum_{i=1}^n t'_i$;
3. *each* k *appears as* k_i *(on the left of the pair) the same number of times it appears as* k'_i *(on the right);*
4. *each* s *appears as* s_i *(on the left of the pair) the same number of times it appears as* s'_i *(on the right):*

The product of prices satisfies that

$$\prod_{i=1}^n \frac{p^{k_i}_{(t_i,s_i)}}{p^{k'_i}_{(t'_i,s'_i)}} \leq 1.$$

Theorem 3. $(x^k, p^k)_{k=1}^K$ *is ED-SEU rational if and only if it satisfies Axiom 2.*

Note that the four requirements in Axiom 2 are the combination of those in Axiom 1 and in Axiom 1 of Echenique and Saito (2013). The Proof of Theorem 3 is omitted; it is similar to the proof of 1 and the proof of Theorem 1 in Echenique and Saito (2013).

4 Proof of Theorem 1

The proof is based on using the first-order conditions for maximizing a utility with the EDU model over a budget set. Our first lemma ensures that we can without loss of generality restrict attention to first order conditions. The proof of the lemma is the same as that of Lemma 3 in Echenique and Saito (2013) with the changes of T to S and $\{\beta^{t-1}\}_{t \in T}$ to $\{\mu_s\}_{s \in S}$.

Lemma 1. *Let* $(x^k, p^k)_{k=1}^K$ *be a dataset. The following statements are equivalent:*

1. $(x^k, p^k)_{k=1}^K$ *is EDU rational.*

2. $(x^k, p^k)_{k=1}^K$ *is EDU rational with a continuously differentiable, strictly increasing and concave utility function.*

3. *There are strictly positive numbers* v_t^k*,* λ^k*, and* $\beta \in (0,1)$*, for* $t = 1, \ldots, T$ *and* $k = 1, \ldots, K$*, such that*

$$\beta^{t-1} v_t^k = \lambda^k p_t^k$$
$$x_t^k > x_{t'}^{k'} \Rightarrow v_t^k \leq v_{t'}^{k'}.$$

We shall use the following lemma, which is a version of the Theorem of the Alternative. This is Theorem 1.6.1 in Stoer and Witzgall (1970). We shall use it here in the cases where F is either the real or the rational numbers.

Lemma 2. *Let* A *be an* $m \times n$ *matrix,* B *be an* $l \times n$ *matrix, and* E *be an* $r \times n$ *matrix. Suppose that the entries of the matrices* A*,* B*, and* E *belong the a commutative ordered field* $\mathbf{F}$*. Exactly one of the following alternatives is true.*

1. *There is* $u \in \mathbf{F}^n$ *such that* $A \cdot u = 0$*,* $B \cdot u \geq 0$*,* $E \cdot u \gg 0$*.*

2. *There is* $\theta \in \mathbf{F}^r$*,* $\eta \in \mathbf{F}^l$*, and* $\pi \in \mathbf{F}^m$ *such that* $\theta \cdot A + \eta \cdot B + \pi \cdot E = 0$*;* $\pi > 0$ *and* $\eta \geq 0$*.*

We also use the following lemma, which follows from Lemma 2 (See Border (2013) or Chambers and Echenique (2011)):

Lemma 3. *Let* A *be an* $m \times n$ *matrix,* B *be an* $l \times n$ *matrix, and* E *be an* $r \times n$ *matrix. Suppose that the entries of the matrices* A*,* B*, and* E *are rational numbers. Exactly one of the following alternatives is true.*

1. *There is* $u \in \mathbf{R}^n$ *such that* $A \cdot u = 0$*,* $B \cdot u \geq 0$*, and* $E \cdot u \gg 0$*.*

2. *There is* $\theta \in \mathbf{Q}^r$*,* $\eta \in \mathbf{Q}^l$*, and* $\pi \in \mathbf{Q}^m$ *such that* $\theta \cdot A + \eta \cdot B + \pi \cdot E = 0$*;* $\pi > 0$ *and* $\eta \geq 0$*.*

We use the following notation in the proofs:

$$\mathcal{X} = \{x_t^k : k = 1, \ldots, K, t = 1, \ldots, T\}.$$

4.1 Necessity

Lemma 4. *If a dataset* $(x^k, p^k)_{k=1}^K$ *is EDU rational, then it satisfies Axiom 1.*

Proof. By Lemma 1, if a dataset is EDU rational then there is a continuously differentiable and concave rationalization u and a strictly positive solution v_t^k, λ^k, β to the system in Statement (3) of Lemma 1 with $u'(x_t^k) = v_t^k$. Let $(x_{t_i}^{k_i}, x_{t'_i}^{k'_i})_{i=1}^n$ be a sequence satisfying the three conditions in Axiom 1. Then $x_{t_i}^{k_i} > x_{t'_i}^{k'_i}$, so

$$1 \geq \frac{u'(x_{t_i}^{k_i})}{u'(x_{t'_i}^{k'_i})} = \frac{\lambda^{k_i}\beta^{t'_i-1}p_{t_i}^{k_i}}{\lambda^{k'_i}\beta^{t_i-1}p_{t'_i}^{k'_i}}.$$

Thus,

$$1 \geq \prod_{i=1}^n \frac{u'(x_{t_i}^{k_i})}{u'(x_{t'_i}^{k'_i})} = \prod_{i=1}^n \frac{\lambda^{k_i}\beta^{t'_i-1}p_{t_i}^{k_i}}{\lambda^{k'_i}\beta^{t_i-1}p_{t'_i}^{k'_i}} = \beta^{\sum_{i=1}^n t'_i - \sum_{i=1}^n t_i} \prod_{i=1}^n \frac{p_{t_i}^{k_i}}{p_{t'_i}^{k'_i}}.$$

The numbers λ^k appear the same number of times in the denominator as in the numerator of this product, as the sequence satisfies (3) in Axiom 1. Then we obtain that

$$\beta^{\sum_{i=1}^n t_i - \sum_{i=1}^n t'_i} \geq \prod_{i=1}^n \frac{p_{t_i}^{k_i}}{p_{t'_i}^{k'_i}}.$$

Since $\beta \leq 1$ and $\sum_{i=1}^n t_i \geq \sum_{i=1}^n t'_i$, and the sequence satisfies (3) in Axiom 1, we obtain that $1 \geq \prod_{i=1}^n \frac{p_{t_i}^{k_i}}{p_{t'_i}^{k'_i}}$. □

4.2 Sufficiency

We proceed to prove the sufficiency direction. Sufficiency follows from the following lemmas as in Echenique and Saito (2013).

We know from Lemma 1 that it suffices to find a solution to the first order conditions. Lemma 5 establishes that Axiom 1 is sufficient when the logarithms of the prices are rational numbers. The role of rational logarithms comes from our use of a version of Farkas's Lemma. Lemma 6 says that we can approximate any data satisfying Axiom 1 with a dataset for which the logs of prices are rational and for which Axiom 1 is satisfied. Finally, Lemma 7 establishes the result. It is worth mentioning that we cannot use Lemma 6 and an approximate solution to obtain a limiting solution.

Lemma 5. *Let data* $(x^k, p^k)_{k=1}^k$ *satisfy Axiom 1. Suppose that* $\log(p_t^k) \in \mathbf{Q}$ *for all* k *and* t*. Then there are numbers* v_t^k*,* λ^k*,* β*, for* $t = 1, \ldots, T$ *and* $k = 1, \ldots, K$ *satisfying* (3) *in Lemma 1.*

Lemma 6. *Let data* $(x^k, p^k)_{k=1}^k$ *satisfy Axiom 1. Then for all positive numbers* $\bar{\varepsilon}$*, there exists* $q_t^k \in [p_t^k - \bar{\varepsilon}, p_t^k]$ *for all* $t \in T$ *and* $k \in K$ *such that* $\log q_t^k \in \mathbf{Q}$ *and the dataset* $(x^k, q^k)_{k=1}^k$ *satisfy Axiom 1.*

Lemma 7. *Let data* $(x^k, p^k)_{k=1}^k$ *satisfy Axiom 1. Then there are numbers* v_t^k*,* λ^k*,* β*, for* $t = 1, \ldots, T$ *and* $k = 1, \ldots, K$ *satisfying* (3) *in Lemma 1.*

4.3 Proof of Lemma 5

We linearize the equation in System (3) of Lemma 1. The result is:

$$\log v(x_t^k) + t \log \beta - \log \lambda^k - \log p_t^k = 0, \tag{3}$$

$$x > x' \Rightarrow \log v(x') \geq \log v(x) \tag{4}$$

$$\log(\beta) \leq 0. \tag{5}$$

In the system comprised by (3) (4) and 5, the unknowns are the real numbers $\log v_t^k$, $\log \beta$, $k = 1, \ldots, K$ and $t = 1, \ldots, T$.

First, we are going to write the system of inequalities (3) and (4) in matrix form.

A system of linear inequalities

We shall define a matrix A such that there are positive numbers v_t^k, λ^k, β the logs of which satisfy Equation (3) if and only if there is a solution $u \in \mathbf{R}^{K \times T + 1 + K + 1}$ to the system of equations

$$A \cdot u = 0,$$

and for which the last component of u is strictly positive.

Let A be a matrix with $K \times T$ rows and $K \times T + 1 + K + 1$ columns, defined as follows: We have one row for every pair (k, t); one column for every pair (k, t); one column for each k; and two additional columns. Organize the columns so that we first have the $K \times T$ columns for the pairs (k, t); then one of the single columns mentioned in last place, which we shall refer to as the β-column; then K columns (one for each k); and finally one last column. In the row corresponding to (k, t) the matrix has zeroes everywhere with the following exceptions: it has a 1 in the column for (k,t); it has a t in the β column; it has a -1 in the column for k; and $-\log p_t^k$ in the very last column.

Thus, matrix A looks as follows:

$$
\begin{array}{c}
\\ (1{,}1) \\ \vdots \\ (k{,}t) \\ \vdots \\ (K{,}T)
\end{array}
\begin{array}{c}
\begin{array}{ccccccccccccc}
(1{,}1) & \cdots & (k{,}t) & \cdots & (K{,}T) & & 1 & \cdots & k & \cdots & K & & p
\end{array}
\\
\left[\begin{array}{ccccc|c|ccccc|c}
1 & \cdots & 0 & \cdots & 0 & 1 & -1 & \cdots & 0 & \cdots & 0 & -\log p_1^1 \\
\vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots & & 0 & \vdots \\
0 & \cdots & 1 & \cdots & 0 & t & 0 & \cdots & -1 & \cdots & 0 & -\log p_t^k \\
\vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots & & 0 & \vdots \\
0 & \cdots & 0 & \cdots & 1 & T & 0 & \cdots & 0 & \cdots & -1 & -\log p_T^K
\end{array}\right]
\end{array}
$$

Consider the system $A \cdot u = 0$. If there are numbers solving Equation (3), then these define a solution $u \in \mathbf{R}^{K\times T+1+K+1}$ for which the last component is 1. If, on the other hand, there is a solution $u \in \mathbf{R}^{K\times T+1+K+1}$ to the system $A \cdot u = 0$ in which the last component is strictly positive, then by dividing through by the last component of u we obtain numbers that solve Equation (3).

In second place, we write the system of inequalities (4) in matrix form. Let B be a matrix B with $(|\mathcal{X}|(|\mathcal{X}|-1)/2)+1$ rows and $K \times T + 1 + K + 1$ columns. Define B as follows: One row for every pair $x, x' \in \mathcal{X}$ with $x > x'$; in the row corresponding to $x, x' \in \mathcal{X}$ with $x > x'$ we have zeroes everywhere with the exception of a -1 in the column for (k,t) such that $x = x_t^k$ and a 1 in the column for (k', t') such that $x' = x_{t'}^{k'}$. These define $|\mathcal{X}|(|\mathcal{X}|-1)/2$ rows. Finally, in the last row, we have zero everywhere with the exception of a -1 at $K \times T + 1$th column. We shall refer to this last row as the β-row.

In third place, we have a matrix E that captures the requirement that the last component of a solution be strictly positive. The matrix E has a single row and $K \times T + 1 + K + 1$ columns. It has zeroes everywhere except for 1 in the last column.

To sum up, there is a solution to system (3), (4) and (5) if and only if there is a vector $u \in \mathbf{R}^{K\times T+1+K+1}$ that solves the system of equations and linear inequalities

$$
S1: \begin{cases} A \cdot u = 0, \\ B \cdot u \geq 0, \\ E \cdot u \gg 0. \end{cases}
$$

Theorem of the Alternative

The entries of A, B, and E are integer numbers, with the exception of the last column of A. Under the hypothesis of the lemma we are proving, the last column consists of rational numbers.

By Lemma 3, then, there is such a solution u to $S1$ if and only if there is no vector $(\theta, \eta, \pi) \in \mathbf{Q}^{K \times T + (|\mathcal{X}|(|\mathcal{X}|-1)/2)+1}$ that solves the system of equations and linear inequalities

$$S2: \begin{cases} \theta \cdot A + \eta \cdot B + \pi \cdot E = 0, \\ \eta \geq 0, \\ \pi > 0. \end{cases}$$

In the following, we shall prove that the non-existence of a solution u implies that the data must violate Axiom 1. Suppose then that there is no solution u and let (θ, η, π) be a rational vector as above, solving system $S2$.

By multiplying (θ, η, π) by any positive integer we obtain new vectors that solve $S2$, so we can take (θ, η, π) to be integer vectors.

Henceforth, we use the following notational convention: For a matrix D with $K \times T + 1 + K + 1$ columns, write D_1 for the submatrix of D corresponding to the first $K \times T$ columns; let D_2 be the submatrix corresponding to the following one column (i.e., β-column); D_3 correspond to the next K columns; and D_4 to the last column. Thus, $D = [D_1 | D_2 | D_3 | D_4]$.

Claim 1. *(i)* $\theta \cdot A_1 + \eta \cdot B_1 = 0$; *(ii)* $\theta \cdot A_2 + \eta \cdot B_2 = 0$; *(iii)* $\theta \cdot A_3 = 0$; *and (iv)* $\theta \cdot A_4 + \pi \cdot E_4 = 0$.

Proof. Since $\theta \cdot A + \eta \cdot B + \pi \cdot E = 0$, then $\theta \cdot A_i + \eta \cdot B_i + \pi \cdot E_i = 0$ for all $i = 1, \ldots, 4$. Moreover, since B_3, B_4, E_1, E_2, and E_3 are zero matrices, we obtain the claim. □

For convenience, we transform the matrices A and B using θ and η.

Transform the matrices A and B

Lets define a matrix A^* from A by letting A^* have the same number of columns as A and including

1. θ_r copies of the rth row when $\theta_r > 0$;

2. omitting row r when $\theta_r = 0$;

3. and θ_r copies of the rth row multiplied by -1 when $\theta_r < 0$.

We refer to rows that are copies of some r with $\theta_r > 0$ as *original* rows, and to those that are copies of some r with $\theta_r < 0$ as *converted* rows.

Similarly, we define the matrix B^* from B by including the same columns as B and η_r copies of each row (and thus omitting row r when $\eta_r = 0$; recall that $\eta_r \geq 0$ for all r).

Claim 2. *For any (k, t), all the entries in the column for (k, t) in A_1^* are of the same sign.*

Proof. By definition of A, the column for (k, t) will have zero in all its entries with the exception of the row for (k, t). In A^*, for each (k, t), there are three mutually exclusive possibilities: the row for (k, t) in A can (i) not appear in A^*, (ii) it can appear as original, or (iii) it can appear as converted. This shows the claim. □

Claim 3. *There exists a sequence of pairs $(x_{t_i}^{k_i}, x_{t_i'}^{k_i'})_{i=1}^{n^*}$ that satisfies (1) in Axiom 1.*

Proof. We define such a sequence by induction. Let $B^1 = B^*$. Given B^i, define B^{i+1} as follows.

Denote by $>^i$ the binary relation on $\mathcal{X}$ defined by $z >^i z'$ if $z > z'$ and there is at least one copy of the row corresponding to $z > z'$ in B^i. The binary relation $>^i$ cannot exhibit cycles because $>^i \subseteq >$. There is therefore at least one sequence $z_1^i, \ldots z_{L_i}^i$ in $\mathcal{X}$ such that $z_j^i >^i z_{j+1}^i$ for all $j = 1, \ldots, L_i - 1$ and with the property that there is no $z \in \mathcal{X}$ with $z >^i z_1^i$ or $z_{L_i}^i >^i z$.

Let the matrix B^{i+1} be defined as the matrix obtained from B^i by omitting one copy of the row corresponding to $z_j^i > z_{j+1}^i$, for all $j = 1, \ldots L_i - 1$.

The matrix B^{i+1} has strictly fewer rows than B^i. There is therefore n^* for which B^{n^*+1} either has no more rows, or $B_1^{n^*+1}$ has only zeroes in all its entries (its rows are copies of the β-row which has only zeroes in its first $K \times T$ columns).

Define a sequence of pairs $(x_{s_i}^{k_i}, x_{t_i'}^{k_i'})_{i=1}^{n^*}$ by letting $x_{s_i}^{k_i} = z_1^i$ and $x_{t_i'}^{k_i'} = z_{L_i}^i$. Note that, as a result, $x_{s_i}^{k_i} > x_{t_i'}^{k_i'}$ for all i. Therefore the sequence of pairs $(x_{s_i}^{k_i}, x_{t_i'}^{k_i'})_{i=1}^{n^*}$ satisfies condition

(1) in Axiom 1. □

We shall use the sequence of pairs $(x^{k_i}_{t_i}, x^{k'_i}_{t'_i})_{i=1}^{n^*}$ as our candidate violation of Axiom 1.

Consider a sequence of matrices A^i, $i = 1, \dots, n^*$ defined as follows. Let $A^1 = A^*$, and

$$C^1 = \begin{bmatrix} A^1 \\ B^1 \end{bmatrix}.$$

Observe that the rows of C^1 add to the null vector by Claim 1.

We shall proceed by induction. Suppose that A^i has been defined, and that the rows of

$$C^i = \begin{bmatrix} A^i \\ B^i \end{bmatrix}$$

add to the null vector.

Recall the definition of the sequence

$$x^{k_i}_{t_i} = z^i_1 > \dots > z^i_{L_i} = x^{k'_i}_{t'_i}.$$

There is no $z \in \mathcal{X}$ with $z >^i z^i_1$ or $z^i_{L_i} >^i z$, so in order for the rows of C^i to add to zero there must be a -1 in A^i_1 in the column corresponding to (k'_i, t'_i) and a 1 in A^i_1 in the column corresponding to (k_i, t_i). Let r_i be a row in A^i corresponding to (k_i, t_i), and r'_i be a row corresponding to (k'_i, t'_i). The existence of a -1 in A^i_1 in the column corresponding to (k'_i, t'_i), and a 1 in A^i_1 in the column corresponding to (k_i, t_i), ensures that r_i and r'_i exist. Note that the row r'_i is a converted row while r_i is original. Let A^{i+1} be defined from A^i by deleting the two rows, r_i and r'_i.

Claim 4. *The sum of r_i, r'_i, and the rows of B^i which are deleted when forming B^{i+1} (corresponding to the pairs $z^i_j > z^i_{j+1}$, $j = 1, \dots, L_i - 1$) add to the null vector.*

Proof. Recall that $z^i_j >^i z^i_{j+1}$ for all $j = 1, \dots, L_i - 1$. So when we add the rows corresponding to $z^i_j >^i z^i_{j+1}$ and $z^i_{j+1} >^i z^i_{j+2}$, then the entries in the column for (k, t) with $x^k_t = z^i_{j+1}$ cancel out and the sum is zero in that entry. Thus, when we add the rows of B^i that are not in B^{i+1} we obtain a vector that is 0 everywhere except the columns corresponding to z^i_1 and $z^i_{L_i}$. This vector cancels out with $r_i + r'_i$, by definition of r_i and r'_i. □

Claim 5. *The matrix A^* can be partitioned into pairs of rows as follows:*

$$A^* = \begin{bmatrix} r_1 \\ r'_1 \\ \vdots \\ r_i \\ r'_i \\ \vdots \\ r_{n^*} \\ r'_{n^*}, \end{bmatrix}$$

in which the rows r'_i are converted and the rows r_i are original.

Proof. For each i, A^{i+1} differs from A^i in that the rows r_i and r'_i are removed from A^i to form A^{i+1}. We shall prove that A^* is composed of the $2n^*$ rows r_i, r'_i.

First note that since the rows of C^i add up to the null vector, and A^{i+1} and B^{i+1} are obtained from A^i and B^i by removing a collection of rows that add up to zero, then the rows of C^{i+1} must add up to zero as well.

By way of contradiction, suppose that there exist rows left after removing r_{n^*} and r'_{n^*}. Then, by the argument above, the rows of the matrix C^{n^*+1} must add to the null vector. If there are rows left, then the matrix C^{n^*+1} is well defined.

By definition of the sequence B^i, however, B^{n^*+1} has all its entries equal to zero, or has no rows. Hence, the rows remaining in $A_1^{n^*+1}$ must add up to zero. By Claim 2, the entries of a column (k,t) of A^* are always of the same sign. Moreover, each row of A^* has a non-zero element in the first $K \times S$ columns. Therefore, no subset of the columns of A_1^* can sum to the null vector. □

Claim 6. *(i) For any k and t, if $x_{t_i}^{k_i} = x_t^k$ for some i, then the row r_i corresponding to (k,t) appears as original in A^*. Similarly, if $x_{t'_i}^{k'_i} = x_t^k$ for some i, then the row corresponding to (k,t) appears converted in A^*.*
(ii) If the row corresponding to (k,t) appears as original in A^*, then there is some i with $x_{t_i}^{k_i} = x_t^k$. Similarly, if the row corresponding to (k,t) appears converted in A^*, then there is i with $x_{t'_i}^{k'_i} = x_t^k$.

Proof. (i) is true by definition of $(x_{t_i}^{k_i}, x_{t'_i}^{k'_i})$. (ii) is immediate from Claim 5 because if the row corresponding to (k,t) appears original in A^* then it equals r_i for some i, and then

$x^k_t = x^{k_i}_{t_i}$. Similarly when the row appears converted. □

Claim 7. *The sequence* $(x^{k_i}_{t_i}, x^{k'_i}_{t'_i})^{n^*}_{i=1}$ *satisfies (2) and (3) in Axiom 1.*

Proof. We first establish (2). Note that A^*_2 is a vector, and in row r the entry of A^*_2 is as follows. There must be a (k,t) of which r is a copy. Then the component at row r of A^*_2 is t if r is original and $-t$ if r is converted. Now, when r appears as original there is some i for which $t = t_i$, when r appears as converted there is some i for which $t = t'_i$. So for each r there is i such that $(A^*_4)_r$ is either t_i or $-t'_i$. By Claim 1 (ii), $\theta \cdot A_2 + \eta \cdot B_2 = 0$. Recall that $\theta \cdot A_2$ equals the sum of the rows of A^*_2. Moreover, B_2 is a vector that has zeroes everywhere except a -1 in the β row (i.e., $K \times T + 1$th row). Therefore, the sum of the rows of A^*_2 equals $\eta_{K\times T+1}$, where $\eta_{K\times T+1}$ is the $K \times T + 1$th element of η. Since $\eta \geq 0$, therefore, $\sum^{n^*}_{i=1} t_i \geq \sum^{n^*}_{i=1} t'_i$, and condition (2) in the axiom is satisfied.

Now we turn to (3). By Claim 1 (iii), the rows of A^*_3 add up to zero. Therefore, the number of times that k appears in an original row equals the number of times that it appears in a converted row. By Claim 6, then, the number of times k appears as k_i equals the number of times it appears as k'_i. Therefore condition (3) in the axiom is satisfied. □

Finally, in the following, we show that

$$\prod_{i=1}^{n^*} \frac{p^{k_i}_{s_i}}{p^{k'_i}_{t'_i}} > 1,$$

which finishes the proof of Lemma 5 as the sequence $(x^{k_i}_{s_i}, x^{k'_i}_{t'_i})^{n^*}_{i=1}$ would then exhibit a violation of Axiom 1.

Claim 8. $\prod^{n^*}_{i=1} \frac{p^{k_i}_{t_i}}{p^{k'_i}_{t'_i}} > 1$.

Proof. By Claim 1 (iv) and the fact that the submatrix E_4 equals the scalar 1, we obtain

$$0 = \theta \cdot A_4 + \pi E_4 = (\sum_{i=1}^{n^*}(r_i + r'_i))_4 + \pi,$$

where $(\sum^{n^*}_{i=1}(r_i + r'_i))_4$ is the (scalar) sum of the entries of A^*_4. Recall that $-\log p^{k_i}_{t_i}$ is the last entry of row r_i and that $\log p^{k'_i}_{t'_i}$ is the last entry of row r'_i, as r'_i is converted and

r_i original. Therefore the sum of the rows of A_4^* are $\sum_{i=1}^{n^*} \log(p_{t_i'}^{k_i'}/p_{t_i}^{k_i})$. Then,

$$\sum_{i=1}^{n^*} \log(p_{t_i'}^{k_i'}/p_{t_i}^{k_i}) = -\pi < 0.$$

Thus

$$\prod_{i=1}^{n^*} \frac{p_{t_i}^{k_i}}{p_{t_i'}^{k_i'}} > 1.$$

□

4.3.1 Proof of Lemma 6

For each sequence $\sigma = (x_{t_i}^{k_i}, x_{t_i'}^{k_i'})_{i=1}^n$ that satisfies conditions (1), (2), and (3) in Axiom 1, and each pair $x_t^k > x_{t'}^{k'}$, define $\tau_\sigma(x_t^k, x_{t'}^{k'})$ to be the number of times the pair $(x_t^k, x_{t'}^{k'})$ appears in the sequence σ. Note that τ_σ is a $\frac{KT(KT-1)}{2}$-dimensional non-negative integer vector. Define

$$T = \left\{ \tau_\sigma \in \mathbf{N}^{\frac{KT(KT-1)}{2}} : \sigma \text{ satisfies (1), (2), (3) in Axiom 1} \right\}.$$

The set T depends only on $(x^k)_{k=1}^K$ in the data set $(x^k, p^k)_{k=1}^K$.

For each pair $x_t^k > x_{t'}^{k'}$, define

$$\hat{\delta}(x_t^k, x_{t'}^{k'}) = \log \frac{p_t^k}{p_{t'}^{k'}}.$$

Then, $\hat{\delta}$ is a $\frac{KT(KT-1)}{2}$-dimensional real-valued vector.

If $\sigma = (x_{t_i}^{k_i}, x_{t_i'}^{k_i'})_{i=1}^n$, then

$$\hat{\delta} \cdot \tau_\sigma = \sum_{(x_t^k, x_{t'}^{k'}) \in \sigma} \hat{\delta}(x_t^k, x_{t'}^{k'}) \tau_\sigma(x_t^k, x_{t'}^{k'}) = \log \left(\prod_{i=1}^n \frac{p_{t_i}^{k_i}}{p_{t_i'}^{k_i'}} \right).$$

So the data satisfy Axiom 1 if and only if $\tau \cdot \hat{\delta} \leq 0$ for all $\tau \in T$.

Enumerate elements in $\mathcal{X}$ in increasing order:

$$x_{t(1)}^{k(1)} < x_{t(2)}^{k(2)} < \cdots < x_{t(N)}^{k(N)}.$$

Fix an arbitrary $\underline{\xi}, \bar{\xi} \in (0,1)$ with $\underline{\xi} < \bar{\xi}$. Due to the denseness of the rational numbers, and the continuity of the exponential function, there exists a positive number $\varepsilon(1)$ such that $\log(p^{k(1)}_{t(1)}\varepsilon(1)) \in \mathbf{Q}$ and $\underline{\xi} < \varepsilon(1) < \bar{\xi}$; Given $\varepsilon(1)$, there exists a positive $\varepsilon(2)$ such that $\log(p^{k(2)}_{t(2)}\varepsilon(2)) \in \mathbf{Q}$ and $\underline{\xi} < \varepsilon(2)/\varepsilon(1) < \bar{\xi}$. More generally, when $\varepsilon(n)$ has been defined, let $\varepsilon(n+1) > 0$ be such that $\log(p^{k(n+1)}_{t(n+1)}\varepsilon(n+1)) \in \mathbf{Q}$ and $\underline{\xi} < \varepsilon(n+1)/\varepsilon(n) < \bar{\xi}$.

In this way have defined $(\varepsilon(n))_{n=1}^N$. Let $q^k_t = p^k_t \varepsilon(n)$, where n is such that $p^k(n)_{t(n)} = p^k_t$. The claim is that the data $(x^k, q^k)_{k=1}^K$ satisfy Axiom 1. Let δ^* be defined from $(q^k)_{k=1}^K$ in the same manner as $\hat{\delta}$ was defined from $(p^k)_{k=1}^K$. For each pair $x^k_t > x^{k'}_{t'}$, if n and m are such that $x^k_t = x^{k(n)}_{t(n)}$ and $x^{k'}_{t'} = x^{k(m)}_{t(m)}$, then $n > m$. By definition of ε, $\varepsilon(n)/\varepsilon(m) < \bar{\xi} < 1$. Hence,

$$\delta^*(x^k_t, x^{k'}_{t'}) = \log \frac{p^k_t \varepsilon(n)}{p^{k'}_{t'} \varepsilon(m)} < \log \frac{p^k_t}{p^{k'}_{t'}} + \log \bar{\xi} < \log \frac{p^k_t}{p^{k'}_{t'}} = \hat{\delta}(x^k_t, x^{k'}_{t'}).$$

Thus, for all $\tau \in T$,

$$\delta^* \cdot \tau \leq \hat{\delta} \cdot \tau \leq 0,$$

as $\tau \geq 0$ and the data $(x^k, p^k)_{k=1}^K$ satisfy Axiom 1. Thus the data $(x^k, q^k)_{k=1}^K$ satisfy Axiom 1.

Note that $\underline{\xi} < \varepsilon(n)$ for all n. So that by choosing $\underline{\xi}$ close enough to 1, we can take the prices (q^k) to be as close to (p^k) as desired.

4.3.2 Proof of Lemma 7

Consider the system comprised by (3), (4), and (5) in the proof of Lemma 5. Let A, B, and E be constructed from the data as in the proof of Lemma 5. The difference with respect to Lemma 5 is that now the entries of A_4 may not be rational. Note that the entries of E, B, and A_i, $i = 1, 2, 3$ are rational.

Suppose, towards a contradiction, that there is no solution to the system comprised by (3), (4), and (5). Then, by the argument in the proof of Lemma 5 there is no solution to System $S1$. By Lemma 2 with $\mathbf{F} = \mathbf{R}$, there is a real vector (θ, η, π) such that

$$\theta \cdot A + \eta \cdot B + \pi \cdot E = 0 \text{ and } \eta \geq 0, \pi > 0.$$

Let $(q^k)_{k=1}^K$ be vectors of prices such that the data set $(x^k, q^k)_{k=1}^K$ satisfies Axiom 1 and $\log q^k_t \in \mathbf{Q}$ for all k and t. (Such $(q^k)_{k=1}^K$ exists by Lemma 6.) Construct matrices A', B', and E' from this data set in the same way as A, B, and E is constructed in the proof of Lemma 5. Note that only the prices are different in (x^k, q^k) compared to (x^k, p^k). So

$E' = E$, $B' = B$ and $A'_i = A_i$ for $i = 1, 2, 3$. Since only prices q^k are different in this dataset, only A'_4 may be different from A_4.

By Lemma 6, we can choose prices q^k such that $|\theta \cdot A'_4 - \theta \cdot A_4| < \pi/2$. We have shown that $\theta \cdot A_4 = -\pi$, so the choice of prices q^k guarantees that $\theta \cdot A'_4 < 0$. Let $\pi' = -\theta \cdot A'_4 > 0$.

Note that $\theta \cdot A'_i + \eta \cdot B'_i + \pi' E_i = 0$ for $i = 1, 2, 3$, as (θ, η, π) solves system $S2$ for matrices A, B and E, and $A'_i = A_i$, $B'_i = B_i$ and $E_i = 0$ for $i = 1, 2, 3$. Finally, $B_4 = 0$ so

$$\theta \cdot A'_4 + \eta \cdot B'_4 + \pi' E_4 = \theta \cdot A'_4 + \pi' = 0.$$

We also have that $\eta \geq 0$ and $\pi' > 0$. Therefore θ, η, and π' constitute a solution $S2$ for matrices A', B', and E'.

By Lemma 2 we know then that there is no solution to $S1$ for matrices A', B', and E', so there is no solution to the system comprised by (3), (4), and (5) in the proof of Lemma 5. However, this contradicts Lemma 5 because the data (x^k, q^k) satisfies Axiom 1 and $\log q^k_t \in \mathbf{Q}$ for all $k = 1, \ldots K$ and $s = 1, \ldots, S$.

5 Proof of Theorem 2

The second statement in the theorem follows from Lemma 1 and the proof of Lemma 4. We proceed to prove the first statement in the theorem. Assume then that $(x^k, p^k)_{k=1}^K$ is a dataset that satisfies Axiom 1.

Recall that $\mathcal{X} = \{x^k_t : k = 1, \ldots, K, t = 1, \ldots, T\}$. Let $\varepsilon > 0$ be s.t.

$$\varepsilon < \min\{|x - x'| : x, x' \in \mathcal{X}, x \neq x'\}.$$

Let $\alpha(x) = \{(k, t) : x = x^k_t\}$ for $x \in \mathcal{X}$.

We shall define a new dataset for which consumptions are not equal, but that still satisfies Axiom 1. Let $(\hat{x}^k, p^k)_{k=1}^K$ be a dataset with the same prices as in $(x^k, p^k)_{k=1}^K$; in which $(\hat{x}^k)_{k=1}^K$ is chosen such that (a) $\hat{x}^k_t \neq \hat{x}^{k'}_{t'}$ when $(k, t) \neq (k', t')$; and (b) for all $x \in \mathcal{X}$

$$|\hat{x}^k_t - x| < \varepsilon,$$

for all $(k, t) \in \alpha(x)$.

Observe that, with this definition of data $(\hat{x}^k, p^k)_{k=1}^K$, if $\hat{x}_t^k > \hat{x}_{t'}^{k'}$ then $x_t^k \geq x_{t'}^{k'}$. The reason is that, either there is x for which $(k,t) \in \alpha(x)$ and $(k',t') \in \alpha(x)$, in which case $x_t^k \geq x_{t'}^{k'}$ because $x = x_t^k = x_{t'}^{k'}$; or there is no x and x', with $x \neq x'$, in which $(k,t) \in \alpha(x)$ and $(k',t') \in \alpha(x')$, which implies that $x > x'$ and thus that $x_t^k > x_{t'}^{k'}$.

With this definition of data, if $(\hat{x}_{s_i}^{k_i}, \hat{x}_{t_i'}^{k_i'})_{i=1}^n$ is a sequence of pairs from dataset $(\hat{x}^k, p^k)_{k=1}^K$ satisfying (1), (2), and (3) in Axiom 1, then $(x_{s_i}^{k_i}, x_{t_i'}^{k_i'})_{i=1}^n$ is a sequence of pairs from dataset $(x^k, p^k)_{k=1}^K$ that also satisfies (1), (2), and (3) in Axiom 1. By hypothesis, $(x^k, p^k)_{k=1}^K$ satisfy Axiom 1, so $(\hat{x}^k, p^k)_{k=1}^K$ satisfy Axiom 1.

Since $(\hat{x}^k, p^k)_{k=1}^K$ satisfies that $x_t^k \neq x_{t'}^{k'}$ if $(k,t) \neq (k',t')$, and Axiom 1, then Lemma 7 implies that there are strictly positive numbers $\hat{v}_t^k$, λ^k, β^{t-1}, for $t = 1, \ldots, T$ and $k = 1, \ldots, K$, such that

$$\beta^{t-1}\hat{v}_t^k = \lambda^k p_t^k$$
$$\hat{x}_t^k > \hat{x}_{t'}^{k'} \Rightarrow \hat{v}_t^k < \hat{v}_{t'}^{k'}.$$

Define the correspondence $v' : \mathcal{X} \rightarrow \mathbf{R}_+$ by

$$v'(x) = \left[\inf\{\hat{v}_t^k(k,t) \in \alpha(x)\}, \sup\{\hat{v}_t^k(k,t) \in \alpha(x)\}\right].$$

Note that if $x > x'$ then $\hat{v}_t^k < \hat{v}_{t'}^{k'}$ for all $(k,t) \in \alpha(x)$ and all $(k',t') \in \alpha(x')$. So as a result of the definition of v', if $x > x'$ then $\sup v'(x) < \inf v'(x')$.

Let $v : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ be

$$v(x) = \{\inf v'(\tilde{x}) : \tilde{x} \in \mathcal{X}, \tilde{x} \leq x\}$$

for $x \geq \inf \mathcal{X}$; and $v(x) = \{\sup v'(\tilde{x}) : \tilde{x} \in \mathcal{X}\}$ for $x < \inf \mathcal{X}$. The correspondence v is monotone. There is therefore a concave function $u : \mathbf{R}_+ \rightarrow \mathbf{R}$ such that

$$\partial u(x) = v(x)$$

for all x (See Rockafellar (1997), Theorem 24.8).

In particular, for all $x \in \mathcal{X}$ and all $(k,t) \in \alpha(x)$ we have $\hat{v}_t^k \in \partial u(x)$. Since $\beta^{t-1}\hat{v}_t^k = \lambda^k p_t^k$, we have

$$\frac{\lambda^k p_t^k}{\beta^{t-1}} \in \partial u(x_t^k).$$

Hence the first-order conditions for EDU maximization are satisfied at x_t^k.

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