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RESPONSE TIME AND UTILITY

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SOCIAL SCIENCE WORKING PAPER 1410

November 2015

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Abstract

Response time is the time an agent needs to make a decision. One fundamental finding in psychology and neuroscience is that, in a binary choice, the response time is shorter as the difference between the utilities of the two options becomes larger. We consider situations in which utilities are not observed, but rather inferred from revealed preferences: meaning they are inferred from subjects' choices. Given data on subjects' choices, and the time to make those choices, we give conditions on the data that characterize the property that response time is decreasing in utility differences.

JEL classification numbers: D01,D10

Key words: Psychometrics, lag times, response times, revealed preference

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1 Introduction

Response time refers to the time an agent needs to make a decision. The relation between utility and response time has been studied extensively in psychology and neuroscience. (See Luce (1986) and Krajbich et al. (2014) for surveys.) One fundamental finding in psychology and neuroscience is that in binary choice experiments, subjects' response time is shorter as the difference between the utility of the two options becomes larger. This finding is based on having estimates of subjects' values for the different options. In classical economic environments, however, one has no access to such "psychometric" estimates of utility (indeed the traditional position is to discourage economists from using them), and must infer utility from choices.

We consider the property that response time is decreasing in utility differences from the perspective of revealed preference theory. Given is data on agents' choices, and the time taken to make them. We give conditions that describe when data is consistent with the theory that response time is decreasing in utility differences.

In a binary choice between x and y, $t(x, y)$ denotes the response time which the agent needs to choose over x to y. We axiomatize the following model:

$$t(x, y) = f(u(x) - u(y)), \tag{1}$$

*We thank Antonio Rangel, Ian Krajbich, and John Clithero for valuable discussions.

where f is strictly decreasing function and $u(x) > u(y)$. The strict decreasingness of f captures the inverse relationship between response times and utility differences.

The model in (1) is captured by a few simple properties, or axioms. The first is that choices must be acyclic; this is standard, as (1) is based on maximizing utility and a cycle in an agents choices would indicate that utility is not being maximized. The other properties capturing model (1) are discussed in detail in Section 3. The basic idea is that, since f is supposed to be decreasing, one obtains restrictions on concatenated pairs of choices. Furthermore, under a rich domain assumption, we show that f and u are uniquely identified from the data.

Our model is a generalization of several well-known models in psychology and neuroeconomics. In osychology, *Pieron's law* states that

$$t(x, y) = r_0 + k\frac{1}{I^\beta},$$

where I is intensity of the choice and it is often assumed that I captures the utility difference between the two options (i.e., $I = u(x) - u(y)$). (See Luce (1986) for details.)

In neuroscience, researchers use a stochastic model, *Drift Diffusion Model* (DDM) proposed by Ratcliff (1978). The model implies:

$$E[t(x, y)] = r_0 + \frac{1}{\mu}\tanh\left(\frac{\mu}{\sigma^2}\right),$$

where the diffusion process is $B(t) = \mu dt + \sigma dW(t)$ and $W(t)$ is a Winer process. Here, the drift μ captures the utility difference between the two options (i.e., $\mu = u(x) - u(y)$).

Both Pieron's law and the DDM formula predict an inverse relationship between response times and utility differences.

There are several papers on response times in economics, but ours is the first axiomatic study on the topic. Wilcox (1993) uses response times to evaluate decision cost in his experiments. Rubinstein (2007) measures response times in his experiments on games, and suggests that choices made instinctively require less response time than choices that require the use of cognitive reasoning. Piovesan and Wengström (2009) experimentally show that egoistic subjects make faster decisions than subjects with social preferences. In experiments on single-person decision problems, Rubinstein (2013) measures response times to study mistakes and biases such as Allais's paradox.

Chabris et al. (2009) measure response times in their experiments on intertemporal choices. By assuming linear utility and a specific form of discounting function, they show an inverse relationship between average response times and utility differences. They claim that their evidence is consistent with the models of Gabaix et al. (2006). In the model of Gabaix et al. (2006), an agent uses partially myopic option-value calculations to select their next cognitive operation.

More recently, Fedenberg et al. (2015) model the joint distribution of choice probabilities and response time in binary choice tasks as the solution to a problem of optimal sequential sampling, where the agent is uncertain of the utility of each action and pays a constant cost per time for gathering information. They do not provide an axiomatization of their model, focusing instead on which aspects of optimal sampling are implicitly assumed by the DDM model.

A general version of our model is discussed in Krantz et al. (1971), but they focus on environments with rich data. We use their main result to prove a version of our theorem under a rich data assumption (see Section 3.1).

Some recent papers use response time to improve prediction rates. Clithero and Rangel (2013) use response times to predict choices in single-person decision problems. Based on DDM, they made predictions by using response times. Schotter and Trevino (2014) use response times to predict threshold behavior in a simple global game experiment.

2 Models

Let X be a finite set and let $\succ \subset X \times X \times \mathbf{R}_+$ be the primitive that describes an agent's choice behavior. We write $x \succ_t y$ to denote $(x, y, t) \in \succ$. This means that the agent chooses x over y, and that he times time t to make that decision. We denote this t by $t(x, y)$. We assume that for every $x, y \in X$ there is at most one $t \in \mathbf{R}_+$ with $(x, y, t) \in \succ$. So we simply write $x \succ y$ to denote $x \succ_t y$.

Definition 1. *A pair (u, f) of a function $u : X \to \mathbf{R}$ and a strictly decreasing function $f : \mathbf{R}_+ \to \mathbf{R}$ is called a response time representation if $x \succ y$ implies that*

$$u(x) > u(y), \tag{2}$$

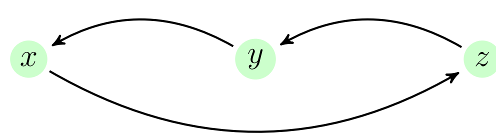
and moreover that

$$u(x) - u(y) = f(t(x, y)). \tag{3}$$

3 Main Results

In this section, we propose one axiom that characterizes response time representations. The pair (u, f) is not uniquely determined from the data. To motivate the axiom, we introduce two weaker axioms, Monotonicity and Additivity. In Section 3.1, we show that Monotonicity and Additivity alone, under a "richness" property, characterize the representation, and the representation has a strong uniqueness property.

It is obvious that any model based on utility maximization must avoid cycles in revealed preference, for example that $x \succ y$, $y \succ z$ and $z \succ x$. A cycle is depicted in the following diagram, where an arrow $y \to x$ means that $x \succ y$.



Think of what is wrong with a cycle. In the cycle $z - y - x - z$, one can start at z, change to y, supposedly for a higher utility as $y \succ z$; then one can change from y to x for a higher utility, and then back to z. Given that we arrive back at z, the option we started from, it is simply not possible that utility would increase with each of those changes.

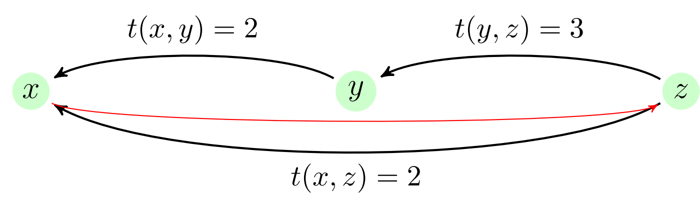
Now we consider cycles in their relation to response time. Information on response time is going to allow us to rule out more than just cycles in $\succ$. Information on response time conveys *cardinal* information on utility gains and losses. This allows us to rule out more than just cycles in $\succ$. It allows us to rule out generalized cycles that we obtain by some times losing utility, as long as the utility loss is compensated by a corresponding utility gain.

Our first axiom captures the inverse relationship between response times and the difference in the values of the two options. Consider three options x, y, z such that $x \succ y \succ z$. Then, given a utility representation, the difference $u(x) - u(z)$ must be larger than the differences $u(x) - u(y)$ and $u(y) - u(z)$. So the inverse relationship implies that the response time $t(x, z)$ must be smaller than $t(x, y)$ and $t(y, z)$. Formally,

Axiom 1. *(Monotonicity): If $x \succ y, y \succ z$, and $x \succ z$ then $t(x, z) < t(x, y)$ and $t(x, z) < t(y, z)$.*

It should be clear than a response-time representation requires that if $u(x) - u(y) > u(x') - u(y') > 0$, then $t(x, y) < t(x', y')$, as the function f is monotone decreasing.

Any violation of Monotonicity can be understood as a kind of cycle. Suppose that x, y and z are as in the axiom, but $t(x, y) \leq t(x, z)$. Consider the following diagram. The diagram does not depict a cycle, because the long line connecting x and z has an arrow pointing towards x, not z.

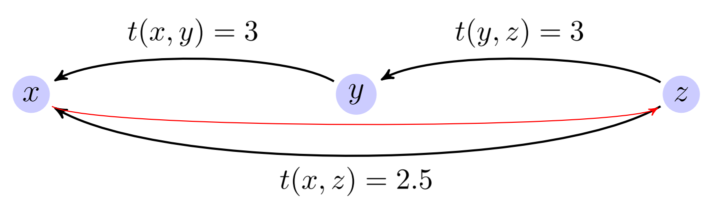


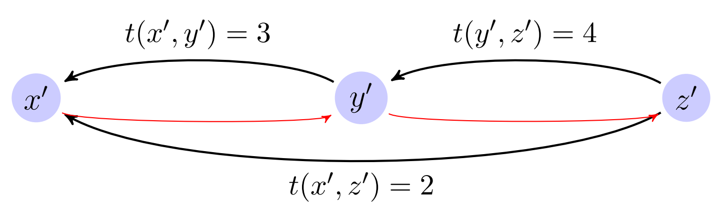
Consider the "cycle" $z - y - x - z$, using the red arrow. When we move from z to y and from y to x we should be gaining utility, as $y \succ z$ and $z \succ x$. But when we move from x to z, using the red arrow in the diagram, we lose utility. However, the loss in utility is *compensated* by the gain when we go from y to x. We know that it is compensated because $t(x, y) \leq t(x, z)$. Therefore the end result is of a net gain in utility when we move along: $z - y - x - z$; this is not possible because we end at z, where we started.

The model has additional implications yet, beyond Monotonicity. These originate from the difference structure. Given any x, y, and z; and x', y', and z', if the response time $t(x, y)$ is smaller than $t(x', y')$; and $t(y, z)$ is smaller than $t(y', z')$, then by the inverse relationship between response time and utility differences, the utility difference $u(x) - u(y)$ must be larger than $u(x') - u(y')$; and $u(y) - u(z)$ must be larger than $u(y') - u(z')$. Therefore, by adding the two differences, we have that the utility difference $u(x) - u(z)$ is larger than $u(x') - u(z')$. Hence, by the inverse relationship again, the response time $t(x, z)$ must be smaller than $t(x', z')$. Our reasoning suggests the following axiom.

Axiom 2. *(Additivity): If* $t(x, y) \leq t(x', y')$ *and* $t(y, z) \leq t(y', z')$*, then* $t(x, z) \leq t(x', z')$*.*

Violations of Additivity can again be understood as exhibiting certain "cycles" in which utility losses are appropriately compensated. For violations of Additivity, we need more than a single cycle. Consider the situation in the following diagram, involving six options: x, y, z, x', y' and z'.





The diagram exhibits two "cycles:" $z - y - x - z$ and $x' - y' - z' - x'$. As before we are using red arrows to signify a movement against a black arrow, agains revealed preference. The red arrow changes from x to z, from x' to y', and $y'$to z', signify utility losses.

But importantly these losses are compensated by utility gains, and at least one of them is *strictly* compensated. The losses $x' - y'$ and $y' - z'$ are compensated by the gains z–y and y–x, as the response times of these utility gains are weakly smaller than the response times of each of the two losses. The loss x–z is strictly compensated by the gain z'–x', as the response time is strictly smaller.

The effect then of traversing both cycles in the diagram is a *net utility gain*. It is not possible to start and end at the same point and gain utility, even when we need to start and end at two different points. In general the axiom that characterizes our model will need to rule out collections of cycles, just as in the diagram, and it will need to keep track of when utility losses can be said to be compensated by utility gains, given the information we have on response times.

These observations suggest the following stronger axiom, which characterises the representation. To show the axiom, we need to introduce some definitions.

Definition 2. *A set*

$$\{(x_i^k)_{i=1}^{n_k} : k = 1, \dots, K\}$$

in which $x_1^k = x_{n_k}^k$ *and either* $x_i \succ x_{i+1}$ *or* $x_{i+1}^k \succ x_i^k$ *is called a* collection of cycles .[1]

Definition 3. *Given a collection of cycles* $\{(x_i^k)_{i=1}^{n_k} : k = 1, \dots, K\}$*, a* compensation *is a one-to-one function* π *that maps any pair* (x_i^k, x_{i+1}^k) *with* $x_i^k \succ x_{i+1}^k$ *into some pair* $(x_{i'}^{k'}, x_{i'+1}^{k'}) = \pi(x_i^k, x_{i+1}^k)$ *with* $x_{i'+1}^{k'} \succ x_{i'}^{k'}$ *and* $t(x_{i'+1}^{k'}, x_{i'}^{k'}) \leq t(x_i^k, x_{i+1}^k)$.

As we have seen in the previous examples, the idea of a compensation is that a utility loss when we go from x_i^k to x_{i+1}^k is compensated by the utility gain when we go from $x_{i'}^{k'}$ to $x_{i'+1}^{k'}$. The pair $(x_{i'}^{k'}, x_{i'+1}^{k'})$ *compensates* (x_i^k, x_{i+1}^k). To see this interpretation notice

[1]Sums are mod n_k. We adopt the convention that any unquantified variable in a formula is universally quantified over the relevant range.

that $t(x^{k'}_{i'+1}, x^{k'}_{i'}) \leq t(x^k_i, x^k_{i+1})$ implies that the uitility loss by moving from x^k_i to x^k_{i+1} is smaller than the utility gain by moving from $x^{k'}_{i'}$ to $x^{k'}_{i'+1}$.

Definition 4. *A compensation π is a* covering *if all inequalities $t(x^{k'}_{i'+1}, x^{k'}_{i'}) \leq t(x^k_i, x^k_{i+1})$ hold with equality, and if any pair $x^{k'}_{i'+1} \succ x^{k'}_{i'}$ is the image of (is used to compensate) some pair $x^k_i \succ x^k_{i+1}$.*

Axiom 3. *(Compensation) For any collection of cycles, any compensation is a covering.*

It should now be clear that all the three diagrams above illustrate violations of the Compensation axiom. In the first there is no pair that needs compensating. The compensation is empty, and, for example, $x \succ y$ is not used to compensate any pair in the cycle. In the second example (the violation of Monotonicity) the pair $y \succ z$ is not used to compensate any pair. Finally, in the third example (the violation of Additivity), what goes wrong is that some response time in a compensating pair is strictly smaller. (In Section 3.1, we show that Compensation implies Monotonicity and Additivity)

Given our discussion so far it should be clear that Compensation is a necessary condition for $\succ$ to have a response-time representation. The main result of our paper is that Compensation is sufficient as well:

Theorem 1. *$\succ$ satisfies Compensation if and only if $\succ$ admits a response time representation.*

Remark. Monotonicity and Additivity are sufficient to characterize response time representations under a richness assumption. We present this result below in 3.1. For finite data, however, Compensation is the characterizing property.

Proof of Theorem 1: Let T be the set of $t \in \mathbf{R}_+$ for which there is $x, y \in X$ such that $(x, y, t) \in \succ$. So we write $t(x, y)$ to mean the $t \in \mathbf{R}$ such that $(x, y, t) \in \succ$.

Define first a matrix A which has $|X| + T$ columns in $| \succ |$ rows. We arrange T columns in a way that the largest t appears at the last column. (This arrangement is made to capture the implication $x \succ y \implies u(x) > u(y)$. This will be clear later.) For each triple $(x, y, t) \in \succ$ there is a row. In the row corresponding to (x, y, t) there are zeros in every entry except for a 1 in the row for x, a -1 in the row for y, and a -1 in the row for t. A looks as follows

$$\begin{array}{c} \\ \vdots \\ {\scriptstyle (x,y,t)\in\succ} \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccccccccc} & \cdots & x & \cdots & y & \cdots & & & \cdots & t & \cdots & \max T \end{array} \\ \left[\begin{array}{ccccccc|ccccc} \vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & -1 & \cdots & 0 & 0 & \cdots & -1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots \end{array}\right] \end{array}$$

Consider the system $A \cdot u = 0$. If there are numbers solving the right hand side of equation (3), then these define a solution $u \in \mathbf{R}^{|X|+|T|}$. If, on the other hand, there is a solution $u \in \mathbf{R}^{|X|+|T|}$ to the system $A \cdot u = 0$, then the vector u defines a solution to (3).

Define the matrix B as a matrix with $|X| + |T|$ columns and one row for every pair (t, t') with $t < t'$. In the row corresponding to (t, t') with $t < t'$ there are zeroes in every entry except for a 1 in the column for t and a -1 in the column for t'. In symbols, the row corresponding to (t, t') is $1_t - 1_{t'}$. B looks as follows

$$\begin{array}{c} \\ \vdots \\ {\scriptstyle t<t'} \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccccccccc} & \cdots & x & \cdots & y & \cdots & & & \cdots & t & \cdots & t' & \cdots & \max T \end{array} \\ \left[\begin{array}{ccccccc|ccccccc} \vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 1 & \cdots & -1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \end{array}\right] \end{array}$$

Then, the system $B \cdot u \gg 0$ captures the requirement that f be strictly decreasing.

In third place, we have a matrix E that captures the requirement that $f(\max T) > 0$. The matrix E has a single row and $|X| + T$ columns. It has zeroes everywhere except for 1 in the last column. Note that since f is strictly decreasing and $f(\max T) > 0$, it follows that $f > 0$. Therefore by equation (3), we have $x \succ y \implies u(x) > u(y)$.

To sum up, there is a solution to system (3) and (2) if and only if there is a vector $u \in \mathbf{R}^{|X|+|T|}$ that solves the following system of equations and linear inequalities

$$(S1): \; A \cdot u = 0, \; (B, E) \cdot u \gg 0.$$

The entries of A, B, and E are either 0, 1 or -1. By Lemma 3, then, there is such a solution u to $S1$ if and only if there is no rational vector (θ, η, λ) that solves the system of equations and linear inequalities

$$(S2):\ \theta \cdot A + \eta \cdot B + \lambda \cdot E = 0,\ (\eta, \lambda) > 0.$$

Suppose that there is no solution u of $S1$. Then, there exists a rational vector (θ, η, λ) that solves the above system of equations and linear inequalities. By multiplying (θ, η, λ) by any positive integer we obtain new vectors that solve $S2$, so we can take (θ, η, λ) to be integer vectors. We transform the matrices A using θ and η. Define a matrix A^* from A by letting A^* have the same number of columns as A and including: (i) θ_r copies of the rth row when $\theta_r > 0$; (ii) omitting row r when $\theta_r = 0$; and (iii) θ_r copies of the rth row multiplied by -1 when $\theta_r < 0$. We refer to rows that are copies of some r with $\theta_r > 0$ as *original* rows, and to those that are copies of some r with $\theta_r < 0$ as *converted* rows.

Similarly, we define the matrix B^* from B by including the same columns as B and η_r copies of each row (and thus omitting row r when $\eta_r = 0$; recall that $\eta_r \geq 0$ for all r).

Henceforth, we use the following notational convention: For a matrix D with $|X|+|T|$ columns, write D_1 for the submatrix of D corresponding to the first $|X|$ columns; let D_2 be T column. Thus, $D = [D_1|D_2]$.

Then, $A_i^* + B_i^* + \lambda E_i = 0$ for each $i = 1, 2$. Moreover, $B_1^* = 0 = E_1$, Hence, we obtain $A_1^* = 0$ and $A_2^* + B_2^* = -\lambda E_2 \leq 0$. The rows of B^* are vectors in $\{-1, 0, 1\}^{|X|+T}$ that we can write as $1_t - 1_{t'}$ with $t < t'$. We can assume without loss of generality that if there is a row associated with (t, t'), then there is no row associated with (t'', t) (i.e. $t'' < t$) or with (t', t'') (i.e., $t' < t''$). The reason that this is without loss is the following. Say that there are two rows associated with (t, t') and (t', t''). Then the sum of the rows will give

$$(1_t - 1_{t'}) + (1_{t'} - 1_{t''}) = 1_t - 1_{t''}.$$

Then the matrix B^* can be replaced with a matrix which omits the rows for (t, t') and (t', t'') and includes instead a row for (t, t'') while preserving the property that $A_2^* + B_2^* \leq 0$.

The matrix A_1^* defines a graph (X, F) in which F is a multiset. The matrix A^* is the incidence matrix of the graph (X, F); note that there is a edge (x, y) in this graph only if $x \succ y$ or $y \succ x$. By the Poincarè-Veblen-Alexandre theorem, since the sum of the rows of A^* is zero, the graph can be decomposed into a collection of cycles $C_1, \ldots, C_K$: each cycle C^k is a sequence $(x_i^k)_{i=1}^{n_k}$ with $x_{n_k}^k = x_1^k$. Each pair (x_l^k, x_{l+1}^k) corresponds to a row $r(x_l^k, x_{l+1}^k)$ of A^*; the function r is one-to-one and onto between the edges in F and the rows of A^*.

The rows of A are identified with a triple $(x, y, t) \in \succ$. For each (x, y) there is at most one row associated to $(x, y, t(x, y))$. So we identify each row r with the corresponding pair (x, y), and write $t(r)$ for $t(x, y)$.

Suppose first that there are no reversed rows in A^*. Then the cycle C^1 implies a violation of Compensation, as it admits the null compensation and the null compensation is not a covering. Suppose then that there is at least one reversed row.

Lemma 1. *For each $k \in K$, for each original row $r(x_l^k, x_{l+1}^k)$ of C^k, there is a reversed row $\pi(r(x_l^k, x_{l+1}^k))$ of A^* with $t(\pi(r(x_l^k, x_{l+1}^k))) \leq t(r(x_l^k, x_{l+1}^k))$. The function π is one-to-one.*

Proof. If the row $r(x_l^k, x_{l+1}^k)$ is reversed, then in the column for $t^* = t(x_l^k, x_{l+1}^k)$ there is a 1 in row $r(x_l^k, x_{l+1}^k)$. Then, $A_2^* + B_2^* \leq 0$ means that in the column for t^* there is some row ρ of A^* or of B^* in which the entry in the column for t^* is -1. There are two cases to consider.

Case 1: Firstly, if row ρ is in A^*, then ρ must be an original row because its entry in column t^* is -1. Let $\pi(r(x_l^k, x_{l+1}^k))$ be equal to ρ. We have

$$t(\pi(r(x_l^k, x_{l+1}^k))) = t^* = t(r(x_l^k, x_{l+1}^k)).$$

Case 2: Secondly, if row ρ is in B^* then ρ corresponds to some pair (t, t^*) with $t < t^*$. The reason is that there is a -1 in row ρ and column t^*, and hence $\rho = 1_t - 1_{t^*}$ for some $t < t^*$. Moreover, by the assumption, there is no row (t', t). Therefore there is a 1 in the column for t and row ρ in B^*, and no -1 in the column for t in B^*. Then, $A_2^* + B_2^* < 0$

implies that there is a row $\hat{\rho}$ in A^* for which the column for t has a -1. The row $\hat{\rho}$ must then be original. Let $\pi(r(x_l^k, x_{l+1}^k)) = \hat{\rho}$. Note that $t = t(\hat{\rho}) < t^*$. We have

$$t(\pi(r(x_l^k, x_{l+1}^k))) < t^* = t(r(x_l^k, x_{l+1}^k)). \tag{4}$$

In this way, we have assigned $r(x_l^k, x_{l+1}^k)$ to $\pi(r(x_l^k, x_{l+1}^k))$. In order to proceed, we define $A^*(1)$ and $B^*(1)$ by deleting the rows used above from A^* and B^*. In particular, in Case 1, we delete $r(x_l^k, x_{l+1}^k)$ and $\pi(r(x_l^k, x_{l+1}^k))$ from A^* to define $A^*(1)$. We define $B^*(1) = B^*$. In Case 2, we delete $r(x_l^k, x_{l+1}^k)$ and $\pi(r(x_l^k, x_{l+1}^k))$ from A^* in order to define $A^*(1)$. From B^*, we delete the row ρ which corresponds to the pair (t, t^*) in order to define $B^*(1)$. Note that in either case, the sum of deleted rows in the last $|T|$ columns is zero. Since $A_2^* + B_2^* \leq 0$, it follows that $A_2^*(1) + B_2^*(1) \leq 0$.

Now with $A^*(1)$ and $B^*(1)$, we proceed to define the function π: we choose a reversed row from $A^*(1)$ and assign an original row exactly in the same way above. After deleting the corresponding rows, we obtains matrices $A^*(2)$ and $B^*(2)$ by the same procedure that produced $A^*(1)$ and $B^*(2)$. We keep this construction until we exhaust all rows in A^*. This ensures that the function π is one-to-one because when a row is selected to be the image of π it is deleted from the corresponding matrix, and unavailable for assignment in the rest of the process. The process must ends after a finite number of steps, say n, because the number of rows in A^* is finite. □

To finish the proof, we need to show the following.

Suppose that the process of the deletion at the end of the proof of Lemma 1 ends after n steps. Now consider matrices $A^*(n)$ and $B^*(n)$. To finish the proof, it suffices to show the following lemma because it will exhibit a compensation that is not a covering.

Lemma 2. *(i) π is not onto; or (ii) there exists an original row $r(x_l^k, x_{l+1}^k)$ such that $t(\pi(r(x_l^k, x_{l+1}^k))) < t(r(x_l^k, x_{l+1}^k))$.*

Proof. There are two cases to consider. The first case is when $\lambda > 0$; the second case is when $\lambda = 0$.

Case 1: Consider the case $\lambda > 0$. Since $A_2^*(n) + B_2^*(n) + \lambda E_2 = 0$ and the last column of E_2 is 1, there must be a row ρ in $A_2^*(n)$ or in $B_2^*(n)$ which has -1 at the last column.

Subcase 1.1: ρ is in $A_2^*(n)$. Then the row is an original row. Since the row is in $A_2^*(n)$ (and not deleted), $\rho \notin$ range π. This completes the proof.

Subcase 1.2: ρ is in $B_2^*(n)$. Then ρ corresponds to some pair $(t, \max T)$ with $t < \max T$. The reason is that there is a -1 in row ρ and column $\max T$, and hence $\rho = 1_t - 1_{\max T}$ for some $t < \max T$. Moreover, by the assumption, there is no row (t', t). Therefore, there is a 1 in the column for t and row ρ in $B^*(n)$, and no -1 in the column for t in $B^*(n)$. Then, $A_2^*(n) + B_2^*(n) < 0$ implies that there is a row $\hat{\rho}$ in $A^*(n)$ for which the column for t has a -1. The row $\hat{\rho}$ must then be original. Since the row is in $A_2^*(n)$ (and not deleted), $\hat{\rho} \notin$ range π. So, π is not onto. This completes the proof for the first case.

Case 2: Consider the case $\lambda = 0$. Since $(\eta, \lambda) > 0$, then $\eta > 0$. So B^* is not zero. Hence, there must be Case 2 in Lemma 1. Therefore, (4) holds. There exists an original row $r(x_l^k, x_{l+1}^k)$ such that $t(\pi(r(x_l^k, x_{l+1}^k))) < t(r(x_l^k, x_{l+1}^k))$. This completes the proof for the second case. □

To finish the proof of Theorem 1, consider the collection $\{(x_i^k)_{i=1}^{n_k}\}_{k=1}^K$ of cycles obtained above. Notice that for each i and k, $(x_i^k, x_{i+1}^k, t(x_i^k, x_{i+1}^k))$ is an original row when $x_{i+1}^k \succ x_i^k$, and $(x_{i+1}^k, x_i^k, t(x_{i+1}^k, x_i^k))$ is a reversed row when $x_i^k \succ x_{i+1}^k$. Notice also that the function π obtained in Lemma 1 is a compensation that is not a covering. Thus we obtain a violation of Compensation.

3.1 Rich Data–Uniqueness

In this section, we obtain an identification result by assuming "rich" datasets. We assume axioms that capture the richness of the dataset, as well as Monotonicity and Additivity, two axioms that we discussed at length in Section 3.

The primitive relation is the same as in the previous section: $\succ \subset X \times X \times \mathbf{R}_+$. The set X may no longer be finite. We require a standard axiom.

Axiom 4. *(Weak Order): $\succ$ on X is asymmetric and negatively transitive.*

Our next two axioms impose Richness of our domain.

Axiom 5. *(Richness): If $t(x,y) < t(x',y')$, then there exist $z, w \in X$ such that $t(x,z) = t(x',y') = t(w,y)$.*

Axiom 6. *(Archimedean): For any $x, y, z, w \in X$ such that $x \succ y$ and $z \succ w$, there exist $x_1, \ldots, x_n \in X$ such that $x_{i+1} \succ x_i$, $t(x_{i+1}, x_i) = t(x,y)$ for all $i \leq n-1$, and $t(z,w) > t(x_n, x_1)$.*

Theorem 2. *$\succ$ satisfies Weak Order, Monotonicity, Additivity, Richness, and Archimedean if and only if $\succ$ admits a response time representation.*

Moreover, if $\succ$ has two different representation (u, f) and (v, g), then there exists a positive number α and a real number β such that $u = \alpha v + \beta$ and $f(d) = g(\alpha d)$ for all $d \in domf$.

Proof of Theorem 2: For all $(x,y), (z,d) \in \succ$, define

$$(x,y) \succsim' (z,w) \text{ if and only if } t(x,y) \leq t(z,w).$$

By definition $\succsim'$ is complete and transitive relation on $\succ$. We will show the following properties of $\succsim'$.

Claim: (i) if $x \succ y, y \succ z$, and $x \succ z$, then $(x,z) \succsim' (x,y)$ and $(x,z) \succsim' (y,z)$;
(ii) if $(x,y) \succsim' (x',y')$ and $(y,z) \succsim' (y',z')$, then $(x,z) \succsim' (x',z')$;
(iii) if $(x,y) \succ' (z,w)$, then there exist $w', w'' \in A$ such that $(x,w') \sim' (z,w) \sim' (w'',y)$;
(iv) for any sequence $x_1, x_2, \cdots \in A$, if $x_{i+1} \succ x_i$, $(x_{i+1}, x_i) \sim' (x_2, x_1)$ for all i, and there exists $(y,z) \in \succ$ such that for all i, $(y,z) \succ' (x_i, x_1)$, then the sequence is finite.

Proof. Note that (i) follows from Monotonicity. (ii) follows from Additivity. (iii) follows from Richness. To see that (iv) follows from Archimedean, suppose (iv) does not hold.

Then, there exists an infinite sequence $\{x_i\}_{i=1}^{\infty} \subset X$ and $(z, w) \in \succ$ such that $x_{i+1} \succ x_i$, $t(x_{i+1}, x_i) = t(x_2, x_1)$ for all i, and $t(z, w) < t(x_i, x_1)$ for all i.

By Archimedean, on the other hand, there exists a sequence $\{y_i\}_{i=1}^{n} \subset X$ such that $y_{i+1} \succ y_i$, $t(y_{i+1}, y_i) = t(x_2, x_1)$ for all i, and $t(z, w) > t(y_n, y_1)$. Since $t(y_{i+1}, y_i) = t(x_2, x_1) = t(x_{i+1}, x_i)$ for all i, by using Additivity repeatedly, we obtain $t(x_n, x_1) = t(y_n, y_1)$. Hence, $t(z, w) < t(x_n, x_1) = t(y_n, y_1) < t(z, w)$, which is a contradiction. □

By Claim, Theorem 3 (Krantz et al. (1971), p147) shows that there exists a function $\phi : X \to \mathbf{R}$ such that $(x, y) \succsim' (z, w)$ if and only if $\phi(x) - \phi(y) \geq \phi(z) - \phi(w)$. Moreover, if there exists another ϕ' satisfies the condition, then there exists a positive number α and a real number β such that $\phi' = \alpha\phi + \beta$. (See Appendix B for the complete statement of the theorem by Krantz et al. (1971).)

For all $x \in X$, define

$$u(x) = \phi(x).$$

For all $x, y \in X$ such that $x \succ y$, define

$$g(t(x, y)) = \phi(x) - \phi(y).$$

Then we have $g(t(x, y)) = u(x) - u(y)$.

To show that g is strictly decreasing, choose any $s, s' \in \text{dom}g$. Then, there exist $x, y, x', y' \in X$ such that $s = t(x, y)$ and $s' = t(x', y')$. Then,

$$\begin{aligned} s \geq s' &\Leftrightarrow t(x, y) \geq t(x', y') \\ &\Leftrightarrow (x', y') \succsim' (x, y) \\ &\Leftrightarrow \phi(x') - \phi(y') \geq \phi(x) - \phi(y) \\ &\Leftrightarrow g(t(x', y')) \geq g(t(x, y)) \\ &\Leftrightarrow g(s') \geq g(s). \end{aligned}$$

So g is strictly decreasing, hence g^{-1} exists. Define $f = g^{-1}$. Then f is strictly decreasing and $t(x, y) = g^{-1}(\phi(x) - \phi(y)) = f(u(x) - u(y))$. ■

Remark: *Compensation implies Monotonicity and Additivity.*

Proof of Remark: To show Monotonicity, suppose by contradiction that $x \succ y, y \succ z, x \succ z$ and $t(x,z) \geq \min\{t(x,y), t(y,z)\}$. Consider the case, $t(x,z) \geq t(x,y)$. Consider a comparable cycle (z,y,x,z). Since $z \prec y \prec x \succ z$, there is only one worsening (x,z). (Here, we are abusing notation $\prec$. For any $x, y \in X$, by $x \prec y$, we mean $y \succ x$. We will abuse the notation in the following.) So we can define a one-to-one function π by $\pi(x,z) = (x,y)$. Since $t(x,z) \geq t(x,y)$, π is a compensation function and not onto, which contradicts Compensation.

Consider the case, $t(x,z) \geq t(y,z)$. Consider a comparable cycle (y,x,z,y). Since $y \prec x \succ z \prec y$, in the comparable cycle, there is only one worsening (x,z). So we can define a one-to-one function π by $\pi(x,z) = (z,y)$. Since $t(x,z) \geq t(y,z)$, π is a compensation function and not onto, which contradicts Compensation. So Monotonicity is satisfied.

To show Additivity, suppose by contradiction that $t(x,y) \leq t(x',y')$, $t(y,z) \leq t(y',z')$, and $t(x,z) > t(x',z')$. We have $x \succ z \prec y \prec x$ and $x' \succ y' \prec z' \prec x'$.

Consider two cases:

Case 1: $x \succ x'$. Consider a comparable cycle:

$$x' \succ y' \succ z' \prec x' \prec x \succ z \prec y \prec x \succ x'.$$

We can define a one to one function π as follows: $\pi(x',y') = (y,x)$, $\pi(y',z') = (z,y)$, $\pi(x,z) = (z',x')$, and $\pi(x,x') = (x',x)$. This function is compensation function. Since $t(x,y) \leq t(x',y')$, $t(y,z) \leq t(y',z')$, and $t(x,z) > t(x',z')$, this contradicts Compensation.

Case 2: $x' \succ x$. Consider a comparable cycle:

$$z \prec y \prec x \prec x' \succ y' \succ z' \prec x' \succ x \succ z.$$

We can define a one to one function π as follows: $\pi(x',y') = (y,x)$, $\pi(y',z') = (z,y)$, $\pi(x,z) = (z',x')$, and $\pi(x',x) = (x,x')$. This function is compensation function. Since

$t(x, y) \leq t(x', y')$, $t(y, z) \leq t(y', z')$, and $t(x, z) > t(x', z')$, this contradicts Compensation. ■

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Appendix A Appendix: Theorem of the Alternative

We make use of the following version of the theorem of the alternative. See Lemma 12 in Chambers and Echenique (2011). It has solutions over the real field in the primal system, and over the rational field in the dual.

Lemma 3. *Let A be an $m \times n$ matrix, B be an $l \times n$ matrix, and E be an $r \times n$ matrix. Suppose that the entries of the matrices A, B, and E are rational numbers. Exactly one of the following alternatives is true.*

1. *There is $u \in \mathbf{R}^n$ such that $A \cdot u = 0$, $B \cdot u \geq 0$, and $E \cdot u \gg 0$.*
2. *There is $\theta \in \mathbf{Q}^r$, $\eta \in \mathbf{Q}^l$, and $\pi \in \mathbf{Q}^m$ such that $\theta \cdot A + \eta \cdot B + \pi \cdot E = 0$; $\pi > 0$ and $\eta \geq 0$.*

Appendix B Appendix: Theorem in Krantz et al. (1971)

Theorem 3. *Let A be a nonempty set, A^* be a nonempty subset of $A \times A$, and $\succsim$ be a binary relation on A^*. Suppose that*

- *$\succsim$ is complete and transitive,*
- *If $ab, bc \in A^*$, then $ac \in A^*$,*
- *If $ab, bc \in A^*$, then $ac \succ ab, bc$,*
- *If $ab \succsim ab'$, $bc \succsim b'c'$, then $ac \succsim a'c'$,*
- *If $ab \succ cd$, there exists $d', d'' \in A$ such that $ad' \sim cd \sim d''b$,*
- *For any sequence $\{a_i\}$, if $a_{i+1}a_i \sim a_2a_1$ for all i and there exist $d'd'' \succ a_ia_1$ for all i, then the sequence is finite.*

Then, there exists $\phi : A^* \to \mathbf{R}$ *such that*

$$ab \succsim cd \Leftrightarrow \phi(a) - \phi(b) \geq \phi(c) - \phi(d)$$

Moreover, if ϕ' *satisfies the above equivalence, then there exists a positive number* α *and real number* β *such that* $\phi = \alpha\phi' + \beta$