

# Nonexistence of Local Self-Similar Blow-up for the 3D Incompressible Navier-Stokes Equations

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## Abstract

We prove the nonexistence of local self-similar solutions of the three dimensional incompressible Navier-Stokes equations. The local self-similar solutions we consider here are different from the global self-similar solutions. The self-similar scaling is only valid in an inner core region which shrinks to a point dynamically as the time,  $t$ , approaches the singularity time,  $T$ . The solution outside the inner core region is assumed to be regular. Under the assumption that the local self-similar velocity profile converges to a limiting profile as  $t \rightarrow T$  in  $L^p$  for some  $p \in (3, \infty)$ , we prove that such local self-similar blow-up is not possible for any finite time.

## 1 Introduction.

In this paper, we study the 3D incompressible Navier-Stokes equations with unit viscosity and zero external force:

$$\begin{cases} u_t + (u \cdot \nabla)u = -\nabla p + \Delta u, \\ \nabla \cdot u = 0, \\ u|_{t=0} = u_0(x). \end{cases} \quad (1.1)$$

We assume that the initial condition  $u_0$  is divergence free and  $u_0 \in L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$  for some  $p \in (3, \infty)$ .

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Many physicists and mathematicians have made a great deal of effort in understanding the physical as well as the mathematical properties of the 3D incompressible Navier-Stokes equations. One of the long standing open questions is whether the solution of the 3D Navier-Stokes equations can develop a finite time singularity from a smooth initial condition [3]. Global existence and regularity of the Navier-Stokes equations have been known in two space dimensions for a long time [7]. One of the main difficulties in obtaining the global regularity of the 3D Navier-Stokes equations is mainly due to the presence of the vortex stretching, which is absent for the 2D problem. Under suitable smallness assumption on the initial condition, the local-in-time existence and regularity results have been obtained for some time [7, 14, 9]. But these results do not give any hint on the question of global existence and regularity for the 3D Navier-Stokes.

In this paper, we prove the nonexistence of local self-similar singular solutions of the 3D Navier-Stokes equations. The local self-similar solutions we consider are very different from the global self-similar solutions considered by Leray [8]. The self-similar scaling is only valid in an inner core region which shrinks to a point dynamically as the time,  $t$ , approaches the singularity time,  $T$ . The solution outside the inner core region is assumed to be regular and does not satisfy self-similar scaling. This type of local self-similar solution is developed dynamically, and has been observed in some numerical studies. Under the assumption that the local self-similar velocity profile converges to a limiting profile as  $t \rightarrow T$  in  $L^p$  for some  $p \in (3, \infty)$ , we prove that such local self-similar blow-up is not possible. We remark that the nonexistence of global self-similar solutions has been proved by Necas, Ruzicka and Sverak in [10]. The result of [10] was further improved by Tsai in [15].

We prove our main result by using a ‘‘Dynamic Singularity Rescaling’’ technique. This technique is simple but effective. Below we give a brief description of this technique. Assume that the solution of the 3D Navier-Stokes develops a local self-similar singularity at  $x = 0$  at time  $T$  for the first time. A typical local self-similar singular solution is of the form

$$u(x, t) = \frac{1}{\sqrt{T-t}}U(y, t), \quad p(x, t) = \frac{1}{T-t}P(y, t), \quad y = \frac{x}{\sqrt{T-t}}, \quad (1.2)$$

for  $0 \leq t < T$ . We assume that  $u$  is smooth outside an inner core region  $\{x, |x| \leq (T-t)^\alpha\}$  for some  $\alpha > 0$  small. In particular,  $u(x, t)$  and  $p(x, t)$  are bounded for any fixed  $|x| > 0$  as  $t \rightarrow T$ . Using this condition, we can easily show that

$$|U(y, t)| \leq C(T)/|y|, \quad |P(y, t)| \leq C(T)/|y|^2, \quad \text{for } |y| \gg 1, \quad t \leq T. \quad (1.3)$$

Thus, it is reasonable to assume that  $U \in L^p$  for some  $p \in (3, \infty)$ . But the  $L^p$  norm of  $U$  may be unbounded for  $0 < p \leq 3$ .

We assume that there exists a limiting profile  $\bar{U}(y) \in L^p$  as  $t \rightarrow T$

$$\lim_{t \rightarrow T} \|U(t) - \bar{U}\|_{L^p} = 0, \quad (1.4)$$

for some  $p$  satisfying  $3 < p < \infty$ .

Next, we introduce the following rescaling in time:

$$\tau = \frac{1}{2} \log \frac{T}{T-t}, \quad (1.5)$$

for  $0 \leq t < T$ . Note that by this time rescaling, we have transformed the original Navier-Stokes equations from  $[0, T)$  in  $t$  to  $[0, \infty)$  in the new time variable  $\tau$ . Since  $u$  is smooth for  $0 < t < T$ ,  $U$  is smooth for  $0 < \tau < \infty$ . It is easy to derive the equivalent evolution equations for the rescaled velocity:

$$U_\tau + U + (y \cdot \nabla)U + 2(U \cdot \nabla)U = -2\nabla P + 2\Delta U, \quad (1.6)$$

with initial condition  $U|_{\tau=0} = \sqrt{T}u_0(y/\sqrt{T})$ , where  $U$  satisfies  $\nabla \cdot U = 0$  for all times. The problem on the possible local self-similar blowup of the Navier-Stokes equations is now converted to the problem on the large time behavior of the rescaled equations (1.6). By assumption (1.4), we know that  $U \rightarrow \bar{U}$  as  $\tau \rightarrow \infty$  in  $L^p$ . We will prove that the limiting velocity profile actually satisfies the steady state equation of (1.6):

$$\bar{U} + (y \cdot \nabla)\bar{U} + 2(\bar{U} \cdot \nabla)\bar{U} = -2\nabla \bar{P} + 2\Delta \bar{U}, \quad (1.7)$$

for some  $\bar{P}$ . Now it follows from the result of [15] that  $\bar{U} \equiv 0$ , which implies that  $\lim_{\tau \rightarrow \infty} \|U(\tau)\|_{L^p} = 0$  for some  $p \in (3, \infty)$ .

The fact that  $\lim_{\tau \rightarrow \infty} \|U(\tau)\|_{L^p} = 0$  is significant because it shows that the rescaled velocity field becomes small dynamically as  $\tau \rightarrow \infty$ . It is easy to show that if the the solution  $U$  is small in the  $L^p$  norm at some time,  $\tau_m$ , the solution must decay exponentially in  $\tau$  for  $\tau \geq \tau_m$ . The exponential decay in  $U$  gives a sharp dynamic growth estimate in terms of the original velocity field. In fact, it exactly cancels the dynamic singular rescaling factor,  $(\sqrt{T-t})^{-1}$ , in the front of  $U$ . This gives us a uniform bound on the growth of  $L^p$  for  $0 < t < T$  with  $p \in (3, \infty)$ , and consequently it rules out the possibility of a finite time blowup at  $T$  [11, 12, 6].

The nonexistence of local self-similar blowup of the 3D Navier-Stokes equations has some interesting implication. First, the assumption on the existence of a limiting self-similar profile,  $\bar{U}$ , can be verified numerically if a local self-similar blow is observed

in a computation. Secondly, this result is related to a recent existence result by one of the authors [4] for the axisymmetric 3D Navier-Stokes equations with swirl. Let  $v^r$  denote the radial component of the velocity field and  $r = \sqrt{x^2 + y^2}$ . The result in [4] shows that if  $\lim_{r \rightarrow 0} |rv^r| = 0$  holds uniformly for  $0 \leq t \leq T$ , then the solution of the Navier-Stokes equations is regular for  $t \leq T$ . By the well-known Caffarelli-Kohn-Nirenberg result [1], if the axisymmetric 3D Navier-Stokes equations develop a finite time singularity, the singularity must lie in the  $z$  axis. One of the most likely scenarios that would violate the condition,  $\lim_{r \rightarrow 0} |rv^r| = 0$ , is the local self-similar blowup of the Navier-Stokes equations. The result presented in this paper would rule out such a possibility. For more discussions regarding other aspects of the Navier-Stokes equations, we refer the reader to [7, 2, 14, 9].

The rest of the paper is organized as follows. In Section 2, we state our main theorem and present its proof. The proof is divided into four subsections. A couple of technical results are deferred to the appendices.

## 2 The main result and regularity analysis.

**Theorem 1.** *Let  $u_0 \in L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$  for some  $p \in (3, \infty)$  and  $T$  be the first local self-similar singularity time. Assume that  $U(y, t)$  defined by (1.2) converges to  $\bar{U}$  in  $L^p$  as  $t \rightarrow T$ . Then we must have  $T = +\infty$ , i.e. there is no finite time local self-similar blowup for the 3D Navier-Stokes equations.*

Before we prove our main theorem, we state the following well-known  $(L^{\tilde{q}}, L^{\tilde{p}})$ -estimates for the heat kernel in  $\mathbb{R}^3$ ,  $e^{-t\Delta}$ , where  $\Delta$  is the Laplacian operator.

$$\|e^{-t\Delta} w\|_{L^{\tilde{q}}} \leq c_0 t^{-\left(\frac{3}{\tilde{p}} - \frac{3}{\tilde{q}}\right)/2} \|w\|_{L^{\tilde{p}}}, \quad (2.1)$$

$$\|\nabla e^{-t\Delta} w\|_{L^{\tilde{q}}} \leq c_0 t^{-\left(1 + \frac{3}{\tilde{p}} - \frac{3}{\tilde{q}}\right)/2} \|w\|_{L^{\tilde{p}}}, \quad (2.2)$$

for  $1 < \tilde{p} \leq \tilde{q} < \infty$ ,  $c_0$  depends on  $\tilde{p}$  and  $\tilde{q}$  only. In our analysis, we take  $\tilde{q} = p$  and  $\tilde{p} = p/2$ . For this particular choice of  $\tilde{p}$  and  $\tilde{q}$ , we can choose a constant,  $c_0$ , such that the above two inequalities hold. Throughout the paper, we will use  $c_0$  and  $c_1$  to denote various constants that do not depend on the individual functions, and use  $C_j$  ( $j = 1, 2$ ) to denote various constants that depend on the initial condition,  $u_0$ . We also define

$$\gamma = 3/p. \quad (2.3)$$

Since  $3 < p < \infty$ , we have  $0 < \gamma < 1$ .

**Proof of Theorem 1.**

We will prove the theorem by contradiction. Suppose that  $T < +\infty$ . This means that the solution  $u$  to problem (1.1) develops a singularity at  $t = T$  for the first time, but  $u$  is the unique smooth solution of (1.1) for  $0 < t < T$  and is bounded in  $L^p$ .

We will divide the proof into four steps, which are given in the following four subsections.

## 2.1 Dynamic singularity rescaling and a priori estimates.

We make the following dynamic singularity rescaling of the 3D Navier-Stoke equations:

$$\begin{cases} \tau = \frac{1}{2} \log \frac{T}{T-t}, & y = \frac{x}{\sqrt{T-t}} \\ u(x, t) = \frac{1}{\sqrt{T-t}} U(y, \tau), \\ p(x, t) = \frac{1}{T-t} P(y, \tau), & \text{for } 0 \leq t < T. \end{cases} \quad (2.4)$$

Note that with this dynamic singularity rescaling, we transform the time interval from  $[0, T)$  in the original time variable  $t$  to  $[0, \infty)$  in the rescaled time variable  $\tau$ . It is easy to derive an evolution equation for the rescale velocity field:

$$\begin{cases} U_\tau + U + (y \cdot \nabla)U + 2(U \cdot \nabla)U = -2\nabla P + 2\Delta U, \\ \nabla \cdot U = 0, \\ U|_{\tau=0} = \sqrt{T}u_0(x). \end{cases} \quad (2.5)$$

Note that since  $u(x, t)$  is the unique smooth solution of the Navier-Stokes equations (1.1) for  $0 < t < T$ ,  $U(x, \tau)$  is the unique smooth solution of the rescaled Navier-Stokes equations (2.5) for  $0 < \tau < \infty$ .

Let  $\phi(y) = (\phi_1, \phi_2, \phi_3)$  be a smooth, compactly supported, divergence free vector field in  $\mathbb{R}^3$  and  $\psi(\tau)$  be a smooth, compactly supported test function in  $(0, 1)$  satisfying  $\int_0^1 \psi(\tau) d\tau = 1$ . Multiplying (2.5) by  $\psi(\tau - n)\phi(y)$  and integrating over  $\mathbb{R}^3 \times [n, n+1]$  for some  $n > 0$ , we obtain after integration by parts

$$\begin{aligned} & \int_n^{n+1} \int_{\mathbb{R}^3} (-\psi_\tau \phi \cdot U + \psi \phi \cdot U - \psi \nabla \cdot (\phi \otimes y) \cdot U - 2\psi \nabla \phi \cdot (U \otimes U)) dy d\tau \\ &= 2 \int_n^{n+1} \int_{\mathbb{R}^3} \psi \Delta \phi \cdot U dy d\tau, \end{aligned} \quad (2.6)$$

where  $\psi$  is evaluated at  $\tau - n$ .

By assumption (1.4), we have

$$\lim_{\tau \rightarrow \infty} \|U(\tau) - \bar{U}\|_{L^p} = 0, \quad (2.7)$$

for some  $p > 3$ . Thus  $\|U(\tau)\|_{L^p}$  is bounded for  $\tau$  sufficiently large, and  $\|\bar{U}\|_{L^p}$  is also bounded. Let  $U(\tau) = \bar{U} + R(\tau)$ . By (2.7), we have  $\lim_{\tau \rightarrow \infty} \|R(\tau)\|_{L^p} = 0$ . Substituting  $U(\tau) = \bar{U} + R(\tau)$  into (2.6) and letting  $n \rightarrow \infty$ , we will show that all the terms involving  $R$  will go to zero as  $n \rightarrow \infty$ . It is sufficient to prove this for the nonlinear term:

$$\int_n^{n+1} \int_{\mathbb{R}^3} \psi \nabla \phi \cdot (R \otimes R) dy d\tau.$$

Let  $q = p/(p-2) > 1$ . Then we have  $2/p + 1/q = 1$ . Using the Hölder inequality, we obtain

$$\begin{aligned} \left| \int_n^{n+1} \int_{\mathbb{R}^3} \psi \nabla \phi \cdot (R \otimes R) dy d\tau \right| &\leq C \sup_{n \leq \tau \leq n+1} \int_{\mathbb{R}^3} |\nabla \phi| |R|^2 dy \\ &\leq C \|\nabla \phi\|_{L^q} \sup_{n \leq \tau \leq n+1} \|R(\tau)\|_{L^p}^2 \rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Other terms can be proved similarly. Therefore, by letting  $n \rightarrow \infty$ , we get

$$\begin{aligned} & - \left( \int_0^1 \psi_\tau(\tau) d\tau \right) \int_{\mathbb{R}^3} \phi(y) \bar{U}(y) dy \\ & + \left( \int_0^1 \psi(\tau) d\tau \right) \left( \int_{\mathbb{R}^3} (\phi \cdot \bar{U} - \nabla \cdot (\phi \otimes y) \cdot \bar{U} - 2\nabla \phi \cdot (\bar{U} \otimes \bar{U})) dy \right) \\ & = 2 \left( \int_0^1 \psi(\tau) d\tau \right) \left( \int_{\mathbb{R}^3} \Delta \phi \cdot \bar{U} dy \right). \end{aligned} \quad (2.8)$$

Since  $\psi$  has compact support in  $[0, 1]$ , we conclude that

$$\int_0^1 \psi_\tau(\tau) d\tau = 0.$$

Moreover, we have  $\int_0^1 \psi(\tau) d\tau = 1$  by assumption on  $\psi$ . Thus, we obtain

$$\int_{\mathbb{R}^3} (\phi \cdot \bar{U} - \nabla \cdot (\phi \otimes y) \cdot \bar{U} - 2\nabla \phi \cdot (\bar{U} \otimes \bar{U}) - 2\Delta \phi \cdot \bar{U}) dy = 0. \quad (2.9)$$

Thus,  $\bar{U}$  is a weak solution of the steady state rescaled Navier-Stokes equations:

$$\bar{U} + (y \cdot \nabla) \bar{U} + 2(\bar{U} \cdot \nabla) \bar{U} = -2\nabla \bar{P} + 2\Delta \bar{U}, \quad (2.10)$$

with  $\nabla \cdot \bar{U} = 0$ . Let  $R_j$  be a Riesz operator with Fourier symbol  $\xi_j/|\xi|$ . One can easily modify the proof of Lemma 3.1 of [10] to show that  $\bar{P} = R_j R_k (\bar{U}_j \bar{U}_k)$ .

Since  $\bar{U} \in L^p$  for some  $p \in (3, \infty)$ , we can apply Theorem 1 of [15] to conclude that  $\bar{U} \equiv 0$ . As a result, we obtain the following *a priori* decay estimate for  $\|U(\tau)\|_{L^p}$ .

**Lemma 1.** *The solution  $U(x, \tau)$  of the rescaled Navier-Stokes equations (2.5) satisfies the following decay estimate:*

$$\lim_{\tau \rightarrow \infty} \|U(\tau)\|_{L^p} = 0. \quad (2.11)$$

For the purpose of our later analysis, we will choose a  $\tau_m$  large enough to satisfy the following inequality:

$$2c_0^2 c_1 \|U(\tau_m)\|_{L^p} \leq \frac{1}{6}, \quad (2.12)$$

where the constant  $c_1$  is defined by

$$c_1 = \left( \frac{2}{1-\gamma} + \frac{1}{2} \right) (1 - e^{-2})^{-\frac{1+\gamma}{2}}. \quad (2.13)$$

The reason for such a choice of  $\tau_m$  will become clear later in our analysis.

## 2.2 Dynamic decay estimates for the rescaled equations.

In this subsection, we will perform estimates for the rescaled Navier-Stokes equations starting from  $\tau = \tau_m$  with the initial value,  $U(x, \tau_m)$ :

$$\begin{cases} V_\tau + V + (y \cdot \nabla)V + 2(V \cdot \nabla)V = -2\nabla P + 2\Delta V \\ \nabla \cdot V = 0 \\ V|_{\tau=0}(x) \equiv V_0(x) = U(x, \tau_m), \end{cases} \quad (2.14)$$

where  $\tau_m$  is defined by (2.12)-(2.13). Since  $U(x, \tau)$  is the unique smooth solution of the rescaled Navier-Stokes equations (2.5) for  $0 < \tau < \infty$ , the function  $V(x, \tau)$  defined by

$$V(x, \tau) = U(x, \tau + \tau_m), \quad \text{for } \tau \geq 0, \quad (2.15)$$

is the unique smooth solution of (2.14).

Next, we perform estimates for the linearized operator in (2.14)

$$\frac{\partial V}{\partial \tau} + V + (y \cdot \nabla_y) V - 2\Delta_y V = 0, \quad (2.16)$$

with initial value  $V|_{\tau=0} = V_0$ .

Let  $y = e^\tau \tilde{y}$  and  $\tilde{V}(\tilde{y}, \tau) \equiv V(y, \tau)$ . Then we have

$$\frac{\partial \tilde{V}}{\partial \tau} + \tilde{V} - 2e^{-2\tau} \Delta_{\tilde{y}} \tilde{V} = 0, \quad (2.17)$$

with initial value  $\tilde{V}|_{\tau=0} = V_0$ .

Taking the Fourier transform of (2.17), we get

$$\frac{\partial \widehat{\tilde{V}}}{\partial \tau} + \widehat{\tilde{V}} + 2e^{-2\tau} |\xi|^2 \widehat{\tilde{V}} = 0, \quad (2.18)$$

where the Fourier transformation is defined as  $\widehat{f}(\xi) \equiv \int f(x) e^{-2\pi i x \cdot \xi} dx$ . Equation (2.18) can be written as

$$\frac{\partial}{\partial \tau} \left( e^{\tau+2|\xi|^2 \int_0^\tau e^{-2s} ds} \widehat{\tilde{V}}(\tau) \right) = 0. \quad (2.19)$$

Integrating from 0 to  $\tau$ , we get

$$\widehat{\tilde{V}}(\tau) = e^{-\tau-|\xi|^2(1-e^{-2\tau})} \widehat{V}_0. \quad (2.20)$$

Using the explicit formula of the Fourier transform of a Gaussian in three space dimensions (see, e.g. [13])

$$\widehat{e^{-\pi\alpha^2|x|^2}} = \frac{1}{\alpha^3} e^{-\pi|\xi|^2/\alpha^2}, \quad (2.21)$$

with  $\alpha^2 = \left( \frac{\pi}{1-e^{-2\tau}} \right)$ , we obtain

$$\mathcal{F}^{-1} \left( e^{-|\xi|^2(1-e^{-2\tau})} \right) = \left( \frac{\pi}{1-e^{-2\tau}} \right)^{\frac{3}{2}} e^{-\pi^2|x|^2/(1-e^{-2\tau})}, \quad (2.22)$$

where  $\mathcal{F}^{-1} f(x) \equiv \int f(\xi) e^{2\pi i x \cdot \xi} d\xi$  is the inverse Fourier transformation. Therefore, we have

$$\tilde{V}(\tilde{y}, \tau) = e^{-\tau} \left( \frac{\pi}{1-e^{-2\tau}} \right)^{\frac{3}{2}} \int V_0(\tilde{x}) \left( e^{-\pi^2|\tilde{y}-\tilde{x}|^2/(1-e^{-2\tau})} \right) d\tilde{x}. \quad (2.23)$$

Denote by  $e^{-\tau A}$  the solution operator of the linearized equations (2.17). Define

$$t_0(\tau) = (1 - e^{-2\tau}), \quad (2.24)$$



and denote  $\Delta$  as the Laplacian operator, then we have

$$e^{-\tau A}V_0 = \tilde{V}(\tilde{y}, \tau) = e^{-\tau} \left( e^{-t_0(\tau)\Delta}V_0 \right). \quad (2.25)$$

Define the following bilinear operator:

$$F(U, V) = 2 \left( 1 - \nabla (-\Delta)^{-1} \nabla \cdot \right) \nabla \cdot (U \otimes V). \quad (2.26)$$

In particular, if we set  $V = U$ , we have

$$\begin{aligned} F(U, U) &= 2 \left( \nabla \cdot (U \otimes U) - \nabla (-\Delta)^{-1} \nabla \cdot \nabla \cdot (U \otimes U) \right) \\ &= 2 (U \cdot \nabla U + \nabla P). \end{aligned} \quad (2.27)$$

The rescaled 3D Navier-Stokes equations (2.14) can be converted into the following integral equation:

$$V(\tau) = e^{-\tau A}V_0 - \int_0^\tau e^{-(\tau-s)A}F(U, U)(s)ds. \quad (2.28)$$

To solve the integral equation (2.28), we construct a successive approximation,  $V^{(n)}$ , using the following iterative scheme (see [5]):  $V^{(0)} = e^{-\tau A}V_0$ ,

$$V^{(n+1)} = V^{(0)} - G(V^{(n)}, V^{(n)}), \quad n \geq 0, \quad (2.29)$$

where the bilinear operator  $G(U, V)$  is defined as follows:

$$G(U, V) = \int_0^\tau e^{-(\tau-s)A}F(U, V)(s)ds. \quad (2.30)$$

To establish the convergence of the approximate solution sequence,  $V^{(n)}$ , we need to use the following lemma, which follows from (2.25) and the well-known  $(L^q, L^p)$ -estimates (2.1)-(2.2) for the heat kernel.

**Lemma 2.** *Let  $V \in L^{\tilde{p}}$  for  $1 < \tilde{p} \leq \tilde{q} < \infty$ . We have*

$$\|e^{-\tau A}V\|_{L^{\tilde{q}}} \leq c_0 e^{-(1-3/\tilde{q})\tau} t_0(\tau)^{-\left(\frac{3}{\tilde{p}} - \frac{3}{\tilde{q}}\right)/2} \|V\|_{L^{\tilde{p}}}, \quad (2.31)$$

$$\|\nabla e^{-\tau A}V\|_{L^{\tilde{q}}} \leq c_0 e^{-(2-3/\tilde{q})\tau} t_0(\tau)^{-\left(1 + \frac{3}{\tilde{p}} - \frac{3}{\tilde{q}}\right)/2} \|V\|_{L^{\tilde{p}}}. \quad (2.32)$$

The lemma can be proved easily by noting that the heat kernel actually acts on the variable  $\tilde{y}$  through the function  $\tilde{V}(\tilde{y}, \tau)$  and  $\tilde{y} = e^{-\tau}y$ . Thus we lose a factor  $e^{3\tau/\tilde{q}}$  when we estimate the  $L^{\tilde{q}}$  norm by changing variables from  $y$  to  $\tilde{y}$ , but we gain a factor of  $e^{-\tau}$  when we differentiate with respect to  $y$ .

Applying (2.31) with  $\tilde{p} = \tilde{q} = p$ , we obtain

$$\|V^{(0)}\|_{L^p}(\tau) = \|e^{-\tau A}V_0\|_{L^p}(\tau) \leq c_0 e^{-(1-\gamma)\tau} \|V_0\|_{L^p}, \quad (2.33)$$

where  $\gamma = 3/p$ . To estimate  $\|G(U, V)\|_{L^p}$ , we use (2.32) with  $\tilde{q} = p$  and  $\tilde{p} = p/2$ :

$$\|G(U, V)\|_{L^p}(\tau) \leq 2c_0 \int_0^\tau e^{-(2-\gamma)(\tau-s)} t_0(\tau-s)^{-\frac{1+\gamma}{2}} \|U\|_{L^p}(s) \|V\|_{L^p}(s) ds, \quad (2.34)$$

where we have used the Hölder inequality  $\|U \otimes V\|_{L^{p/2}} \leq \|U\|_{L^p} \|V\|_{L^p}$  and the fact that  $(-\Delta)^{-1} \nabla \cdot \nabla \cdot$  is a Riesz operator of degree zero, which is a bounded operator from  $L^p$  to  $L^p$ . In particular, we obtain by setting  $V = U$  that

$$\|G(U, U)\|_{L^p}(\tau) \leq 2c_0 \int_0^\tau e^{-(2-\gamma)(\tau-s)} t_0(\tau-s)^{-\frac{1+\gamma}{2}} \|U\|_{L^p}^2(s) ds. \quad (2.35)$$

Now, applying (2.33) and (2.35) to the iterative scheme (2.29), we get

$$\|V^{(n+1)}\|_{L^p}(\tau) \leq c_0 e^{-(1-\gamma)\tau} \|V_0\|_{L^p} + 2c_0 \int_0^\tau e^{-(2-\gamma)(\tau-s)} t_0(\tau-s)^{-\frac{1+\gamma}{2}} \|V^{(n)}\|_{L^p}^2(s) ds. \quad (2.36)$$

Define

$$K_n = \sup_{0 \leq \tau < \infty} \|e^{(1-\gamma)\tau} V^{(n)}(\tau)\|_{L^p}. \quad (2.37)$$

Multiplying (2.36) by  $e^{(1-\gamma)\tau}$  on both sides and using (2.37), we obtain

$$e^{(1-\gamma)\tau} \|V^{(n+1)}\|_{L^p}(\tau) \leq c_0 \|V_0\|_{L^p} + 2c_0 e^{-\tau} K_n^2 \int_0^\tau e^{\gamma s} (1 - e^{-2(\tau-s)})^{-\frac{1+\gamma}{2}} ds. \quad (2.38)$$

In Appendix II, we will prove that

$$e^{-\tau} \int_0^\tau e^{\gamma s} (1 - e^{-2(\tau-s)})^{-\frac{1+\gamma}{2}} ds \leq c_1, \quad \text{for all } \tau \geq 0, \quad (2.39)$$

where  $c_1$  is defined in (2.13). Now, take the supremum of the both sides of (2.38) for all  $\tau \geq 0$ , we obtain the following recurrence inequalities:

$$K_{n+1} \leq K_0 + MK_n^2, \quad \text{for } n \geq 0, \quad (2.40)$$

with  $K_n|_{n=0} = K_0$ , where

$$M = 2c_0 c_1, \quad K_0 = c_0 \|V_0\|_{L^p}. \quad (2.41)$$

We will prove the following lemma in Appendix I.

**Lemma 3.** *Let  $K_0$  and  $M$  be two positive constants satisfying*

$$K_0 M \leq \frac{1}{6}, \quad (2.42)$$

*then there exists a positive constant  $K_{\max}$ , such that*

$$K_n \leq K_{\max}, \quad \text{for all } n \geq 1 \quad (2.43)$$

*holds for the recurrence sequence  $K_n$  satisfying (2.40). Moreover the upper bound  $K_{\max}$  satisfies*

$$2MK_{\max} \leq \frac{1}{2}. \quad (2.44)$$

Recall that in (2.12), we have chosen  $\tau_m$  such that

$$2c_0^2 c_1 \|U(\tau_m)\|_{L^p} \leq 1/6. \quad (2.45)$$

Therefore, we have

$$K_0 M \leq 2c_0^2 c_1 \|U(\tau_m)\|_{L^p} \leq \frac{1}{6}. \quad (2.46)$$

Thus, for our choice of  $\tau_m$  defined in (2.12), the recurrence sequence  $K_n$  has an upper bound  $K_{\max}$  for all  $n$ . That is

$$\|V^{(n)}\|_{L^p}(\tau) \leq K_{\max} e^{-(1-\gamma)\tau}, \quad \text{for } n \geq 1. \quad (2.47)$$

### 2.3 Convergence of the approximate solution sequence.

In this subsection, we will establish the convergence of the approximate solution sequence,  $\{V^{(n)}\}$ , and study the property of its limiting solution. We will first show that the approximate solution sequence  $\{V^{(n)}\}$  is a Cauchy sequence in  $L^p$ . By subtracting (2.29) with index  $n$  from that with index  $n - 1$ , we obtain

$$\begin{aligned} & \|V^{(n+1)} - V^{(n)}\|_{L^p} \\ &= \|G(V^{(n)}, V^{(n)}) - G(V^{(n-1)}, V^{(n-1)})\|_{L^p} \\ &= \|G(V^{(n)}, V^{(n)} - V^{(n-1)}) + G(V^{(n)} - V^{(n-1)}, V^{(n-1)})\|_{L^p}. \end{aligned}$$

Using (2.34), (2.47) and (2.39), we obtain

$$\begin{aligned}
& e^{(1-\gamma)\tau} \|V^{(n+1)} - V^{(n)}\|_{L^p(\tau)} \\
& \leq 2c_0 e^{(1-\gamma)\tau} \int_0^\tau e^{-(2-\gamma)(\tau-s)} t_0 (\tau-s)^{-\frac{1+\gamma}{2}} \left( \|V^{(n)}\|_{L^p} + \|V^{(n-1)}\|_{L^p} \right) \|V^{(n)} - V^{(n-1)}\|_{L^p(s)} ds \\
& \leq 4c_0 K_{\max} e^{-\tau} \int_0^\tau e^{\gamma s} t_0 (\tau-s)^{-\frac{1+\gamma}{2}} ds \left( \sup_{0 \leq s < \infty} e^{(1-\gamma)s} \|V^{(n)} - V^{(n-1)}\|_{L^p(s)} \right) \\
& \leq 4c_0 c_1 K_{\max} \sup_{0 \leq s < \infty} e^{(1-\gamma)s} \|V^{(n)} - V^{(n-1)}\|_{L^p(s)} \\
& \leq \frac{1}{2} \sup_{0 \leq s < \infty} e^{(1-\gamma)s} \|V^{(n)} - V^{(n-1)}\|_{L^p(s)},
\end{aligned}$$

where we have used (2.41) and (2.44) in deriving the last inequality. Taking the supremum on the left hand side would yield

$$\sup_{0 \leq \tau < \infty} e^{(1-\gamma)\tau} \|V^{(n+1)} - V^{(n)}\|_{L^p(\tau)} \leq \frac{1}{2} \sup_{0 \leq \tau < \infty} e^{(1-\gamma)\tau} \|V^{(n)} - V^{(n-1)}\|_{L^p(\tau)}, \quad (2.48)$$

which implies

$$\sup_{0 \leq \tau < \infty} e^{(1-\gamma)\tau} \|V^{(n+m)} - V^{(n)}\|_{L^p(\tau)} \leq C_1 \left( \frac{1}{2} \right)^n, \quad \text{for any } n, m \geq 1, \quad (2.49)$$

where  $C_1$  depends on  $V^{(0)}$  only. Thus  $\{V^{(n)}\}$  is a Cauchy sequence in  $\text{BC}([0, \infty); L^p(\mathbb{R}^3))$ . Here  $\text{BC}([0, \infty); L^p(\mathbb{R}^3))$  denotes the class of bounded and continuous function from  $[0, \infty)$  to  $L^p(\mathbb{R}^3)$ . As a result, we have proved that  $V^{(n)}(\tau)$  converges uniformly to a limiting function  $\bar{V}(\tau)$  in  $\text{BC}([0, \infty); L^p(\mathbb{R}^3))$ . Taking the limit  $n \rightarrow \infty$  in (2.47), we obtain

$$\|\bar{V}\|_{L^p(\tau)} \leq K_{\max} e^{-(1-\gamma)\tau}. \quad (2.50)$$

Next, we will show that  $\bar{V}$  is a solution of the integral equation (2.28). To this end, we define  $R^{(n)}(x, \tau) \equiv V^{(n)}(x, \tau) - \bar{V}(x, \tau)$ . We have just shown that

$$\sup_{0 \leq \tau < \infty} e^{(1-\gamma)\tau} \|R^{(n)}\|_{L^p(\tau)} = \sup_{0 \leq \tau < \infty} e^{(1-\gamma)\tau} \|V^{(n)} - \bar{V}\|_{L^p(\tau)} \rightarrow 0, \quad (2.51)$$

as  $n \rightarrow \infty$ . Now substituting  $V^{(n)} = \bar{V} + R^{(n)}$  into the iterative scheme (2.29) and using the bilinearity of operator  $G(U, V)$ , we get

$$\bar{V} - V^{(0)} + G(\bar{V}, \bar{V}) = -(R^{(n+1)} + G(R^{(n)}, \bar{V}) + G(\bar{V}, R^{(n)}) + G(R^{(n)}, R^{(n)})). \quad (2.52)$$

We will prove that the error terms on the right hand side of (2.52) tend to zero uniformly for all  $\tau \geq 0$ . It is obvious that  $\|R^{(n+1)}\|_{L^p} \rightarrow 0$  uniformly as  $n \rightarrow \infty$  from (2.51).

To show that the error terms which are linear in  $R^{(n)}$  tend to zero uniformly, we use (2.34) and the *a priori* bound on  $\bar{V}$  given by (2.50). Specifically, we have

$$\begin{aligned}
& \|G(R^{(n)}, \bar{V}) + G(\bar{V}, R^{(n)})\|_{L^p(\tau)} \\
& \leq 4c_0 \int_0^\tau e^{-(2-\gamma)(\tau-s)} t_0 (\tau-s)^{-\frac{1+\gamma}{2}} \|R^{(n)}\|_{L^p(s)} \|\bar{V}\|_{L^p(s)} ds \\
& \leq 4c_0 K_{\max} e^{-(1-\gamma)\tau} e^{-\tau} \int_0^\tau e^{\gamma s} t_0 (\tau-s)^{-\frac{1+\gamma}{2}} ds \left( \sup_{0 \leq s < \infty} e^{(1-\gamma)s} \|R^{(n)}\|_{L^p(s)} \right) \\
& \leq 4c_0 c_1 e^{-(1-\gamma)\tau} K_{\max} \sup_{0 \leq s < \infty} e^{(1-\gamma)s} \|R^{(n)}\|_{L^p(s)} \\
& \leq \sup_{0 \leq s < \infty} e^{(1-\gamma)s} \|R^{(n)}\|_{L^p(s)} \rightarrow 0,
\end{aligned}$$

uniformly for all  $\tau$  as  $n \rightarrow \infty$ , where we have used  $M = 2c_0 c_1$  and (2.44).

To show that the nonlinear error term  $G(R^{(n)}, R^{(n)})$  also tends to zero uniformly, we note that the *a priori* bounds on  $V^{(n)}$  and  $\bar{V}$  also provide the following *a priori* bound for  $R^{(n)}$ :

$$\|R^{(n)}\|_{L^p(\tau)} \leq 2K_{\max} e^{-(1-\gamma)\tau}, \quad \text{for } n \geq 1. \quad (2.53)$$

Using (2.53) and applying the same argument as above, we can prove that

$$\|G(R^{(n)}, R^{(n)})\|_{L^p(\tau)} \leq \sup_{0 \leq s < \infty} e^{(1-\gamma)s} \|R^{(n)}\|_{L^p} \rightarrow 0,$$

uniformly for  $0 \leq \tau < \infty$  as  $n \rightarrow \infty$ .

Now, passing the limit  $n \rightarrow \infty$  in the  $L^p$  norm, we obtain

$$\bar{V}(\tau) = V^{(0)} - G(\bar{V}, \bar{V}), \quad \text{for all } \tau \geq 0, \quad (2.54)$$

which shows that  $\bar{V}$  is a solution of the integral equation (2.28), satisfying the decay property (2.50).

## 2.4 The non-blowup estimates in the original variables.

In this subsection, we will complete the regularity analysis in the original physical variable. By the uniqueness of strong solutions in  $L^p$  with  $p > 3$ , we have

$$\|\bar{V}\|_{L^p(\tau)} = \|U\|_{L^p(\tau + \tau_m)}, \quad \text{for } 0 \leq \tau < \infty. \quad (2.55)$$

Now we can use the decay estimate for  $\bar{V}$  in (2.50) to obtain a decay estimate for  $U$ , which in turn will rule out the possibility of a finite time singularity for the 3D Navier-Stokes equations.

Using (2.50) and (2.55), we immediately obtain a decay estimate for  $U$ :

$$\|U\|_{L^p}(\tau) \leq K_{\max} e^{-(1-\gamma)(\tau-\tau_m)}, \quad \text{for } \tau \geq \tau_m. \quad (2.56)$$

This proves the following decay estimate for  $U$ .

**Lemma 4.** *The solution  $U(x, \tau)$  of rescaled Navier-Stokes equations (2.5) with  $\tau_m$  defined by (2.12) has a uniform decay rate in  $\tau$  as follows:*

$$\|U\|_{L^p}(\tau) \leq K_{\max} e^{-(1-\gamma)(\tau-\tau_m)}, \quad \text{for } \tau \geq \tau_m. \quad (2.57)$$

Substituting the relation

$$u(x, t) = \frac{1}{\sqrt{T-t}} U(y, \tau) \quad (2.58)$$

into (2.57), we obtain for  $t_m \leq t < T$  with  $t_m = T(1 - e^{-2\tau_m})$ ,

$$\begin{aligned} \|u\|_{L^p}(t) &= \frac{(T-t)^{\gamma/2}}{(T-t)^{1/2}} \|U\|_{L^p}(\tau) \\ &\leq \frac{K_{\max}}{(T-t)^{(1-\gamma)/2}} e^{-(1-\gamma)(\tau-\tau_m)} = \frac{K_{\max} e^{(1-\gamma)\tau_m}}{(T-t)^{(1-\gamma)/2}} e^{-(1-\gamma)\tau} \\ &= \frac{K_{\max} e^{(1-\gamma)\tau_m}}{(T-t)^{(1-\gamma)/2}} \left(\frac{T-t}{T}\right)^{(1-\gamma)/2} \\ &\leq \frac{K_{\max} e^{(1-\gamma)\tau_m}}{T^{(1-\gamma)/2}}, \quad \text{for } t_m \leq t < T. \end{aligned} \quad (2.59)$$

Since  $u_0 \in L^p$  for some  $p \in (3, \infty)$ , it is easy to show that there is a local-in-time smooth solution whose  $L^p$  norm is bounded [5] (This can also be proved directly by using the same iterative scheme applied to the original Navier-Stokes equations for a short time). Moreover, since  $T$  is the first singularity time, we conclude that  $u$  is smooth for  $0 < t \leq t_m < T$  and has a bounded  $L^p$  norm for  $t \leq t_m$ . Thus,  $\|u\|_{L^p}(t)$  is uniformly bounded for  $0 \leq t < T$ .

Now, we can apply the so-called Ladyzhenskaya-Prodi-Serrin condition (see [6], [11] and [12]), which is also known as the  $L^{p,q}$  criteria. The so-called  $L^{p,q}$  criteria state that if a suitable weak solution of (1.1) satisfies

$$u \in L^q([0, T]; L^p(\mathbb{R}^3)) \quad (2.60)$$

with

$$\frac{3}{p} + \frac{2}{q} \leq 1, \quad p \in [3, \infty], \quad (2.61)$$

then  $u$  is a smooth solution of the 3D Navier-Stokes equation up to  $t = T$ . In our case, we have obtained a uniform bound in  $L^p$  for  $u$  with  $p \in (3, \infty)$  for  $0 \leq t < T$ . Thus the  $L^{p,q}$  criterion is satisfied with  $q = \infty$ . Therefore, we conclude that  $u$  is a smooth function in  $\mathbb{R}^3 \times (0, T]$ .

This conclusion contradicts with our assumption that  $u$  would cease to be regular at time  $T$  for the first time. This contradiction implies that  $u$  can not develop a local self-similar singularity in any finite time. This completes the proof of Theorem 1.

## Appendix I.

In this appendix, we prove Lemma 3.

**Proof of Lemma 3.** It is sufficient to obtain an upper bound for the recurrence equalities

$$\tilde{K}_{n+1} = \tilde{K}_0 + M\tilde{K}_n^2, \quad \tilde{K}_0 = K_0. \quad (2.62)$$

It is easy to see that  $K_n \leq \tilde{K}_n$ , for all  $n \geq 1$ . To simplify the notation, we will drop the tilde in  $\tilde{K}_n$  in the following. Define  $l_n = K_{n+1} - K_n$ , then we have

$$l_n = M(K_{n-1} + K_n)l_{n-1}. \quad (2.63)$$

It is easy to see that  $l_n > 0$  for all  $n \geq 0$  and  $K_n$  is a monotonely increasing sequence. We claim that

$$M(K_{j-1} + K_j) \leq \frac{1}{2}, \quad \text{for all } j \geq 1. \quad (2.64)$$

We will prove (2.64) by an induction argument.

1. For  $j = 1$ , we have

$$M(K_0 + K_1) = M(K_0 + K_0 + MK_0^2) \leq \frac{1}{2} \quad (2.65)$$

from the assumption  $K_0M \leq \frac{1}{6}$ .

2. Assume that (2.64) holds for all  $j \leq n$ , we will prove that it also hold for  $j = n+1$ . Let  $\alpha = 1/2$ . It follows from (2.63) and the induction assumption that

$$l_j \leq \alpha l_{j-1}, \quad \text{for all } 1 \leq j \leq n, \quad (2.66)$$

which implies that

$$l_j \leq \alpha^j l_0. \quad (2.67)$$

Applying  $K_{n+1} = K_n + l_n$  recursively and using (2.67), we obtain

$$\begin{aligned}
K_{n+1} &= K_0 + \sum_{j=0}^n l_j \\
&\leq K_0 + l_0 \sum_{j=0}^n \alpha^j \\
&= K_0 + l_0 \frac{1 - \alpha^{n+1}}{1 - \alpha} \\
&\leq K_0 + 2MK_0^2 \leq \frac{4}{3}K_0,
\end{aligned} \tag{2.68}$$

where we have used  $MK_0 \leq 1/6$ . Define  $K_{\max} = \frac{4}{3}K_0$ . Then we have

$$2MK_{\max} = \frac{8}{3}MK_0 \leq \frac{4}{9} < \frac{1}{2}. \tag{2.69}$$

Thus, we obtain

$$M(K_n + K_{n+1}) \leq 2MK_{\max} < \frac{1}{2}. \tag{2.70}$$

This proves the claim (2.64) by induction, and we obtain

$$K_n \leq K_{\max}, \quad \text{for all } n \geq 0. \tag{2.71}$$

We have already shown that  $2MK_{\max} < \frac{1}{2}$  in (2.69). This completes the proof of Lemma 3.

## Appendix II. Proof of estimate (2.39)

In this appendix, we prove estimate (2.39). First, we state a useful inequality

$$|1 - e^{-2x}| \geq (1 - e^{-2})|x|, \quad \text{for } 0 \leq x \leq 1, \tag{2.72}$$

which is a consequence of the fact that  $(1 - e^{-x})/x$  is a monotonely decreasing function for  $x > 0$ . We consider two cases. If  $\tau > 1$ , we divide the integral into two parts as follows:

$$\begin{aligned}
&\int_0^\tau e^{\gamma s} (1 - e^{-2(\tau-s)})^{-\frac{1+\gamma}{2}} ds \\
&= \int_0^{\tau-1} e^{\gamma s} (1 - e^{-2(\tau-s)})^{-\frac{1+\gamma}{2}} ds + \int_{\tau-1}^\tau e^{\gamma s} (1 - e^{-2(\tau-s)})^{-\frac{1+\gamma}{2}} ds \\
&\leq \int_0^{\tau-1} e^{\gamma s} (1 - e^{-2})^{-\frac{1+\gamma}{2}} ds + \int_{\tau-1}^\tau e^{\gamma \tau} ((1 - e^{-2})(\tau - s))^{-\frac{1+\gamma}{2}} ds \\
&= (1 - e^{-2})^{-\frac{1+\gamma}{2}} \frac{e^{\gamma(\tau-1)} - 1}{\gamma} + e^{\gamma \tau} (1 - e^{-2})^{-\frac{1+\gamma}{2}} \frac{2}{1-\gamma} (\tau - s)^{\frac{1-\gamma}{2}} \Big|_{s=\tau-1}^{s=\tau} \\
&= (1 - e^{-2})^{-\frac{1+\gamma}{2}} \frac{e^{\gamma(\tau-1)} - 1}{\gamma} + \frac{2}{1-\gamma} e^{\gamma \tau} (1 - e^{-2})^{-\frac{1+\gamma}{2}} \\
&\leq \left( \frac{2}{1-\gamma} + \frac{1}{\gamma} \right) (1 - e^{-2})^{-\frac{1+\gamma}{2}} e^{\gamma \tau},
\end{aligned} \tag{2.73}$$



where we have used (2.72). Thus we prove

$$e^{-\tau} \int_0^\tau e^{\gamma s} \left(1 - e^{-2(\tau-s)}\right)^{-\frac{1+\gamma}{2}} ds \leq c_1 e^{-(1-\gamma)\tau} < c_1, \quad \text{for all } \tau > 1,$$

where  $c_1$  is defined in (2.13). For  $\tau \leq 1$ , we have by using (2.72)

$$\begin{aligned} e^{-\tau} \int_0^\tau e^{\gamma s} \left(1 - e^{-2(\tau-s)}\right)^{-\frac{1+\gamma}{2}} ds &\leq e^{-(1-\gamma)\tau} \int_0^\tau \left((1 - e^{-2})(\tau - s)\right)^{-\frac{1+\gamma}{2}} ds \\ &\leq \frac{2e^{-(1-\gamma)\tau}}{1-\gamma} (1 - e^{-2})^{-\frac{1+\gamma}{2}} \tau^{\frac{1-\gamma}{2}} \\ &\leq c_1 \tau^{\frac{1-\gamma}{2}} e^{-(1-\gamma)\tau} \leq c_1. \end{aligned} \tag{2.74}$$

This proves (2.39).

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