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TESTABLE IMPLICATIONS OF BARGAINING THEORIES

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SOCIAL SCIENCE WORKING PAPER 1348

June 2011

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Abstract

We develop the testable implications of well-known theories of bargaining over money. Given a finite data set of bargaining outcomes, where utility functions are unknown, we ask if a given theory could have generated the observations. When the data come with a fixed disagreement point, we show that the Nash, utilitarian, and the egalitarian max-min bargaining solutions are all observationally equivalent. These theories are in turn characterized by a simple test of comonotonicity of bargaining outcomes.

When the disagreement point is allowed to vary, we characterize the testable implications of the equal gain/loss egalitarian solution. The main application of our result is to testing the tax code for compliance with the principle of equal loss. For other theories, we introduce a general method based on the study of real solutions to systems of polynomial inequalities.

JEL classification numbers: C71,C78

Key words: Revealed Preference, Nash Bargaining, Utilitarian Solution, Egalitarian Solution

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1 Introduction

The purpose of this paper is to investigate the testable implications of theories of fair allocation. We have in mind the allocation of a single-dimensional resource: we can essentially focus on the allocation of money amongst a set of agents. We suppose that we have available certain data on how money was divided amongst a fixed number of agents, but have no such data on agents' preferences and the method or protocol that lead to the division. We want to know when observed allocations are consistent with standard bargaining theory.

There are several well-known theories that could explain a given division of money. One might imagine that the resource is divided in a way to maximize a sum of utilities, what we term the **utilitarian** model. One might instead assume that the resource is divided according to a process of Nash bargaining, what we call the **Nash** model (after Nash (1950)). Finally, the classical egalitarian paradigm of maximizing the utility of the worst off agents leads to what we call the **maxmin** model. Our goal here is to ask whether such theories place any testable restrictions on observable data.

Each of these three theories is commonly assumed in economic applications. Probably the most common is the utilitarian model. Aside from the simplicity of working with sums, the motivation is that in a model where utility can be transferred (but transfers might not be observed), there are transfers for which an allocation $(x_1, ..., x_n)$ Pareto dominates $(y_1, ..., y_n)$ if and only if $\sum_{i=1}^n u_i(x_i) \geq \sum_{i=1}^n u_i(y_i)$.

The Nash solution is also used in applied modeling, from macroeconomics to contract theory and applied mechanism design. Assuming identical linear utilities over wealth, the

Nash solution is simply the recommendation to split surpluses equally. In fact, while this solution is often justified by Nash's argument, essentially any symmetric social welfare function would make the same recommendation. For example, framing the example as a transferable utility game (which is possible by linearity of utility), Shapley's value recommends the same solution.

Finally, the maxmin approach, commonly identified with Rawls (1999), finds less application, but is a favorite topic of study for welfare theorists. And Young (1988) has suggested that many methods of taxation are based on a kind of maxmin principle.

We have three main results.

Firstly (Section 2), data comes in the form of observed income shares only. If negotiations break down, the disagreement outcome for all individuals is known to be zero. We observe a finite collection of data points. For example, these might be the outcome of union wage bargaining, bankruptcy liquidation proceedings, or government subsidies. We investigate the restrictions that each of the three models, utilitarian, Nash, and maxmin, place on the allocations of shares. We want to test the hypotheses that shares are allocated according to these theories when utility functions are assumed to be concave, but otherwise can differ across individuals.

We discover that the empirical content of the three models is identical. No data set will ever allow us to distinguish between them. A dataset either refutes all three or is consistent with all three. Furthermore, the theories have very weak predictive power. The only empirical prediction of any of these theories is that data should be perfectly ordinally correlated, or comonotonic. This means that if one individual's share rises across data points, so must all individuals'. The only refutations of the models are observations in which one individual's income share rises while another's falls. Comonotonicity, sometimes called "resource monotonicity," is such a basic principle of distributive justice in the single-dimensional commodity model, that there are essentially no normative theories violating it.

In second place, in Section 3 we turn to data in which the disagreement point can vary (yet remains observable). We focus on the classical model of equal gains or equal losses–essentially the maxmin model discussed above. Here, we imagine that there is some utility function, common to all individuals, and that division of money is chosen to equalize gains in utility from the individual endowments. Our results are a direct application of the Theorem of the Alternative, and as a consequence our test is easily operationalized using linear programming methods.

A standard application of this model is to taxation–the "disagreement" point is the vector of ex-post incomes, while the observed shares are ex-ante incomes. Our theory is then a classical egalitarian method of taxation (Young, 1988): the theory of equal loss due to taxation. When utility over money is concave, one may expect poor agents to be taxed less than rich agents. We present a test of this theory in the case when utility is concave but unknown. Young (1990) studies the same problem, but using a parametric estimation approach to find the best-fitting utility index to tax data in the United States.

The results of this section have an interesting byproduct–the testable implications of Hotelling's model of spatial competition (Hotelling, 1929). Subsection 3.1 demonstrates how this problem is a special case of the environment studied in section 3.

In third place, we analyze other models under data with a variable disagreement point. The equal gains/losses model is simple to analyze with a variable disagreement point, but other theories, such as the Nash theory, or even a utilitarian theory, are much more difficult. Data were consistent with the equal losses model if and only if they satisfied a system of linear inequalities–this is where our application of the Theorem of the Alternative comes in. By contrast, data are consistent with the Nash model if and only if they satisfy a system of polynomial inequalities. The Theorem of the Alternative no longer applies in this case. To this end, we describe a result which has been recently popular in the mathematics literature–the Positivstellensatz. The Positivstellenstz is a kind of Theorem of the Alternative for polynomial inequalities. For any system of polynomial inequalities, that system is infeasible if and only if some dual system is feasible. Practically speaking, the dual system has polynomials as its own variables, and hence, can be quite difficult to apply. On the other hand, recent numerical advances based on techniques of Parrilo (2003) often allow such infeasibility certificates to be obtained. These results apply semidefinite programming (Vandenberghe and Boyd, 1996) to the problem of determining whether a given polynomial is a sum of squares.

The Positivstellensatz technique can be compared with the Tarski-Seidenberg elimination procedure, first studied in the economics literature by Brown and Matzkin (1996). Tarski-Seidenberg provides an algorithm that one can perform on finite systems of polynomial inequalities in order to determine whether or not those inequalities are consistent (that is, the theory of the reals is decidable).

1.1 Related literature

The recent contribution of Chiappori, Donni, and Komunjer (2010) investigates the empirical content of Nash bargaining. There are several important differences between that work and ours. The main difference is that their framework assumes disagreement points are unobserved. Instead, they suppose that some vector of underlying, observable Euclidean characteristics uniquely determine both the utility functions of agents, as well as the disagreement point. Without assuming any kind of structure on the joint dependence of disagreement point and utility on these underlying characteristics, their model obviously has no testable implications (this is their Proposition 2). To have any empirical content, they must assume some structure on the dependence of the utility function and disagreement point on these characteristics. They assume that this dependence is known to satisfy certain properties (differentiability and "exclusion restrictions") both within and across characteristics. By contrast, in our model, disagreement point observations are part of the observed data, and this leads to the falsifiability of the model.

The other main distinction between their work and ours is that they are concerned with understanding the testable implications of the model in a continuous sense–the implications of the model if we could observe the division across *all* possible problems. Our work, on the other hand, assumes only that a finite number of possible division problems are observed (with their solutions). The distinction in the two approaches can be best understood by considering the classical demand model: their approach is analogous to characterizing rationalizability by conditions on the Slutsky matrix; while our approach is analogous to Afriat's (Afriat, 1967) discussion of finite data sets which are rationalizable.

Earlier works discussing the empirical content of Nash bargaining, usually assuming all individuals are identical and risk neutral, include Hamermesh (1973) and Bowlby and Schriver (1978). Svejnar (1980) provides a critique of these ideas.

As earlier noted, Young (1990) constructs a test of the maxmin hypothesis, using empirical data on US income taxes from 1957-1987. His approach is estimation-based, and he finds that tax data are reasonably close to predicted data from the maxmin model in most years (there are exceptions). He assumes specific parametric forms for the utility function. By contrast, we provide an exact test of the maxmin model, assuming no parametric functional form. Young (1988) provides a kind of exact empirical test of the maxmin model, assuming the solution to all possible problems is observed, and further assuming observations *across* different populations.

2 A single test for all theories

In this section, we consider an environment for which $d = 0$ throughout; that is, the disagreement point is fixed and symmetric. Our aim will be to understand the testable implications of three different social choice models when utility indices can be different, but are required to be strictly concave and strictly increasing. We establish that the only principle of justice that can be tested when preferences are allowed to differ across individuals is a basic solidarity principle. And the principle is extremely weak: it solely requires that if one agent's consumption increases, then so does the consumption of all remaining agents (it says nothing about how much). The punchline is twofold. First, these three models have identical testable implications. Thus, among the three most popular models of social welfare, we would have no way of identifying which one is being used based on data alone. Second, the empirical predictions of these models are very weak. It is hard to think of any kind of environment where this principle would ever be refuted, or any justifiable normative reason for violating the principle. The principle that all agents should share in marginal gains is so basic it can hardly even be called a fairness principle.

The available data takes the following form. We have K observations, each one describing an allocation $x = (x_1, \ldots, x_N)$ of an aggregate monetary quantity $\sum_{i=1}^N x_i$. We assume that the disagreement point is normalized to 0, so all observations here are of strictly positive quantities.[1] A **data set** then takes the form $\{(x^k)\}_{k=1}^K$, where $x^k \in \mathbb{R}^N_{++}$.

There are three basic models to consider: first, the utilitarian model. Data $\{(x^k)\}_{k=1}^K$ are **utilitarian rationalizable** if there exist strictly monotonic and strictly concave u_i for which $\sum_{i\in N} u_i(x_i^k) \geq \sum_{i\in N} u_i(y_i)$ for all allocations $(y_1, \ldots, y_N)$ with $\sum_i y_i = \sum_i x_i^k$. Data $\{(x^k)\}_{k=1}^K$ are **Nash rationalizable** if there exist strictly monotonic and strictly concave u_i, normalized so that $u_i(0) = 0$, for which $\prod_i u_i(x_i^k) \geq \prod_i u_i(y_i)$ for all $\sum_i y_i = \sum_i x_i^k$. Finally, data $\{(x^k)\}_{k=1}^K$ are **maxmin rationalizable** if there exist strictly monotonic and strictly concave u_i, normalized so that $u_i(0) = 0$, for which $\min_{i\in N} u_i(x_i^k) \geq \min_{i\in N} u_i(y_i)$ for all $\sum_i y_i = \sum_i x_i^k$.

[1]Importantly, the disagreement point must be the same for all observations. This assumption can be interpreted in two ways. First, we can suppose that the disagreement point is observed, and normalize the data accordingly. Second, we can suppose that the disagreement point is unobserved but fixed.

Note that data are Nash rationalizable if and only if there exist u_i strictly concave and positive for which $\sum_i \log(u_i(x_i^k)) \geq \sum_i \log(u_i(y_i))$ for all $\sum_i y_i = \sum_i x_i^k$.

Finally, say that the data $\{x^k\}_{k=1}^K$ are **comonotonic** if for all $i, j \in N$ and all k, l, $x_i^k < x_i^l$ implies $x_j^k < x_j^l$. Comonotonicity requires that outcomes are perfectly ordinally correlated.

Theorem 1. *Given data $\{x^k\}_{k=1}^K$, the following are equivalent.*

1. *The data are comonotonic.*
2. *The data are utilitarian rationalizable.*
3. *The data are Nash rationalizable.*
4. *The data are maxmin rationalizable.*

Proof. It is easy to see that if the data are either utilitarian or Nash rationalizable, then they must be comonotonic. This follows as in each case, the data solve a basic concave optimization problem.

Step 1: The data are comonotonic imply the data are utilitarian rationalizable.

Suppose that the data are comonotonic. We claim that there exist u_i strictly concave, strictly increasing, satisfying $u_i(0) = 0$, and for which for all $k \in \{1, ..., K\}$, $\sum_i u_i(x_i^k) \geq \sum_i u_i(y_i)$ for all $\sum_i y_i = \sum_i x_i^k$. In fact, we show that we can take u_i to be twice continuously differentiable.

Consider the following claim:

Claim 2. *Given a finite collection of positive real numbers $\{x^1, ..., x^K\} \subseteq \mathbf{R}_{++}$, where $x_k < x_{k+1}$ for all k, there is a smooth, decreasing, function $f > 0$ such that $f(x^k) = 1/k$, $k = 1, \ldots, K$.*

Step 1 follows from the claim in the following way. By comonotonicity, we may without loss of generality assume the data are ordered so that $k < l$ implies that $x_i^k < x_i^l$ for all i.[2] Then let f_i be the constructed function in Claim (2) for the set $\{x_i^1, ..., x_i^K\}$.

[2]By comonotonicity, it is without loss of generality to assume that $k \neq l$ implies $x_i^k \neq x_i^l$.

Let $u_i(x) = \int_0^x f_i(s)ds$. Since f_i is strictly positive and decreasing, u_i has the desired properties. On the other hand, for each k,

$$\left.\frac{\partial \sum_{i=1}^N u_i(y_i)}{\partial y_i}\right|_{(y_1,\ldots,y_N)=(x_1^k,\ldots,x_N^k)} = f_i(x_i^k) = 1/k.$$

Then the first-order conditions for maximization of $\sum_{i=1}^N u_i(y_i)$ are satisfied at x^k, and thus the functions u_i rationalize the data.

To prove Step 1, let $z^k = \frac{x^k+x^{k+1}}{2}$, $k = 1,\ldots,K-1$. Let $a^1, b^1 > 0$ be the solution to the equation $a^1 - b^1x^1 = 1$ and $a^1 - b^1z^1 = (1+(1/2))(1/2)$.

Let $a^k, b^k > 0$ be the solution to the equation $a^k - b^kx^k = 1/k$ and $a^k - b^kz^{k-1} = (1/(k-1)+1/k)(1/2)$, $k = 2,\ldots,K$.

Chose $\theta > 0$ such that $\theta/x^K < 1/K$.

Let

$$l(x) = \begin{cases} a^1 - b^1x & \text{if } x \leq z^1 \\ a^k - b^kx & \text{if } z^{k-1} < x \leq z^k \\ \max\left\{a^K - b^Kx, \theta/x\right\} & \text{if } z^{K-1} \leq x \end{cases}$$

Note that l is strictly monotonically decreasing, $l > 0$, and that $l(x^k) = 1/k$, $k = 1,\ldots,K$.

Let $\Delta > 0$ be such that

$$\max\left\{x^k - z^{k-1}, z^k - x^k\right\} < \Delta$$

for all k.

Let $\Psi : \mathbf{R} \to \mathbf{R}$ be defined by

$$\Psi(x) = \begin{cases} Ke^{-\frac{1}{1-x^2}} & \text{if } |x| \leq \Delta \\ 0 & \text{if } |x| > \Delta, \end{cases}$$

where K is chosen such that $\int \Psi = 1$.

Let $f : \mathbf{R}_+ \to \mathbf{R}$ be defined by

$$f(x) = \int_{|z|<\Delta} \Psi(z) l(x - z) dz.$$

It is easy to see that f is smooth and strictly decreasing.

Note that

$$\begin{aligned} f(x^k) &= \int_{|z|<\Delta} \Psi(z)(a^k - b^k x^k + b^k z) dz \\ &= (1/k) \int_{|z|<\Delta} \Psi(z) dz + b^k \int_{|z|<\Delta} z\Psi(z) dz = 1/k; \end{aligned}$$

where we used the definition of Δ and that $\int_{|z|<\Delta} z\Psi(z)dz = 0$.

Step 2: If the data are utilitarian rationalizable, then they are Nash rationalizable.

Suppose the data are utilitarian rationalizable. We have shown in Step 1 that we can assume that the rationalizing utility functions are twice continuously differentiable. Denote by u_i the rationalizing utility functions. We now consider the following functions:

$$u_i^\lambda(x) = \exp(\lambda u_i(x))$$

where $\lambda > 0$. Consider any compact interval X large enough to contain all data points, 0, and all points $\sum_i x_i^k$.

Note that $(u_i^\lambda)'(x) = \lambda u_i'(x) \exp(\lambda u_i(x))$ and

$$(u_i^\lambda)''(x) = \lambda u_i''(x) \exp(\lambda u_i(x)) + (\lambda u_i'(x))^2 \exp(\lambda u_i(x)).$$

Define

$$g_i^\lambda(x) = \frac{(u_i^\lambda)''(x)}{(\lambda \exp(\lambda u_i(x)))} = u_i''(x) + \lambda (u_i'(x))^2.$$

Note that $g_i^\lambda(x)$ converges monotonically to $u_i''(x)$ as $\lambda \to 0$, so it converges uniformly (on the compact interval X). Consequently, there exists $\lambda^* > 0$ small for which for all $i \in N$, $g^{\lambda^*}(x) < 0$ for all $x \in X$. Therefore, $(u_i^{\lambda^*})''(x) < 0$ for all $x \in X$ and all $i \in N$, and clearly, $(u_i^{\lambda^*})'(x) > 0$.

We now extend each $u_i^{\lambda^*}$ to all of $\mathbf{R}$ arbitrarily, in order to preserve concavity (for example, we can choose an extension which is affine outside of X).

Then the functions $(u_i^{\lambda^*})_{i\in N}$ Nash-rationalize the data: by construction, the functions $(u_i)_{i\in N}$ utilitarian-rationalize the data, so that the problem $\max \sum_i \ln(u_i^{\lambda^*}(y_i))$ subject to $\sum_i y_i = \sum_i x_i^k$ is solved at $(x_i^k)_{i\in N}$.

Step 3: The data are comonotonic if and only if they are maxmin rationalizable

That maxmin rationalizable data are comonotonic is obvious. Conversely, suppose that we have comonotonic data. By comonotonicity, we may without loss of generality suppose that $x_i^1 < x_i^2 < ... < x_i^K$ for all i (we may remove repeated observations, and by comonotonicity there are no observations k, l for which for some i, j, $x_i^k = x_i^l$ and $x_j^k < x_j^l$). We construct, for each i, a strictly decreasing positive function f_i (potentially discontinuous), whose integrals will rationalize the data.

We will illustrate a construction of the functions f_i by induction. For each $i \in N$, let g_i be any positive, strictly decreasing (affine) function on $[0, x_i^1]$. Then for each $i \in N$, choose $\alpha_i > 0$ so that $\int_0^{x_i^1} \alpha_i g_i(x)dx = \int_0^{x_j^1} \alpha_j g_j(x)dx$. Set $f_i(x) = \alpha_i g_i(x)$ here.

Let $k < K$. Assume that f_i has been constructed on the interval $[0, x_i^k]$, and is positive and strictly decreasing (and that $\int_{x_i^l}^{x_i^{l+1}} f_i(x)dx = \int_{x_j^l}^{x_j^{l+1}} f_j(x)dx$ for all $l < k$ and all i and j). Choose g_i again on $(x_i^k, x_i^{k+1}]$ so that g_i is affine, decreasing and positive. For all $i \in N$, choose $\beta_i > 0$ so that $\int_{x_i^k}^{x_i^{k+1}} \beta_i g_i(x)dx = \int_{x_j^k}^{x_j^{k+1}} \beta_j g_j(x)dx$, *and* so that $\beta_i g_i(x_i^k) \leq f_i(x_i^k)$ for all $i \in N$. Define $f_i(x)$ on $(x_i^k, x_i^{k+1}]$ as $\beta_i g_i(x)$.

Finally, we define, for each $i \in N$, f_i on (x_i^k, ∞) so that $f_i(x) = \frac{\gamma}{x}$, where $\gamma > 0$ is chosen small enough that $f_i(x)$ remains everywhere decreasing.

Now it is enough to define $u_i(x) = \int_0^x f_i(x)dx$, and note that u_i is strictly increasing and strictly concave. Moreover, by construction, u_i maxmin rationalize the data. □

While we study three of the most common social welfare functions existing in the literature, the result can be shown to hold more generally. We conjecture that the result can be shown to hold for a broad class of social welfare functions which are separable across agents (both the utilitarian and Nash rules are separable, while maxmin is weakly separable).

3 The classical equal gains model

Let N be a finite set of agents, we may without loss of generality assume it to be finite. An **observation** consists of a pair $(d, x) \in \mathbf{R}^N \times \mathbf{R}^N$. We write observations in the form (d, x). The pair (d, x) may represent many things; for example, d may represent a vector of investments and x a vector of returns. Or, d might represent a profile of post-tax incomes whereas x represents a profile of pre-tax incomes.

A dataset is a finite set of observations $D = \left\{\left(d^k, x^k\right)\right\}_{k=1}^{K}$.

$$D = \left\{(d^k, x^k) : k = 1, \dots, K\right\}.$$

where $(d^k, x^k) \in \mathbf{R}^{2n}$ for all k.

A **utility** is a strictly increasing function $u : \mathbf{R} \to \mathbf{R}$.

A dataset D is *rationalized* by the utility $u : \mathbf{R} \to \mathbf{R}$ if, for all $k \in \{1, ..., K\}$ and all $i, j \in N$,

$$u(x_i^k) - u(d_i^k) = u(x_j^k) - u(d_j^k).$$

Say that dataset D is *rationalizable* if there is a utility that rationalizes it.

The notion of rationalization is compatible with the notion of an "equal standard," applied to all agents, represented by the utility function u. It is consistent with the maxmin story, applied to gains in utility. This can be contrasted with Young (1990), who essentially provides the same model, but under a taxation interpretation.

To begin to understand what rationalizability here entails, let us suppose we have two agents, so that $N = \{1, 2\}$, and that we observe the data points $\{((0, 5), (7, 8)), ((1, 2), (3, 8))\}$. These data correspond to observations $((d^1, x^1), (d^2, x^2))$ We claim that these data cannot be rationalized. To see why, suppose that u were a utility function rationalizing these data. This requires that $u(5) - u(0) = u(8) - u(7)$, and that $u(2) - u(1) = u(8) - u(3)$. Therefore, we must have

$$[u(7) - u(8)] + [u(8) - u(3)] + [u(5) - u(0)] + [u(1) - u(2)] = 0. \tag{1}$$

But we can regroup terms in this expression, obtaining the following:

$$[-u(2) + u(7)] + [-u(8) + u(8)] + [-u(3) + u(5)] + [-u(0) + u(1)] = 0. \tag{2}$$

The contradiction arises because in equation (2), each term in brackets is nonnegative, and there is at least one strictly positive term (in fact, each of the terms $[-u(2)+u(7)]$, $[-u(3)+u(5)]$, and $[-u(0)+u(1)]$ are strictly positive). Therefore, the terms cannot add up to zero.

We have shown that data such as these cannot be rationalized. Before we provide the general condition that data must satisfy to be rationalizable, one more example may help to understand. Again, let us consider two agents, to keep the analysis simple. Let us suppose we observe the data $\{((1,3),(8,9)),((2,5),(8,9)),((2,4),(9,10)),((0,4),(9,10))\}$. Now, again, by appropriately adding and subtracting, we obtain:

$$([u(1)-u(3)]+[u(5)-u(2)]+[u(2)-u(4)]+[u(4)-u(0)])+$$
$$([u(8)-u(8)]+[u(9)-u(10)]+[u(10)-u(9)]+[u(9)-u(8)])=0.$$

But note again, by regrouping, we obtain:

$$([-u(0)+u(1)]+[-u(3)+u(5)]+[-u(2)+u(2)]+[-u(4)+u(4)])+$$
$$([-u(8)+u(8)]+[-u(9)+u(9)]+[-u(10)+u(10)]+[-u(9)+u(9)]=0.$$

And again, each of the terms inside of the brackets is nonnegative, and some are strictly positive. This results in another contradiction.

In each of these two cases, what we have done is the following. We have taken data points that, if rationalizable, should force a certain expression to add to zero. By regrouping the terms, the monotonicity of u forces a contradiction, in that the expression could not possibly add to zero. It turns out that the inability to regroup data in this sense is necessary and sufficient for the data to be rationalizable. To make sense of this, we have to be more specific in what we mean by “regrouping data.” It is easiest to think of this in graph theoretic terms. In equation (1), we can think of edges pointing from 7 to 8, from 8 to 3, from 5 to 0, and from 1 to 2. Note that these edges come in “pairs,” namely, the edge pointing up from 7 to 8 comes from the data point $((0,5),(7,8))$, and is naturally paired with the edge pointing down from 5 to 0. Likewise, the edge pointing up from 1 to 2 is naturally paired with the edge pointing down from 8 to 3. The interesting point is that when we put these edges together in the appropriate sequence, they form a kind of a cycle. That is, consider the “edges” $(7,8),(8,3),(5,0),(1,2)$. The endpoints of adjacent edges here are ordered, where we treat $(1,2)$ and $(7,8)$ to be adjacent. That is, the terminal node of edge $(7,8)$ is less than or equal (in fact, equal) to the first note in

$(8, 3)$. And so forth, for each pair of adjacent edges. In fact, the terminal node of $(8, 3)$ is strictly less than the first note in $(5, 0)$, as $3 < 5$. And, returning to equation (2), we see that when we regrouped the data, the term $-u(3) + u(5)$ appeared.

To this end, we define a *cycle* to be a finite sequence of ordered pairs of real numbers, $\{(z_l^1, z_l^2)\}_{l=1}^L$, for which for all $l = 1, \ldots, L-1$, $z_l^2 \leq z_{l+1}^1$ and $z_L^2 \leq z_1^1$. A *strict cycle* is a cycle $\{(z_l^1, z_l^2)\}_{l=1}^L$, for which for some l, $z_l^2 < z_{l+1}^1$ or $z_L^2 < z_1^1$. A finite sequence $\{(z_l^1, z_l^2)\}_{l=1}^L$ *defines a (strict) cycle* if there exists a bijection $\sigma : L \to L$ for which $\left\{\left(z_{\sigma(l)}^1, z_{\sigma(l)}^2\right)\right\}_{l=1}^L$ is a (strict) cycle.

Then the ordered pairs $\{(7, 8), (8, 3), (5, 0), (1, 2)\}$ from our first example form a strict cycle. We could conjecture that for data not to be rationalizable, we should be able to pair "up" edges with "down" edges in a way that forms a strict cycle. But this is not quite enough. If we look at the regrouping in the second example, we again paired up edges with down edges. But we did not end up with a single cycle, in fact, we ended up with *two* cycles, only one of which was strict. Namely, the edges $\{(1, 3), (5, 2), (2, 4), (4, 0), (8, 9), (9, 10), (10, 9), (9, 8)\}$do not themselves form a cycle, but the two sets of edges $\{(1, 3), (5, 2), (2, 4), (4, 0)\}$, $\{(8, 9), (9, 10), (10, 9), (9, 8)\}$ each form a cycle. Only the first cycle here is strict, but that is all we need.

In general, we can see there is no reason that a sequence of paired edges need correspond to one, two, or even k cycles. All that we need to obtain a contradiction is that data can be grouped into paired edges which can be partitioned into cycles, at least one of which is strict. These observations motivate the following definitions.

Let L be a natural number, and let $\{(a_l, b_l)\}_{l=1}^L$ and $\{(a'_l, b'_l)\}_{l=1}^L$ be two sequences of L ordered pairs. Say that $\{(a_l, b_l)\}_{l=1}^L$ and $\{(a'_l, b'_l)\}_{l=1}^L$ can be *partitioned into cycles* if there exists a natural number T, and for each $t \leq T$, a collection of finite sequences $\{(z_{tl}^1, z_{tl}^2)\}_{l=1}^{L_t}$ which define cycles (at least one cycle of which is strict), for which there exists a bijection $f : \{(t, l) : t \leq T, l \leq L_t\} \to \{(l, i) : l \leq L, i = \{1, 2\}\}$ for which $(z_{tl}^1, z_{tl}^2) = (a_{f_1(t,l)}, b_{f_1(t,l)})$ if $f_2(t, l) = 1$, and $(z_{tl}^1, z_{tl}^2) = (a'_{f_1(t,l)}, b'_{f_1(t,l)})$ if $f_2(t, l) = 2$.

The inability to partition paired data points into cycles is exactly the necessary and sufficient condition needed to guarantee that data are rationalizable.

Proposition 3. *The data* $D = \left\{(d^k, x^k) : k = 1, \ldots, K\right\}$ *are rationalizable if and only if there are no sequences of data points* $(d^l, x^l)_{l=1}^L$ *in* D*, and agents* $i_l \neq j_l$ *for all* l*, such that* $\left(\left[d_{i_l}^l, x_{i_l}^l\right]\right)_{l=1}^L$ *and* $\left(\left[x_{j_l}^l, d_{j_l}^l\right]\right)_{l=1}^L$ *can be partitioned into cycles, at least one of which is strict.*

Two points are worth mentioning. The definition of cycle does not preclude repetition of elements; nor does the notion of "sequence of data points" referred to in the statement of the Proposition.

Before stating the proof, we point out the following version of the theorem of the alternative (or Farkas' Lemma).

Lemma 4. *(Integer-Real Farkas) Let $\{A_i\}_{i=1}^K$ be a finite collection of vectors in $\mathbf{Q}^n$. Then one and only one of the following statements is true:*

i) There exists $y \in \mathbf{R}^n$ such that for all $i = 1, ..., L$, $A_i \cdot y \geq 0$ and for all $i = L+1, ..., K$, $A_i \cdot y > 0$.

ii) There exists $z \in \mathbf{Z}_+^K$ such that $\sum_{i=1}^K z_i A_i = 0$, where $\sum_{i=L+1}^K z_i > 0$.

Proof. It is clear that both i) and ii) cannot simultaneously hold. We therefore establish that if ii) does not hold, i) holds. By Theorem 3.2 of Fishburn (1973), if ii) does not hold, there exists $q \in \mathbf{Q}^n$ such that for all $i = 1, ..., L$, $A_i \cdot q \geq 0$ and for all $i = L+1, ..., K$, $A_i \cdot q > 0$. Hence, $q \in \mathbf{Z}^n$. □

Proof. Let $X \subseteq \mathbf{R}^n$ be a finite set such that $d^k, x^k \in X$ for all k.

There is a rationalizing u if and only if there is a solution to the system of linear inequalities

$$((1_{x_i^k} - 1_{d_i^k}) + (1_{d_j^k} - 1_{x_j^k})) \cdot u \geq 0 \tag{3}$$

$$(1_{z'} - 1_z) \cdot u > 0. \tag{4}$$

There is an inequality (3) for each i, j and k, and an inequality (4) for each $z', z \in X$ with $z < z'$.

Once a solution to the linear inequalities has been obtained, the function u can be completed by linear interpolation.

By the Lemma 4, there is no solution to system (3)-(4) iff there are vectors $\lambda \in \mathbf{Z}_+^{KN^2}$ and $\theta \in \mathbf{Z}_+^{|X|^2}$ with

$$\sum_{k,i,j} \lambda_{k,i,j}((1_{x_i^k} - 1_{d_i^k}) + (1_{d_j^k} - 1_{x_j^k})) + \sum_{(z,z'):z'>z} \theta_{z,z'}(1_{z'} - 1_z) = 0$$

and $\sum_{(z,z'):z'>z} \theta_{z,z'} > 0$.

Without loss of generality, we can assume that $d_i^k \neq d_j^k$ and $x_i^k \neq x_j^k$ for all k and all $i \neq j$. To see this note that, if $d_i^k = d_j^k$ then $x_i^k < x_j^k$ implies that there is no rationalizing monotonic u; but then the intervals $\left[d_i^k, x_i^k\right]$ and $\left[d_j^k, x_j^k\right]$ define a strict cycle: $\langle d_i^k, x_i^k\rangle$ $\langle x_j^k, d_j^k\rangle$. Similarly if $x_j^k < x_i^k$. On the other hand, $x_i^k = x_j^k$ implies that the inequalities corresponding to k, i, j in (3) are always satisfied. So these inequalities are irrelevant to the existing of a rationalizing u. The argument is analogous when $d_i^k \neq d_j^k$ and $x_i^k = x_j^k$.

Let the vectors $(\lambda_{k,i,j})$ and $(\theta_{z,z'})$ be as above. Consider the following collections of vectors in $\{-1,0,1\}^X$: Let A_D be the collection of vectors with $\lambda_{k,i,j}$ copies of $(1_{d_j^k} - 1_{x_j^k})$; let A_U be the collection with $\lambda_{k,i,j}$ copies of $(1_{x_i^k} - 1_{d_i^k})$. Let $f : A_D \to A_U$ be the bijection which associates each $(1_{d_j^k} - 1_{x_j^k})$ with a different copy of $(1_{x_i^k} - 1_{d_i^k})$.

Let A_M be the collection with $\theta_{z,z'}$ copies of $1_{z'} - 1_z$ for each $z, z' \in X$ with $z' > z$. By definition of λ and θ, we know that the sum of the elements of A_D, A_U, and A_M equals the null vector. We also have that $A_M \neq \emptyset$.

Let $G = (X, E)$ be the graph obtained by letting there be an edge pointing from x to x' iff there is a vector $1_{x'} - 1_x$ in one of the collections A_D, A_U or A_M. By the Poincaré-Veblen-Alexander Theorem (see Berge (2001), p. 148, Theorem 5), G can be partitioned into circuits $C_1, \ldots C_T$. Note that, if $e = (v, v') \in A_U \cup A_M$ then $v \leq v'$. If $e = (v, v') \in A_D$, then $v \geq v'$.

Consider the edges in circuit C_t: Let $\left[d_{i_l}^l, x_{i_l}^l\right]$, $l = 1, \ldots L_t^U$ be the set of intervals defined by edges $(d_{i_l}^l, x_{i_l}^l) \in A_U$ and $\left[d_{j_l}^l, x_{j_l}^l\right]$ $l = 1, \ldots L_t^D$ be the set of intervals defined by edges $(x_{j_l}^l, d_{j_l}^l) \in A_D$. For any edge $e = (v, v') \in A_U \cup A_D$ in C_t, let (v'', v''') be the first edge in C_t after e that is in $A_U \cup A_D$. Then either $v' = v''$ or there are edges in A_M between e and (v'', v''') in C_t; so $v' \leq v''$. Hence, for any $e = (v, v') \in A_U \cup A_D$ in C_t, the successor edge $(v'', v''') \in A_U \cup A_D$ satisfies that $v' \leq v''$. Hence the intervals $(d_{i_l}^l, x_{i_l}^l)$ $l = 1, \ldots L_t^U$ and $(x_{j_l}^l, d_{j_l}^l)$ $l = 1, \ldots L_t^D$ define a cycle.

In addition, since $A_M \neq \emptyset$, at least one of the sets of intervals defined by a circuit C_t defines a strict cycle.

Finally, since there is a bijection between the edges in A_U and in A_D, we have that $\sum_t L_t^U = \sum_t L_t^D = L$. So if we let $(\left[d_{i_l}^l, x_{i_l}^l\right])_{l=1}^L$ collect the sequences $\left[d_{i_l}^l, x_{i_l}^l\right]$, $l = 1, \ldots L_t^U$, and $(\left[x_{j_l}^l, d_{j_l}^l\right])_{l=1}^L$ collect the sequences $\left[d_{j_l}^l, x_{j_l}^l\right]$ $l = 1, \ldots L_t^D$, then we have a sequence of intervals in the condition in the statement of the proposition. $\square$

We have here asked for data to be rationalized by a single utility function, common to all $i \in N$. If, instead, we ask that for each i, there exists $u_i : \mathbf{R} \to \mathbf{R}$ for which for all $i, j \in N$

$$u_i(x_i^k) - u_i(d_i^k) = u_j(x_j^k) - u_j(d_j^k),$$

we obviously get a weaker condition. The weakening required here is simply that when partitioning data into cycles, each cycle can only contain edges corresponding to a *single* agent. The proof is similar to the proceeding and is hence omitted.

3.1 An application to spatial competition

In our version of Hotelling's model, we observe a finite collection of intervals $([a^k, b^k]) \subseteq [0,1]$, and for each observed interval, a location $m^k \in (a^k, b^k)$. We want to know, when does there exist a full-support distribution μ of agents on $[0,1]$ such that for each k, m^k is the median of μ conditional on $[a^k, b^k]$? This provides us with the testable implications of the Hotelling model when the distribution of agents is unobserved, but when the boundaries of spatial competition can vary.

The relation to section 3 is as follows. A distribution μ satisfying the properties exists if and only if there is a strictly increasing $F : [0,1] \to \mathbb{R}$ (a cdf) for which for all k, $F(b^k) - F(m^k) = F(m^k) - F(a^k)$. Now, imagine that in the previous section we had only two agents ($|N| = 2$), and $d^k = (m^k, a^k)$, $x^k = (b^k, m^k)$.

This leads us directly to the following corollary:

Corollary 5. *A finite list of intervals $[a^k, b^k]$ and locations m^k is consistent with the Hotelling model if and only if there are no sequences of data points $[a^l, b^l]_{l=1}^{L}$, $[a^l, b^l]_{l=L+1}^{L'}$ for which $\{(a^l, m^l)\}_{l=1}^{L}$, $\{(b^l, m^l)\}_{l=1}^{L}$, $\{(m^l, a^l)\}_{l=L+1}^{L'}$,$\{(m^l, b^l)\}_{l=L+1}^{L'}$ can be partitioned into cycles.*

4 General results

Section 2 assumed a fixed disagreement point, and claimed that there was little in the way of testable implications of the Nash (or utilitarian model). Now we turn to an analysis of these two theories when the disagreement point can vary. We claim that there are clear testable implications of the Nash model, for example, when the disagreement point can vary; and in fact, these testable implications come in an easily refutable form. That is, a refutation of the model can be provided by demonstrating a solution to a collection of polynomial inequalities. Our observations will follow immediately from a deep result in mathematics known as the **Positivstellensatz** (Stengle (1973)). See in particular Bochnak, Coste, and Roy (1998) or Marshall (2008).

Before beginning a discussion of the Positivstellensatz, refer back to Lemma 4. Generally speaking, y is some unknown, and we would like to find whether or not y with the stated properties exists. The vectors A_i represent vectors which are somehow generated from observed data. For example, if we observe revealed preference choices, then K may be the number of observations, and n the number of possible alternatives from which an agent chooses. Then every observation corresponds to a vector A_i. Say that we observe object j chosen over object l, then there is an A_i of the form $1_i - 1_j$. The existence of an $y \in \mathbf{R}^n$ satisfying the inequalities then translates into the existence of a utility function rationalizing the data.

The existence of a solution to the prescribed linear inequalities is verifiable: once we have a solution, we can check that the inequalities are satisfied. The role of Lemma 4 is to demonstrate that existence of a solution is also *falsifiable*. Together with Eran Shmaya, we have shown in an earlier work Chambers, Echenique, and Shmaya (2010) (building on the philosophical ideas of Popper and the mathematical ideas of Tarksi), that falsifiability of a theory is equivalent to a form of universal axiomatizability.[3] The two statements in Lemma 4 are existential, and in this sense verifiable. But what the theorem says is that if our hypothesized theory is false (that there does not exist y), then this falsity can be demonstrated, by establishing the existence of z. In other words, the statement "There exists $y \in \mathbf{R}^n$ such that for all $i = 1, ..., L$, $A_i \cdot y \geq 0$ and for all $i = L+1, ..., K$, $A_i \cdot y > 0$" is equivalent, by Lemma 4, to the *universal* statement: "For all $z \in \mathbf{Z}_+^K$ for which $\sum_{i=L+1}^K z_i > 0$, we have $\sum_{i=1}^K z_i A_i \neq 0$."

The latter statement is universal, and hence, in a sense falsifiable. The real issue is that the universal quantifier does not typically operate on observables (here, z is simply a vector–but in our example, observed data were revealed preference comparisons). It turns out though, that since z is integer-valued, this universal quantifier can be translated directly into observables. For example, in the revealed preference example, the fact that for all for all $z \in \mathbf{Z}_+^K$ for which $\sum_{i=L+1}^K z_i > 0$, we have $\sum_{i=1}^K z_i A_i \neq 0$ is the same as saying there are no preference cycles. In general, Lemma 4 allows us to find the exact

[3]See also Chambers, Echenique, and Shmaya (2011), where we give a general existence results of universal and effective revealed preference tests. These papers focus on the abstract properties of the revealed preference exercise, while the present paper is about specific tests for specific economic theories.

empirical content of many linear models (Scott (1964) is a classic reference). In fact, it is often the case that one can require the universal quantifier on z to operate over a *finite* number of z.

The Positivstellensatz is a related statement for *polynomial* inequalities. While the Theorem of the Alternative does not appear to be a direct corollary, the statements are related. To understand the statement, we need a bit of notation. Given is a collection of variables, say $\{x_1, ..., x_n\}$. We assume the notion of polynomial is understood. We will describe one variant of the Positivstellensatz (there are variants corresponding to strict inequalities as well).

Given a collection of polynomials $\{f_1, ..., f_m\}$, we define the **ideal** of $\{f_1, ..., f_m\}$ to be the collection of all polynomials which can be written in the form:

$$\sum_{i=1}^{m} g_i f_i,$$

where g_i is a polynomial. We define the **cone** generated by $f_1, ..., f_m$ to be the smallest set of polynomials including all sums of squares of polynomials, all polynomials $f_1, ..., f_m$, and which is closed under addition and multiplication. It is easy to see that any such element can be written as

$$\sum_{S \subseteq \{1,...,m\}} \left(g_S \prod_{i \in S} f_i \right),$$

where g_S is a sum of squares of polynomials. Finally, we define the **multiplicative monoid** generated by $f_1, ..., f_m$ to be the collection of polynomials of the form $\prod_{i=1}^{m} f_i^{a_i}$, where each a_i is a nonnegative integer. The following can be found, for example, in Bochnak, Coste, and Roy (1998), Theorem 4.4.2.

Theorem 6 (Positivstellensatz)**.** *A collection of inequalities $f_i(x) = 0$, $i = 1, ..., m$, $g_i(x) \geq 0$, $i = 1, ..., k$, $h_i(x) \neq 0$, $i = 1, ..., j$ is inconsistent if and only if there exist polynomials f in the ideal of $\{f_1, ..., f_m\}$, g in the cone generated by $\{g_1, ..., g_k\}$, and h in the multiplicative monoid generated by $\{h_1, ..., h_j\}$ for which $f + g + h = 0$.*

The Positivstellensatz thus provides a "dual" system of polynomial inequalities that must be satisfied for satisfaction of some primal system to be possible. Thus, if a system of polynomial inequalities cannot be satisfied, it is possible to demonstrate this. Practically speaking, however, this may be quite difficult. In an interesting recent field of research in the mathematics literature, it has been shown that if one is willing only to search for demonstrations of violations *which have bounded degree*, then the problem becomes much simpler. Indeed, it can be shown to revert to a classical semidefinite programming problem. This approach is outlined in Parrilo (2003), a shorter introduction is provided in Parrilo (2004); see especially Example 1 there. Marshall (2008) Chapter 10 provides a detailed explanation. Thus, there are practical techniques for demonstrating the infeasibility of given list of polynomial inequalities. However, the fact that there are no "bounded" degree polynomials solving the dual system of polynomial inequalities does not prove that the primal list can be satisfied.

It is interesting that, while many economists know and apply the Theorem of the Alternative, there are almost no applications of the Positivstellensatz (with the notable exception of the much-overlooked work of Richter (1975), which itself builds on the work of Tversky (1967)). The theorem is potentially very useful to applied economists, who would use the algorithms in Parrilo (2004) to carry out tests on actual data sets.

To get a sense of how these ideas might be applied in economics, let us consider an environment where we observe several bargaining problems: fix $N = \{1, 2\}$, and suppose we observe $\{(d^t, x^t)\}$. We want to know if these data can be rationalized by the Nash model, in the sense that there exist u_i for each $i \in N$, strictly concave and monotonic, for which x^t solves $\max \prod_{i \in N} (u_i(y_i) - u_i(d_i^t))$ subject to $y \geq d^t$ and $\sum_i y_i = \sum_i x_i^t$. In fact, it is enough to be able to find numbers $u_i(x_i^t), u_i'(x_i^t), u_i(d_i^t)$ such that for all t,

$$u_1'(x_1^t)\left(u_2(x_2^t) - u_2(d_2^t)\right) = u_2'(x_2^t)\left(u_1(x_1^t) - u_1(d_1^t)\right),$$

and such that the numbers are consistent with concavity (for example, if $x_1^t \leq x_1^{t'} < d_1^{t''}$, we require $u_1'(x_1^t) > \frac{u_1(d_1^{t''}) - u_1(x_1^{t'})}{d_1^{t''} - x_1^{t'}}$). These inequalities are all polynomial (in fact, they are all quadratic), so in principle, if they cannot be satisfied, then we can find the dual polynomials in accordance with Theorem 6. Note that, in principle, terms such as $d_1^{t''} - x_1^{t'}$ can be irrational.

The Positivstellensatz is closely related to, but distinct from, the Tarski-Seidenberg theorem, which has seen few but important applications in economics. Brown and Matzkin (1996) (see also Brown and Kubler (2008)) exploit this technique to find testable implications of equilibrium behavior (they also explain how the well-known equivalence of the strong axiom of revealed preference and rationality is a special case of Tarski-Seidenberg). Testing whether a system of polynomial inequalities is feasible turns out to be equivalent to testing whether another (dual) list of polynomial inequalities *in the coefficients* is satisfied for the particular choice of coefficients. The canonical example of this is that there exists x for which $ax^2 + bx + c \geq 0$ if and only if $b^2 - 4ac \geq 0$.

5 Conclusion

We consider finite sets of observations of bargaining outcomes, and develop the testable implications of some of the best-known models in bargaining theory.

We consider two basic frameworks. Our results are sharpest for the case where we assume that disagreement points are fixed across observations. We show that the utilitarian, Nash bargaining, and egalitarian max-min models are all observationally equivalent. Further, we show that a simple test for these models consists in checking that the observed allocations are comonotonic.

When disagreement points can vary, we present a characterization of the data that are consistent with a form of egalitarianism, namely the model of equal gains/losses. The main application of these results are to tax data, where we can check for consistency of the tax code with the principle of equal loss when the utility function is unknown but concave.

Finally, we introduce the Positivstellensatz for testing Nash bargaining and the utilitarian model on data with a variable disagreement point. We do not have a "closed form" test for this case, but the optimization literature has developed useful practical tools that are readily applicable to testing for consistency of a data set with Nash bargaining, for example. We hope that one contribution of our paper will be to draw the attention of economists to these new tools.

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