Identifying Dynamic Games with Serially-Correlated Unobservables*

Yingyao Hu
Dept. of Economics, Johns Hopkins University

Matthew Shum
Division of Humanities and Social Sciences, Caltech

Abstract

In this paper we consider the nonparametric identification of Markov dynamic games models in which each firm has its own unobserved state variable, which is persistent over time. This class of models includes most models in the Ericson and Pakes (1995) and Pakes and McGuire (1994) framework. We provide conditions under which the joint Markov equilibrium process of the firms' observed and unobserved variables can be nonparametrically identified from data. For stationary continuous action games, we show that only three observations of the observed component are required to identify the equilibrium Markov process of the dynamic game. When agents' choice variables are discrete, but the unobserved state variables are continuous, four observations are required.

1   Introduction

In this paper, we consider nonparametric identification in Markovian dynamic games models where which each agent may have its own serially-correlated unobserved state variable. This class of models includes most models in the Ericson and Pakes (1995) and Pakes and McGuire (1994) framework. These models have been the basis for much of the recent

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*The authors can be reached at `yhu@jhu.edu` and `mshum@caltech.edu`.

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empirical applications of dynamic game models. Throughout, by "unobservable", we mean variables which are commonly observed by all agents, and condition their actions, but are unobserved by the researcher.

Consider a dynamic duopoly game in which two firms compete. It is straightforward to extend our assumptions and arguments to the case of N firms. A dynamic duopoly is described by the sequence of variables $\left(W_{t+1}, X_{t+1}^*\right), \left(W_t, X_t^*\right), ..., \left(W_1, X_1^*\right)$ where

$$
\begin{aligned}
W_t &= \left(W_{1,t}, W_{2,t}\right), \\
X_t^* &= \left(X_{1,t}^*, X_{2,t}^*\right).
\end{aligned}
$$

$W_{i,t}$ stands for the observed information on firm i and $X_{i,t}^*$ denote the unobserved heterogeneity of firm i at period t, which we allow to vary over time and be serially-correlated.

In empirical dynamic games model, the observed variables $W_{i,t}$ consists of two variables:

$$
W_{i,t} \equiv (Y_{i,t}, M_{i,t}),
$$

where $Y_{i,t}$ denotes firm i's choice, or control variable in period t, and $M_{i,t}$ denotes the state variables of firm i which are observed by both the firms and the researcher. We assume that the serially-correlated variables $X_{1,t}^*$ and $X_{2,t}^*$ are observed by both firms prior to making their choices of $Y_{1,t}, Y_{2,t}$ in period t, but the researcher never observes X_t^*. For simplicity, we assume that each firm's variables $Y_{i,t}, M_{i,t}, X_{i,t}^*$ are scalar-valued.

Main Results: Our goal is to identify the density

$$
f_{W_t, X_t^* \mid W_{t-1}, X_{t-1}^*}, \tag{1}
$$

which corresponds to the equilibrium transition density of the choice and state variables along the Markov equilibrium path of the dynamic game. In Markovian dynamic settings, the transition density can be factored into two components of interest:

$$
\begin{aligned}
f_{W_t, X_t^* \mid W_{t-1}, X_{t-1}^*} &= f_{Y_t, M_t, X_t^* \mid Y_{t-1}, M_{t-1}, X_{t-1}^*} \\
&= \underbrace{f_{Y_t \mid M_t, X_t^*}}_{\text{CCP}} \cdot \underbrace{f_{M_t, X_t^* \mid Y_{t-1}, M_{t-1}, X_{t-1}^*}}_{\text{state transition}}.
\end{aligned} \tag{2}
$$

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The first term denotes the conditional choice probabilities (CCP) for the firms' actions in period t, conditional on the current state (M_t, X_t^*). In the Markov equilibrium, firms' optimal strategies typically depends just on the current state variables (M_t, X_t^*), but not past values. The second term denotes is the equilibrium Markovian transition probabilities for the state variables (M_t, X_t^*). As shown in Hotz and Miller (1993) and Magnac and Thesmar (2002), once these two structural components are known, it is possible to recover the "deep" structural elements of the model, including the period utility functions.

In an earlier paper (Hu and Shum (2008)), we focused on nonparametric identification of Markovian single-agent dynamic optimization models. There, we showed that in stationary models, four periods of data $W_{t+1}, \ldots, W_{t-2}$ were enough to identify the Markov transition $W_t, X_t^* | W_{t-1}, X_{t-1}^*$, while five observations $W_{t+1}, \ldots, W_{t-3}$ were required for the nonstationary case. In this paper, we focus on Markovian dynamic games. We show that, once additional features of the dynamic optimization framework are taken into account, only three observations $W_t, \ldots, W_{t-2}$ are required to identify $W_t, X_t^* | W_{t-1}, X_{t-1}^*$ in the stationary case, when Y_t is a continuous choice variable. If Y_t is a discrete choice variable (while X_t^* is continuous), then four observations are required for identification.

Related literature Recently, there has been a growing literature related to identification and estimation of dynamic games. Papers include Aguirregabiria and Mira (2007), Pesendorfer and Schmidt-Dengler (2007), Bajari, Benkard, and Levin (2007), Pakes, Ostrovsky, and Berry (2007), and Bajari, Chernozhukov, Hong, and Nekipelov (2007). Our main contribution relative to this literature is to provide nonparametric identification results for the case where there are firm-specific unobserved state variables, which are serially correlated over time. Allowing for firm-specific and serially-correlated unobservables is important, because the dynamic game models in Ericson and Pakes (1995) and Pakes and McGuire (1994) (see also Doraszelski and Pakes (2007)), which provide an important framework for much of the existing empirical work in dynamic games, explicitly contain firm-specific "product quality" variables which are typically unobserved by researchers.

A few recent papers have considered estimation methodologies for games with serially-correlated unobservables. Arcidiacono and Miller (2006) develop an EM-algorithm for estimating dynamic games where the unobservables are assumed to follow a discrete Markov process. Siebert and Zulehner (2008) extend the Bajari, Benkard, and Levin (2007) approach to estimate a dynamic product choice game for the computer memory industry where each firm experiences a serially-correlated productivity shock. Finally, Blevins (2008) de-

velops simulation estimators for dynamic games with serially-correlated unobservables, utilizing state-of-the-art recursive importance sampling ("particle filtering") techniques. However, all these papers focus on estimation of parametric models in which the parameters are assumed to be identified, whereas this paper concerns nonparametric identification.

The class of models considered in this paper also resemble models analyzed in the dynamic treatment effects literature in labor economics (eg. Cunha, Heckman, and Schennach (2006), Abbring and Heckman (2007), Heckman and Navarro (2007)). Specifically, Cunha, Heckman, and Schennach (2006) apply the result of Hu and Schennach (2008) to show nonparametric identification of a nonlinear factor model consisting of $(W_t, W_t', W_t'', X_t^*)$, where the observed processes $\{W_t\}_{t=1}^T$, $\{W_t'\}_{t=1}^T$, and $\{W_t''\}_{t=1}^T$ constitute noisy measurements of the latent process $\{X_t^*\}_{t=1}^T$, contaminated with random disturbances. In contrast, we consider a different setting where (W_t, X_t^*) jointly evolves as a dynamic first-order Markov process.

2 Examples of Dynamic Duopoly Games

To make things concrete, we present two examples of a dynamic duopoly problem, both of which are in the "dynamic investment" framework of Ericson and Pakes (1995) and Pakes and McGuire (1994), but simplified without an entry decision.

Example 1 is a model of learning by doing in a durable goods market, similar to Benkard (2004). There are two heterogeneous firms $i = 1, 2$, with each firm described by two time-varying state variables $(M_{i,t}, X_{i,t}^*)$. $M_{i,t}$ denotes the "installed base" of firm i, which are the share of consumers who have previously bought firm i's product. $X_{i,t}^*$ is firm i's marginal cost, which is unobserved to the econometrician, and is an unobserved state variable. There is learning by doing, in the sense that increases in the installed base will lower future marginal costs. In each period, each firm's choice variable $Y_{i,t}$ is its period t price, which affects the demand for its product in period t and thereby the future installed base, which in turn affects future production costs.

In the following, let $Y_t \equiv (Y_{1,t}, Y_{2,t})$, and similarly for M_t and X_t^*. Let $S_t \equiv (M_t, X_t^*)$ denote the payoff-relevant state variables of the game in period t. $S_{i,t} \equiv (M_{i,t}, X_{i,t}^*)$, for $i = 1, 2$, denotes firm i's state variables. Each period, firms earn profits by selling their products to

consumers who have not yet bought the product. The demand curve for firm i's product is

$$q_i(Y_t, M_t, \eta_{i,t})$$

which depends on the price and installed base of both firms' products. Firm i's demand also depends on $\eta_{i,t}$, a firm-specific demand shock. As in Aguirregabiria and Mira (2007) and Pesendorfer and Schmidt-Dengler (2007), we assume that $\eta_{i,t}$ is privately observed by each firm; that is, only firm 1, but not firm 2, observes $\eta_{1,t}$, making this a game of incomplete information. Furthermore, we assume that the demand shocks $\eta_{i,t}$ are i.i.d. across firm and periods, and distributed according to a distribution K which is common knowledge to both firms. The main role of the variable $\eta_{i,t}$ is to generate randomness in $Y_{i,t}$, even after conditioning on (M_t, X_t^*).

The period t profits of firm i can then be written:

$$\Pi_i(Y_t, S_t, \eta_{i,t}) = q_i(Y_t, M_t, \eta_{i,t}) * (Y_{i,t} - X_{i,t}^*)$$

where $Y_{i,t} - X_{i,t}^*$ is firm i's margin from each unit that it sells.

Installed base evolves according to the conditional distribution:

$$M_{i,t+1} \sim G(\cdot | M_{i,t}, Y_{i,t}). \tag{3}$$

One example is to model the incremental change $M_{i,t+1} - M_{i,t}$ as a log-normal random variable

$$\log(M_{i,t+1} - M_{i,t}) \sim q_i(Y_t, M_t, \eta_{i,t}) + \epsilon_{i,t}, \quad \epsilon_{i,t} \sim N(0, \sigma_\epsilon^2), \text{i.i.d.-}(i, t).$$

Marginal cost evolves according to the conditional distribution

$$X_{i,t+1}^* \sim H(\cdot | X_{i,t}^*, M_{i,t+1}). \tag{4}$$

One example is

$$X_{i,t+1}^* = X_{i,t}^* - N(\gamma(M_{i,t+1} - M_{i,t}), \sigma_k^2)$$

where γ and σ_k are unknown parameters. This encompasses learning-by-doing because the incremental reduction in marginal cost $(X_{i,t+1}^* - X_{i,t}^*)$ depends on the incremental increase

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in installed base $(M_{i,t+1} - M_{i,t})$.

In the dynamic Markov-perfect equilibrium, each firm's optimal pricing strategy will also be a function of the current S_t, and the current demand shock $\eta_{i,t}$:

$$Y_{i,t} = Y_i^*(S_t, \eta_{i,t}), \quad i = 1, 2 \tag{5}$$

where the strategy satisfies the equilibrium Bellman equation:

$$
\begin{aligned}
Y_i^*(S_t, \eta_{i,t}) = \mathrm{argmax}_y E_{\eta_{-i,t}} \bigg\{ &\Pi_i(S_t, y, Y_{-i,t} = Y_{-i}^*(S_t, \eta_{-i,t})) + \\
&\beta E\left[V_i(S_{t+1}, \eta_{i,t+1})|y, Y_{-i,t} = Y_{-i}^*(S_t, \eta_{-i,t})\right]\bigg\}
\end{aligned}
\tag{6}
$$

subject to Eqs. (4) and (3). In the above equation, $V_i(S_t, \eta_{it})$ denotes the equilibrium value function for firm i, which is equal to the expected discounted future profits that firm i will make along the equilibrium path, starting at the current state (S_t, η_{it}). ∎

Example 2   is a simplified version of the dynamic investment models estimated in the productivity literature. (See Ackerberg, Benkard, Berry, and Pakes (2007) for a detailed survey of this literature.) In this model, firms' state variables are $(M_{i,t}, X_{i,t}^*)$, where $M_{i,t}$ denotes firm i's capital stock, and $X_{i,t}^*$ denotes its productivity shock in period t. $Y_{i,t}$, firm i's choice variable, denotes new capital investment in period t.

Capital stock $M_{i,t}$ evolves deterministically, as a function of $(Y_{i,t-1}, M_{i,t-1})$:

$$M_{i,t} = (1 - \delta) \cdot M_{i,t-1} + Y_{i,t-1}. \tag{7}$$

The productivity shock is serially correlated, and evolves according to the conditional distribution

$$X_{i,t+1}^* \sim H(\cdot|X_{i,t}^*, M_{i,t}). \tag{8}$$

Each period, firms earn profits by selling their products. Let $q_i(p_{i,t}, p_{-i,t}, \eta_{i,t})$ denote the demand curve for firm i's product, which depends on the quality and prices of both firms' products. As in Example 1, $\eta_{i,t}$ denotes the privately observed demand shock for firm i in period t, which is distributed i.i.d. across firms and time periods.

The period t profits of firm i are:

$$q_i(p_{i,t}, p_{-i,t}, \eta_{i,t}) * (p_{i,t} - c_i(S_{i,t})) - K(Y_{i,t})$$

where $c_i(\cdot)$ is the marginal cost function for firm i (we assume constant marginal costs) and $K(Y_{it})$ is the investment cost function.

Following the literature, we assume that each firm's price in period t are determined by a static equilibrium, given the current values of the state variables S_t, and the firm-specific demand shock $\eta_{i,t}$. Let $p_i^*(S_t, \eta_{i,t})$ denote the static equilibrium prices for each firm in period t. By substituting in the equilibrium prices in firm's profit function, we obtain each firm's "reduced-form" expected profits:

$$\Pi_i(S_t, Y_t, \eta_{i,t}) = E_{\eta_{-i,t}} q_i\big(p_1^*(S_t, \eta_{1,t}), p_2^*(S_t, \eta_{2,t}), \eta_{i,t}\big) * \big[p_i^*(S_t, \eta_{i,t}) - c_i(S_{i,t})\big] - K(Y_{i,t}), \quad i = 1, 2$$

As in Example 1, the Markov equilibrium investment strategy for each firm just depends on the current state variables S_t, and the current shock $\eta_{i,t}$:

$$Y_t = Y_i^*(S_t, \eta_{it}), \quad i = 1, 2.$$

subject to the Bellman equation (6) and the transitions (7) and (8). ■

The substantial difference between examples 1 and 2 is that in example 2, the evolution of the observed state variable $M_{i,t}$ is deterministic, whereas in example 1 there is randomness in $M_{i,t}$ conditional on $(M_{i,t-1}, Y_{i,t-1})$ (i.e., compare Eqs. (3) and (7)). As we will see below, this has important implications for nonparametric identification.

Moreover, as illustrated in these two examples, for the first part of the paper we focus on games with continuous actions, so that Y_t are continuous variables. Later, we will consider the important alternative case of discrete-action games, where Y_t is discrete-valued.

3   Nonparametric identification

In this section, we present the assumptions for nonparametric identification in the dynamic game model. The assumption we make here are different than those in our earlier paper (Hu and Shum (2008)), and are geared specifically for the dynamic games literature, and

motivated directly by existing applied work utilizing dynamic games. We assume that for each market j, $\left\{ \left(W_{t+1}, X_{t+1}^*\right), \left(W_t, X_t^*\right), ..., \left(W_1, X_1^*\right)\right\}_j$ is an independent random draw from the identical distribution $f_{W_{t+1},W_t,...,W_1,X_{t+1}^*,X_t^*,...,X_1^*}$. This rules out across-market effects and spillovers. For each market j, $\{W_1, \ldots, W_T\}_j$ is observed, for $T \geq 4$.

After presenting each assumption, we relate it to the examples in the previous section. Define $\Omega_{<t} = \left\{W_{t-1}, ..., W_1, X_{t-1}^*, ..., X_1^*\right\}$. We assume the dynamic process satisfies:

Assumption 1 *First-order Markov:*

$$f_{W_t, X_t^*|W_{t-1}, X_{t-1}^*, \Omega_{<t-1}} = f_{W_t, X_t^*|W_{t-1}, X_{t-1}^*}; \tag{9}$$

Remark: The first-order Markov assumption is satisfied along the Markov-equilibrium path of both examples given in the previous section. ∎

Without loss of generality, we assume that $W_t = (Y_t, M_t) \in \mathbb{R}^2$. We assume

Assumption 2

$$
\begin{aligned}
&(i) \quad f_{Y_t|M_t, X_t^*, Y_{t-1}, M_{t-1}, X_{t-1}^*} = f_{Y_t|M_t, X_t^*}, \\
&(ii) \quad f_{X_t^*|M_t, Y_{t-1}, M_{t-1}, X_{t-1}^*} = f_{X_t^*|M_t, M_{t-1}, X_{t-1}^*}.
\end{aligned}
$$

Assumption 2(i) is motivated completely by the state-contingent aspect of the optimal policy function in dynamic optimization models. It turns out that this assumption is stronger than necessary for our identification, but it allows us to achieve identification only using three periods of data. Assumption 2(ii) implies that X_t^* is independent of Y_{t-1} conditional on M_t, M_{t-1} and X_{t-1}^*. This is consistent with the setup above.

Remarks: Assumption 2 is satisfied in both examples 1 and 2. ∎

The conditional independence assumptions 1-2 imply that the Markov transition density (1) can be factored into

$$
\begin{aligned}
f_{W_t, X_t^*|W_{t-1}, X_{t-1}^*} &= f_{Y_t, M_t, X_t^*|Y_{t-1}, M_{t-1}, X_{t-1}^*} \\
&= f_{Y_t|M_t, X_t^*} \cdot f_{X_t^*|M_t, M_{t-1}, X_{t-1}^*} \cdot f_{M_t|Y_{t-1}, M_{t-1}, X_{t-1}^*}. \tag{10}
\end{aligned}
$$

In the identification procedure, we will identify these three components of $f_{W_t, X_t^*|W_{t-1}, X_{t-1}^*}$

in turn.

Next, we restrict attention to stationary equilibria in the dynamic game, which is natural given our focus on Markov equilibria. In stationary equilibria, the Markov transition density $f_{W_t, X_t^* | W_{t-1}, X_{t-1}^*}$ is time-invariant.

Assumption 3 *Stationarity of Markov kernel:*

$$f_{W_t, X_t^* | W_{t-1}, X_{t-1}^*} = f_{W_2, X_2^* | W_1, X_1^*}.$$

Consider the joint density of $\{Y_t, M_t, Y_{t-1}, M_{t-1}, Y_{t-2}\}$. Let $supp\,(Y_t, M_t, Y_{t-1}, M_{t-1}, Y_{t-2})$ denote for the support of the random vector $\{Y_t, M_t, Y_{t-1}, M_{t-1}, Y_{t-2}\}$, together with $\mathcal{Y}_t = supp\,(Y_t)$, $\mathcal{M}_t = supp\,(M_t)$, and $\mathcal{X}_t^* = supp\,(X_t^*)$.

We show in the Appendix, that Assumptions 1-2 imply that

$$
\begin{aligned}
& f_{Y_t, M_t, Y_{t-1} | M_{t-1}, Y_{t-2}} \\
= \ & \int f_{Y_t | M_t, M_{t-1}, X_{t-1}^*} f_{M_t, Y_{t-1} | M_{t-1}, X_{t-1}^*} f_{X_{t-1}^* | M_{t-1}, Y_{t-2}} dx_{t-1}^* \\
= \ & \int f_{Y_t | M_t, M_{t-1}, X_{t-1}^*} f_{M_t, Y_{t-1}, X_{t-1}^* | M_{t-1}, Y_{t-2}} dx_{t-1}^* \qquad (11)
\end{aligned}
$$

where the final line follows from Assumptions 1-2.

Let $\omega\,(\cdot)$ be a user-specified function. For example, $\omega\,(\cdot)$ may be $\omega\,(x) = x$, x^2, c, $I(x > c)$, or $I(x = c)$ for some constant c. In practice, the choice of $\omega(\cdot)$ will be model-specific. As will be clear below, one reason we introduce the $\omega(\cdot)$ function is in order to accommodate games where firms' choice variables Y_{it} are deterministic functions of the current state (M_t, X_t^*). (For both examples in the previous section, this would correspond to the case without the firm-specific shocks η_{it}.) Given Eq. (11), it follows that, for any $(x, m_t, m_{t-1}, z) \in supp\,(Y_t, M_t, M_{t-1}, Y_{t-2})$

$$
\begin{aligned}
& \int \omega\,(y_{t-1})\, f_{Y_t, M_t, Y_{t-1} | M_{t-1}, Y_{t-2}}\,(x, m_t, y_{t-1} | m_{t-1}, z)\, dy_{t-1} \qquad (12) \\
= \ & \int \Big[f_{Y_t | M_t, M_{t-1}, X_{t-1}^*}\,\big(x | m_t, m_{t-1}, x_{t-1}^*\big)\, E\left[\omega\,(Y_{t-1})\, | m_t, m_{t-1}, x_{t-1}^*\right] \cdot \\
& \qquad f_{M_t, | M_{t-1}, X_{t-1}^*}\,\big(m_t, | m_{t-1}, x_{t-1}^*\big)\, f_{X_{t-1}^* | M_{t-1}, Y_{t-2}}\,\big(x_{t-1}^* | m_{t-1}, z\big) \Big] dx_{t-1}^*,
\end{aligned}
$$

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where we use the shorthand notation

$$E\left[\omega\left(Y_{t-1}\right)|m_t, m_{t-1}, x_{t-1}^*\right] = E\left[\omega\left(Y_{t-1}\right)|M_t = m_t, M_{t-1} = m_{t-1}, X_{t-1}^* = x_{t-1}^*\right].$$

In order to identify the unknown densities on the right hand side, we use the identification strategy for the nonclassical measurement error models in Hu and Schennach (2008). Their results imply that two measurements and a dependent variable of a latent explanatory variable are enough to achieve identification. For fixed values of (M_t, M_{t-1}), we see that (Y_t, Y_{t-1}, Y_{t-2}) enter equation (11) separately in, respectively, the first, second, and third terms. This implies that we can use (Y_t, Y_{t-2}) as the two measurements and Y_{t-1} as the dependent variable of the latent variable X_{t-1}^*.

Since the function $\omega(\cdot)$ is user-specified, the density on the left hand side of equation (12) is observed in the data. Let $\mathcal{L}^p(\mathcal{X})$, $1 \le p < \infty$ stand for the space of functions $h(\cdot)$ with $\int_{\mathcal{X}} |h(x)|^p dx < \infty$, and let $\mathcal{L}^\infty(\mathcal{X})$ denote the space of functions $h(\cdot)$ with $\sup_{x \in \mathcal{X}} |h(x)| < \infty$. For any $1 \le p \le \infty$, We define the integral operator $L_{Y_t, m_t, \omega|m_{t-1}, Y_{t-2}} : \mathcal{L}^p(\mathcal{Y}_{t-2}) \to \mathcal{L}^p(\mathcal{Y}_t))$ for any given $(m_t, m_{t-1}) \in supp(M_t, M_{t-1})$ and any $h \in \mathcal{L}^p(\mathcal{Y}_{t-2})$,

$$\left(L_{Y_t, m_t, \omega|m_{t-1}, Y_{t-2}} h\right)(x) = \int k(x, m_t|m_{t-1}, z) h(z) dz,$$

$$k(x, m_t|m_{t-1}, z) = \int \omega(y_{t-1}) f_{Y_t, M_t, Y_{t-1}|M_{t-1}, Y_{t-2}}(x, m_t, y_{t-1}|m_{t-1}, z) dy_{t-1}.$$

We also define the operator corresponding to the unobserved density $f_{Y_t|M_t, M_{t-1}, X_{t-1}^*}$, i.e., $L_{Y_t|m_t, m_{t-1}, X_{t-1}^*} : \mathcal{L}^p(\mathcal{X}_{t-1}^*) \to \mathcal{L}^p(\mathcal{Y}_t)$, as follows:

$$\left(L_{Y_t|m_t, m_{t-1}, X_{t-1}^*} h\right)(x) = \int f_{Y_t|M_t, M_{t-1}, X_{t-1}^*}(x|m_t, m_{t-1}, x_{t-1}^*) h(x_{t-1}^*) dx_{t-1}^*.$$

As shown in Hu and Schennach (2008), the identification of an operator, e.g., $L_{Y_t|m_t, m_{t-1}, X_{t-1}^*}$, is equivalent to that of its corresponding density, e.g., $f_{Y_t|M_t, M_{t-1}, X_{t-1}^*}$. Identification of $L_{Y_t|m_t, m_{t-1}, X_{t-1}^*}$ from the observed $L_{Y_t, m_t, \omega|m_{t-1}, Y_{t-2}}$ requires

Assumption 4 *For a known $\omega(\cdot)$ and any $(m_t, m_{t-1}) \in supp(M_t, M_{t-1})$,*

(i) $L_{Y_t, m_t, \omega|m_{t-1}, Y_{t-2}}$ is one-to-one;

(ii) $L_{Y_t|m_t, m_{t-1}, X_{t-1}^*}$ is one-to-one.

Assumption 4(ii) rules out cases where Y_t is discrete-valued, but X_{t-1}^* is continuous. Hence,

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in this section, we are restricting attention to games with continuous action spaces, so that (Y_t, X_{t-1}^*) are both continuous.[1]

Remark: Instead of proving that the operators are one-to-one in examples 1 and 2, here we only discuss whether the variation in Y_{t-2} or X_{t-1}^* will imply variation in Y_t for fixed (m_t, m_{t-1}), which is necessary in order for the one-to-one assumptions to hold. In both examples 1 and 2, equation (12) implies that, just holding m_t, y_{t-1}, and and m_{t-1} fixed, variation in Y_{t-2} will affect X_{t-1}^*. In turn, changes in X_{t-1}^* will induce changes in X_t^*, which will directly affect Y_t. This reasoning implies that Assumption 4(i) should hold as long as $E\left[\omega\left(Y_{t-1}\right)|m_t, m_{t-1}, x_{t-1}^*\right]$ does not vanish for a nontrivial set of x_{t-1}^*.

Similarly, variation in X_{t-1}^* implies changes in X_t^*, which directly affects Y_t, even when m_t and m_{t-1} are fixed. Therefore, Assumption 4(ii) should hold. However, note that in order for one-to-one to hold, the supports of Y_t and X_t^* should be of the same dimension. If, for example, Y_t is discrete (as in entry games), but X_t^* is continuous, then the condition would fail. ∎

As shown in the Appendix, equation (12) can be written in operator notation as (for fixed (m_t, m_{t-1})):

$$L_{Y_t, m_t, \omega|m_{t-1}, Y_{t-2}} = L_{Y_t|m_t, m_{t-1}, X_{t-1}^*} D_{\omega|m_t, m_{t-1}, X_{t-1}^*} D_{m_t|m_{t-1}, X_{t-1}^*} L_{X_{t-1}^*|m_{t-1}, Y_{t-2}}, \quad (13)$$

where $D_{\omega|m_t, m_{t-1}, X_{t-1}^*}$ and $D_{m_t|m_{t-1}, X_{t-1}^*}$ are diagonal operators.

Since $\omega(\cdot)$ is a user-specified function, we can evaluate equation (12) with $\omega(x) = 1$, which yields

$$f_{Y_t, M_t|M_{t-1}, Y_{t-2}} = \int f_{Y_t|M_t, M_{t-1}, X_{t-1}^*} f_{M_t|M_{t-1}, X_{t-1}^*} f_{X_{t-1}^*|M_{t-1}, Y_{t-2}} dx_{t-1}^*. \quad (14)$$

The above can be expressed in operator notation as

$$L_{Y_t, m_t|m_{t-1}, Y_{t-2}} = L_{Y_t|m_t, m_{t-1}, X_{t-1}^*} D_{m_t|m_{t-1}, X_{t-1}^*} L_{X_{t-1}^*|m_{t-1}, Y_{t-2}} \quad (15)$$

for any given $(m_t, m_{t-1}) \in supp(M_t, M_{t-1})$. (See the appendix for the full definition of these operators.)

---

[1]On the other hand, if X_{t-1}^* is discrete, one may allow Y_t to be discrete with a support no smaller than that of X_{t-1}^*.

As in Hu and Schennach (2008), if the latter operator relation can be inverted (which is ensured by Assumption 4), we can combine Eqs. (13) and (15) to get

$$
\begin{aligned}
& L_{Y_t, m_t, \omega | m_{t-1}, Y_{t-2}} L_{Y_t, m_t | m_{t-1}, Y_{t-2}}^{-1} \\
\equiv\ & L_{Y_t | m_t, m_{t-1}, X_{t-1}^*} \cdot D_{\omega | m_t, m_{t-1}, X_{t-1}^*} \cdot L_{Y_t | m_t, m_{t-1}, X_{t-1}^*}^{-1}.
\end{aligned}
\tag{16}
$$

This representation shows that an eigenvalue-eigenfunction decomposition of the observed operator $L_{Y_t, m_t, \omega | m_{t-1}, Y_{t-2}} L_{Y_t, m_t | m_{t-1}, Y_{t-2}}^{-1}$ yields the unknown density functions $f_{Y_t | m_t, m_{t-1}, X_{t-1}^*}$ as the eigenfunctions and $E\left[\omega\left(Y_{t-1}\right) | m_t, m_{t-1}, x_{t-1}^*\right]$ as the eigenvalues.

The following assumption ensures the uniqueness of this decomposition, and restricts the choice of the $\omega(\cdot)$ function.

Assumption 5 *(i) For any $(m_t, m_{t-1}, x_{t-1}^*) \in supp\left(M_t, M_{t-1}, X_{t-1}^*\right)$, there exists a known $\omega(\cdot)$ such that*

$$
\left| E\left[\omega\left(Y_{t-1}\right) | m_t, m_{t-1}, x_{t-1}^*\right] \right| < \infty.
\tag{17}
$$

(ii) For any $(m_t, m_{t-1}, x_{t-1}^*)$ and $(m_t, m_{t-1}, \widetilde{x}_{t-1}^*)$ in $supp\left(M_t, M_{t-1}, X_{t-1}^*\right)$ with $x_{t-1}^* \neq \widetilde{x}_{t-1}^*$, there exists a known $\omega(\cdot)$ such that

$$
E\left[\omega\left(Y_{t-1}\right) | m_t, m_{t-1}, x_{t-1}^*\right] \neq E\left[\omega\left(Y_{t-1}\right) | m_t, m_{t-1}, \widetilde{x}_{t-1}^*\right],
\tag{18}
$$

where both expectations are nonzero.

Assumption 5(i) implies that the conditional expectation $E\left[\omega\left(Y_{t-1}\right) | m_t, m_{t-1}, x_{t-1}^*\right]$ is always bounded. Assumption 5(ii) implies that a change in x_{t-1}^* can be detected by the conditional expectation of $\omega\left(Y_{t-1}\right)$ for some known function $\omega(\cdot)$.

Remark: For example 1, as long as the density $f_{Y_{1,t-1} | M_t, M_{t-1}, X_{t-1}^*}$ has a finite mean, we may use $\omega\left(y_{t-1}\right) = y_{1,t-1}$ to satisfy assumption 5(i). When we use $\omega\left(y_{t-1}\right) = I\left(y_{1,t-1} > c\right)$, $E\left[\omega\left(Y_{t-1}\right) | m_t, m_{t-1}, x_{t-1}^*\right] = \Pr\left(Y_{1,t-1} > c | m_t, m_{t-1}, x_{t-1}^*\right)$ also satisfies assumption 5(i).

On the other hand, $M_t | M_{t-1}, Y_{t-1}$ is deterministic for example 2. Hence, for fixed (m_t, m_{t-1}), the density $f_{Y_{t-1} | M_t, M_{t-1}, X_{t-1}^*}$ is only defined at a single point $y_{t-1} = m_t - (1-\delta)m_{t-1}$, which is finite. Therefore, in this example, $E\left[\omega\left(Y_{t-1}\right) | m_t, m_{t-1}, x_{t-1}^*\right]$ is trivially bounded, for $\omega(Y_{t-1}) = Y_{t-1}$.

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Assumption 5(ii) requires that the conditional expectation $E\left[\omega\left(Y_{t-1}\right)|m_t, m_{t-1}, x^*_{t-1}\right]$ varies in X^*_{t-1} given any fixed (m_t, m_{t-1}), so that the "eigenvalues" in the decomposition (16) are distinctive. For example 1, given the preceding discussion, if assumption 5(i) is satisfied, then assumption 5(ii) will hold if, for any $(m_t, m_{t-1}, x^*_{t-1})$ in $supp\left(M_t, M_{t-1}, X^*_{t-1}\right)$,

$$0 < \left|E\left[\omega\left(Y_{t-1}\right)|m_t, m_{t-1}, x^*_{t-1}\right]\right| < \infty \text{ and } \frac{\partial E\left[\omega\left(Y_{t-1}\right)|m_t, m_{t-1}, x^*_{t-1}\right]}{\partial x^*_{t-1}} \neq 0.$$

For example 2, the capital stock M_t evolves deterministically, so that $E\left[\omega\left(Y_{t-1}\right)|m_t, m_{t-1}, x^*_{t-1}\right] = \omega\left(m_t - (1-\delta)m_{t-1}\right)$. Since this does not change with x^*_{t-1} for any fixed (m_t, m_{t-1}), Therefore, assumption 5(ii) fails. ∎

Remark (Deterministic choices): In some models, the choice variable Y_{it} is a deterministic function of the current state variables, i.e.,

$$Y_{i,t-1} = g_i(M_{t-1}, X^*_{t-1}), \quad i = 1, 2. \tag{19}$$

In examples 1 and 2, this would be the case if we eliminated the privately-observed demand shocks η_{1t} and η_{2t}.

With $\omega\left(y_{t-1}\right) = y_{1,t-1}$, we have

$$E\left[\omega\left(Y_{t-1}\right)|m_t, m_{t-1}, x^*_{t-1}\right] = g_1(m_{t-1}, x^*_{t-1}).$$

Therefore, a sufficient condition for assumption 5 is that for any m_{t-1} and x^*_{t-1},

$$0 < \left|g_1(m_{t-1}, x^*_{t-1})\right| < \infty \text{ and } \frac{\partial g_1(m_{t-1}, x^*_{t-1})}{\partial x^*_{t-1}} \neq 0.$$

That means Y_{t-1} is monotonic in X^*_{t-1} for any given m_{t-1}. ∎

Remark: Notice that in the decomposition (16), the function $\omega\left(\cdot\right)$ only appears in the eigenvalues. Therefore, if there are several $\omega(\cdot)$ functions which satisfy Assumption (5), the decompositions (16) using these different $\omega(\cdot)$'s should yield the same eigenfunctions. Hence, depending on the specific model, it may be possible to use this feature as a general specification check for Assumptions (1) and (2). We do not explore this possibility here. ∎

Under the foregoing assumptions, the density $Y_t, m_t, y_{t-1}|m_{t-1}, Y_{t-2}$ and the function $\omega\left(\cdot\right)$

(for fixed m_t, m_{t-1}) can form a unique eigenvalue-eigenvector decomposition. In this decomposition, the eigenfunction corresponds to the density $f_{Y_t|m_t,m_{t-1},X^*_{t-1}}(\cdot|m_t, m_{t-1}, x^*_{t-1})$ which can be written as

$$f_{Y_t|m_t,m_{t-1},X^*_{t-1}}(\cdot|m_t, m_{t-1}, x^*_{t-1}) = f_{Y_{1,t},Y_{2,t}|m_t,m_{t-1},X^*_{1,t-1},X^*_{2,t-1}}(\cdot, \cdot|m_t, m_{t-1}, x^*_{1,t-1}, x^*_{2,t-1}).$$
(20)

The eigenvalue-eigenfunction decomposition only identifies this eigenfunction up to some arbitrary ordering of the $(x^*_{1,t-1}, x^*_{2,t-1})$ argument. Hence, in order to pin down the right ordering of x^*_{t-1}, an additional ordering assumption is required. In our earlier paper (Hu and Shum (2008)), where x^*_t was scalar-valued, a monotonicity assumption sufficed to pin down the ordering of x^*_t. However, in dynamic games, x^*_{t-1} is multivariate, so that monotonicity is no longer well-defined.

Consider the marginal density

$$f_{Y_{i,t}|m_t,m_{t-1},X^*_{1,t-1},X^*_{2,t-1}}(\cdot|m_t, m_{t-1}, x^*_{1,t-1}, x^*_{2,t-1}),$$

which can be computed from Eq. (20) above. We make the following ordering assumption:

Assumption 6 *(i) for any given $(m_t, m_{t-1}) \in supp(M_t, M_{t-1})$ and $x^*_{2,t}$, there exist a known functional G such that $G\left[f_{Y_{1,t}|M_t,M_{t-1},X^*_{t-1}}(\cdot|m_t, m_{t-1}, x^*_{1,t-1}, x^*_{2,t-1})\right]$ is monotonic in $x^*_{1,t-1}$;*
(ii) Without loss of generality, we normalize $x^*_{1,t-1}$ as

$$x^*_{1,t-1} = G\left[f_{Y_{1,t}|M_t,M_{t-1},X^*_{t-1}}(\cdot|m_t, m_{t-1}, x^*_{1,t-1}, x^*_{2,t-1})\right];$$

(iii) $x^*_{2,t-1}$ also satisfies (i) and (ii), i.e.,

$$x^*_{2,t-1} = G\left[f_{Y_{2,t}|M_t,M_{t-1},X^*_{t-1}}(\cdot|m_t, m_{t-1}, x^*_{1,t-1}, x^*_{2,t-1})\right]$$

for any given (m_t, m_{t-1}) and $x^*_{1,t-1}$.

Remark: Consider example 1, and firm 1. A sufficient condition for this assumption is that $f_{Y_{1,t}|m_t,m_{t-1},X^*_{1,t-1},X^*_{2,t-1}}$ is stochastically increasing (in the sense of first-order stochastic dominance) in $x^*_{1,t-1}$, for fixed $x^*_{2,t-1}$ and m_t, m_{t-1}. Note that, in the Markov equilibrium,

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$Y_{1,t}$ is a function of $M_t, X_t^*, \eta_{i,t}$. Hence, the distribution of $Y_{1,t}|M_t, M_{t-1}, X_{1t-1}^*, X_{2,t-1}^*$ can be written as:

$$Y_i^*(X_{1,t}^*, X_{2,t}^*, m_t, \eta_{i,t})|m_t, m_{t-1}, X_{1,t-1}^*, x_{2,t-1}^*.$$

We see from the above that, once $m_t, m_{t-1}, x_{2,t-1}^*$ are fixed then $X_{1,t-1}^*$ affects $Y_{1,t}$ only through the conditional distribution of $X_{1,t}^*|m_{1,t-1}, X_{1,t-1}^*$. Hence, assumption 6(i) is satisfied if (i) $Y_{i,t}$ is increasing in $X_{1,t}^*$, for fixed $m_t, x_{2,t}^*, \eta_{1,t}$; and (ii) the distribution $X_{1,t}^*|m_{1,t-1}, X_{1,t-1}^*$ is stochastically increasing in $X_{1,t-1}^*$.[2] Both (i) and (ii) should be confirmed on a model-by-model basis, but is not unreasonable given the interpretation of $Y_{i,t}$ as a price and $X_{1,t}^*$ as a marginal cost variable. ∎

From the eigenvalue-eigenvector decomposition in Eq. (16), we can identify the densities $f_{Y_t|m_t, m_{t-1}, X_{t-1}^*}$. Eq. (11) and assumption 4(ii) then imply the identification of the joint density $f_{m_t, y_{t-1}, X_{t-1}^*|m_{t-1}, Y_{t-2}}$. This can be factored into the components $f_{m_t, y_{t-1}|m_{t-1}, X_{t-1}^*}$, and $f_{X_{t-1}^*|m_{t-1}, Y_{t-2}}$, for all (y_{t-1}, m_t, m_{t-1}) with positive support. From the factorization

$$f_{M_t, Y_{t-1}|M_{t-1}, X_{t-1}^*} = f_{M_t|Y_{t-1}, M_{t-1}, X_{t-1}^*} \cdot f_{Y_{t-1}|M_{t-1}, X_{t-1}^*}$$

we can recover $f_{M_t|Y_{t-1}, M_{t-1}, X_{t-1}^*}$ and $f_{Y_{t-1}|M_{t-1}, X_{t-1}^*}$. Given stationarity, the latter density is identical to $f_{Y_t|M_t, X_t^*}$, so that from $f_{M_t, Y_{t-1}|M_{t-1}, X_{t-1}^*}$ we have recovered the first two components of $f_{W_t, X_t^*|W_{t-1}, X_{t-1}^*}$ in Eq. (10).

All that remains now is to identify the third component $f_{X_t^*|M_t, M_{t-1}, X_{t-1}^*}$. To obtain this, note that the following operator relation holds (see the Appendix for details):

$$f_{Y_t|m_t, m_{t-1}, X_{t-1}^*} = L_{Y_t|m_t, X_t^*} f_{X_t^*|m_t, m_{t-1}, X_{t-1}^*}$$

for given (m_t, m_{t-1}), and where we define the operator $L_{Y_t|m_t, X_t^*} : \mathcal{L}^p(\mathcal{X}_t^*) \to \mathcal{L}^p(\mathcal{Y}_t)$,

$$\left(L_{Y_t|m_t, X_t^*} h\right)(x) = \int f_{Y_t|M_t, X_t^*}(x|m_t, x_t^*) h(x_t^*) dx_t^*.$$

Identification of $f_{X_t^*|M_t, M_{t-1}, X_{t-1}^*}$ then requires

Assumption 7 *for any $m_t \in \mathcal{M}_t$, $L_{Y_t|m_t, X_t^*}$ is one-to-one.*

---

[2]Note that we do not have to worry about an indirect effect of X_{1t-1}^* on Y_{1t} through the X_{2t-1}^*. This is because, for example 1, the evolution of X_{2t}^* depends just on M_t and X_{2t-1}^* (cf. Eq. (4)), which are both held fixed for Assumption 5.

Similarly to assumption 4(ii), this one-to-one assumption rules out the case where the support of Y_t is discrete, but X_t^* is continuous.

Given assumption (7), the final component in Eq. (10) can be recovered as:

$$f_{X_t^*|m_1,m_{t-1},X_{t-1}^*} = \left(L_{Y_t|m_t,X_t^*}\right)^{-1} f_{Y_t|m_t,m_{t-1},X_{t-1}^*} \tag{21}$$

where both terms on the right-hand-side have already been identified in previous steps.

Remark (deterministic choices, continued): For the case where Y_{it} is determinisitc given the current state variables (X_t^*, M_t), and $y_{it} = g_i(m_t, x_t^*)$, for both firms $i = 1, 2$, we can take:

$$f_{Y_t|m_t,X_t^*} = \mathbf{1}\left(Y_{it} = g_i(m_t, X_t^*), \ i = 1, 2\right).$$

Given knowledge of $f_{m_t,y_{t-1}|m_{t-1},X_{t-1}^*}$, the functions $g_1(\cdots)$ and $g_2(\cdots)$ can be mapped out by varying (y_{t-1}, m_t, m_{t-1}) along points with positive support. In this case, Assumption 7 requires that the mapping $(Y_{1t}, Y_{2t})' = [g_1(m_t, X_t^*), g_2(m_t, X_t^*)]'$ is invertible in $X_t^* = (X_{1t}^*, X_{2t}^*)$. Consequently, equation (21) is analogous to the usual multivariate change of variables formula between Y_t and X_t^*. ∎

Finally, we summarize the identification results as follows:

Theorem 1 *(Stationary case) Under the assumptions 1, 2, 3, 4, 5, 6, and 7, the density $f_{W_t,W_{t-1},W_{t-2}}$, for any $t \in \{3, \ldots T\}$, uniquely determines the time-invariant Markov equilibrium transition density $f_{W_2,X_2^*|W_1,X_1^*}$.*

Proof. See the appendix. ∎

This theorem implies that we may identify the Markov kernel density with three periods of data.

Without stationarity, the desired density $f_{Y_t|M_t,X_t^*}$ is not the same as $f_{Y_{t-1}|M_{t-1},X_{t-1}^*}$, which can be recovered from the three observations $f_{W_t,W_{t-1},W_{t-2}}$. However, in this case, we can repeat the whole foregoing argument for the three observations $f_{W_{t+1},W_t,W_{t-1}}$ to identify $f_{Y_t|M_t,X_t^*}$. Hence, the following corollary is immediate:

Corollary 1 *(Nonstationary case) Under the assumptions 1, 2, 4, 5, 6, and 7, the den-*

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sity $f_{W_{t+1},W_t,W_{t-1},W_{t-2}}$ *uniquely determines the time-varying Markov equilibrium transition density* $f_{W_t,X_t^*|W_{t-1},X_{t-1}^*}$*, for every period* $t \in \{3, \ldots T - 1\}$*.*

4 Extensions

4.1 Games with discrete actions

Up to this point, we have focused on games where Y_t and X_{t-1}^* are both continuous, so that assumption 4(ii) is satisfied. In this section, we consider discrete-choice games, where Y_t is discrete-valued, but the unobserved state variables X_t^* are continuous.[3] In this case, Assumption 4(ii) is violated, so that (Y_t, Y_{t-2}) cannot be used as the two measurements in the Hu and Schennach (2008) setup.

However, under assumptions 1-2, we may use (M_{t+1}, M_{t-2}) to play the role of (Y_t, Y_{t-2}) in the identification procedure, especially when Y_t is discrete. Consider the joint density of $\{M_{t+1}, Y_t, M_t, Y_{t-1}, M_{t-1}, Y_{t-2}, M_{t-2}\}$. One can show that assumptions 1-2 imply that

$$f_{M_{t+1},Y_t,M_t,Y_{t-1},M_{t-1},Y_{t-2},M_{t-2}} \tag{22}$$
$$= \int f_{M_{t+1},Y_t|M_t,M_{t-1},X_{t-1}^*} f_{M_t,Y_{t-1}|M_{t-1},X_{t-1}^*} f_{X_{t-1}^*,M_{t-1},Y_{t-2},M_{t-2}} dx_{t-1}^*.$$

The identification strategy in Hu and Schennach (2008) implies that two measurements and a dependent variable of a latent explanatory variable are enough to achieve identification. For fixed (m_t, m_{t-1}), Equation (22) implies that, when X_{t-1}^*, Y_t and M_t are all continuous, we may use either (Y_t, Y_{t-2}) or (M_{t+1}, M_{t-2}) as the two measurements of X_{t-1}^*. The former case was considered in the previous section.

However, when Y_t is discrete and M_t is continuous, we have to use (M_{t+1}, M_{t-2}) as the two measurements of X_{t-1}^*.[4]

---

[3] This is also the setup in Heckman and Navarro's (2007) single-agent dynamic treatment effect model, where Y_t is an indicator for school attendance and X_t^* denotes a student's unobserved ability. In that model, Heckman and Navarro allow X_t^* to be arbitrarily correlated over time, but assume that it enters the model additively. They demonstrate identification using large support assumptions. In contrast, we restrict the joint process of W_t, X_t^* to be first-order Markov, but allow the unobservables X_t^* to enter in nonseparable manner. Consequently, our identification arguments are quite different from theirs.

[4] In general, one can use (U, V) as the two measurements of X_{t-1}^*, where $U = g(M_{t+1}, Y_t)$ and $V = q(M_{t-2}, Y_{t-2})$ for some known functions $g(\cdot, \cdot)$ and $q(\cdot, \cdot)$.

Integrating out (Y_t, Y_{t-2}) in Eq. (22), we obtain

$$\int \omega\left(y_{t-1}\right) f_{M_{t+1},M_t,Y_{t-1}|M_{t-1},M_{t-2}}\left(x,m_t,y_{t-1}|m_{t-1},z\right) dy_{t-1} \tag{23}$$

$$= \int \left[f_{M_{t+1}|M_t,M_{t-1},X_{t-1}^*}\left(x|m_t,m_{t-1},x_{t-1}^*\right) E\left[\omega\left(Y_{t-1}\right)|m_t,m_{t-1},x_{t-1}^*\right] \cdot \right.$$

$$\left. f_{M_t,|M_{t-1},X_{t-1}^*}\left(m_t,|m_{t-1},x_{t-1}^*\right) f_{X_{t-1}^*|M_{t-1},M_{t-2}}\left(x_{t-1}^*|m_{t-1},z\right) \right] dx_{t-1}^*$$

for any $(x,m_t,m_{t-1},z) \in supp\left(M_{t+1},M_t,M_{t-1},M_{t-2}\right)$. We may achieve identification under the assumptions as follows. Assumption 4 is replaced by the following assumption.

Assumption 4' *For a known $\omega\left(\cdot\right)$ and any $(m_t,m_{t-1}) \in supp\left(M_t,M_{t-1}\right)$,*

(i) $L_{M_{t+1},m_t,\omega|m_{t-1},M_{t-2}}$ is one-to-one;

(ii) $L_{M_{t+1}|m_t,m_{t-1},X_{t-1}^*}$ is one-to-one.

The operators $L_{M_{t+1},m_t,\omega|m_{t-1},M_{t-2}}$ and $L_{M_{t+1}|m_t,m_{t-1},X_{t-1}^*}$ are defined in the appendix. In the setting where Y_{t-1} is discrete, but X_{t-1}^* is continuous, Assumption 4'(ii) essentially requires that the observed state variable M_t be informative on X_{t-1}^*, i.e.,

$$f_{M_t|Y_{t-1},M_{t-1},X_{t-1}^*} \neq f_{M_t|Y_{t-1},M_{t-1}}. \tag{24}$$

This is because, once (m_t,m_{t-1}) are fixed, the effect of X_{t-1}^* on M_{t+1} must come either through Y_t (which depends on X_t^*, and which in turn is serially correlated with X_{t-1}^*)), or through X_t^* (which is serially correlated with X_{t-1}^*). When Y_t is discrete, but X_t^* is continuous, then Y_t cannot fully "transmit" all the information in X_{t-1}^*, which is required to satisfy Assumption 4'(ii). Hence, M_t must transmit the information in X_{t-1}^*, as in equation (24).

The eigenfunction corresponds to the density $f_{M_{t+1}|m_t,m_{t-1},X_{t-1}^*}\left(\cdot|m_t,m_{t-1},\cdot\right)$ which can be written as

$$f_{M_{t+1}|m_t,m_{t-1},X_{t-1}^*}\left(\cdot|m_t,m_{t-1},\cdot\right) = f_{M_{1,t+1},M_{2,t+1}|m_t,m_{t-1},X_{1,t-1}^*,X_{2,t-1}^*}\left(\cdot,\cdot|m_t,m_{t-1},\cdot,\cdot\right).$$

We then need the following ordering assumption instead of assumption 6.

Assumption 6' *(i) for any given $(m_t,m_{t-1}) \in supp\left(M_t,M_{t-1}\right)$ and $x_{2,t}^*$, there exist a*

known functional G *such that* $G\left[f_{M_{1,t+1}|M_t,M_{t-1},X^*_{t-1}}(\cdot|m_t,m_{t-1},x^*_{1,t-1},x^*_{2,t-1})\right]$ *is mono-tonic in* $x^*_{1,t-1}$;

(ii) Without loss of generality, we normalize $x^*_{1,t-1}$ *as*

$$x^*_{1,t-1} = G\left[f_{M_{1,t+1}|M_t,M_{t-1},X^*_{t-1}}(\cdot|m_t,m_{t-1},x^*_{1,t-1},x^*_{2,t-1})\right];$$

(iii) $x^*_{2,t-1}$ *also satisfies (i) and (ii), i.e.,*

$$x^*_{2,t-1} = G\left[f_{M_{2,t+1}|M_t,M_{t-1},X^*_{t-1}}(\cdot|m_t,m_{t-1},x^*_{1,t-1},x^*_{2,t-1})\right]$$

for any given (m_t, m_{t-1}) *and* $x^*_{1,t-1}$.

The following assumption is needed to replace assumption 7.

Assumption 7' *for any* $m_t \in \mathcal{M}_t$, $L_{M_{t+1}|m_t,X^*_t}$ *is one-to-one.*

The operator $L_{M_{t+1}|m_t,X^*_t}$ is defined in the appendix. This assumption also requires that equation (24) holds. Finally, we summarize the identification results as follows:

Theorem 2 *Under the assumptions 1, 2, 3, 4', 5, 6', and 7', the density* $f_{W_{t+1},W_t,W_{t-1},W_{t-2}}$ *for any* $t \in \{3, \ldots T-1\}$, *uniquely determines the time-invariant Markov equilibrium tran-sition density* $f_{W_2,X^*_2|W_1,X^*_1}$.

Proof. See the appendix. ∎

4.2 Alternatives to Assumption 2(ii)

In this section, we consider alternative conditions of assumption 2(ii). Assumption 2(ii) implies that X^*_t is independent of Y_{t-1} conditional on M_t, M_{t-1} and X^*_{t-1}. There are other alternative "limited feedback" assumptions, which may be suitable for different empirical

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settings. Assumptions 1 and 2(i) imply

$$f_{W_{t+1},W_t,W_{t-1},W_{t-2}}$$
$$= f_{Y_{t+1},M_{t+1},Y_t,M_t,Y_{t-1},M_{t-1},Y_{t-2},M_{t-2}}$$
$$= \int \int \left[f_{Y_{t+1},M_{t+1}|Y_t,M_t,X_t^*} f_{Y_t|M_t,X_t^*} f_{X_t^*,M_t|Y_{t-1},M_{t-1},X_{t-1}^*} \cdot \right.$$
$$\left. f_{Y_{t-1}|M_{t-1},X_{t-1}^*} f_{X_{t-1}^*,M_{t-1},Y_{t-2},M_{t-2}} \right] dx_t^* dx_{t-1}^*.$$

Assumption 2(ii) implies that the state transition density satisfies

$$f_{X_t^*,M_t|Y_{t-1},M_{t-1},X_{t-1}^*} = f_{X_t^*|M_t,M_{t-1},X_{t-1}^*} f_{M_t|Y_{t-1},M_{t-1},X_{t-1}^*}.$$

Alternative "limited feedback" assumptions may be imposed on the density $f_{X_t^*,M_t|Y_{t-1},M_{t-1},X_{t-1}^*}$.
One alternative to assumption 2(ii) is

$$f_{X_t^*,M_t|Y_{t-1},M_{t-1},X_{t-1}^*} = f_{X_t^*|M_t,Y_{t-1},X_{t-1}^*} f_{M_t|Y_{t-1},M_{t-1},X_{t-1}^*}, \tag{25}$$

which implies that M_{t-1} does not have a direct effect on X_t^* conditional on M_t, Y_{t-1}, and X_{t-1}^*. A second alternative is

$$f_{X_t^*,M_t|Y_{t-1},M_{t-1},X_{t-1}^*} = f_{M_t|X_t^*,Y_{t-1},M_{t-1}} f_{X_t^*|Y_{t-1},M_{t-1},X_{t-1}^*}. \tag{26}$$

which is the "limited feedback" assumption used in our earlier study (Hu and Shum (2008)) of identification on single-agent dynamic optimization problems. Both alternatives (25) and (26) can be handled using identification arguments similar to the one in Hu and Shum (2008).

A third alternative to assumption 2(ii) is

$$f_{X_t^*,M_t|Y_{t-1},M_{t-1},X_{t-1}^*} = f_{X_t^*|M_t,Y_{t-1},M_{t-1},X_{t-1}^*} f_{M_t|M_{t-1},X_{t-1}^*}. \tag{27}$$

This alternative can be handled in an identification framework similar to the one used in this paper.

5 Conclusions

In this paper, we show several results regarding nonparametric identification in a general class of Markov dynamic games, including many models in the Ericson and Pakes (1995) and Pakes and McGuire (1994) framework. We show that only three observations $W_t, \ldots, W_{t-2}$ are required to identify $W_t, X_t^* | W_{t-1}, X_{t-1}^*$ in the stationary case, when Y_t is a continuous choice variable. If Y_t is a discrete choice variable (while X_t^* is continuous), then four observations are required for identification.

In ongoing work, we are working on developing estimation procedures for dynamic games which utilize these identification results.

Proof. (theorem 1) First, assumptions 1-2 imply that the density of interest becomes

$$
\begin{aligned}
f_{W_t,X_t^*|W_{t-1},X_{t-1}^*} &= f_{Y_t,M_t,X_t^*|Y_{t-1},M_{t-1},X_{t-1}^*} \\
&= f_{Y_t|M_t,X_t^*,Y_{t-1},M_{t-1},X_{t-1}^*} f_{X_t^*|M_t,Y_{t-1},M_{t-1},X_{t-1}^*} f_{M_t|Y_{t-1},M_{t-1},X_{t-1}^*} \\
&= f_{Y_t|M_t,X_t^*} f_{X_t^*|M_t,M_{t-1},X_{t-1}^*} f_{M_t|Y_{t-1},M_{t-1},X_{t-1}^*}.
\end{aligned}
\tag{28}
$$

We consider the observed density $f_{W_{t+1},W_t,W_{t-1},W_{t-2}}$. One can show that assumptions 1 and 2(i) imply

$$
\begin{aligned}
&f_{W_{t+1},W_t,W_{t-1},W_{t-2}} \\
={}& \int\int f_{W_{t+1},W_t,W_{t-1},W_{t-2},X_t^*,X_{t-1}^*} \, dx_t^* dx_{t-1}^* \\
={}& \int\int f_{W_{t+1}|W_t,W_{t-1},W_{t-2},X_t^*,X_{t-1}^*} f_{W_t,X_t^*|W_{t-1},W_{t-2},X_{t-1}^*} f_{W_{t-1},W_{t-2},X_{t-1}^*} \, dx_t^* dx_{t-1}^* \\
={}& \int\int f_{W_{t+1}|W_t,X_t^*} f_{W_t,X_t^*|W_{t-1},X_{t-1}^*} f_{W_{t-1},W_{t-2},X_{t-1}^*} \, dx_t^* dx_{t-1}^* \\
={}& \int\int f_{Y_{t+1},M_{t+1}|Y_t,M_t,X_t^*} f_{Y_t|M_t,X_t^*} f_{X_t^*|M_t,Y_{t-1},M_{t-1},X_{t-1}^*} f_{M_t|Y_{t-1},M_{t-1},X_{t-1}^*} f_{X_{t-1}^*,Y_{t-1},M_{t-1},Y_{t-2},M_{t-2}} \, dx_t^* dx_{t-1}^* \\
={}& \int\int f_{Y_{t+1},M_{t+1}|Y_t,M_t,X_t^*} f_{Y_t|M_t,X_t^*} f_{X_t^*|M_t,Y_{t-1},M_{t-1},X_{t-1}^*} f_{M_t|Y_{t-1},M_{t-1},X_{t-1}^*} \times \\
&\times f_{Y_{t-1}|M_{t-1},X_{t-1}^*} f_{X_{t-1}^*,M_{t-1},Y_{t-2},M_{t-2}} \, dx_t^* dx_{t-1}^* \\
={}& \int\int f_{Y_{t+1},M_{t+1}|Y_t,M_t,X_t^*} f_{Y_t|M_t,X_t^*} f_{X_t^*|M_t,Y_{t-1},M_{t-1},X_{t-1}^*} f_{M_t,Y_{t-1}|M_{t-1},X_{t-1}^*} f_{X_{t-1}^*,M_{t-1},Y_{t-2},M_{t-2}} \, dx_t^* dx_{t-1}^*.
\end{aligned}
$$

Assumption 2(ii) then implies

$$
\begin{aligned}
&f_{W_{t+1},W_t,W_{t-1},W_{t-2}} \\
={}& \int\int f_{Y_{t+1},M_{t+1}|Y_t,M_t,X_t^*} f_{Y_t|M_t,X_t^*} f_{X_t^*|M_t,M_{t-1},X_{t-1}^*} f_{M_t,Y_{t-1}|M_{t-1},X_{t-1}^*} f_{X_{t-1}^*,M_{t-1},Y_{t-2},M_{t-2}} \, dx_t^* dx_{t-1}^* \\
={}& \int\left(\int f_{Y_{t+1},M_{t+1}|Y_t,M_t,X_t^*} f_{Y_t|M_t,X_t^*} f_{X_t^*|M_t,M_{t-1},X_{t-1}^*} \, dx_t^*\right) f_{M_t,Y_{t-1}|M_{t-1},X_{t-1}^*} f_{X_{t-1}^*,M_{t-1},Y_{t-2},M_{t-2}} \, dx_{t-1}^*
\end{aligned}
$$

The expression in the parenthesis can be simplified as follows:

$$\int f_{Y_{t+1},M_{t+1}|Y_t,M_t,X_t^*} f_{Y_t|M_t,X_t^*} f_{X_t^*|M_t,M_{t-1},X_{t-1}^*} dx_t^*$$

$$= \int f_{Y_{t+1},M_{t+1}|Y_t,M_t,M_{t-1},X_t^*,X_{t-1}^*} f_{Y_t|M_t,M_{t-1},X_t^*,X_{t-1}^*} f_{X_t^*|M_t,M_{t-1},X_{t-1}^*} dx_t^*$$

$$= \int f_{Y_{t+1},M_{t+1},Y_t,X_t^*|M_t,M_{t-1},X_{t-1}^*} dx_t^*$$

$$= f_{Y_{t+1},M_{t+1},Y_t|M_t,M_{t-1},X_{t-1}^*}.$$

We then have

$$f_{W_{t+1},W_t,W_{t-1},W_{t-2}} \tag{29}$$

$$= f_{Y_{t+1},M_{t+1},Y_t,M_t,Y_{t-1},M_{t-1},Y_{t-2},M_{t-2}}$$

$$= \int f_{Y_{t+1},M_{t+1},Y_t|M_t,M_{t-1},X_{t-1}^*} f_{M_t,Y_{t-1}|M_{t-1},X_{t-1}^*} f_{X_{t-1}^*,M_{t-1},Y_{t-2},M_{t-2}} dx_{t-1}^*.$$

Integrating out Y_{t+1}, M_{t+1}, and M_{t-2} leads to

$$f_{Y_t,M_t,Y_{t-1}|M_{t-1},Y_{t-2}} \tag{30}$$

$$= \int f_{Y_t|M_t,M_{t-1},X_{t-1}^*} f_{M_t,Y_{t-1}|M_{t-1},X_{t-1}^*} f_{X_{t-1}^*|M_{t-1},Y_{t-2}} dx_{t-1}^*.$$

Furthermore, we may have for a user-specified function $\omega(\cdot)$

$$\int \omega(y_{t-1}) f_{Y_t,M_t,Y_{t-1}|M_{t-1},Y_{t-2}} dy_{t-1} \tag{31}$$

$$= \int f_{Y_t|M_t,M_{t-1},X_{t-1}^*} E\left[\omega(Y_{t-1})|m_t,m_{t-1},x_{t-1}^*\right] f_{M_t,|M_{t-1},X_{t-1}^*} f_{X_{t-1}^*|M_{t-1},Y_{t-2}} dx_{t-1}^*,$$

One can show that the equation above is equivalent to

$$L_{Y_t,m_t,\omega|m_{t-1},Y_{t-2}} = L_{Y_t|m_t,m_{t-1},X_{t-1}^*} D_{\omega|m_t,m_{t-1},X_{t-1}^*} D_{m_t|m_{t-1},X_{t-1}^*} L_{X_{t-1}^*|m_{t-1},Y_{t-2}}, \tag{32}$$

where for any given $(m_t,m_{t-1}) \in supp(M_t,M_{t-1})$

$$L_{Y_t,m_t,\omega|m_{t-1},Y_{t-2}} \quad : \quad \mathcal{L}^p(\mathcal{Y}_{t-2}) \to \mathcal{L}^p(\mathcal{Y}_t),$$

$$\left(L_{Y_t,m_t,\omega|m_{t-1},Y_{t-2}} h\right)(x) \quad = \quad \int k(x,m_t|m_{t-1},z) h(z) dz,$$

$$\text{with } k(x,m_t|m_{t-1},z) \quad = \quad \int \omega(y_{t-1}) f_{Y_t,M_t,Y_{t-1}|M_{t-1},Y_{t-2}}(x,m_t,y_{t-1}|m_{t-1},z) dy_{t-1},$$

$$L_{Y_t|m_t,m_{t-1},X^*_{t-1}} \quad : \quad \mathcal{L}^p\left(\mathcal{X}^*_{t-1}\right) \to \mathcal{L}^p\left(\mathcal{Y}_t\right),$$

$$\left(L_{Y_t|m_t,m_{t-1},X^*_{t-1}}h\right)(x) \; = \; \int f_{Y_t|M_t,M_{t-1},X^*_{t-1}}(x|m_t,m_{t-1},x^*_{t-1})h(x^*_{t-1})dx^*_{t-1},$$

$$D_{\omega|m_t,m_{t-1},X^*_{t-1}} \quad : \quad \mathcal{L}^p\left(\mathcal{X}^*_{t-1}\right) \to \mathcal{L}^p\left(\mathcal{X}^*_{t-1}\right),$$

$$\left(D_{\omega|m_t,m_{t-1},X^*_{t-1}}h\right)(x^*_{t-1}) \; = \; E\left[\omega\left(Y_{t-1}\right)|m_t,m_{t-1},x^*_{t-1}\right]h\left(x^*_{t-1}\right),$$

$$D_{m_t|m_{t-1},X^*_{t-1}} \quad : \quad \mathcal{L}^p\left(\mathcal{X}^*_{t-1}\right) \to \mathcal{L}^p\left(\mathcal{X}^*_{t-1}\right),$$

$$\left(D_{m_t|m_{t-1},X^*_{t-1}}h\right)(x^*_{t-1}) \; = \; f_{M_t|M_{t-1},X^*_{t-1}}(m_t|m_{t-1},x^*_{t-1})h\left(x^*_{t-1}\right),$$

$$L_{X^*_{t-1}|m_{t-1},Y_{t-2}} \quad : \quad \mathcal{L}^p\left(\mathcal{Y}_{t-2}\right) \to \mathcal{L}^p\left(\mathcal{X}^*_{t-1}\right),$$

$$\left(L_{X^*_{t-1}|m_{t-1},Y_{t-2}}h\right)(x^*_{t-1}) \; = \; \int f_{X^*_{t-1}|M_{t-1},Y_{t-2}}(x^*_{t-1}|m_{t-1},z)h(z)dz.$$

Integrating out Y_{t-1} in equation (30) leads to

$$f_{Y_t,M_t|M_{t-1},Y_{t-2}} \tag{33}$$
$$= \int f_{Y_t|M_t,M_{t-1},X^*_{t-1}}f_{M_t|M_{t-1},X^*_{t-1}}f_{X^*_{t-1}|M_{t-1},Y_{t-2}}dx^*_{t-1}.$$

This equation is equivalent to

$$L_{Y_t,m_t|m_{t-1},Y_{t-2}} = L_{Y_t|m_t,m_{t-1},X^*_{t-1}}D_{m_t|m_{t-1},X^*_{t-1}}L_{X^*_{t-1}|m_{t-1},Y_{t-2}} \tag{34}$$

for any given $(m_t,m_{t-1}) \in supp\left(M_t,M_{t-1}\right)$, where

$$L_{Y_t,m_t|m_{t-1},Y_{t-2}} \quad : \quad \mathcal{L}^p\left(\mathcal{Y}_{t-2}\right) \to \mathcal{L}^p\left(\mathcal{Y}_t\right),$$

$$\left(L_{Y_t,m_t|m_{t-1},Y_{t-2}}h\right)(x) \; = \; \int f_{Y_t,M_t|M_{t-1},Y_{t-2}}(x,m_t|m_{t-1},z)h(z)dz.$$

Assumption 4 then implies

$$L_{Y_t,m_t,y_{t-1}|m_{t-1},Y_{t-2}}L^{-1}_{Y_t,m_t|m_{t-1},Y_{t-2}}$$
$$\equiv \; L_{Y_t|m_t,m_{t-1},X^*_{t-1}}D_{\omega|m_t,m_{t-1},X^*_{t-1}}L^{-1}_{Y_t|m_t,m_{t-1},X^*_{t-1}}.$$

With assumptions 3, 4, 5, 6, and 7, theorem 1 in Hu and Schennach (2008) implies that $f_{Y_t|M_t,M_{t-1},X_{t-1}^*}$ is nonparametrically identified. By equation (30) and assumption 4, identification of $L_{Y_t|m_t,m_{t-1},X_{t-1}^*}$ implies that of $f_{M_t,Y_{t-1}|M_{t-1},X_{t-1}^*}$ and $f_{X_{t-1}^*|M_{t-1},Y_{t-2}}$. We may then identify $f_{Y_{t-1}|M_{t-1},X_{t-1}^*}$ and $f_{M_t|Y_{t-1},M_{t-1},X_{t-1}^*}$ from

$$f_{Y_{t-1}|M_{t-1},X_{t-1}^*} = \int f_{M_t,Y_{t-1}|M_{t-1},X_{t-1}^*} dm_t,$$

$$f_{M_t|Y_{t-1},M_{t-1},X_{t-1}^*} = \frac{f_{M_t,Y_{t-1}|M_{t-1},X_{t-1}^*}}{f_{Y_{t-1}|M_{t-1},X_{t-1}^*}}.$$

By the stationarity assumption 3, we identify $f_{Y_t|M_t,X_t^*} = f_{Y_{t-1}|M_{t-1},X_{t-1}^*}$. In order to identify $f_{X_t^*|M_t,M_{t-1},X_{t-1}^*}$, we consider

$$f_{Y_t|M_t,M_{t-1},X_{t-1}^*} = \int f_{Y_t,X_t^*|M_t,M_{t-1},X_{t-1}^*} dx_t^*$$

$$= \int f_{Y_t|M_t,X_t^*} f_{X_t^*|M_t,M_{t-1},X_{t-1}^*} dx_t^*.$$

Notice that $f_{Y_t|M_t,M_{t-1},X_{t-1}^*}$ on the left hand side has been identified. We define for a given m_t

$$L_{Y_t|m_t,X_t^*} : \mathcal{L}^p(\mathcal{X}_t^*) \to \mathcal{L}^p(\mathcal{Y}_t),$$

$$\left(L_{Y_t|m_t,X_t^*}h\right)(x) = \int f_{Y_t|M_t,X_t^*}(x|m_t,x_t^*)h(x_t^*)dx_t^*.$$

We then have

$$f_{Y_t|M_t=m_t,M_{t-1}=m_{t-1},X_{t-1}^*} = L_{Y_t|m_t,X_t^*} f_{X_t^*|M_t=m_t,M_{t-1}=m_{t-1},X_{t-1}^*}.$$

Then the invertibility of $L_{Y_t|m_t,X_t^*}$ in assumption 7 implies the identification of $f_{X_t^*|M_t,M_{t-1},X_{t-1}^*}$. In summary, we have identified $f_{Y_t|M_t,X_t^*}$, $f_{M_t|Y_{t-1},M_{t-1},X_{t-1}^*}$, and $f_{X_t^*|M_t,M_{t-1},X_{t-1}^*}$. Therefore, the density of interest $f_{Y_t,M_t,X_t^*|Y_{t-1},M_{t-1},X_{t-1}^*}$, i.e., $f_{W_t,X_t^*|W_{t-1},X_{t-1}^*}$ is identified in equation (28). ∎

Proof. (theorem 2) Integrating out Y_{t+1}, Y_t, and Y_{t-2} in equation (29) leads to

$$f_{M_{t+1},M_t,Y_{t-1}|M_{t-1},M_{t-2}}$$
$$= \int f_{M_{t+1}|M_t,M_{t-1},X_{t-1}^*} f_{M_t,Y_{t-1}|M_{t-1},X_{t-1}^*} f_{X_{t-1}^*,M_{t-1},M_{t-2}} dx_{t-1}^*.$$

We may then have

$$\int \omega\left(y_{t-1}\right) f_{M_{t+1},M_t,Y_{t-1}|M_{t-1},M_{t-2}} dy_{t-1} \tag{35}$$

$$= \int f_{M_{t+1}|M_t,M_{t-1},X^*_{t-1}} E\left[\omega\left(Y_{t-1}\right)|m_t,m_{t-1},x^*_{t-1}\right] f_{M_t,|M_{t-1},X^*_{t-1}} f_{X^*_{t-1}|M_{t-1},M_{t-2}} dx^*_{t-1}.$$

The equation above is equivalent to

$$L_{M_{t+1},m_t,\omega|m_{t-1},M_{t-2}} = L_{M_{t+1}|m_t,m_{t-1},X^*_{t-1}} D_{\omega|m_t,m_{t-1},X^*_{t-1}} D_{m_t|m_{t-1},X^*_{t-1}} L_{X^*_{t-1}|m_{t-1},M_{t-2}},$$

where for any given $(m_t, m_{t-1}) \in supp\left(M_t, M_{t-1}\right)$

$$L_{M_{t+1},m_t,\omega|m_{t-1},M_{t-2}} \quad : \quad \mathcal{L}^p\left(\mathcal{M}_{t-2}\right) \to \mathcal{L}^p\left(\mathcal{M}_{t+1}\right)),$$

$$\left(L_{M_{t+1},m_t,\omega|m_{t-1},M_{t-2}}h\right)(x) \quad = \quad \int k(x, m_t|m_{t-1}, z)h(z)dz$$

$$k(x, m_t|m_{t-1}, z) \quad = \quad \int \omega\left(y_{t-1}\right) f_{M_{t+1},M_t,Y_{t-1}|M_{t-1},M_{t-2}}\left(x, m_t, y_{t-1}|m_{t-1}, z\right) dy_{t-1}.$$

$$L_{M_{t+1}|m_t,m_{t-1},X^*_{t-1}} \quad : \quad \mathcal{L}^p\left(\mathcal{X}^*_{t-1}\right) \to \mathcal{L}^p\left(\mathcal{M}_{t+1}\right),$$

$$\left(L_{M_{t+1}|m_t,m_{t-1},X^*_{t-1}}h\right)(x) \quad = \quad \int f_{M_{t+1}|M_t,M_{t-1},X^*_{t-1}}(x|m_t,m_{t-1},x^*_{t-1})h(x^*_{t-1})dx^*_{t-1},$$

$$L_{X^*_{t-1}|m_{t-1},M_{t-2}} \quad : \quad \mathcal{L}^p\left(\mathcal{M}_{t-2}\right) \to \mathcal{L}^p\left(\mathcal{X}^*_{t-1}\right),$$

$$\left(L_{X^*_{t-1}|m_{t-1},M_{t-2}}h\right)(x^*_{t-1}) \quad = \quad \int f_{X^*_{t-1}|M_{t-1},M_{t-2}}(x^*_{t-1}|m_{t-1}, z)h(z)dz.$$

Letting $\omega\left(\cdot\right) = 1$ in equation (35) leads to

$$f_{M_{t+1},M_t|M_{t-1},M_{t-2}}$$

$$= \int f_{M_{t+1}|M_t,M_{t-1},X^*_{t-1}} f_{M_t|M_{t-1},X^*_{t-1}} f_{X^*_{t-1},M_{t-1},M_{t-2}} dx^*_{t-1}.$$

This equation is equivalent to

$$L_{M_{t+1},m_t|m_{t-1},M_{t-2}} = L_{M_{t+1}|m_t,m_{t-1},X^*_{t-1}} D_{m_t|m_{t-1},X^*_{t-1}} L_{X^*_{t-1}|m_{t-1},M_{t-2}}$$

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for any given $(m_t, m_{t-1}) \in supp\,(M_t, M_{t-1})$, where

$$L_{M_{t+1},m_t|m_{t-1},M_{t-2}} \quad : \quad \mathcal{L}^p\,(\mathcal{M}_{t-2}) \to \mathcal{L}^p\,(\mathcal{M}_{t+1})),$$

$$\left(L_{M_{t+1},m_t|m_{t-1},M_{t-2}}h\right)(x) \quad = \quad \int f_{M_{t+1},M_t|M_{t-1},M_{t-2}}(x, m_t|m_{t-1}, z)h(z)dz.$$

The same identification procedure as in the proof of theorem 1 implies the identification of $f_{M_{t+1}|M_t,M_{t-1},X^*_{t-1}}$, $f_{Y_{t-1}|M_{t-1},X^*_{t-1}}$ and $f_{M_t|Y_{t-1},M_{t-1},X^*_{t-1}}$. By the stationarity, we identify $f_{M_{t+1}|Y_t,M_t,X^*_t} = f_{M_t|Y_{t-1},M_{t-1},X^*_{t-1}}$ and $f_{Y_t|M_t,X^*_t} = f_{Y_{t-1}|M_{t-1},X^*_{t-1}}$, which implies identification of $f_{M_{t+1}|M_t,X^*_t}$ through

$$f_{M_{t+1}|M_t,X^*_t} = \int_{Y_t} f_{M_{t+1}|Y_t,M_t,X^*_t} f_{Y_t|M_t,X^*_t}.$$

In order to identify $f_{X^*_t|M_t,M_{t-1},X^*_{t-1}}$, we consider

$$
\begin{aligned}
f_{M_{t+1}|M_t,M_{t-1},X^*_{t-1}} &= \int_{Y_t} \int f_{M_{t+1},Y_t,X^*_t|M_t,M_{t-1},X^*_{t-1}} dx^*_t \\
&= \int_{Y_t} \int f_{M_{t+1}|Y_t,M_t,X^*_t} f_{Y_t|M_t,X^*_t} f_{X^*_t|M_t,M_{t-1},X^*_{t-1}} dx^*_t \\
&= \int \left(\int_{Y_t} f_{M_{t+1}|Y_t,M_t,X^*_t} f_{Y_t|M_t,X^*_t} \right) f_{X^*_t|M_t,M_{t-1},X^*_{t-1}} dx^*_t \\
&= \int f_{M_{t+1}|M_t,X^*_t} f_{X^*_t|M_t,M_{t-1},X^*_{t-1}} dx^*_t.
\end{aligned}
$$

Notice that $f_{M_{t+1}|M_t,M_{t-1},X^*_{t-1}}$ on the left hand side has been identified. We define for a given m_t

$$L_{M_{t+1}|m_t,X^*_t} \quad : \quad \mathcal{L}^p\,(\mathcal{X}^*_t) \to \mathcal{L}^p\,(\mathcal{M}_{t+1}),$$

$$\left(L_{M_{t+1}|m_t,X^*_t}h\right)(x) \quad = \quad \int f_{M_{t+1}|M_t,X^*_t}(x|m_t, x^*_t)h(x^*_t)dx^*_t.$$

We then have

$$f_{M_{t+1}|M_t=m_t,M_{t-1}=m_{t-1},X^*_{t-1}} = L_{M_{t+1}|m_t,X^*_t} f_{X^*_t|M_t=m_t,M_{t-1}=m_{t-1},X^*_{t-1}}.$$

Then the invertibility of $L_{M_{t+1}|m_t,X^*_t}$ in assumption 7' implies the identification of $f_{X^*_t|M_t,M_{t-1},X^*_{t-1}}$. In summary, we have identified $f_{Y_t|M_t,X^*_t}$, $f_{M_t|Y_{t-1},M_{t-1},X^*_{t-1}}$, and $f_{X^*_t|M_t,M_{t-1},X^*_{t-1}}$. Therefore, the density of interest $f_{Y_t,M_t,X^*_t|Y_{t-1},M_{t-1},X^*_{t-1}}$, i.e., $f_{W_t,X^*_t|W_{t-1},X^*_{t-1}}$ is identified. ∎

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