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LETTER TO THE EDITOR

Statistical properties of solutions to the Navier–Stokes equation in the limit of vanishing viscosity

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Abstract. In this letter, we present a new approach to the Navier-Stokes turbulence. With the Gaussian soft constraint on the Navier-Stokes equation, we derive the energy spectrum to be $E(k) \sim k^{-3}$ and k^{-2} in two and three spatial dimensions respectively. We also point out the possible future developments.

This is the first of a series of articles investigating the Navier-Stokes turbulence. The objective is to find out what is the relationship, if any, between properties of the fluid turbulence and statistical dynamics of the Navier-Stokes equation. The major contribution of this letter is to present a new viewpoint and hence a new approach to the problem of fully developed turbulence.

Our starting point is the incompressible Navier-Stokes equation. We take as granted that the incompressible Navier-Stokes equation is the underlying evolution equation appropriate for the fully developed turbulence. Even if this is true, there is still the problem of the existence and uniqueness of solutions to the Navier-Stokes equation in three dimensions for large time in the limit of vanishing viscosity. People worry about the possibility of a finite-time singularity in solutions of Euler's equation in three dimensions†. However, there is no definitive proof of this scenario. It might be possible that the vortex line zigzags around as it stretches so as to avoid a finite-time singularity. For the purpose of this calculation, we have to assume that solutions of the Navier-Stokes equation exist and are unique for all time in three dimensions when the viscosity becomes small.

The Navier-Stokes equation is first order in time. If we assume the existence and uniqueness of the solution, then when the boundary condition and the initial velocity profile are specified exactly, we can follow the subsequent evolution of the solution and predict its future behaviour exactly. However, in the experiment of the fluid turbulence, we cannot control the initial condition precisely. When we repeat the experiment, our initial velocity profile is almost the same as before. But there is certain probability of deviating from the previous set-up. Nevertheless, we observe similar turbulent behaviour. Hence, our measurement is in a sense the average of solutions to the Navier-Stokes equation, due to our inability in setting up the precise initial condition.

I believe that the main task of the theoretical investigation into the fluid turbulence is to find a way to translate the experimental situation into a suitable mathematical

† For two dimensions, one has the existence and uniqueness of solutions for all time, see [1].

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language. Experimentally, we can measure the simultaneous n-point correlation of velocity fluctuations. In particular, the most often quoted quantity is the velocity-velocity correlation in Fourier space. One typical experiment is to set up a mesh wire at the upstream of the wind tunnel and then measure the n-point velocity correlations at various places downstream. As we continue the experiment, our upstream initial velocity profile will have a certain probability distribution, because we cannot control the initial condition exactly. Hence, if we fix our probes at certain places downstream, the measurement we make is an average over solutions of the Navier–Stokes equation. Now, it is found that even though there are complications due to the presence of the boundary wall, the region far away from the boundary shows remarkable universal statistical behaviours.

It is the major challenge for theorists to understand these universal behaviours. Thus, as a first step towards the understanding of the fully developed turbulence, we assume that we have a fluid occupying the whole space and let the velocity be zero at infinity. From experiments, we know that all fluids show the same turbulent behaviour when the Reynold's number is large. Hence it is reasonable to assume that in the limit of vanishing viscosity, almost any random initial condition will lead to the same turbulent behaviour. In a more realistic calculation, this assumption has to be verified. That will be the second phase of the investigation. For the moment, let us assume that this is true. Then the problem is to find the scaling behaviour of an n-point velocity correlation function, averaged over all solutions of the Navier–Stokes equation. We hope that the correct Kolmogorov scaling region will be recovered in the limit of vanishing viscosity.

We start with the incompressible Navier–Stokes equation (with $\rho = 1$):

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla} P + \nu \boldsymbol{\nabla}^2 \boldsymbol{v} \tag{1}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 \qquad \text{and} \qquad \lim_{|\boldsymbol{x}| \to \infty} \boldsymbol{v}(\boldsymbol{x}, t) = 0. \tag{2}$$

The generating functional of correlation functions of the velocity fields, averaged over all solutions to the Navier–Stokes equation with certain initial distribution of velocity profile, is:

$$Z[J] = \int \mathscr{D}\boldsymbol{v}(\boldsymbol{x}, t)\, \mathrm{d}\mu(\boldsymbol{A}(\boldsymbol{x})) \exp\left(\int_a \mathrm{d}^d \boldsymbol{x} \int_0^\infty \mathrm{d}t \boldsymbol{J}(\boldsymbol{x}, t) \cdot \boldsymbol{v}(\boldsymbol{x}, t) \right)$$
$$\times \prod_{\boldsymbol{x}} \prod_{t \geq 0} \delta\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} + \boldsymbol{\nabla} P - \nu \boldsymbol{\nabla}^2 \boldsymbol{v} \right) \prod_{\boldsymbol{x}} \delta[\boldsymbol{v}(\boldsymbol{x}, 0) - \boldsymbol{A}(\boldsymbol{x})] \tag{3}$$

where $\mu(\boldsymbol{A}(\boldsymbol{x}))$ is the probability distribution function of the initial velocity profile $\boldsymbol{A}(\boldsymbol{x})$. We have put a lattice cutoff of order a in the spatial integral. We need an ultraviolet cutoff because the Navier–Stokes equation is a hydrodynamic equation.

From various experiments on the fluid turbulence, the data seem to suggest that in the fully developed isotropic and homogeneous turbulence, the two-point velocity correlation in Fourier space displays the scaling behaviour with an exponent independent of initial conditions and energy inputs. If this is true, then in the limit of vanishing viscosity, we can weight all possible initial conditions equally if we are interested only in the derivation of the scaling exponent. This is what we are going to do. The generating functional of n-point velocity correlation functions in equation (3) is hard to compute when one has to deal with the initial velocity profile.

Next, we reason that all solutions of the Navier–Stokes equation for $t \geq 0$ evolve from almost all possible initial conditions at $t = 0$. In order for this reasoning to work, we must demand that solutions to the Navier–Stokes equation as initial value problems are unique for all time. We also have to assume that solutions never blow up in finite time. Since eventually we are only interested in the limit of vanishing viscosity, we do not have to worry about the possibility that all solutions will be damped out by the viscosity.

Therefore we replace the averaging over all possible initial conditions by the averaging over all solutions of the Navier–Stokes equation with equal weight:

$$
\begin{aligned}
Z[J] = \int & \mathscr{D}\boldsymbol{v}(\boldsymbol{x}, t) \exp\left(\int_a \mathrm{d}^d\boldsymbol{x} \int_0^\infty \mathrm{d}t\, \boldsymbol{J}(\boldsymbol{x}, t) \cdot \boldsymbol{v}(\boldsymbol{x}, t)\right) \\
& \times \prod_{\boldsymbol{x}} \prod_{t \geq 0} \prod_{\alpha=1}^{d} \delta\left(\frac{\partial v_\alpha}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) v_\alpha + \partial_\alpha P - \nu \boldsymbol{\nabla}^2 v_\alpha\right) \\
= \int & \prod_{\alpha=1}^{d} \mathscr{D}v_\alpha(\boldsymbol{x}, t) \mathscr{D}\psi_\alpha(\boldsymbol{x}, t) \\
& \times \exp\left\{\int_a \mathrm{d}^d\boldsymbol{x} \int_0^\infty \mathrm{d}t \left[\mathrm{i}\psi_\alpha \left(\frac{\partial v_\alpha}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) v_\alpha + \partial_\alpha P - \nu \boldsymbol{\nabla}^2 v_\alpha\right)\right.\right. \\
& \left.\left. + J_\alpha(\boldsymbol{x}, t) v_\alpha(\boldsymbol{x}, t)\right]\right\}. \qquad (4)
\end{aligned}
$$

However, there are certain technical difficulties with the above formulation. First of all, the usual power counting in determining the dimensions of the fields $\boldsymbol{v}$ and $\boldsymbol{\psi}$ is not possible here. The dimensions of the fields are indeterminate. Secondly, we do not know how to deal with the free case. In the free case, the Lagrangian is linear in both $\boldsymbol{v}$ and $\boldsymbol{\psi}$. I have not found a way to handle the linear Lagrangian.

We have to emphasise that the reason we arrive at the linear Lagrangian as the free part is because that we must restrict the field configurations to those satisfying the Navier–Stokes equation. The reason is obvious: if we believe that the Navier–Stokes equation is the underlying master equation for the fluid, then any realisable velocity field configuration must evolve through the Navier–Stokes equation.

In order to make further progress with the calculation, let us try the following trick. Since we are interested in the limit of viscosity, we can soften the δ-function constraint by the following procedure:

$$
\begin{aligned}
Z[J] = \lim_{\nu \to 0} \int & \prod_{\alpha=1}^{d} \mathscr{D}v_\alpha(\boldsymbol{x}, t) \exp \int_a \mathrm{d}^d\boldsymbol{x} \int_0^\infty \mathrm{d}t\, J_\alpha(\boldsymbol{x}, t) v_\alpha(\boldsymbol{x}, t)\Big) \\
& \times \exp\left[-\frac{1}{\nu^2} \int_a \mathrm{d}^d\boldsymbol{x} \int_0^\infty \mathrm{d}t \left(\frac{\partial v_\alpha}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) v_\alpha + \partial_\alpha P - \nu \boldsymbol{\nabla}^2 v_\alpha\right)^2\right]. \qquad (5)
\end{aligned}
$$

Next we perform the Fourier transform in space and the cosine transform in time on the velocity fields. We can do this because the domain of iterest of $\boldsymbol{v}(\boldsymbol{x}, t)$ is $t > 0$ and hence we are free to extend the definition of $\boldsymbol{v}(\boldsymbol{x}, t)$ to the range $t < 0$ in such a way that $\boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{v}(\boldsymbol{x}, -t)$.

After the following transformation:

$$v_\alpha(\boldsymbol{x}, t) = \int^1 \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} \int_0^\infty \sqrt{\frac{2}{\pi}}\, \mathrm{d}w \cos(wt) \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}) v_\alpha(\boldsymbol{k}, w)$$
$$v_\alpha(\boldsymbol{k}, w) = \int_a \mathrm{d}^d \boldsymbol{x} \int_0^\infty \sqrt{\frac{2}{\pi}}\, \mathrm{d}t \cos(wt) \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}) v_\alpha(\boldsymbol{x}, t) \tag{6}$$

the generating functional becomes

$$\begin{aligned} Z[J(\boldsymbol{k}, w)] = \lim_{\nu\to 0} \int \prod_{\alpha=1}^{d} \mathscr{D} v_\alpha(\boldsymbol{k}, w) \exp\left(\int^1 \frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^d} \int_0^\infty \mathrm{d}w J_\alpha(-\boldsymbol{k}, w) v_\alpha(\boldsymbol{k}, w)\right) \\ \times \exp\left[-\int^1 \frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^d} \int_0^\infty \mathrm{d}w v_\alpha(-\boldsymbol{k}, w)\left(\frac{w^2}{\nu^2} - \frac{2wk^2}{\nu} + k^4\right) v_\alpha(\boldsymbol{k}, w)\right] \\ \times \exp\left(-\frac{1}{\nu^2}\int^1 \frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^2}\int_0^\infty \mathrm{d}w\, 2(\mathrm{i}w + \nu k^2) v_\alpha(-\boldsymbol{k}, w) \mathscr{F}(\alpha; \boldsymbol{k}w; \boldsymbol{q}_1\Omega_1, \boldsymbol{q}_2\Omega_2)\right) \\ \times \exp\left(-\frac{1}{\nu^2}\int^1 \frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^d}\int_0^\infty \mathrm{d}w\, \mathscr{F}(\alpha; \boldsymbol{k}w; \boldsymbol{q}_1\Omega_1, \boldsymbol{q}_2\Omega_2) \right. \\ \left. \times \mathscr{F}(\alpha; -\boldsymbol{k}w; \boldsymbol{q}_3\Omega_3, \boldsymbol{q}_4\Omega_4)\right) \end{aligned} \tag{7}$$

$$\begin{aligned} \mathscr{F}(\alpha; \boldsymbol{k}w; \boldsymbol{q}_1\Omega_1, \boldsymbol{q}_2\Omega_2) \\ = ik_n\left(\delta_{m\alpha} - \frac{k_m k_\alpha}{k^2}\right)(2\pi)^d \sqrt{\frac{2}{\pi}} \int^1 \frac{\mathrm{d}^d\boldsymbol{q}_1}{(2\pi)^d}\frac{\mathrm{d}^d\boldsymbol{q}_2}{(2\pi)^d} \\ \times \int_0^\infty \sqrt{\frac{2}{\pi}}\mathrm{d}\Omega_1 \int_{-\infty}^\infty \frac{1}{2}\sqrt{\frac{2}{\pi}}\,\mathrm{d}\Omega_2\, v_n(\boldsymbol{q}_1, \Omega_1) v_m(\boldsymbol{q}_2, \Omega_2)\delta(\boldsymbol{q}_1 + \boldsymbol{q}_2 - \boldsymbol{k}) \\ \times[\delta(\Omega_1 + \Omega_2 - w) + \delta(\Omega_1 + \Omega_2 + w)] \end{aligned} \tag{8}$$

where we have conveniently set the upper momentum cutoff to be 1.

Now, it is well known that when the Lagrangian is multiplied by a constant, then in the limit of that constant becoming infinite, the only surviving graphs are those without loops [2]. Hence, with the softening of the δ-function constraint by the procedure described in equation (5), the current theory is on the same level as the Ginzburg-Landau theory of phase transitions of the second kind. However, the three-point and four-point vertex functions (1PI proper functions) diverge as $\nu \to 0$. I suppose this is not very surprising. The limit of vanishing viscosity is a very singular limit.

Since we have to take the limit $\nu \to 0$ first, the only non-zero two-point Green function is:

$$\langle v_\alpha(\boldsymbol{k}, 0) v_\alpha(-\boldsymbol{k}, 0)\rangle \propto 1/k^4. \tag{9}$$

Because we are averaging over all solutions equally and taking the limit $\nu \to 0$, there is no timescale in the problem. The correlation function is time invariant:

$$\mathrm{Tr} k^{d-1}\langle v_\alpha(\boldsymbol{k}, 0) v_\alpha(-\boldsymbol{k}, 0)\rangle = \int E(\boldsymbol{k})\,\mathrm{d}t \propto E(\boldsymbol{k}). \tag{10}$$

Thus:

$$\begin{aligned} E(\boldsymbol{k}) \propto 1/k^2 \qquad \text{in 3D} \\ E(\boldsymbol{k}) \propto 1/k^3 \qquad \text{in 2D.} \end{aligned} \tag{11}$$

Perhaps I should compare my approach to previous attempts on the Navier-Stokes turbulence. In the literature, the best reference in the subject of applying the field-theoretic method to the Navier-Stokes turbulence is the paper by De Dominicis and Martin [3]. However, their approach is quite different. De Dominicis and Martin considered the Navier-Stokes equation with a Gaussian-generated random force. They had to adjust the correlation of the random force properly in order to match the Kolmogrov $\frac{5}{3}$ spectrum. As a matter of fact, the majority of the work in applying the renormalisation group method to the Navier-Stokes turbulence starts with the Gaussian-generated random force. There, the force-force scaling exponent, which is a free parameter in the theory, actually determines the scaling exponent of the velocity-velocity correlation in k-space. In the current approach, on the contrary, there is absolutely no adjustable parameter.

In summary, I have presented a new approach to the Navier-Stokes turbulence. In this approach, one does not have to consider separately the energy cascade in three dimensions and the enstrophy cascade in two dimensions in order to derive the right energy spectrum. We feel strongly that one should not need further information besides the Navier-Stokes equation to derive the correct energy spectrum. Of course, the answers in equation (11) are not quite correct. We believe that the problem lies with the softening of the δ-function constraint on the Navier-Stokes equation. It is well known in the graph-bipartitioning problem that one would arrive at the wrong solution if one does not enforce the constraint exactly [4]. We think this is what happens here. We are currently investigating equation (4) more rigorously. The result will be published soon. We hope then to shed light on the validity of the assumption that solutions of the Navier-Stokes equation exist and are unique for all time in three dimensions in the limit of vanishing viscosity.

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