Supplemental Material for:
Explicit derivation of duality between a free Dirac cone and
quantum electrodynamics in $(2+1)$ dimensions

DISCRETE SYMMETRIES FOR SINGLE DIRAC CONE/$N=1$ QED$_3$ DUAL THEORIES

We defined the action of the antiunitary symmetries $\mathcal{T}$ and $\mathcal{C}$ on the chiral electrons $\psi_y = \eta_y e^{i\phi_y}$ as

$$\mathcal{T}\psi_y(x)\mathcal{T}^{-1} = (-1)^y \psi_{y+1}(x) \,, \tag{S.1}$$

$$\mathcal{C}\psi_y(x)\mathcal{C}^{-1} = \psi_{y+1}^\dagger(x) \,. \tag{S.2}$$

We note that charge conjugation is frequently defined differently, i.e., implemented as a unitary operator $\mathcal{C}_{\text{unitary}}$. Antiunitary charge conjugation $\mathcal{C}$ may be viewed as unitary charge conjugation composed with time reversal $\mathcal{C} = \mathcal{T}\mathcal{C}_{\text{unitary}}$ (this is reflected in its property of transforming electrons into holes of opposite chirality). The updoped single Dirac cone without magnetic field exhibits both kinds of particle-hole symmetry. The significance of antiunitary particle-hole symmetry for the half-filled Landau level was emphasized in Refs. [1–3].

The wire model is additionally symmetric under inversion $\mathcal{I}$ relative to a point half-way between neighboring wires and mirror $\mathcal{M}$ defined by

$$\mathcal{I}\psi_y(x)\mathcal{I}^{-1} = \psi_{-y+1}(-x) \,, \tag{S.3}$$

$$\mathcal{M}\psi_y(x)\mathcal{M}^{-1} = (-1)^y \psi_{-y}(x) \,. \tag{S.4}$$

Using a convention where Klein factors transform as $\mathcal{T}\eta_y\mathcal{T}^{-1} = \mathcal{C}\eta_y\mathcal{C}^{-1} = \eta_{y+1}$, as well as $\mathcal{I}\eta_y\mathcal{I}^{-1} = \eta_{-y+1}$ and $\mathcal{M}\eta_y\mathcal{M}^{-1} = \eta_{-y}$, we conclude that the chiral boson fields transform according to

$$\mathcal{T}\phi_y(x)\mathcal{T}^{-1} = -\phi_{y+1}(x) + \alpha_y \,, \tag{S.5}$$

$$\mathcal{C}\phi_y(x)\mathcal{C}^{-1} = \phi_{y+1}(x) \,, \tag{S.6}$$

$$\mathcal{I}\phi_y(x)\mathcal{I}^{-1} = \phi_{-y+1}(-x) \,, \tag{S.7}$$

$$\mathcal{M}\phi_y(x)\mathcal{M}^{-1} = \phi_{-y}(x) + \alpha_y \,, \tag{S.8}$$

where $\alpha_y = y\pi$. The dual modes $\tilde{\phi}_y$ defined via

$$\tilde{\phi}_y(x) \equiv \sum_{y' \neq y} \text{sgn}\,(y - y')\,(-1)^{y'} \phi_{y'}(x) \tag{S.9}$$

therefore transform as

$$\mathcal{T}\tilde{\phi}_y(x)\mathcal{T}^{-1} = \sum_{y' \neq y} \text{sgn}\,(y - y')\,(-1)^{y'}\left(-\phi_{y'+1}(x) + \alpha_{y'}\right) = \tilde{\phi}_{y+1}(x) + \alpha_y + \sum_{y'} \alpha_{y'} \,, \tag{S.10}$$

$$\mathcal{C}\tilde{\phi}_y(x)\mathcal{C}^{-1} = \sum_{y' \neq y} \text{sgn}\,(y - y')\,(-1)^{y'} \phi_{y'+1}(x) = -\tilde{\phi}_{y+1}(x) \,, \tag{S.11}$$

$$\mathcal{I}\tilde{\phi}_y(x)\mathcal{I}^{-1} = \sum_{y' \neq y} \text{sgn}\,(y - y')\,(-1)^{y'} \phi_{-y'+1}(-x) = \tilde{\phi}_{-y+1}(-x) \,, \tag{S.12}$$

$$\mathcal{M}\tilde{\phi}_y(x)\mathcal{M}^{-1} = \sum_{y' \neq y} \text{sgn}\,(y - y')\,(-1)^{y'}\left(\phi_{-y'}(x) + \alpha_{y'}\right) = -\tilde{\phi}_{-y}(x) + \alpha_y + \sum_{y'} \alpha_{y'} \,. \tag{S.13}$$

In the first and last equations we used that the phase fields are defined modulo 2π and $\alpha_{y'} = -\alpha_{y'} \mod 2\pi$ for any y'; we will henceforth omit the unimportant constant $\sum_{y'} \alpha_{y'}$. Table I provides a full list of symmetry transformations for all involved fields.

TABLE I. Action of discrete symmetries for single Dirac cone/$N = 1$ QED$_3$ duality.

	$\Psi(x,y)$	$\psi_y(x)$	$\phi_y(x)$	$\tilde{\phi}_y(x)$	$\tilde{\psi}_y(x)$	$\tilde{\Psi}(x,y)$
$\mathcal{T}\ldots\mathcal{T}^{-1}$	$i\sigma^y\Psi(x,y)$	$(-1)^y\psi_{y+1}(x)$	$-\phi_{y+1}(x)+\alpha_y$	$\tilde{\phi}_{y+1}(x)+\alpha_y$	$-\tilde{\psi}_{y+1}^\dagger(x)$	$\sigma^x\tilde{\Psi}(x,y)$
$\mathcal{C}\ldots\mathcal{C}^{-1}$	$\sigma^x\Psi^\dagger(x,y)$	$\psi_{y+1}^\dagger(x)$	$\phi_{y+1}(x)$	$-\tilde{\phi}_{y+1}(x)$	$-(-1)^y\tilde{\psi}_{y+1}(x)$	$i\sigma^y\tilde{\Psi}^\dagger(x,y)$
$\mathcal{I}\ldots\mathcal{I}^{-1}$	$\sigma^x\Psi(-x,-y)$	$\psi_{-y+1}(-x)$	$\phi_{-y+1}(-x)$	$\tilde{\phi}_{-y+1}(-x)$	$-\tilde{\psi}_{-y+1}(-x)$	$-\sigma^x\tilde{\Psi}(-x,-y)$
$\mathcal{M}\ldots\mathcal{M}^{-1}$	$\sigma^z\Psi(x,-y)$	$(-1)^y\psi_{-y}(x)$	$\phi_{-y}(x)+\alpha_y$	$-\tilde{\phi}_{-y}(x)+\alpha_y$	$\tilde{\psi}_{-y}^\dagger(x)$	$\tilde{\Psi}^\dagger(x,-y)$

GAUGE INVARIANCE

Here we explicitly demonstrate gauge invariance of the bosonized wire model with staggered Chern-Simons term. Consider the action $\mathcal{S} = \mathcal{S}_{\text{wire}} + \mathcal{S}'_{\text{staggered-CS}}$

$$\mathcal{S}_{\text{wire}} = \sum_y \int_{t,x} \left\{ \frac{(-1)^y}{4\pi} \partial_x\phi_y \left(\partial_t\phi_y - 2A_{0,y}\right) - \frac{v}{4\pi}\left(\partial_x\phi_y - A_{1,y}\right)^2 \right\} , \tag{S.14}$$

$$\mathcal{S}'_{\text{staggered-CS}} = -\sum_y \frac{\nu_y}{4\pi} \int_{t,x} \left\{ \Delta A_{0,y}\left(A_{1,y+1} + A_{1,y}\right) - \partial_t A_{2,y+1/2}\left(A_{1,y+1} + A_{1,y}\right) + \partial_x A_{2,y+1/2}\left(A_{0,y+1} + A_{0,y}\right) \right\} , \tag{S.15}$$

where $\Delta A_{\mu,y} = A_{\mu,y+1} - A_{\mu,y}$ as before. One can view $\mathcal{S}'_{\text{staggered-CS}}$ as a lattice analogue of Eq. (7) of the main text; for now we leave the coefficient ν_y general but will fix it momentarily by imposing gauge invariance. Note also that here we do not commit to the $A_2 = 0$ gauge (in contrast to the main text) and label the location of A_2 by the midpoint $y + 1/2$ between two wires y and $y + 1$.

Suppose that we now perform the gauge transformation

$$\phi_y \to \phi_y + f_y , \tag{S.16}$$
$$A_{0,y} \to A_{0,y} + \partial_t f_y , \tag{S.17}$$
$$A_{1,y} \to A_{1,y} + \partial_x f_y , \tag{S.18}$$
$$A_{2,y+1/2} \to A_{2,y+1/2} + \Delta f_y . \tag{S.19}$$

This transformation yields the following anomalies in the separate $\mathcal{S}_{\text{wire}}$ and $\mathcal{S}'_{\text{staggered-CS}}$ parts:

$$\delta\mathcal{S}_{\text{wire}} = \sum_y \int_{t,x} \frac{-(-1)^y}{4\pi} \left[2A_{0,y}\partial_x f_y + \partial_t f_y \partial_x f_y\right] , \tag{S.20}$$

$$\delta\mathcal{S}'_{\text{staggered-CS}} = \sum_y \int_{t,x} \frac{\nu_y - \nu_{y-1}}{4\pi} \left[2A_{0,y}\partial_x f_y + \partial_t f_y \partial_x f_y\right] . \tag{S.21}$$

The total action is gauge invariant when $\delta\mathcal{S}_{\text{wire}} + \delta\mathcal{S}'_{\text{staggered-CS}} = 0$, which is satisfied for

$$\nu_y - \nu_{y-1} = (-1)^y \quad \Rightarrow \quad \nu_y = \nu_0 + \frac{1}{2}(-1)^y . \tag{S.22}$$

On the topological insulator surface, the uniform contribution must vanish ($\nu_0 = 0$) by $\mathcal{T}$ symmetry; we assume that this is the case for the remainder of this Appendix.

EXPLICIT DUALITY IN THE PRESENCE OF NON-ZERO ELECTROMAGNETIC VECTOR POTENTIAL A_μ

In the main text, we introduced the operators D and Δ whose matrix representation is

$$D_{y,y'} = (1 - \delta_{y,y'})\,\text{sgn}(y - y')(-1)^{y'} , \tag{S.23}$$
$$\Delta_{y,y'} = \delta_{y+1,y'} - \delta_{y,y'} . \tag{S.24}$$

It is convenient to further define

$$S_{y,y'} = \delta_{y+1,y'} + \delta_{y,y'} \,, \tag{S.25}$$

$$P_{y,y'} = (-1)^y \delta_{y,y'} \,, \tag{S.26}$$

which satisfy the following relations:

$$DPD^T = -P \,, \tag{S.27}$$

$$\Delta D = -P\Delta \,, \tag{S.28}$$

$$S^T \Delta = -\Delta^T S \,, \tag{S.29}$$

$$PSP = -\Delta \,. \tag{S.30}$$

The first of these encodes the reversed chirality of $\tilde{\psi}_y$ compared to ψ_y while the second one is responsible for the locality of dual-fermion hopping when expressed in terms of electrons. The third relation is a lattice analogue of integration by parts. These matrices allow us to express $\mathcal{S}_{\text{wire}}$ and $\mathcal{S}'_{\text{staggered-CS}}$ concisely (choosing the gauge $A_2 = 0$ for convenience) as

$$\mathcal{S}_{\text{wire}} = \int_{t,x} \left[\frac{P\partial_x \phi\, (\partial_t \phi - 2A_0)}{4\pi} - \frac{v}{4\pi}(\partial_x \phi - A_1)^2 - h_{\text{hop}}[\phi] \right] \,, \tag{S.31}$$

$$\mathcal{S}'_{\text{staggered-CS}} = - \int_{t,x} \frac{P}{8\pi} \Delta A_0 S A_1 \,, \tag{S.32}$$

where $\phi = (\ldots, \phi_{y-1}, \phi_y, \phi_{y+1}, \ldots)$, $A_\mu = (\ldots, A_{\mu,y-1}, A_{\mu,y}, A_{\mu,y+1}, \ldots)$ and a scalar product is implied, i.e.,

$$B\phi C\phi = \sum_y (B\phi)_y (C\phi)_y = \sum_{y,y',y''} B_{yy'}\phi_{y'} C_{yy''}\phi_{y''} = \phi B^T C\phi \tag{S.33}$$

for matrices B,C. The main text derived the duality with $A_\mu = 0$ by mapping the electron action

$$\mathcal{S}[A_\mu = 0] = \int_{t,x} \left[\frac{P}{4\pi}\partial_x \phi \partial_t \phi - \frac{v}{4\pi}(\partial_x \phi)^2 - h_{\text{hop}}[\phi] \right] \tag{S.34}$$

onto the dual action

$$\mathcal{S}_{\text{dual}}[A_\mu = 0] = \int_{t,x} \left[\frac{-P}{4\pi}\partial_x \tilde{\phi}\left(\partial_t \tilde{\phi} - 2a_0 \right) - \frac{u}{4\pi}(\partial_x \tilde{\phi} - a_1)^2 - h_{\text{hop}}[\tilde{\phi}] + \frac{1}{16\pi v}(\Delta a_0)^2 - \frac{v}{16\pi}(\Delta a_1)^2 + \frac{P}{8\pi}\Delta a_0 S a_1 \right] \,. \tag{S.35}$$

When restoring the external vector potential, the free-electron action must be supplemented by the A_μ-dependent terms of Eqs. (S.31) and (S.32). Duality then maps

$$\mathcal{S} = \mathcal{S}[A_\mu = 0] + \int_{t,x} \left[-\frac{\partial_x \phi P A_0}{2\pi} + \frac{v\partial_x \phi A_1}{2\pi} - \frac{vA_1^2}{4\pi} - \frac{P\Delta A_0 S A_1}{8\pi} \right] \tag{S.36}$$

$$\rightarrow \mathcal{S}_{\text{dual}}[A_\mu = 0] + \int_{t,x} \left[-\frac{\partial_x \tilde{\phi} D^T P A_0}{2\pi} + \frac{v\partial_x \tilde{\phi} D^T A_1}{2\pi} - \frac{vA_1^2}{4\pi} - \frac{P\Delta A_0 S A_1}{8\pi} \right] \,. \tag{S.37}$$

Shifting

$$a_0 \rightarrow a_0 - DA_0 + vDPA_1 \tag{S.38}$$

decouples A_μ from $\tilde{\phi}$. Using $\Delta(vDPA_1 - DA_0) = vSA_1 + P\Delta A_0$, see Eqs. (S.30) and (S.28), we find

$$\mathcal{S}_{\text{dual}} = \mathcal{S}_{\text{dual}}[A_\mu = 0] + \int_{t,x} \left[\frac{\Delta a_0 \Delta(vDPA_1 - DA_0)}{8\pi v} + \frac{[\Delta(vDPA_1 - DA_0)]^2}{16\pi v} + \frac{P\Delta(vDPA_1 - DA_0)Sa_1}{8\pi} - \frac{vA_1^2}{4\pi} - \frac{P\Delta A_0 S A_1}{8\pi} \right]$$

$$= \mathcal{S}_{\text{dual}}[A_\mu = 0] + \mathcal{S}_{\text{CS}} + \mathcal{S}_{\text{staggered-}aA} + \int_{t,x} \left[\frac{1}{16\pi v}(\Delta A_0)^2 - \frac{v}{16\pi}(\Delta A_1)^2 \right] \,.$$

Here $\mathcal{S}_{\text{staggered-}aA} = \int_{t,x} P\left(\frac{1}{v}\Delta a_0 \Delta A_0 - v\Delta A_1 \Delta a_1 \right)/8\pi$ drops out in the continuum limit due to the oscillatory factor P while the final term simply renormalizes the dielectric properties of A_μ. The important A_μ contribution that survives the continuum limit is the mutual Chern-Simons term $\mathcal{S}_{\text{CS}} = \int_{t,x} (\Delta a_0 S A_1 + \Delta A_0 S a_1)/8\pi$ as quoted in the main text.

ALTERNATIVE INTERPRETATION OF DUAL FERMIONS

For an alternative physical interpretation of dual fermion, imagine combining pairs of counterpropagating chiral wires into non-chiral ones as in Fig. 1. For each non-chiral wire we define canonical boson fields $\varphi_{2y}, \theta_{2y} = (\phi_{2y} \pm \phi_{2y+1})/2$, where $e^{2i\theta_{2y}}$ introduces a 2π phase slip in φ_{2y}. Viewing each non-chiral wire as a (power-law) superconductor, $e^{2i\theta_{2y}}$ equivalently produces a 4π phase slip in the phase of the Cooper pair $e^{2i\varphi_{2y}}$ and thus hops an hc/e vortex across the wire; see Fig. 1. Dual fermions are expressed in this language as

$$\tilde{\psi}^{\dagger}_{2y/2y+1} \sim \psi^{\dagger}_{2y+1/2y} \prod_{y'>y} e^{2i\theta_{2y'}} \prod_{y'<y} e^{-2i\theta_{2y'}} . \tag{S.39}$$

The two products each describe an hc/e vortex of the same sign moving in from (opposite) infinity. Thus, the dual fermion may be viewed as two hc/e vortices tied to an electron, as in the very different approaches from [2, 4].
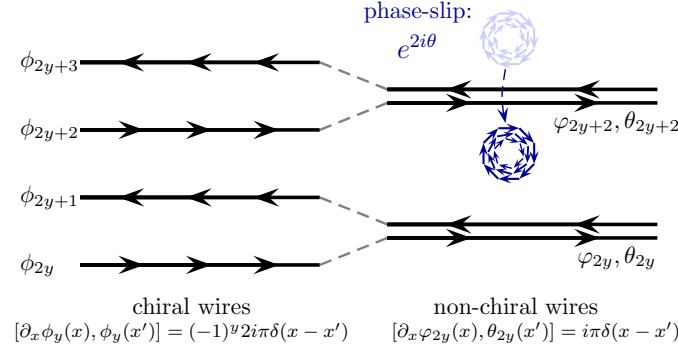

FIG. 1. Array of wires with alternating chirality (left) viewed as an array of non-chiral wires (right). In the non-chiral basis $\psi^{\dagger}_{2y+1}\psi_{2y} \sim e^{2i\theta_{2y}}$ is the familiar phase-slip operator that one can view as tunneling a vortex.

DUALITY BETWEEN THE T-PFAFFIAN AND FU-KANE SUPERCONDUCTOR

We consider a generalized coupled-wire model where only $\mathcal{T}^3$ and $\mathcal{C}^3$ symmetries are enforced, and inter-wire hopping is described by the Hamiltonian density

$$h_{\text{hop}} = w_3^{(0)}(-1)^y \psi^{\dagger}_{3y}\psi_{3y+3} + w_3^{(1)}(-1)^y \psi^{\dagger}_{3y+1}\psi_{3y+4} + w_3^{(2)}(-1)^y \psi^{\dagger}_{3y+2}\psi_{3y+5} + \text{H.c.} . \tag{S.40}$$

These terms only mix electrons between wires whose separation is a multiple of three, and hence give rise to three independent Dirac cones. Two of these may be viewed as forming a strictly two-dimensional system (see Fig. 2) and can be gapped out by the symmetry-preserving mass term

$$h_{\text{mass}} = m(-1)^y \psi^{\dagger}_{3y}\psi_{3y+1} + \text{H.c.}, \tag{S.41}$$

while third Dirac cone remains gapless provided charge is conserved and $\mathcal{T}^3$ or $\mathcal{C}^3$ is enforced.

Instead of being gapped by Eq. (S.41), the same two Dirac cones can form a conventional 2D superconductor that spontaneously breaks charge conservation when the interaction

$$h_{\text{Pair}} = g_{\text{Pair}} \psi^{\dagger}_{3y}\psi^{\dagger}_{3y+1}\psi_{3y+3}\psi_{3y+4} + \text{H.c.} \tag{S.42}$$

is relevant. The remaining massless Dirac cone is clearly unaffected by either h_{mass} or h_{Pair}. However, in the latter case the superconductor formed by the first two Dirac cones can proximity-induce pairing into the final Dirac cone via

$$h_{\text{proximity}} = g_{\text{proximity}} \psi^{\dagger}_{3y+2}\psi^{\dagger}_{3y+5}\psi_{3y+3}\psi_{3y+4} + \text{H.c.}, \tag{S.43}$$

without breaking $\mathcal{T}^3$ or $\mathcal{C}^3$ symmetry. The inter-wire Hamiltonian density

$$h_{\text{FK}} = h_{\text{Pair}} + h_{\text{proximity}} + w_3^{(2)}(-1)^y(\psi^{\dagger}_{3y+2}\psi_{3y+5} + \text{H.c.}) \tag{S.44}$$

conventional 2D Layer

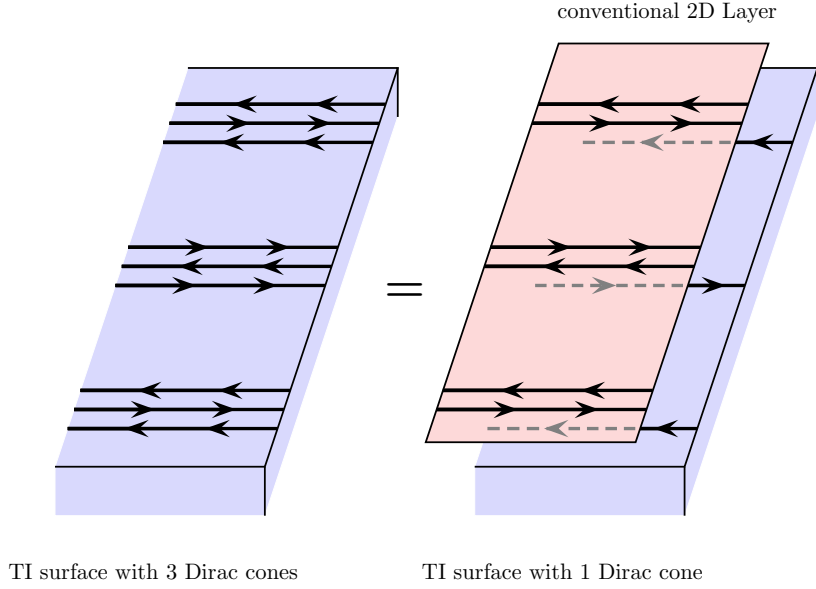TI surface with 3 Dirac cones   TI surface with 1 Dirac cone

FIG. 2. Generalized model of the TI surface containing three Dirac cones, only one of which is protected by time-reversal symmetry and charge conservation. As sketched on the right, this setup may equivalently be viewed as a conventional 2D layer containing two Dirac cones, deposited on top of a topological insulator with a single Dirac cone.

thus describes a Fu-Kane superconductor [5]. Here the surface electrons are gapped modulo the superfluid mode, present since we are not explicitly breaking charge conservation. One readily verifies its familiar properties, such as Majorana zero-modes trapped in vortex cores.

Consider now the Hamiltonian of a *dual* Fu-Kane superconductor $\tilde{h}_{\mathrm{FK}} \equiv h_{\mathrm{FK}}[\psi_y \to \tilde{\psi}_y]$, translated back into the original electron variables ψ using our explicit duality mapping, i.e.,

$$\tilde{h}_{\mathrm{FK}} = g_{\mathrm{Pair}} \, \psi_{3y}^\dagger \psi_{3y+1}^3 \left(\psi_{3y+3}^\dagger \right)^4 \psi_{3y+3}^3 \psi_{3y+4}^\dagger \tag{S.45}$$

$$+ \, g_{\mathrm{proximity}} \, \psi_{3y+2}^\dagger \psi_{3y+5}^\dagger \psi_{3y+3} \psi_{3y+4}. \tag{S.46}$$

$$+ \, w_3^{(2)} \, (-1)^y \psi_{3y+2}^\dagger (\psi_{3y+3} \psi_{3y+4}^\dagger)^2 \psi_{3y+5} + \mathrm{H.c.}. \tag{S.47}$$

Precisely this Hamiltonian was derived in Ref. 6 for the T-Pfaffian topological order. The g_{Pair}-term gaps charge fluctuations, while the other two terms gap the remaining neutral degrees of freedom.

CONTINUUM LIMIT AND THE DUAL-FERMION VELOCITY

The dual action for $A_\mu = 0$ is once again given by

$$\mathcal{S}_{\mathrm{dual}} = \int_{t,x} \sum_y \left[\frac{-(-1)^y \partial_x \tilde{\phi}_y \left(\partial_t \tilde{\phi}_y - 2a_{0,y} \right)}{4\pi} - \frac{u}{4\pi} \left(\partial_x \tilde{\phi}_y - a_{1,y} \right)^2 \tag{S.48} \right.$$

$$\left. + \frac{1}{16\pi v} (\Delta a_{0,y})^2 - \frac{v}{16\pi} (\Delta a_{1,y})^2 + \frac{(-1)^y}{8\pi} \Delta a_{0,y}(a_{1,y+1} + a_{1,y}) \right]. \tag{S.49}$$

We see that the gauge field is massless only near zero momentum in the y-direction, and hence the most naive continuum limit would simply drop the last staggered Chern-Simons term in the action. Here we will pursue a more accurate procedure that formally integrates out such high-momentum massive modes, which yields useful insight into parameters for the dual theory.

To this end note that the Chern-Simons term couples zero-momentum modes with gapped modes at momentum π. Hence, we decompose a_μ into smooth and rapidly oscillatory pieces

$$a_{\mu,y} \approx a_{\mu,y}^{(0)} + (-1)^y a_{\mu,y}^{(\pi)}, \tag{S.50}$$

where both $a_{\mu,y}^{(0)}$ and $a_{\mu,y}^{(\pi)}$ are slowly varying functions of y. The action becomes

$$\mathcal{S}_{\text{dual}} \approx \int_{t,x} \sum_y \left[\frac{-(-1)^y \partial_x \tilde{\phi}_y \left(\partial_t \tilde{\phi}_y - 2a_{0,y}^{(0)}\right)}{4\pi} - \frac{u}{4\pi} \left(\partial_x \tilde{\phi}_y - a_{1,y}^{(0)}\right)^2 - \frac{v}{16\pi} \left(\Delta a_{1,y}^{(0)}\right)^2 + \frac{1}{16\pi v} \left(\Delta a_{0,y}^{(0)}\right)^2 \right] \tag{S.51}$$

$$+ \int_{t,x} \sum_y \left[\frac{a_{0,y}^{(\pi)} \left(\partial_x \tilde{\phi}_y - a_{1,y}^{(0)}\right)}{2\pi} + \frac{u}{2\pi} \left(\partial_x \tilde{\phi}_y - a_{1,y}^{(0)}\right)(-1)^y a_{1,y}^{(\pi)} - \frac{u+v}{4\pi} \left(a_{1,y}^{(\pi)}\right)^2 + \frac{1}{4\pi v} \left(a_{0,y}^{(\pi)}\right)^2 - \frac{\Delta a_{0,y}^{(0)} \Delta a_{1,y}^{(\pi)}}{8\pi} \right],$$

where we dropped rapidly oscillating terms (note however that we are not making any assumptions about $\tilde{\phi}_y$ and all terms containing these fields are kept). At this point, integrating out the massive $a_\mu^{(\pi)}$ modes yields

$$\mathcal{S}_{\text{dual, eff.}} \approx \int_{t,x} \sum_y \left[\frac{-(-1)^y \partial_x \tilde{\phi}_y \left(\partial_t \tilde{\phi}_y - 2a_{0,y}^{(0)}\right)}{4\pi} - \frac{\tilde{v}}{4\pi} \left(\partial_x \tilde{\phi}_y - a_{1,y}^{(0)}\right)^2 - \frac{v}{16\pi} \left(\Delta a_{1,y}^{(0)}\right)^2 + \frac{1}{16\pi v} \left(\Delta a_{0,y}^{(0)}\right)^2 \right], \tag{S.52}$$

up to terms containing higher derivatives. In this more systematic approach the dual-fermion velocity has been renormalized from u to $\tilde{v} = v + \frac{uv}{u+v}$. We note that while the microscopic theory $\mathcal{Z} \sim \int e^{i\mathcal{S}_{\text{dual}}}$ is independent of u (up to a constant multiplying $\mathcal{Z}$ and assuming for certainty non-negative u), this appears to be no longer the case in Eq. (S.52). This is because keeping only $a_\mu^{(0)}$ and $a_\mu^{(\pi)}$ modes in the above analysis is an approximation; recovering the u-independence would demand a more careful treatment of the massive modes, which however would modify only short-range interactions. Universal properties will not depend on such short-range details, and it is sufficient to observe that $\tilde{v} \sim v$ (with weak u dependence). That is, the dual fermions have robust effective kinetic energy along the wires, even if it appears that the bare velocity parameter u was introduced arbitrarily.

STABILITY OF $N = 1$ LATTICE QED$_3$ WITH GENERIC MAXWELL TERM

Our explicit derivation showed that free Dirac electrons are dual to QED$_3$ where the Maxwell term of the emergent photon

$$\mathcal{S}_{\text{MW}} \sim \lambda_\mu \left(\epsilon_{\mu\nu\kappa} \partial_\nu a_\kappa\right)^2 \tag{S.53}$$

has the bare parameter $\lambda_y = 0$. Although a finite λ_y is generated under renormalization, it is instructive to add by hand a non-zero bare λ_y and discuss its effect on the original Dirac electrons. This is most conveniently done by applying the duality transformation to $\mathcal{S}_{\text{dual}}$ of Eq. (18) (now with $\lambda_y \neq 0$), resulting in

$$\mathcal{S} = \int_{t,x} \sum_y \left\{ \frac{(-1)^y \partial_x \phi_y \partial_t \phi_y}{4\pi} - h_{\text{wire}}[\phi, a_\mu'] - h_{\text{hop}}[\phi] \right\} \tag{S.54}$$

$$+ \mathcal{S}_{\text{MW}}[a_\mu]\big|_{\lambda_y \neq 0} + \mathcal{S}_{\text{MW}}[a_\mu']\big|_{\lambda_y = 0} + \mathcal{S}_{\text{CS}}[a_\mu, a_\mu']. \tag{S.55}$$

Here a_μ' is a new emergent gauge field resulting from applying duality a second time, and we suppressed all terms that vanish in the continuum limit. Because of the mutual Chern-Simons term $\mathcal{S}_{\text{CS}}$, integrating out a_μ renders a_μ' massive. Massive a_μ' bosons only mediate short-range interactions that are irrelevant at the charge neutrality point and hence do not destabilize the electronic Dirac liquid. Thus, $N = 1$ lattice QED$_3$ with general bare λ_y maps onto a weakly correlated Dirac liquid of electrons whose short-range interaction strength is determined by λ_y.

MORE PROPERTIES OF $N = 1$ QED$_3$: DUALS OF FERMION CURRENTS

Here we show another interesting example of relating operators in $N = 1$ QED$_3$ to operators in free Dirac theory. Proceeding similarly to our discussion of the fermion mass term in Eq. (19), we can use the microscopic Eq. (12) to also obtain

$$\int_x \sum_y (i\tilde{\psi}_y^\dagger \tilde{\psi}_{y+1} + \text{H.c.}) = \int_x \sum_y (-1)^{y+1} (i\psi_y^\dagger \psi_{y+1} + \text{H.c.}). \tag{S.56}$$

Taking the continuum limit of the two sides gives the identification (omitting numerical factors)

$$\tilde{\Psi}^\dagger \sigma^x i \partial_y \tilde{\Psi} \sim \Psi^\dagger \sigma^y \Psi \equiv j_2, \tag{S.57}$$

where j_2 is an electron current in the $\hat{y}$ direction. A similar equation with $\tilde{\psi}$ and ψ interchanged gives

$$\Psi^\dagger \sigma^x i \partial_y \Psi \sim \tilde{\Psi}^\dagger \sigma^y \tilde{\Psi} \equiv \tilde{j}_2 \,, \tag{S.58}$$

where $\tilde{j}_2$ is a fermion current in the $\hat{y}$ direction in the dual QED$_3$ theory. Thus, we can relate a fermion current in one theory to a fermion bilinear with a derivative in the other theory and vice versa. We will see that these unusual identifications are consistent with the expected properties of QED$_3$.

Let us first consider the second relation identifying the current $\tilde{j}_2$ in the dual theory and the specific electron bilinear with one derivative. The scaling dimension of the latter in the free Dirac theory is 3. This implies that correlations of the $\tilde{j}_2$ currents decay as $1/|\boldsymbol{r}|^6$, with $\boldsymbol{r}$ a space-time coordinate, corresponding to a Fourier space non-analyticity $\langle \tilde{j}_2(\boldsymbol{q}) \tilde{j}_2(-\boldsymbol{q}) \rangle_{\text{sing}} \sim |\boldsymbol{q}|^3$. As we will see, this is expected because the currents $\tilde{j}$ have long-range interactions mediated by the gauge field in the QED$_3$ theory. Indeed, let us remind ourselves how this works in a generic matter-gauge theory (and we will also see an often-stated relation between direct and dual "conductivities"). Consider a (2+1)D path integral for matter fields $\tilde{\Psi}_{\text{matter}}$ coupled to a gauge field a_μ

$$Z[c_\mu] = \int \mathcal{D}\tilde{\Psi}_{\text{matter}} \mathcal{D}a_\mu \, \exp\left[-S_{\text{matter}}[\tilde{\Psi}_{\text{matter}}] - \frac{\kappa}{2}\int_{\boldsymbol{r}} (\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda)^2 - i\int_{\boldsymbol{r}} \tilde{j}_\mu a_\mu - i\int_{\boldsymbol{r}} \tilde{j}_\mu c_\mu\right] \,. \tag{S.59}$$

To simplify equations, in this Appendix only, we work in imaginary time (i.e., Euclidean path integral) and consider a space-time-isotropic Maxwell term for the gauge field and space-time-isotropic full action. We do not need any details about the matter field $\tilde{\Psi}_{\text{matter}}$ other than the exhibited coupling of its current $\tilde{j}_\mu$ to the dynamical gauge field a_μ. Note that we use tilde over the matter-field objects only to match our earlier notation on the QED$_3$ side of the duality.

When defining the above path integral, we have also included a probing gauge field $c_\mu(\boldsymbol{r})$, which allows us to calculate current correlations by functional differentiation:

$$\langle \tilde{j}_\mu(\boldsymbol{r}) \tilde{j}_{\mu'}(\boldsymbol{0}) \rangle = -\frac{1}{Z}\frac{\delta^2 Z}{\delta c_\mu(\boldsymbol{r})\delta c_{\mu'}(\boldsymbol{0})}\Big|_{c_\mu = 0} \,. \tag{S.60}$$

Upon shifting the integration variables $a_\mu \to a_\mu - c_\mu$, we can recast the path integral as

$$Z[c_\mu] = \int \mathcal{D}\tilde{\Psi}_{\text{matter}} \mathcal{D}a_\mu \, \exp\left[-S_{\text{matter}}[\tilde{\Psi}_{\text{matter}}] - \frac{\kappa}{2}\int_{\boldsymbol{r}} (\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda - \epsilon_{\mu\nu\lambda}\partial_\nu c_\lambda)^2 - i\int_{\boldsymbol{r}} \tilde{j}_\mu a_\mu\right] \,. \tag{S.61}$$

By applying Eq. (S.60) to this form, upon differentiation and setting $c_\mu = 0$, we obtain the following relation

$$\langle \tilde{j}_\mu(\boldsymbol{r}) \tilde{j}_{\mu'}(\boldsymbol{0}) \rangle = \kappa(\nabla_\mu \nabla_{\mu'} - \delta_{\mu\mu'}\boldsymbol{\nabla}^2)\delta(\boldsymbol{r}) - \kappa^2 \langle \epsilon_{\mu\nu\lambda}\partial_\nu J_\lambda(\boldsymbol{r}) \, \epsilon_{\mu'\nu'\lambda'}\partial_{\nu'} J_{\lambda'}(\boldsymbol{0}) \rangle \,, \tag{S.62}$$

where $J_\mu(\boldsymbol{r}) \equiv \epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda(\boldsymbol{r})$ denotes the gauge flux. The first term on the right-hand-side is local, so the long-distance behavior is determined by the second term. Specializing to critical field theories, the gauge flux J_μ in the strongly-coupled CFT has scaling dimension 2: In the $N = 1$ QED$_3$ theory, we know this from the duality since J_μ gives the original free Dirac electron current (more precisely, $j_\mu = \frac{J_\mu}{4\pi}$); for general N, the scaling dimension is fixed by the fact that J_μ is a conserved current with short-range interactions and can be also established using large-N treatments. Hence, using Eq. (S.62) we conclude that the $\tilde{j}$ correlations show $1/|\boldsymbol{r}|^6$ power law decay, in agreement with our discussion of Eq. (S.58). Note, however, that this scaling dimension is in principle already known within the QED$_3$ theory, so in this sense we did not find a qualitatively new property.

Let us consider now Eq. (S.57), which relates a specific dual fermion bilinear with a derivative to the electron current j_2 in the free Dirac theory. The latter has scaling dimension 2. Hence $\tilde{\Psi}^\dagger \sigma^x i\partial_y \tilde{\Psi}$ in the QED$_3$ theory also has the same scaling dimension, which is reduced (corresponding to enhanced correlations) compared to non-interacting fermions and is a non-trivial effect due to the gauge field in the QED$_3$ theory. However, in retrospect we could have anticipated this just within the QED$_3$ theory, since this operator has the same symmetry transformation properties as the gauge flux J_2, which as discussed has scaling dimension 2. Thus, this is also not a qualitatively new prediction in the QED$_3$ theory, but provides a nice consistency check for the duality.

Continuing a bit more with general considerations, let us also examine the current correlation in Fourier space. For divergenceless currents $\tilde{j}_\mu$ and J_μ, we can write

$$\langle \tilde{j}_\mu(\boldsymbol{q}) \tilde{j}_{\mu'}(-\boldsymbol{q}) \rangle \equiv K_{\tilde{j}}(\boldsymbol{q})\left(\delta_{\mu\mu'} - \frac{q_\mu q_{\mu'}}{\boldsymbol{q}^2}\right) \,, \qquad \langle J_\mu(\boldsymbol{q}) J_{\mu'}(-\boldsymbol{q}) \rangle \equiv K_J(\boldsymbol{q})\left(\delta_{\mu\mu'} - \frac{q_\mu q_{\mu'}}{\boldsymbol{q}^2}\right) \,. \tag{S.63}$$

Equation (S.62) then gives

$$K_{\tilde{j}}(\boldsymbol{q}) = \kappa \boldsymbol{q}^2 - \kappa^2 \boldsymbol{q}^2 K_J(\boldsymbol{q}) \,. \tag{S.64}$$

Since the leading non-analyticity in $K_J(\boldsymbol{q})$ is $|\boldsymbol{q}|$ (corresponding to J_μ's scaling dimension 2), we conclude that the leading non-analyticity in $K_{\tilde{j}}(\boldsymbol{q})$ is $|\boldsymbol{q}|^3$, as claimed earlier. When considering this matter-gauge system, it is convenient to introduce an "irreducible" current correlation function [7–9] via

$$K_{\tilde{j}}^{\text{irred}}(\boldsymbol{q}) \equiv \frac{K_{\tilde{j}}(\boldsymbol{q})}{1 - \frac{1}{\kappa \boldsymbol{q}^2} K_{\tilde{j}}(\boldsymbol{q})} . \tag{S.65}$$

Physically, while $K_{\tilde{j}}(\boldsymbol{q})$ describes the current response relative to the probing field c_μ, $K_{\tilde{j}}^{\text{irred}}(\boldsymbol{q})$ describes the response relative to the total field $c_\mu + \langle a_\mu \rangle_{c_\mu}$ seen by the matter, where $\langle a_\mu \rangle_{c_\mu}$ is the induced non-zero average of the internal gauge field a_μ due to the external probing field c_μ. It is easy to check that

$$K_{\tilde{j}}^{\text{irred}}(\boldsymbol{q}) = \frac{\boldsymbol{q}^2}{K_J(\boldsymbol{q})} - \kappa \boldsymbol{q}^2 . \tag{S.66}$$

Defining "conductivity" σ_J corresponding to the J_μ currents via the small-momentum behavior of $K_J(\boldsymbol{q})$, we have

$$K_J(\boldsymbol{q}) = \sigma_J |\boldsymbol{q}| \quad \implies \quad K_{\tilde{j}}^{\text{irred}}(\boldsymbol{q}) = \frac{1}{\sigma_J} |\boldsymbol{q}| \quad \text{for small } |\boldsymbol{q}| . \tag{S.67}$$

The last equation is often stated as dual "conductivity" being inverse of the direct conductivity. The difference in the treatments of the J_μ currents and $\tilde{j}_\mu$ currents is that the latter have long-range interactions while the former have only short-range interactions, which affects their scaling dimensions for the total correlation functions.

EMERGENT PARTICLE-HOLE SYMMETRY AND SINGLE DIRAC CONE IN A 2D ELECTRON GAS

A very important application of the duality is the derivation of Son's proposed field theory for the particle-hole-symmetric composite Fermi liquid in the lowest Landau level (LLL) at filling factor $\nu = 1/2$. The starting point for this derivation is the observation that adding a magnetic field to our single Dirac fermion breaks $\mathcal{T}$ but not $\mathcal{C}$, and that the problem of such a Dirac fermion with particle-hole symmetry in magnetic field maps to the 2D electron gas (2DEG) half-filled LLL problem with emergent particle-hole symmetry [2, 4]. The degeneracies associated with the magnetic field (i.e., frustration of the kinetic energy) are "resolved" by going to the dual fermions, which are again Dirac fermions but now at finite density as permitted by the action of $\mathcal{C}$ on these dual fields.

Dirac fermions also arise from different direct approaches to the 2DEG, particularly in the physics between fillings $\nu = 0$ and $\nu = 1$. A single massless Dirac fermion arises at a transition between $\nu = 0$ and $\nu = 1$ states in the Haldane model [10] and also at a plateau transition in the Chalker-Coddington model [11]. In both cases, however, these often-useful Dirac fermions are at *zero* magnetic field, and thus they are not related to the particle-hole-symmetric Dirac fermions used in the derivation of the Son's theory above.
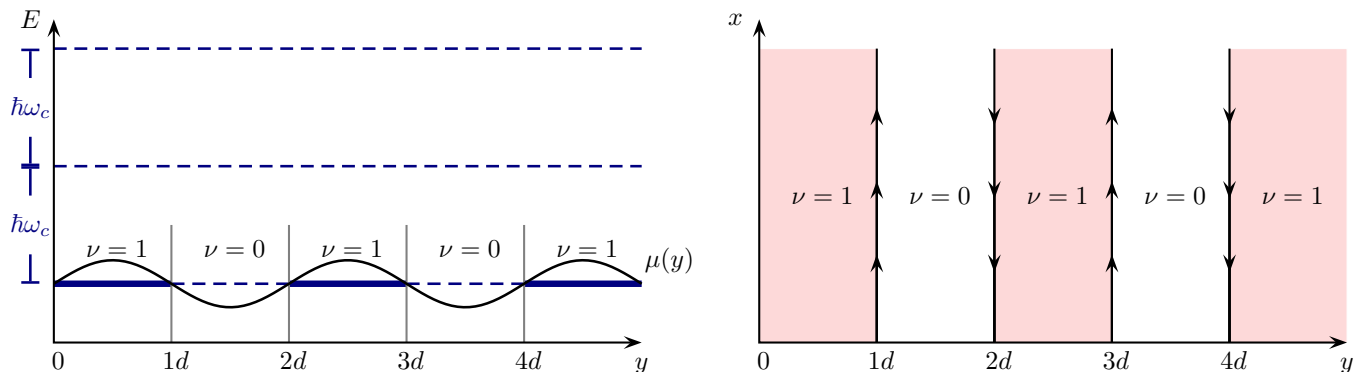

FIG. 3. Left: Electrons in a magnetic field form Landau levels separated by an energy $\hbar\omega_c$. A modulated chemical potential $\mu(y)$ produces regions alternating in space between filling factors $\nu = 0$ and $\nu = 1$. Right: The boundaries of these regions form an array of 1D electron 'wires' of staggered chirality. In the $\hbar\omega_c \to \infty$ limit, hybridizing these modes yields a single massless Dirac cone in a magnetic field with an emergent particle-hole symmetry.

It is nevertheless possible to use our methods—in particular the quasi-1D construction—to derive Son's theory beginning from a 2DEG in magnetic field. To explicitly realize the Hamiltonian of Eqs. (3,4) from the main text in this setting, consider

electrons moving in a vector potential A_μ where $\partial_1 A_2 - \partial_2 A_1 = B$ and $A_0 = \mu + \delta\mu \sin(\frac{\pi}{d}y)$ with the Landau level separation $\hbar\omega_c \gg \delta\mu$ and magnetic length $\ell_B \ll d$. When μ is tuned such that the electrons are at half-filling, we obtain alternating $\nu = 0$ and $\nu = 1$ quantum Hall strips as shown in Fig. 3. The system therefore hosts an array of staggered-chirality edge modes precisely as utilized earlier. Let ψ_y denote the edge field at interface y (to better match our earlier notation we now set $d = 1$); odd and even y respectively correspond to right- and left-movers. In the limit $\hbar\omega_c/\delta\mu \to \infty$, an exact symmetry emerges,

$$\mathcal{C}\psi_y\mathcal{C}^{-1} = \psi_{y+1}^\dagger , \tag{S.68}$$

corresponding to the LLL particle-hole transformation—which interchanges $\nu = 0$ and $\nu = 1$ fillings—composed with a translation. Importantly, $\mathcal{C}$ is *antiunitary*, which can be understood due to the generally complex LLL wavefunctions (see, e.g., [12]). Because of the antiunitarity, the most general $\mathcal{C}$-symmetric, charge-conserving single-particle Hamiltonian that hybridizes the edge modes describes Dirac electrons with zero chemical potential but with a finite a magnetic field, which we analyzed in the main text. We emphasize that the emergent symmetry implemented by $\mathcal{C}$ has the same status as the particle-hole symmetry in a spatially isotropic $\nu = 1/2$ system: it only emerges upon projection into the LLL.

SINGLE (MASSIVE) DIRAC CONE IN MAGNETIC FIELD FROM COUPLED WIRES

It is instructive to see how a (massive) Dirac cone in magnetic field arises more microscopically from a strictly 2D electron system. We therefore consider a two-dimensional system formed by an array of $N_{\rm w}$ wires separated by a distance a. The wires are enumerated by integers j and contain (non-chiral) one-dimensional electron liquids at densities n_j. If the system is subjected to a perpendicular magnetic field of strength B, then the electrons are at half-filling when

$$\nu \equiv \frac{2\pi}{BN_{\rm w}a} \sum_j n_j = \frac{1}{2} . \tag{S.69}$$

For example, this filling is realized when $M + 1$ consecutive wires with density $n = \frac{Ba}{2\pi}$ alternate with $M + 1$ wires with $n = 0$ (shown in Fig. 4 for $M = 1$). To analyze tunneling between the gapless edges of these strips, it is convenient to adopt the gauge

An array of one-dimensional electron wires with modulated density, separated by a distance a.

Electron tunneling leads to $\nu = 1$ integer quantum Hall strips, each of width a and separated by a distance $3a$.

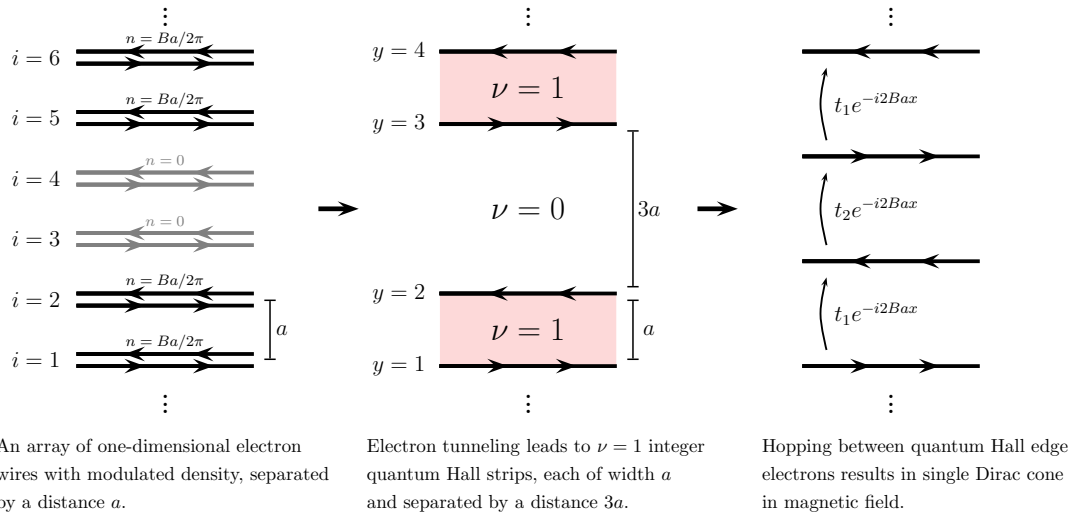Hopping between quantum Hall edge electrons results in single Dirac cone in magnetic field.

FIG. 4. A Dirac dispersion for electrons at $\nu = 1/2$ can be obtained from strips of $\nu = 1$ integer quantum Hall states. In this strict 2D construction there is no local symmetry that prohibits a mass term $m \sim t_1 - t_2$.

$A_x = -Baj, A_y = 0$, and enumerate the edge states by integers y, where odd and even integers denote the bottom and top edges, respectively. Electrons in the j-th wire with non-zero density $n_j = n$ expanded in long-wavelength right- and left-moving fields are expressed as

$$c_j(x) \sim c_{j,R}e^{i(\pi n - Baj)x} + c_{j,L}e^{i(-\pi n - Baj)x} . \tag{S.70}$$

Hopping of electrons between neighboring wires with $n_j \neq 0$ leads to $\nu = 1$ integer quantum Hall strips of width Ma separated by a distance $(M + 2)a$ [13]. We denote the surviving electronic edge modes of such a $\nu = 1$ region formed by wires $j, j +$

$1, \ldots, j + M$ as $c_{jR}, c_{j+M,L} \to \psi_{2y+1}, \psi_{2y+2}$. Tunneling between neighboring edges is then described by

$$h_1 = t_1 \psi_{2y+1}^\dagger \psi_{2y+2} e^{-i2\pi nx - iMaBx} + \text{H.c.} = t_1 \psi_{2y+1}^\dagger \psi_{2y+2} e^{-i(M+1)aBx} + \text{H.c.} \,, \tag{S.71}$$

$$h_2 = t_2 \psi_{2y+2}^\dagger \psi_{2y+3} e^{i2\pi nx - i(M+2)aBx} + \text{H.c.} = t_2 \psi_{2y+2}^\dagger \psi_{2y+3} e^{-i(M+1)aBx} + \text{H.c.} \,, \tag{S.72}$$

where h_1 hops electrons across a given $\nu = 1$ strip with amplitude t_1, while h_2 hops electrons between adjacent $\nu = 1$ strips with amplitude t_2. Importantly, both kinds of hopping carry identical x-dependent phase factors, which correspond to an orbital magnetic field acting on the ψ fermions. For $t_1 = t_2$ we therefore obtain precisely a single massless Dirac cone in a uniform magnetic field set by B. (We now see explicitly that the difference from the Chalker-Coddington network model [11], where the plateau transition yields a Dirac theory at *zero* magnetic field, arises because of kinematic restrictions imposed by the external magnetic field in our setup.) Within the present setup, however, $t_1 = t_2$ is not enforced by any microscopic symmetry—contrary to our surface construction—and requires fine-tuning. This is expected, since the exact particle-hole symmetry of the half-filled Landau level is not a local symmetry in any strictly two-dimensional system and requires projection into the lowest Landau level.

DISCRETE SYMMETRIES FOR BOSONIC TI SURFACE/$N = 2$ QED$_3$ DUAL THEORIES

The action for $N = 2$ QED$_3$ introduced in the main text exhibits the same discrete symmetries $\mathcal{T}, \mathcal{C}, \mathcal{I}, \mathcal{M}$ as a free Dirac cone. In addition there is a local $\mathbb{Z}_2$ symmetry that interchanges the two fermion species,

$$\mathcal{R} \tilde{\psi}_{\sigma,y}(x) \mathcal{R}^{-1} = \tilde{\psi}_{-\sigma,y}(x) \,. \tag{S.73}$$

Table II enumerates how these symmetries act on the various operators involved in the duality mapping. When considering specific bosonic TIs, one typically enforces only a subset of these symmetries. This dictionary may then be used to translate the symmetries of interest on the bosonic TI surface to symmetries in the dual gauge theory.

TABLE II. Action of discrete symmetries for bosonic TI surface/$N = 2$ QED$_3$ duality (omitting phase factors).

	$\tilde{\psi}_{\sigma,y}(x)$	$\tilde{\phi}_{\sigma,y}(x)$	$\tilde{\phi}_{n,y}(x)$	$\tilde{\phi}_{c,y}(x)$	$\phi_{c,y}(x)$	$b_{\sigma,y}(x)$
$\mathcal{T} \ldots \mathcal{T}^{-1}$	$\tilde{\psi}_{\sigma,y+1}(x)$	$-\tilde{\phi}_{\sigma,y+1}(x)$	$-\tilde{\phi}_{n,y+1}(x)$	$-\tilde{\phi}_{c,y+1}(x)$	$\phi_{c,y+1}(x)$	$b_{-\sigma,y+1}^\dagger(x)$
$\mathcal{C} \ldots \mathcal{C}^{-1}$	$\tilde{\psi}_{\sigma,y+1}^\dagger(x)$	$\tilde{\phi}_{\sigma,y+1}(x)$	$\tilde{\phi}_{n,y+1}(x)$	$\tilde{\phi}_{c,y+1}(x)$	$-\phi_{c,y+1}(x)$	$b_{-\sigma,y+1}(x)$
$\mathcal{I} \ldots \mathcal{I}^{-1}$	$\tilde{\psi}_{\sigma,-y+1}(-x)$	$\tilde{\phi}_{\sigma,-y+1}(-x)$	$\tilde{\phi}_{n,-y+1}(-x)$	$\tilde{\phi}_{c,-y+1}(-x)$	$\phi_{c,-y+1}(-x)$	$b_{\sigma,-y+1}(-x)$
$\mathcal{M} \ldots \mathcal{M}^{-1}$	$\tilde{\psi}_{\sigma,-y}(x)$	$\tilde{\phi}_{\sigma,-y}(x)$	$\tilde{\phi}_{n,-y}(x)$	$\tilde{\phi}_{c,-y}(x)$	$-\phi_{c,-y}(x)$	$b_{-\sigma,-y}^\dagger(x)$
$\mathcal{R} \ldots \mathcal{R}^{-1}$	$\tilde{\psi}_{-\sigma,y}(x)$	$\tilde{\phi}_{-\sigma,y}(x)$	$-\tilde{\phi}_{n,y}(x)$	$\tilde{\phi}_{c,y}(x)$	$\phi_{c,y}(x)$	$b_{-\sigma,y}(x)$

KLEIN FACTORS FOR $N = 2$ QED$_3$ DUALITY

For the duality between the fermionic $N = 2$ QED$_3$ and bosonic TI surface theory, it is natural to ask how the Klein factors present in the former are manifested in the latter. We show that the non-local duality transformation consistently maps between the two representations even at the level of exchange statistics. Rather than employing Klein factors η_y as above, here it is convenient to equivalently ensure anticommutation between different fermion species by taking

$$[\tilde{\phi}_{\sigma,y}(x), \tilde{\phi}_{y',\sigma'}(x')] = i\pi(-1)^y \delta_{y,y'}[\delta_{\sigma,\sigma'}\text{sgn}(x - x') + \epsilon_{\sigma,\sigma'}] + i\pi(1 - \delta_{y,y'})\text{sgn}(y - y'), \tag{S.74}$$

where $\epsilon_{\sigma,\sigma'}$ is antisymmetric with $\epsilon_{+-} = 1$. Charge and neutral modes $\tilde{\phi}_{c/n} = (\tilde{\phi}_+ \pm \tilde{\phi}_-)/2$ then obey

$$[\tilde{\phi}_{c,y}(x), \tilde{\phi}_{c,y'}(x')] = i\frac{\pi}{2}(-1)^y \delta_{y,y'}\text{sgn}(x - x') + i\pi(1 - \delta_{y,y'})\text{sgn}(y - y') \tag{S.75}$$

$$[\tilde{\phi}_{n,y}(x), \tilde{\phi}_{n,y'}(x')] = i\frac{\pi}{2}(-1)^y \delta_{y,y'}\text{sgn}(x - x') \tag{S.76}$$

$$[\tilde{\phi}_{c,y}(x), \tilde{\phi}_{n,y'}(x')] = -i\frac{\pi}{2}(-1)^y \delta_{y,y'} \,, \tag{S.77}$$

while fields dual to $\tilde{\phi}_{c,y}$ satisfy

$$[\phi_{c,y}(x), \phi_{c,y'}(x')] = -i\,\frac{\pi}{2}(-1)^y \delta_{y,y'} \mathrm{sgn}(x-x') - i\,\pi(1-\delta_{y,y'})\mathrm{sgn}(y-y') \, . \tag{S.78}$$

Finally, the phases of dual bosons $\varphi_{\sigma,y} = (\phi_{c,y} + \sigma\tilde{\phi}_{n,y})$ exhibit commutators

$$[\varphi_{\sigma,y}(x), \varphi_{\sigma,y'}(x')] = 0 \mod 2i\pi \tag{S.79}$$

$$[\varphi_{\sigma,y}(x), \varphi_{-\sigma,y'}(x')] = -i\,\pi(-1)^y \delta_{y,y'} \mathrm{sgn}(x-x') - i\,\pi(1-\delta_{y,y'}) \tag{S.80}$$

Note that the mutual phase of π in the last line does not affect the interpretation of $b_\sigma \sim e^{i\varphi_{\sigma,y}}$ as bosons. Since bosons of species $\sigma = +$ and $\sigma = -$ are distinguishable, only a full braid (rather than a single exchange) is meaningful; there the acquired phase is an integer multiple of 2π as required for bosons.

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