The N Extra Element Theorem

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***Abstract*— The N Extra Element Theorem (NEET) is an alternative means of analysis for any transfer function of any linear system model, not restricted to electrical systems. Its principal distinction from conventional loop or node analysis is that a simpler reference system model in the absence of N designated "extra" elements is solved first, and the N extra elements are then restored via a correction factor.**

Parameters in the correction factor are various single injection and null double injection driving point immittances seen by the extra elements, and are all calculated upon the reference model. Thus, no calculation is performed upon a model containing any of the designated extra elements, and the final result is obtained by assembly of sequentially obtained results in a "divide and conquer" approach that is potentially easier, shorter, and which produces lower entropy forms than does the conventional approach.

The NEET correction factor is a simultaneous bilinear representation of the extra elements, which can be immittances or dependent generators in any combination, and thus exposes explicitly the contribution of each extra element.

An especially useful implementation of the NEET is to designate all the reactances as extra elements. The frequency response of the transfer function is then contained entirely in the NEET correction factor, which emerges directly as a ratio of polynomials in complex frequency s. The zeros as well as the poles can thus be obtained directly from the driving point resistances seen by the reactances, and it can also be determined whether any of the zeros or poles are exactly factorable.

The approach throughout is to show how the NEET theorem can be useful in practical Design-Oriented Analysis, and emphasis is on the criteria by which the designer-analyst can take maximum advantage of the numerous choices of which elements to designate as "extra," and which of the many versions of the theorem to adopt.

NOMENCLATURE

EE's	Extra Element(s).
EEi	ith EE.
EET	Extra Element Theorem (single, or general for NEET).
2EET	Two Extra Element Theorem.
NEET	N Extra Element Theorem.
ref state	Reference state of EE (short or open).
ref	Reference circuit, model, transfer function, gain, when all EE's are in their *ref* states.
opref	Opposite of reference state of EE (open or short).
dpi, r, a, c	Driving point impedance, resistance, admittance, conductance.
si	Single injection.
ndi	Null double injection.
Z_{di}, Y_{di}	*si dpi*, *dpa* seen by EEi with all other EE's in their *ref* states.
$Z_{di}^{(j,k)}, Y_{di}^{(j,k)}$	*si dpi*, *dpa* seen by EEi with EE's j, k in their *opref* states.
Z_{ni}, Y_{ni}	*ndi dpi*, *dpa* seen by EEi with all other EE's in their *ref* states.
$Z_{ni}^{(j,k)}, Y_{ni}^{(j,k)}$	*ndi dpi*, *dpa* seen by EEi with EE's j, k in their *opref* states.

Equivalent opref products: Equivalent products of *si dpi*'s or *ndi dpi*'s having different EE's in their *opref* states.

Interaction ratio: Ratio of *dpi*'s seen by one EE with another EE in its *ref* state and in its *opref* state.

Reciprocity equality: The interaction ratios of any two EE's are equal, when the states of all other EE's are unchanged.

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I. INTRODUCTION

CONVENTIONAL loop/node analysis for the transfer function of a linear system model requires evaluation of a determinant or inversion of a matrix, and leads to a result in the form of a ratio of sums of products of the various system elements.

The N Extra Element Theorem (NEET) offers to the designer-analyst the following several potential advantages over the conventional approach.

(1) The result for the transfer function, instead of being obtained from a single analysis procedure on the complete system model, is assembled from the results of multiple but separate analyses.

(2) These separate analyses are conducted only on a system model, the "reference" model, that is smaller than the original by the absence of N elements designated as "extra" elements.

(3) The assembled result is automatically in low-entropy form [1], in that the relative contributions of each of the N elements to the overall transfer function are explicitly exposed.

Conventional loop/node analysis is appropriate when only a numerical result is required, since the necessary numerical matrix inversion methods are well developed. However, from the perspective of Design-Oriented Analysis [1], an analytic result in low-entropy form is desired, since this is the only way in which element values can be adjusted in an informed manner with a given specification as the goal for the transfer function.

As discussed in [1], from a design point of view, the only analysis worth doing is Design-Oriented Analysis in terms of low-entropy expressions. Advantage number 3 above arises because the format of the NEET is such that the required transfer function is calulated on the simpler reference network, and then multiplied by a correction factor involving the N extra elements which are thereby reinstated. The key to forcing the transfer function to come out in a particular desired form therefore lies in the choice of which elements to designate as "extra" in the first place.

An especially useful implementation of this choice is to designate all the reactances of the original network as the extra elements. The required transfer function, such as the gain, is thus calculated on a purely resistive reference network, and can be made to correspond, for example, to the zero-frequency or infinite-frequency gain of the original network. The frequency response is then contained entirely in the NEET correction factor, which emerges directly as a ratio of polynomials in complex frequency s. It can also be determined in advance whether any of the pole/zero roots of these polynomials are exactly factorable.

The N Extra Element Theorem may be referred to attractively, if somewhat redundantly, as "the NEET theorem."

The precursors of the NEET, the Extra Element Theorem for one and for two extra elements [2], [3][1] are reviewed in Sections II and III, to establish the general format in which the numerator and denominator of the correction factor are seen to have identical formats—the former being expressed in terms of null double injection driving point impedances, and the latter in terms of single injection driving point impedances.

In Section IV, more compact notation and specific terminology are introduced in preparation for extension of the result to the NEET in Section V, which is done on an intuitive basis with the proof relegated to the Appendix. An *LR* ladder network illustrates application of the basic version of the NEET.

In Section VI, various redundancies lead to other versions of the NEET, applications of which are illustrated by a BJT common-emitter amplifier stage and an LCC low-pass filter.

Strategies for use of the NEET in Design-Oriented Analysis are discussed in Section VII, including the questions of how many and which elements to designate as extra, and which of the various alternative forms of the NEET to adopt.

The NEET apart from the name, has a long history in the literature [4]–[11]. Most of its foundation has already been established, and the principal objective here is to present an exposition of how useful the theorem can be in the everyday work of a designer-analyst. To this end, there is no matrix or determinant in sight, even in the formal proof in the Appendix. Instead, the most "advanced" analysis technique required is calculation of driving point immittances, with the only unfamiliar concept being that of a null double injection driving point immittance.

Emphasis throughout is on a step-by-step approach that exhibits the formulas and strategies for their use in an easily accessible format.

[1] Note: (*Errata*: In [3], immediately above eq. (A.2), Z_D and Z_N should read $Z_D = Z_{d1}|_{Z_2=\infty}$ and $Z_N = Z_{d1}|_{Z_2=0}$; K_n and K_d should be absent from eq. (A.5).)

II. EET Rederivation

The NEET originates from the (single) Extra Element Theorem (EET) [2]. A review and interpretation of the EET will be given here, from a slightly different perspective than in [2].

Consider a linear time-invariant (LTI) network in which u_i is an independent excitation, a voltage or a current source, and $u_o = \alpha_1 u_i$ is an arbitrary response, a voltage across a node pair or a current in a branch. An arbitrary impedance Z_1 in the LTI network, having a voltage v across it and a current i out of the positive terminal, is designated an "extra" element (EE).

It is easily shown, by the substitution theorem, or as in [2], or otherwise, that

$$u_o = \alpha_1 u_i + \alpha_2 v \tag{1a}$$

$$i = \beta_1 u_i + \beta_2 v \tag{1b}$$

where the α's and β's are properties of the system excluding Z_1. Since also $i = -v/Z_1$, simultaneous solution of these equations gives

$$\frac{u_o}{u_i} = \alpha_1 \frac{1 + \dfrac{(\alpha_1\beta_2 - \alpha_2\beta_1)}{\alpha_1} Z_1}{1 + \beta_2 Z_1}. \tag{2}$$

Equation (2), known as the Bilinear Theorem established by Bode [4], has been used in more recent literature to determine network sensitivities and as an alternative circuit analysis approach [5]–[8].

Equation (2) is also the EET whose usefulness, as developed in [2], stems from specific interpretations of its various components. In EET format, (2) is written

$$A = A|_{Z_1=0} \frac{1 + \dfrac{Z_1}{Z_{n1}}}{1 + \dfrac{Z_1}{Z_{d1}}} \tag{3}$$

in which $A|_{Z_1=0} = \alpha_1$ is a transfer function (such as the "gain") of the "reference," or *ref*, circuit which is the original circuit with the EE Z_1 removed and replaced by a short ($Z_1 = 0$), and A is the corresponding transfer function of the circuit with Z_1 restored.

The other two parameters in (3) are defined, initially, directly in terms of the α's and β's of (2)

$$Z_{d1} \equiv 1/\beta_2 \tag{4}$$

$$Z_{n1} \equiv \alpha_1/(\alpha_1\beta_2 - \alpha_2\beta_1). \tag{5}$$

The useful EET interpretation of (3) is that a transfer function u_o/u_i of a linear system model can be expressed as the transfer function of a reference circuit in which a designated impedance Z_1 is zero (that is, "absent"), multiplied by a correction factor involving the extra element Z_1 and two parameters Z_{d1} and Z_{n1}, which also are properties of the reference circuit only.

There is a "dual" form of the EET in which the reference circuit is formed by making the extra element an open ($Z_1 =$

∞) instead of a short:

$$A = A|_{Z_1=\infty} \frac{1 + \dfrac{Z_{n1}}{Z_1}}{1 + \dfrac{Z_{d1}}{Z_1}}. \tag{6}$$

Since the two forms of the EET must give the same result for the same system, it follows that

$$\frac{Z_{n1}}{Z_{d1}} = \frac{A|_{Z_1=0}}{A|_{Z_1=\infty}}. \tag{7}$$

The principal difference between the applications of the EET treated in [2] and those of the Bilinear Theorem treated in other literature is in the interpretation, and also in the method of calculation, of the two parameters Z_{d1} and Z_{n1}.

From (1a) and (1b), the definitions of the α's and β's are

$$\alpha_1 \equiv \left.\frac{u_o}{u_i}\right|_{v=0} \tag{8a}$$

$$\alpha_2 \equiv \left.\frac{u_o}{v}\right|_{u_i=0} \tag{8b}$$

$$\beta_1 \equiv \left.\frac{i}{u_i}\right|_{v=0} \tag{9a}$$

$$\beta_2 \equiv \left.\frac{i}{v}\right|_{u_i=0}. \tag{9b}$$

Hence, from (4) and (9b),

$$Z_{d1} = \left.\frac{v}{i}\right|_{u_i=0} \tag{10}$$

which is interpreted as the single-injection *(si)* driving-point impedance *(dpi)* "seen" looking into the *ref* network at the port where Z_1 is to be connected, under the condition that the independent excitation (transfer function input signal) u_i is zero. As such, Z_{d1} can easily be calculated directly on the *ref* network by replacing Z_1 by a second independent excitation ("test" source) and calculating the resulting *dpi* with $u_i = 0$. It is often convenient, although not necessary, to choose the test source to be a current source i. Single injection, or *si*, is specified because the only independent excitation is the test source, for the purpose of calculating the *dpi* seen by Z_1, and the transfer function input signal is zero.

The parameter Z_{n1}, unlike Z_{d1}, is not equivalent to any one of the α's or β's, and apparently must be calculated by substitution of (8) and (9) into (5), a process which, because it involves the difference of two terms, can lead to considerable difficulty, both analytic and numerical [4]–[7]. Although this difficulty can be bypassed [5]–[7] by invoking (7) instead of (5), so that Z_{n1} is calculated by only products and quotients, a more direct method is developed in [2].

As shown in [2], an expression for Z_{n1}, that is analogous to (10) for Z_{d1}, is

$$Z_{n1} = \left.\frac{v}{i}\right|_{u_o=0} \tag{11}$$

so that Z_{n1} is interpreted as the null double injection driving-point impedance (*ndi dpi*) seen looking into the *ref* network at the port where Z_1 is connected, under the condition that the transfer function input signal u_i is not zero, but is adjusted to null the transfer function output signal u_o. As such, Z_{n1} can easily be calculated directly on the *ref* circuit as the *dpi* seen by Z_1, by replacing Z_1 with a test source and calculating the resulting *ndi dpi* v/i with the transfer function input signal u_i and the test source mutually adjusted to null the transfer function output signal, $u_o = 0$.

This technique of null double injection, or *ndi*, is very powerful, and considerable effort is devoted in [2] to illustrating how easy it is to calculate Z_{n1} directly from the *ref* network via (11). The ease with which Z_{n1} can be determined directly from the circuit model is the key to the practical application of the EET, and hence of the NEET. Moreover, when a problem becomes more complicated, it is the *si* calculation of Z_{d1} that becomes more complicated, whereas the *ndi* calculation of Z_{n1} remains simple.

It is no accident that an *ndi* calculation is usually simpler than an *si* calculation: when one quantity is nulled, usually others are also, so the null "propagates," and the more signals are nulled the easier it is to calculate the consequence, namely the value of Z_{n1}. Indeed, an example in [2] illustrates that, since the *ndi* Z_{n1} is an easier calculation than the *si* Z_{d1}, the best use of (7) is to calculate Z_{d1} from Z_{n1}, rather than the other way around as is usual.

Although the concept of null double injection may seem strange, the calculation of an *ndi* transfer function is actually a very familiar process. In determining the gain of a feedback system, the assumption is often made that the amplifier forward gain is infinite. In this case, the same signal conditions exist as if a "test" source were injected into the amplifier forward path and adjusted relative to the input signal to null the error signal. In either case, the system gain is calculated using the condition of nulled error signal, and is the same process by which an *ndi dpi* is calculated.

The advantages of the EET presented in [2] stem from the fact that solution of a complete linear system model for a certain transfer function is replaced by solution of a *ref* model, in the absence of a designated EE, for the same transfer function plus two *dpi*'s seen by the EE. The EE is said to have either a short or an open *ref* state, depending upon the condition that defines the reference model. If a single application of this process leads to advantages, then presumably greater benefits might accrue from extension of the same process.

III. 2EET Rederivation

A second EE can be incorporated by multiplying the original *ref* transfer function and correction factor for the first EE by the correction factor for the second EE. However, the *dpi*'s for the second EE must be calculated with the first EE already in place, and are therefore more complicated than the *dpi*'s for the first EE. The same procedure can be used to incorporate additional EE's, with the *dpi*'s becoming progressively more complicated.

There is benefit in proceeding this way, but an alternative is to develop a single correction factor in which all the *dpi*'s are calculated with all the EE's absent, so that all the *dpi*'s are calculated on the *ref* circuit.

This was accomplished for two EE's in [3], leading to the Two Extra Element Theorem (2EET) which, for a transfer function H, and for both EE's Z_1 and Z_2 having short as the *ref* state, can be expressed as

$$H = H|_{\substack{Z_1=0\\Z_2=0}} \times \frac{1 + \dfrac{Z_1}{Z_{n1}|_{Z_2=0}} + \dfrac{Z_2}{Z_{n2}|_{Z_1=0}} + \dfrac{Z_1}{Z_{n1}|_{Z_2=0}} \dfrac{Z_2}{Z_{n2}|_{Z_1=\infty}}}{[\text{same as } Num \text{ with sub } d \text{ instead of sub } n]}. \tag{12}$$

The *dpi*'s having the other element in its *ref* state are the same as those that would appear if only one of the two EE's were being incorporated. However, a "new" *dpi* appears, one having the other element in the opposite of its *ref* state, which will be designated as its *opref* state. Because the correction factor must be the same regardless of which EE is incorporated first, it must be symmetric with respect to subscript interchange. This requires that

$$Z_{n1}|_{Z_2=0} Z_{n2}|_{Z_1=\infty} = Z_{n1}|_{Z_2=\infty} Z_{n2}|_{Z_1=0} \tag{13}$$

or

$$\frac{Z_{n1}|_{Z_2=0}}{Z_{n1}|_{Z_2=\infty}} = \frac{Z_{n2}|_{Z_1=0}}{Z_{n2}|_{Z_1=\infty}}. \tag{14}$$

Equation (13) or (14) is described in [3] as a "redundancy relation," and either ratio in (14) is designated as an "interaction parameter" K_n. If the *dpi* for one EE is independent of whether the other is open or short, the interaction parameter is unity and the numerator of the correction factor in (12) factors exactly into the product of the two single correction factors that would occur if each element were incorporated independently.

The above discussion applies analogously to the denominator of (12), with all *dpi* sub n's replaced by sub d's.

Just as there are two versions of the EET, represented by (3) or (6), there are four versions of the 2EET corresponding to the four combinations of two EE's each having two possible *ref* states.

The 2EET version having $Z_1 = \infty$, $Z_2 = 0$ as *ref* states can be found from (12) by the initial step of extracting $Z_1/Z_{n1}|_{Z_2} = 0$ from the numerator and $Z_1/Z_{d1}|_{Z_2=0}$ from the denominator:

$$H = H|_{\substack{Z_1=0\\Z_2=0}} \frac{Z_{d1}|_{Z_2=0}}{Z_{n1}|_{Z_2=0}} \times \frac{\dfrac{Z_{n1}|_{Z_2=0}}{Z_1} + 1 + \dfrac{Z_{n1}|_{Z_2=0}}{Z_1} \dfrac{Z_2}{Z_{n2}|_{Z_1=0}} + \dfrac{Z_2}{Z_{n2}|_{Z_1=\infty}}}{[\text{same as } Num \text{ with sub } d \text{ instead of sub } n]}. \tag{15}$$

Since the second fraction goes to unity when $Z_1 \to \infty$ and $Z_2 \to 0$, a new *ref* gain $H|_{Z_1=\infty, Z_2=0}$ can be defined as

$$H|_{\substack{Z_1=\infty\\Z_2=0}} = H|_{\substack{Z_1=0\\Z_2=0}} \frac{Z_{d1}|_{Z_2=0}}{Z_{n1}|_{Z_2=0}}. \tag{16}$$

With reordering of the numerator (and denominator) of the second fraction, (15) becomes

$$H = H|_{\substack{Z_1=\infty\\Z_2=0}} \times \frac{1 + \dfrac{Z_{n1}|_{Z_2=0}}{Z_1} + \dfrac{Z_2}{Z_{n2}|_{Z_1=\infty}} + \dfrac{Z_{n1}|_{Z_2=0}}{Z_1} \dfrac{Z_2}{Z_{n2}|_{Z_1=0}}}{[\text{same as } Num \text{ with sub } d \text{ instead of sub } n]}. \tag{17}$$

This is the 2EET with *ref* states $Z_1 = \infty$, $Z_2 = 0$. The change of Z_1 *ref* from short to open has resulted in the Z_1 ratios being inverted in (17) relative to (12); the Z_2 ratios remain uninverted. The *dpi*'s within these ratios should be carefully noted: the *dpi*'s for Z_1 still have $Z_2 = 0$ as *ref*, but the *dpi*'s for Z_2 now have $Z_1 = \infty$ as a new *ref* state. However, the *dpi* for Z_2 in the final product term has Z_1 in what is now its *opref* state, namely $Z_1 = 0$, just as the corresponding term in (12) has Z_1 in its *opref* state $Z_1 = \infty$ for the 2EET version with short Z_1 *ref*.

A third 2EET version, with *ref* states $Z_1 = 0$, $Z_2 = \infty$ can obviously be obtained from (17) simply by subscript interchange. The fourth 2EET version, with *ref* states $Z_1, Z_2 = \infty$ can be obtained from (17) by extraction of $Z_2/Z_{n2}|_{Z_1=\infty}$ from the numerator and $Z_2/Z_{d2}|_{Z_1=\infty}$ from the denominator, in a repetition of the process by which (17) was obtained from (12). With incorporation of the redundancy relation (14), the result is

$$H|_{\substack{Z_1=\infty\\Z_2=\infty}} = H|_{\substack{Z_1=\infty\\Z_2=0}} \frac{Z_{d2}|_{Z_1=\infty}}{Z_{n2}|_{Z_1=\infty}} \tag{18}$$

and

$$H = H|_{\substack{Z_1=\infty\\Z_2=\infty}} \times \frac{1 + \dfrac{Z_{n1}|_{Z_2=\infty}}{Z_1} + \dfrac{Z_{n2}|_{Z_1=\infty}}{Z_2} + \dfrac{Z_{n1}|_{Z_2=\infty}}{Z_1} \dfrac{Z_{n2}|_{Z_1=0}}{Z_2}}{[\text{same as } Num \text{ with sub } d \text{ instead of sub } n]}. \tag{19}$$

Both EE ratios are now inverted, relative to (12). The *dpi*'s within these ratios both have the other element in its (new) open *ref* state, except that, again, the *dpi* for Z_2 in the final product term has Z_1 in its *opref* state.

As is required by the symmetry of subscript interchange, in any of the four 2EET versions, the *opref dpi* in the Z_2 ratio can be moved to the Z_1 ratio in the product term by means of the redundancy relation (13) or (14).

IV. Condensed Notation and New Definitions

For development of an N extra element theorem, it is desirable to introduce a more compact notation. Indeed, this is almost mandatory if the NEET is to be practically useful.

The key is to remove explicit statement of the EE *ref* states from the symbols that appear in the NEET expression, and to indicate explicitly only deviations from the *ref* states. Therefore, the first step is to choose a *ref* state, short or open, for each EE. The *ref* linear circuit model has all the designated

EE's in their respective *ref* states, and the *ref* transfer function $H_{\rm ref}$ is calculated from the *ref* model.

The *ndi dpi* for the ith EE (EEi) is written simply as Z_{ni}, it being understood that all other EE's are in their *ref* states, and likewise for the *si dpi* Z_{di}, and both are calculated on the *ref* model.

If any other EEj is in the opposite of its *ref* state, its *opref* state, this is indicated by a superscript j in parenthesis. Thus, $Z_{ni}^{(j,k)}$ is the *ndi dpi* for the ith EE when EE's j and k are in their *opref* states. Identical definitions apply to the *si dpi*'s, with sub n replaced by sub d.

Equations (12)–(14) for the 2EET with both EE's having short *ref* states can now be written in this condensed notation. Despite the fact that (13) and (14) are actually the same, it will be convenient to refer to, and to use, the product or the ratio form specifically, and so henceforth (13) will be renamed as an "equivalent *opref* product," and (14) as a "reciprocity equality" between the "interaction ratios" of the two EE's.

Equations (12)–(14) thus become

$$H = H_{\rm ref}\frac{1+\dfrac{Z_1}{Z_{n1}}+\dfrac{Z_2}{Z_{n2}}+\dfrac{Z_1}{Z_{n1}}\dfrac{Z_2}{Z_{n2}^{(1)}}}{[\text{same as } \textit{Num} \text{ with } d \text{ instead of } n]}. \tag{20}$$

Equivalent opref Product:

$$Z_{n1}Z_{n2}^{(1)} = Z_{n1}^{(2)}Z_{n2}. \tag{21}$$

Reciprocity Equality Between Interaction Ratios:

$$\frac{Z_{n1}}{Z_{n1}^{(2)}} = \frac{Z_{n2}}{Z_{n2}^{(1)}}. \tag{22}$$

The second step is to introduce the dual admittance notation to distinguish EE's and *dpi*'s that have open *ref* states from those with short *ref* states.

Thus, (17) is rewritten as

$$H = H_{\rm ref}\frac{1+\dfrac{Y_1}{Y_{n1}}+\dfrac{Z_2}{Z_{n2}}+\dfrac{Y_1}{Y_{n1}}\dfrac{Z_2}{Z_{n2}^{(1)}}}{[\text{same as } \textit{Num} \text{ with sub } d \text{ instead of sub } n]} \tag{23}$$

with an equivalent *opref* product

$$Y_{n1}Z_{n1}^{(1)} = Y_{n2}^{(2)}Z_{n2} \tag{24}$$

and a reciprocity equality

$$\frac{Y_{n1}}{Y_{n1}^{(2)}} = \frac{Z_{n2}}{Z_{n2}^{(1)}}. \tag{25}$$

Likewise, (19) is rewritten

$$H = H_{\rm ref}\frac{1+\dfrac{Y_1}{Y_{n1}}+\dfrac{Y_2}{Y_{n2}}+\dfrac{Y_1}{Y_{n1}}\dfrac{Y_2}{Y_{n2}^{(1)}}}{[\text{same as } \textit{Num} \text{ with sub } d \text{ instead of sub } n]} \tag{26}$$

with an equivalent *opref* product

$$Y_{n1}Y_{n2}^{(1)} = Y_{n1}^{(2)}Y_{n2} \tag{27}$$

and a reciprocity equality

$$\frac{Y_{n1}}{Y_{n1}^{(2)}} = \frac{Y_{n2}}{Y_{n2}^{(1)}}. \tag{28}$$

The benefit of this condensed notation is that all the versions of the 2EET have the same format, the only difference being that Z ratios appear for EE's having short *ref* states, and Y ratios appear for EE's having open *ref* states. Further, the abbreviation *dpi*. a, r, c will be used to refer to a driving-point impedance, admittance, resistance, or conductance, respectively.

It must be remembered that the $H_{\rm ref}$'s and the *dpi*'s are *not the same* in the various versions, because they are defined for different EE *ref* states. For example, Z_{n2} in (23)–(25) is actually $Z_{n2}|_{Y_1=0}$, and is not the same as Z_{n2} in (20)–(22), which is actually $Z_{n2}|_{Z_1=0}$.

Of course, all versions of the 2EET represent the same result in different forms, and correspondingly the equivalent *opref* products and the reciprocity equalities are also really the same, which can be verified by restoration of the EE *ref* subscripts.

V. The NEET Theorem

The proof of the NEET, in the present condensed notation, is relegated to the Appendix. However, the above review of the EET, the 2EET, and particularly the introduction of the condensed notation, render the extension of the 2EET to incorporate more EE's almost obvious.

A. The NEET Theorem in Words

The NEET theorem states that any transfer function H of a linear system model can be expressed as the transfer function $H_{\rm ref}$ of the *ref* model when N EE's have their *ref* states short, multiplied by a correction factor consisting of a numerator and a denominator, each of which contains ratios of each EE to one of its *dpi*'s. The structure of the numerator and denominator is identical, each being 1 plus the sum of the products of the ratios 1 at a time (or simply the sum of the ratios), plus the sum of products of the ratios 2 at a time, plus the sum of products 3 at a time, and so on up to the single product N at a time that involves all the EE's.

To write down a specific version of the NEET, one needs to know which *dpi*,a appears in each ratio. This can be found from a "NEET construction algorithm," which surfaces as one of the steps in the NEET proof given in the Appendix. The approach quantifies the intuitive steps of extending the theorem from $N-1$ EE's to N EE's, as follows.

B. The NEET Construction Algorithm

Write down the *Num* of the $(N-1)$EET, for example, the 2EET. All the terms in the *Num* of the $(N-1)$EET appear in the *Num* of the NEET. Terms to be added to form the sum of the products n at a time for the$N$EET are those for the sum of the products $(n-1)$ at a time for the $(N-1)$EET, each multiplied by the ratio of the Nth EE to its *dpi*,a having the other EE's in the same product in their *opref* states. The *Denom* of the NEET is formed in the same way.

To illustrate the NEET construction algorithm, let us construct the *Num* of the 3EET. We begin by writing the *Num* for

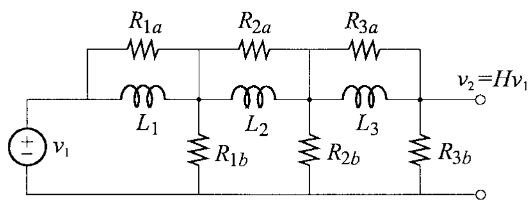

Fig. 1. Example 1: LR ladder network.

the 2EET from (20):

$$\textit{Num} = 1+\left[\frac{Z_1}{Z_{n1}}+\frac{Z_2}{Z_{n2}}\cdots\right]+\left[\frac{Z_1}{Z_{n1}}\frac{Z_2}{Z_{n2}^{(1)}}\cdots\right]+[\cdots]. \quad (29)$$

The ellipses indicate where extra terms are to be added. In the sum of products 1 at a time, there is one term to be added, Z_3/Z_{n3}. In the sum of products 2 at a time, terms to be added are the two terms in the above sum of products 1 at a time, the first multiplied by $Z_3/Z_{n3}^{(1)}$ and the second by $Z_3/Z_{n3}^{(2)}$. There is a new product 3 at a time to be added, which is the term in the above sum of products 2 at a time multiplied by $Z_3/Z_{n3}^{(1,2)}$. The result for the 3EET, for all *ref* states short, is as follows.

C. "Basic" NEET Version, for $N = 3$ and All Ref States Short

$$H = H_{\mathrm{ref}}\,\frac{\textit{Num}}{\textit{Denom}} \quad (30a)$$

where

$$\begin{aligned}\textit{Num} = 1 &+ \left[\left(\frac{Z_1}{Z_{n1}}+\frac{Z_2}{Z_{n2}}\right)+\frac{Z_3}{Z_{n3}}\right]\\ &+\left[\left\{\frac{Z_1}{Z_{n1}}\frac{Z_2}{Z_{n2}^{(1)}}\right\}+\left(\frac{Z_1}{Z_{n1}}\frac{Z_3}{Z_{n3}^{(1)}}+\frac{Z_2}{Z_{n2}}\frac{Z_3}{Z_{n3}^{(2)}}\right)\right]\\ &+\left[\left\{\frac{Z_1}{Z_{n1}}\frac{Z_2}{Z_{n2}^{(1)}}\frac{Z_3}{Z_{n3}^{(1,2)}}\right\}\right] \quad (30b)\end{aligned}$$

$$\textit{Denom} = [\text{same as } \textit{Num} \text{ with sub } d \text{ instead of sub } n].$$

The two pairs of internal brackets reveal how the terms originate according to the NEET construction algorithm: thus, the curved bracket term in line 2 of (30b) comes from the curved bracket term in line 1, and the curly bracket term in line 3 comes from the curly bracket term in line 2.

Equation (30) is designated the "basic" version because, in each *dpi* product, the *opref* superscripts accumulate in the same order as the subscripts.

Before considering other versions and redundancy relations of the NEET, let us examine an example.

D. Example 1: LR Ladder Network

Suppose it is desired to find the voltage transfer function $H = v_2/v_1$ of the ladder network of Fig. 1. To apply the NEET, one would want to take advantage of the special case in which all reactances are designated as EE's so that the *ref* model is purely resistive and all the *dpi*'s are driving-point resistances (*dpr*'s).

Further, since all the EE's are inductances, if the *ref* state for each EE is taken to be short, the *ref* transfer function corresponds to the zero-frequency response of the circuit.

Hence, we decide to apply the 3EET with the three inductances designated as the three EE's, with all *ref* states short, which is the version displayed in (30).

The analyses, in any order, for the *ref* transfer function and for the $2^N - 1 = 7$ *ndi dpr*'s and 7 *si dpr*'s (see the Appendix) are conducted on the *ref* model, which is the circuit of Fig. 1 with the three inductances replaced by shorts. Immediately, we find

$$H_{\mathrm{ref}} = 1. \quad (31)$$

The denominator *si dpr*'s for (30c) are found by setting $v_1 = 0$ and applying a test source in place of each inductance with the other inductances short or open, as appropriate. The results are:

$$\begin{aligned}R_{d1} &= R_{1a}||R_{1b}||R_{2b}||R_{3b} && (32a)\\ R_{d2} &= R_{2a}||R_{2b}||R_{3b} && (32b)\\ R_{d2}^{(1)} &= R_{2a}||(R_{1a}||R_{1b}+R_{2b}||R_{3b}) && (32c)\\ R_{d3} &= R_{3a}||R_{3b} && (32d)\\ R_{d3}^{(1)} &= R_{3a}||(R_{3b}+R_{1a}||R_{1b}||R_{2b}) && (32e)\\ R_{d3}^{(2)} &= R_{3a}||(R_{2a}||R_{2b}+R_{3b}) && (32f)\\ R_{d3}^{(1,2)} &= R_{3a}||[(R_{1a}||R_{1b}+R_{2a})||R_{2b}+R_{3b}]. && (32g)\end{aligned}$$

The numerator *ndi dpi*'s for (30a) are found by restoring the transfer function input signal v_1, applying a test source in place of each inductance with the other inductances short or open as appropriate, and supposing that in each case the two sources are mutually adjusted to null the transfer function output signal v_2. Here, as is usual, the numerator *ndi dpr*'s are easier to calculate than the denominator *si dpr*'s because the null requires that the current from each test source flows only through the resistance in parallel with it; hence,

$$\begin{aligned}R_{n1} &= R_{1a}\\ R_{n2} &= R_{n2}^{(1)} = R_{2a}\\ R_{n3} &= R_{n3}^{(1)} = R_{n3}^{(2)} = R_{n3}^{(1,2)} = R_{3a}. \quad (33)\end{aligned}$$

It remains only to substitute $Z_i = sL_i$ into (30b) and (30c):

$$\begin{aligned}\textit{Denom} = 1 &+ \left[\frac{L_1}{R_{d1}}+\frac{L_2}{R_{d2}}+\frac{L_3}{R_{d3}}\right]s\\ &+\left[\frac{L_1}{R_{d1}}\frac{L_2}{R_{d2}^{(1)}}+\frac{L_1}{R_{d1}}\frac{L_3}{R_{d3}^{(1)}}+\frac{L_2}{R_{d2}}\frac{L_3}{R_{d3}^{(2)}}\right]s^2\\ &+\left[\frac{L_1}{R_{d1}}\frac{L_2}{R_{d2}^{(1)}}\frac{L_3}{R_{d3}^{(1,2)}}\right]s^3 \quad (34)\end{aligned}$$

$$\textit{Num} = \left(1+\frac{L_1}{R_{1a}}s\right)\left(1+\frac{L_2}{R_{2a}}s\right)\left(1+\frac{L_3}{R_{3a}}s\right). \quad (35)$$

The *Num* cubic in s factors exactly because each *ndi dpr* is the same regardless of whether the other EE's are short or open; that is, the interaction ratios are each unity.

Results for incorporation of additional sections into the *LR* filter of Fig. 1 could be accomplished by extension of the 3EET of (30) to the 4EET, and so on, by use of the NEET construction algorithm.

VI. Other NEET Versions

In the 2EET, there is one *ndi dpi* or *si dpi* equivalent *opref* product, expressed in (21) for the *ndi dpi*'s, which derives from the requirement that each term in the NEET should be the same regardless of EE enumeration. Rearrangement of (21) into (22) identifies the reciprocity equality between the two interaction ratios.

When more than two EE's are present, the reciprocity equality holds for any two EE's i and j, and so (22) can be generalized to

$$\frac{Z_{ni}}{Z_{ni}^{(j)}} = \frac{Z_{nj}}{Z_{nj}^{(i)}}. \tag{36}$$

Each such ratio indicates the degree to which a certain *ndi dpi* is influenced by whether another EE is replaced by a short or an open, and the equality of the two ratios is a reciprocity relation required by the arbitrariness of EE enumeration. As already mentioned, if the interaction ratio is unity, each *ndi dpi* is unaffected by the value of the other EE, and there is no interaction between the two EE's.

When more than two EE's are under consideration, the notation of (36) implies that all EE's other than i and j are in their *ref* states. However, (36) still applies if any or all of the other EE's are in their *opref* states:

$$\frac{Z_{ni}^{(m,k,\cdots)}}{Z_{ni}^{(j,m,k,\cdots)}} = \frac{Z_{nj}^{(m,k,\cdots)}}{Z_{nj}^{(i,m,k,\cdots)}}. \tag{37}$$

The constituent *ndi dpi*'s, and their interaction ratios, in (37) may be different from those in (36), but the reciprocity equality still holds.

Even for all *ref* states short, the number of EET versions for more EE's than two increases rapidly with N. Successive use of the reciprocity equalities (37) leads to equivalent *opref* products that develop equivalent versions of the NEET.

For example, in (30b) for the *Num* of the 3EET, each *ndi dpi* product 2 at a time can be replaced by the equivalent *opref* products formed by the appropriate version of (36).

Also in (30b), the *ndi dpi* product 3 at a time can be replaced by any of the following equivalent *opref* products:

$$\begin{aligned} Z_{n1}Z_{n2}^{(1)}Z_{n3}^{(1,2)} &= Z_{n1}^{(2)}Z_{n2}Z_{n3}^{(1,2)} \\ &= Z_{n1}^{(2,3)}Z_{n2}Z_{n3}^{(2)} \\ &= Z_{n1}^{(2,3)}Z_{n2}^{(3)}Z_{n3} \\ &= Z_{n1}^{(3)}Z_{n2}^{(1,3)}Z_{n3} \\ &= Z_{n1}Z_{n2}^{(1,3)}Z_{n3}^{(1)}. \end{aligned} \tag{38}$$

Each successive equality in (38) is generated by the following versions of (37):

$$\begin{aligned} \frac{Z_{n1}}{Z_{n1}^{(2)}} &= \frac{Z_{n2}}{Z_{n2}^{(1)}} \\ \frac{Z_{n1}^{(2)}}{Z_{n1}^{(2,3)}} &= \frac{Z_{n3}^{(2)}}{Z_{n3}^{(1,2)}} \\ \frac{Z_{n2}}{Z_{n2}^{(3)}} &= \frac{Z_{n3}}{Z_{n3}^{(2)}} \\ \frac{Z_{n1}^{(3)}}{Z_{n1}^{(2,3)}} &= \frac{Z_{n2}^{(3)}}{Z_{n2}^{(1,3)}} \\ \frac{Z_{n1}}{Z_{n1}^{(3)}} &= \frac{Z_{n3}}{Z_{n3}^{(1)}}. \end{aligned} \tag{39}$$

It may be noted in passing that the remaining version of (37) for three EE's, which is

$$\frac{Z_{n2}^{(1)}}{Z_{n2}^{(1,3)}} = \frac{Z_{n3}^{(1)}}{Z_{n3}^{(1,2)}} \tag{40}$$

recreates the first version of (38) from the last version.

The equations of (38) represent all the possible *opref* superscript combinations for the equivalent *opref* products for three EE's. In general, for any *dpi* product n at a time, the number of such equivalent *opref* products is $n!$. Thus, in (30b), each *dpi* product 2 at a time has 2 versions, and the *dpi* product 3 at a time has the six versions of (38).

Although the various reciprocity equalities are of interest in themselves, a simple short-cut algorithm enables any of the $n!$ equivalent *opref* products, such as those in (38), to be written down directly. This "equivalent *opref* product algorithm" is presented next.

A. Equivalent opref Product Algorithm

In any *dpi* product, a "first" *dpi* has all the other EE's in their *ref* states (no *opref* superscript); a "second" *dpi* has the EE for the previous *dpi* in its *opref* state; a "third" *dpi* has the EE's for the two previous *dpi*'s in their *opref* states, and so on.

In the "basic" version of the 3EET in (30b), the "first," "second," etc., *dpi*'s are enumerated in the same order as the EE subscripts. The first form of the equivalent *opref* products (38) corresponds to this basic version.

However, the *dpi*'s can be enumerated in any order of the EE subscripts. Thus, for example, in the fourth *opref* redundancy form of (38), the "first" *dpi* is Z_{n3}, the "second" is $Z_{n2}^{(3)}$, and the "third" is $Z_{n1}^{(2,3)}$.

A different allocation of "first," "second," "third," etc., can be selected for each *dpi* product, even within each sum of products, and even for the *Num* and *Denom*.

Moreover, the same procedure applies if one or more of the ratios are admittance ratios rather than impedance ratios.

Let us return to Example 1 to explore a different choice of *opref* redundancy.

B. Example 1 Revisited

In the previous treatment of the *LR* ladder network of Fig. 1, (34) for the *Denom* of the 3EET correction factor was expressed in the "basic" form with the *dpr opref* superscripts accumulating in the same order as the subscripts.

Suppose, in (34), that in the sum of *dpr* products 2 at a time, the product $R_{d2}R_{d3}^{(2)}$ is replaced by the alternative form $R_{d2}^{(3)}R_{d3}$. A new calculation for $R_{d2}^{(3)}$ is required, which from Fig. 1 is

$$R_{d2}^{(3)} = R_{2a}||R_{2b}||(R_{3a}+R_{3b}). \tag{41}$$

With R_{d3} from (32d), the equivalent product is

$$R_{d2}^{(3)}R_{d3} = [R_{2a}||R_{2b}||(R_{3a}+R_{3b})][R_{3a}||R_{3b}]. \tag{42}$$

In contrast, the original ("basic") product from (32b) and (32f) is

$$R_{d2}R_{d3}^{(2)} = [R_{2a}||R_{2b}||R_{3b}][R_{3a}||(R_{2a}||R_{2b}+R_{3b})]. \tag{43}$$

These are two different forms of the same product: despite superficial differences of appearance, they are indeed the same (otherwise a mistake has been made!), as can be checked by algebraic manipulation of one form into the other.

The important point is that two different low-entropy forms of the same result (different series-parallel resistance combinations) are obtained from different *dpi,a opref* equivalent products.

As an aside, it may be noted that the four *dpr*'s involved in (42) and (43) can each be evaluated with EE1 in its *opref* state. Three of these have already been displayed in (32c), (32e), and (32g); the fourth, $R_{d2}^{(1,3)}$, can be evaluated directly from Fig. 1 as

$$R_{d2}^{(1,3)} = R_{2a}||[R_{1a}||R_{1b}+R_{2b}||(R_{3a}+R_{3b})]. \tag{44}$$

The resulting counterparts to (42) and (43) are

$$\begin{aligned} R_{d2}^{(1,3)}R_{d3}^{(1)} = &\{R_{2a}||[R_{1a}||R_{1b}+R_{2b}||(R_{3a}+R_{3b})]\} \\ &\times [R_{3a}||(R_{3b}+R_{1a}||R_{1b}||R_{2b})] \end{aligned} \tag{45}$$

and

$$\begin{aligned} R_{d2}^{(1)}R_{d3}^{(1,2)} = &[R_{2a}||(R_{1a}||R_{1b}+R_{2b}||(R_{3b})] \\ &\times \{R_{3a}||[R_{1a}||R_{1b}+R_{2a})||R_{2b}+R_{3b}]\}. \end{aligned} \tag{46}$$

Again, the apparently different equivalent products in (45) and (46) are in fact the same.

The important point is the following. Either equivalent *opref* product (42) or (43) can be used in the appropriate term of (34), and the corresponding reciprocity equality is

$$\frac{R_{d2}}{R_{d2}^{(3)}} = \frac{R_{d3}}{R_{d3}^{(2)}} \tag{47}$$

which is the *Denom* version of (36) with $i=2$, $j=3$. The reciprocity equality corresponding to (45) and (46) is

$$\frac{R_{d2}^{(1)}}{R_{d2}^{(1,3)}} = \frac{R_{d3}^{(1)}}{R_{d3}^{(1,2)}} \tag{48}$$

which is the *Denom* version of (37) with $i=2$, $j=3$, $m=1$.

This means, as discussed in relation to (36) and (37), that the reciprocity equality holds between EE's 2 and 3 regardless of whether EE1 is in its *ref* or its *opref* state. However, the equivalent *opref* product of (45) or (46) is *not* equal to that of (42) or (43), since any of the *dpr*'s is obviously larger when another EE is open than when it is short, and so neither (45) nor (46) can be substituted for the product (42) or (43) in (34).

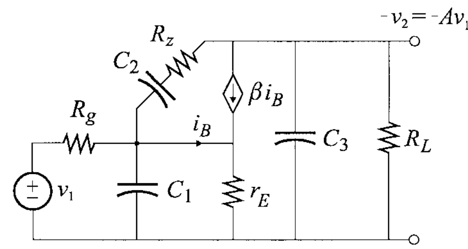

Fig. 2. Example 2: BJT common-emitter amplifier stage.

When several EE's are under consideration, the equivalent *opref* products offer many choices for *dpi,a*'s to be evaluated. The next example illustrates some criteria for making these choices.

C. Example 2: BJT Common-Emitter Amplifier Stage

Consider the circuit of Fig. 2, which represents a basic BJT common-emitter amplifier stage with inclusion of three capacitances.

Suppose we wish to analyze the circuit for the voltage gain $A = v_2/v_1$ by use of the NEET; the theorem would be equally applicable to analysis for the input or output impedance, the power-supply-to-output gain, or for any other transfer function of interest.

We first identify which elements in the model to designate as "extra." The obvious choice is to designate the three capacitances as EE's with open *ref* states; the circuit of Fig. 2 with the capacitances replaced by opens becomes the *ref* circuit, whose *ref* gain corresponds to the "midband" gain A_m, and the NEET correction factor will be a ratio of polynomials in complex frequency s.

The basic form of the NEET is

$$A = A_m \frac{\mathit{Num}}{\mathit{Denom}} \tag{49}$$

where A_m is the *ref* gain given by

$$A_m = \frac{\alpha R_L}{r_E + R_g/(1+\beta)} \tag{50}$$

and $\alpha \equiv \beta/(1+\beta)$.

The appropriate 3EET correction factor is as in the "basic" version of (30b), but in terms of admittance ratios, since all EE *ref* states are opens:

$$\begin{aligned} \mathit{Num} = 1 &+ \left(\frac{Y_1}{Y_{n1}} + \frac{Y_2}{Y_{n2}} + \frac{Y_3}{Y_{n3}}\right) \\ &+ \left(\frac{Y_1}{Y_{n1}}\frac{Y_2}{Y_{n2}^{(1)}} + \frac{Y_1}{Y_{n1}}\frac{Y_3}{Y_{n3}^{(1)}} + \frac{Y_2}{Y_{n2}}\frac{Y_3}{Y_{n3}^{(2)}}\right) \\ &+ \left(\frac{Y_1}{Y_{n1}}\frac{Y_2}{Y_{n2}^{(1)}}\frac{Y_3}{Y_{n3}^{(1,2)}}\right) \end{aligned} \tag{51a}$$

$$\mathit{Denom} = [\text{same as } \mathit{Num} \text{ with sub } d \text{ instead of sub } n]. \tag{51b}$$

The next step is to choose which, if any, of the *dpa* products in the above basic 3EET version to replace by equivalent *opref* products. The first criterion is simply which *dpa*'s are easiest to determine from the particular *ref* circuit under consideration. Since each *dpa* product contains at least one *dpa* with at least one EE in its *opref* state, the choice comes down to which *dpa*'s are easier to determine with which EE's in their *opref* states.

In the case of the circuit of Fig. 2, EE1 and EE3 each shunt the signal to ground, whereas EE2 is in a "feedback" position. It is to be expected, therefore, that calculation of the *dpa* for C_2 would be easier with either or both C_1 and C_3 in their *opref* states (short) than in their *ref* states (open), but that calculation of the *dpa* for either C_1 or C_3 would be easier with C_2 in its *ref* state (open).

We discuss first the *Denom* of (51b). Consider the *dpa* product $Y_{d1}Y_{d2}^{(1)}$: this is the product of *si dpa*'s for C_1 and C_2, both having C_3 in its (open) *ref* state, but one of them having the other in its (short) *opref* state. Hence, according to the expectation of the preceding paragraph, $Y_{d1}Y_{d2}^{(1)}$ is easier to calculate than $Y_{d1}^{(2)}Y_{d2}$, and so the basic *opref* product form should be retained in (51b).

Consider the *dpa* product $Y_{d1}^{(3)}Y_{d3}$. This is the same as $Y_{d1}Y_{d3}^{(1)}$, because $Y_{d3} = Y_{d3}^{(1)}$ and $Y_{d1} = Y_{d1}^{(3)}$; in other words, both interaction ratios are unity, because there is no interaction between C_1 and C_3 when C_2 is in its (open) *ref* state. Hence, again, the basic product form in (51b) can be retained.

Consider the *dpa* product $Y_{d2}Y_{d3}^{(2)}$. Because $Y_{d2}^{(3)}$ is easier to calculate than $Y_{d3}^{(2)}$, the alternative form $Y_{d2}^{(3)}Y_{d3}$ should be substituted for the basic form $Y_{d2}Y_{d3}^{(2)}$ in (51b).

Last, consider $Y_{d1}Y_{d2}^{(1)}Y_{d3}^{(1,2)}$. This is a triple product, one factor of which is an *si dpa* with both other EE's in their (short) *opref* states, and the best choice for this *dpa* is $Y_{d2}^{(1,3)}$. The other two factors are then either $Y_{d1}Y_{d3}^{(1)}$ or $Y_{d1}^{(3)}Y_{d3}$, which have already been seen to be equal. Hence in (51b), the basic triple product form should be replaced by the alternative form $Y_{d1}Y_{d2}^{(1,3)}Y_{d3}^{(1)}$, established by application of the equivalent opref product algorithm in which the "first" *dpa* is Y_{d1}, the "second" is $Y_{d3}^{(1)}$, and the "third" is $Y_{d2}^{(1,3)}$.

From Fig. 2, the denominator *dpa*'s required in (51b) are

$$1/Y_{d1} = R_g \| (1+\beta)r_E \equiv R_p \tag{52a}$$
$$1/Y_{d2} = R_z + mR_L \tag{52b}$$
$$1/Y_{d2}^{(1)} = R_z + R_L \tag{52c}$$
$$1/Y_{d2}^{(3)} = R_z + R_g \| (1+\beta)r_E \equiv R_z + R_p \tag{52d}$$
$$1/Y_{d2}^{(1,3)} = R_z \tag{52e}$$
$$1/Y_{d3} = R_L \tag{52f}$$
$$1/Y_{d3}^{(1)} = R_L \tag{52g}$$

where $R_p = R_g \| (1+\beta)r_E$ is introduced merely to condense the notation.

Most of the above *dpa*'s can be determined directly by inspection of the circuit of Fig. 2; the only one not thus easily determined is Y_{d2}. However, this result can be obtained by extension of an example in [2]. That example is a special case of Fig. 2, in which R_z is absent (short), C_1 and C_3 are absent (opens), and C_2 is designated C_t, which was the (single) EE under consideration. In [2], the numerator *ndi dpr* was found to be

$$R_n = -r_E/\alpha \tag{53}$$

and the denominator *si dpr* was

$$R_d = mR_L \tag{54}$$

where

$$m = \frac{R_g \| (1+\beta)r_E}{R_g \| r_E \| R_L} \tag{55}$$

is the "Miller multiplier." It is worth noting here that this result for R_d was obtained in [2] by two different methods: first, *directly*, which is a fairly lengthy process; and second, *indirectly*, and much more simply, from R_n by use of the redundancy relation $Z_d = Z_n(A|_{Z=\infty}/A|_{Z=0})$, which is (7) for a single EE without subscript.

In the present example, the *dpa* for C_2 with C_1 and C_3 open is obviously $1/R_d$ from (54) in series with $1/R_z$, which leads to (52b) above.

Regarding the *Num* of (51a), the same criterion applies for choice among the various equivalent *opref* products, namely, which *dpa*'s are easiest to determine. In lieu of any reason to do otherwise, the same choices suffice as for the *Denom*, although it should be remembered that it is not necessary to make the same choices. For the present example, determination of the same choices for the *Num* gives

$$1/Y_{n1} = 0 \tag{56a}$$
$$1/Y_{n2} = R_z - r_E/\alpha \tag{56b}$$
$$1/Y_{n2}^{(1)} = R_z - r_E/\alpha \tag{56c}$$
$$1/Y_{n2}^{(3)} = R_z - r_E/\alpha \tag{56d}$$
$$1/Y_{n2}^{(1,3)} = R_z - r_E/\alpha \tag{56e}$$
$$1/Y_{n3} = 0 \tag{56f}$$
$$1/Y_{n3}^{(1)} = 0. \tag{56g}$$

As often happens, there are fewer zeros than poles in the transfer function under consideration, in this case, because C_1 and C_3 short the signal to ground at infinite frequency. This is the reason why $1/Y_{n1}$ and $1/Y_{n3}$ are both zero, above. This also means that, since correspondingly there are no s^2 or s^3 terms in the *Num*, all the double and triple product terms in (51a) must be zero, and so (56c)–(56e) above are superfluous.

All the work has now been done for the analysis for the voltage gain A of the circuit in Fig. 2, and it remains only to find the *Num* and *Denom* in (49) by substitution of all the *dpa*'s from (52) and (56) into the chosen *dpa* equivalent *opref* products, together with identification of the three EE's as $Y_i = sC_i$. The results are

$$\mathit{Num} = 1 + sC_2(R_z - r_E/\alpha) \tag{57}$$
$$\begin{aligned}\mathit{Denom} = 1 &+ s[C_1R_p + C_2(R_z + mR_L) + C_3R_L] \\ &+ s^2[C_1C_2R_p(R_z+R_L) + C_2C_3R_L(R_z+R_p) \\ &+ C_3C_1R_LR_p] + s^3[C_1C_2C_3R_LR_pR_z].\end{aligned} \tag{58}$$

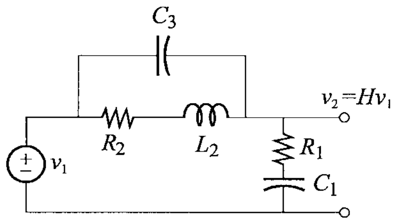

Fig. 3. Example 3: LCC low-pass filter.

Two special cases of the general result are of interest.

First, if $R_z = 0$, C_1, C_2, and C_3 could be identified as the three internal BJT capacitances—the base-emitter diffusion, collector-base transition layer, and substrate capacitances, respectively. The result for (49) with substitution of the reduced forms of (57) and (58) is then shown in (59) at the bottom of the page. It may be noted that if $C_2 = 0$, the denominator of (59) factors exactly, which is consistent with the previous observation that there is no interaction between C_1 and C_3 when $C_2 = 0$.

This shows the degenerate result in which one pole disappears because the three capacitances form a loop; the collector-emitter (substrate) capacitance C_3 hardly affects either of the two remaining poles as long as it is smaller than the base-emitter diffusion capacitance C_1 or the Miller-multiplied collector-base capacitance mC_2. The only zero is the right half-plane (rhp) zero caused by the collector-base capacitance C_2.

The second special case of interest is that in which C_2 is an intentionally added element external to the BJT device, to create a controllable dominant pole or an integrator, and is much larger than any of the three BJT internal capacitances. In this case, C_1 and C_3 can be omitted, and (49) with substitution of the reduced forms of (57) and (58) becomes

$$A = A_m \frac{1 + sC_2(R_z - r_E/\alpha)}{1 + sC_2(R_z + mR_L)}. \tag{60}$$

This shows how R_z, increasing from zero, moves the rhp zero to infinite frequency and brings it back in the left half-plane, while hardly affecting the pole as long as R_z is much less than the Miller-multiplied load resistance mR_L.

D. Example 3: LCC Low-Pass Filter

A final example illustrates application of the NEET with EE's having both open and short *ref* states, and also introduces another criterion for choice of *opref* redundancy.

In the low-pass filter circuit of Fig. 3, there are three reactances that produce three poles in the voltage transfer function (gain) $H = v_2/v_1$, and also three zeros because both the zero-frequency and infinite-frequency gains are flat. To expose the frequency domain response only in the NEET correction factor, we choose the three reactances as the three EE's, whose subscripts have been chosen in Fig. 3 to anticipate their designation as the three EE's.

The *ref* circuit therefore contains only the two resistances R_1 and R_2, and to select the zero-frequency gain as the *ref* gain of the complete circuit, we choose the *ref* state of the inductance to be short, and the *ref* states of the capacitances to be opens.

The basic form of the NEET is

$$H = H_{\text{ref}} \frac{\textit{Num}}{\textit{Denom}} \tag{61}$$

where H_0, the *ref* gain, is given by

$$H_0 = 1. \tag{62}$$

The appropriate 3EET correction factor, in "basic" form, is

$$\begin{aligned} \textit{Num} = 1 &+ \left(\frac{Y_1}{Y_{n1}} + \frac{Z_2}{Z_{n2}} + \frac{Y_3}{Y_{n3}}\right) \\ &+ \left(\frac{Y_1}{Y_{n1}}\frac{Z_2}{Z_{n2}^{(1)}} + \frac{Y_1}{Y_{n1}}\frac{Y_3}{Y_{n3}^{(1)}} + \frac{Z_2}{Z_{n2}}\frac{Y_3}{Y_{n3}^{(2)}}\right) \\ &+ \left(\frac{Y_1}{Y_{n1}}\frac{Z_2}{Z_{n2}^{(1)}}\frac{Y_3}{Y_{n3}^{(1,2)}}\right) \end{aligned} \tag{63a}$$

$$\textit{Denom} = [\text{same as } \textit{Num} \text{ with sub } d \text{ instead of sub } n]. \tag{63b}$$

With both Z ratios and Y ratios present, we must be careful to remember the *ref* states: Y_{n1} is the *dpa* for EE1 with $Z_2, Y_3 = 0; Z_{n2}$ is the *dpi* for EE2 with $Y_1, Y_3 = 0; Y_{n3}$ is the *dpa* for EE3 with $Y_1, Z_2 = 0$.

The circuit of Fig. 3 is sufficiently simple that all of the *dpi*,a's can be determined with essentially equal ease. Therefore, there is no *a priori* reason to substitute any equivalent *opref* products for the basic *dpi*,a products in (63a). In listing the *dpi*,a's for substitution into these equations, for clarity the full *ref* conditions will be displayed.

For the *Denom* (63b), the 7 equations are

$$1/Y_{d1} \equiv 1/Y_{d1}|_{Z_2,Y_3=0} = R_1 + R_2 \tag{64a}$$
$$Z_{d2} \equiv Z_{d2}|_{Y_1,Y_3=0} = \infty \tag{64b}$$
$$Z_{d2}^{(1)} \equiv Z_{d2}|_{Y_1=\infty,Y_3=0} = R_1 + R_2 \tag{64c}$$
$$1/Y_{d3} \equiv 1/Y_{d3}|_{Y_1,Z_2=0} = R_2 \tag{64d}$$
$$1/Y_{d3}^{(1)} \equiv 1/Y_{d3}|_{Y_1=\infty,Z_2=0} = R_1 \| R_2 \tag{64e}$$
$$1/Y_{d3}^{(2)} \equiv 1/Y_{d3}|_{Y_1=0,Z_2=\infty} = \infty \tag{64f}$$
$$1/Y_{d3}^{(1,2)} \equiv 1/Y_{d3}|_{Y_1,Z_2=\infty} = R_1. \tag{64g}$$

When these equations are substituted into (63b), a difficulty emerges with the $Z_{d2}Y_{d3}^{(2)}$ product, which is ∞/∞ and is therefore indeterminate. The obvious next step is to try the equivalent *opref* product $Z_{d2}^{(3)}Y_{d3}$, which requires that (64f) be replaced by a new equation for $Z_{d2}^{(3)}$:

$$Z_{d2}^{(3)} \equiv Z_{d2}|_{Y_1=0,Y_3=\infty} = R_2. \tag{65}$$

The *opref* product $Z_{d2}^{(3)}Y_{d3}$ then gives a determinate result.

A second criterion for choice of equivalent *opref* products has thus emerged: not only do we wish to choose *dpi*,a's

$$A = A_m \frac{(1 - sC_2 r_E/\alpha)}{1 + s[C_1R_p + (mC_2 + C_3)R_L] + s^2[C_1(C_2 + C_3) + C_2C_3]R_LR_p} \tag{59}$$

that are the easiest to determine, but we also wish to avoid indeterminacies.

For the *Num* (63a), the 7 equations are

$$1/Y_{n1} \equiv 1/Y_{n1}|_{Z_2,Y_3=0} = R_1 \tag{66a}$$
$$Z_{n2} \equiv Z_{n2}|_{Y_1,Y_3=0} = \infty \tag{66b}$$
$$Z_{n2}^{(1)} \equiv Z_{n2}|_{Y_1=\infty,Y_3=0} = \infty \tag{66c}$$
$$1/Y_{n3} \equiv 1/Y_{n3}|_{Y_1,Z_2=0} = R_2 \tag{66d}$$
$$1/Y_{n3}^{(1)} \equiv 1/Y_{n3}|_{Y_1=\infty,Z_2=0} = R_2 \tag{66e}$$
$$1/Y_{n3}^{(2)} \equiv 1/Y_{n3}|_{Y_1=0,Z_2=\infty} = \infty \tag{66f}$$
$$1/Y_{n3}^{(1,2)} \equiv 1/Y_{n3}|_{Y_1,Z_2=\infty} = \infty. \tag{66g}$$

An indeterminacy occurs in the $Z_{n2}Y_{n3}^{(2)}$ product of (63a), so we try the equivalent *opref* product $Z_{n2}^{(3)}Y_{n3}$, which requires that (66f) be replaced by a new equation for $Z_{n2}^{(3)}$:

$$Z_{n2}^{(3)} \equiv Z_{n2}|_{Y_1=0,Y_3=\infty} = R_2. \tag{67}$$

The *opref* product $Z_{n2}^{(3)}Y_{n3}$ then gives a determinate result.

An indeterminacy also occurs in the triple *dpi*,a product of (63a), because of $Z_{n2}^{(1)}Y_{n3}^{(1,2)}$ being ∞/∞. However, the equivalent *opref* product $Z_{n2}^{(1,3)}Y_{n3}^{(1)}$ contains $Y_{n3}^{(1)}$ which is already known to be finite from (66e). With $Z_{n2}^{(1)}$ from (66c) replaced by a new equation for $Z_{n2}^{(1,3)}$, which is

$$Z_{n2}^{(1,3)} \equiv Z_{n2}|_{Y_1,Y_3=\infty} = R_2 \tag{68}$$

the equivalent *opref* product $Y_{n1}Z_{n2}^{(1,3)}Y_{n3}^{(1)}$ of the triple *dpi*,a product in (63a) can now be evaluated.

All the work has now been done for the analysis for the voltage gain H of the circuit in Fig. 3.

Since by (62) $H_0 = 1$, (61) becomes

$$H = H_{\text{ref}}\frac{\mathit{Num}}{\mathit{Denom}}. \tag{69}$$

With EE's $Y_1 = sC_1$, $Z_2 = sL_2$, $Y_3 = sC_3$, and evaluation of the chosen *dpi*,a equivalent *opref* product, (63a) and (63b) become

$$\begin{aligned}\mathit{Num} = 1 &+ s\left[C_1R_1 + \frac{L_2}{\infty} + C_3R_2\right]\\ &+ s^2\left[\frac{C_1L_2R_1}{\infty} + C_1C_3R_1R_2 + \frac{L_2C_3R_2}{R_2}\right]\\ &+ s^3\left[\frac{C_1L_2C_3R_1R_2}{R_2}\right]\end{aligned} \tag{70a}$$

$$\begin{aligned}\mathit{Denom} = 1 &+ s\left[C_1(R_1+R_2) + \frac{L_2}{\infty} + C_3R_2\right]\\ &+ s^2\left[\frac{C_1L_2(R_1+R_2)}{R_1+R_2} + C_1C_3R_1R_2 + \frac{L_2C_3R_2}{R_2}\right]\\ &+ s^3\left[\frac{C_1L_2C_3(R_1+R_2)R_1}{(R_1+R_2)}\right].\end{aligned} \tag{70b}$$

In (70), the infinite *dpi*,a's have been retained to indicate the origin of the various terms; however, further simplification gives

$$\begin{aligned}\mathit{Num} = 1 &+ s[C_1R_1 + C_3R_2] + s^2[L_2C_3 + C_1C_3R_1R_2]\\ &+ s^3[C_1L_2C_3R_1]\end{aligned} \tag{71a}$$

$$\begin{aligned}\mathit{Denom} = 1 &+ s[C_1(R_1+R_2) + C_3R_2]\\ &+ s^2[C_1L_2 + C_1C_3R_1R_2 + L_2C_3]\\ &+ s^3[C_1L_2C_3R_1].\end{aligned} \tag{71b}$$

The *Num* factors exactly as

$$\mathit{Num} = (1 + sC_1R_1)(1 + sC_3R_2 + s^2L_2C_3) \tag{71c}$$

which is a consequence of no interaction between EE1 and EE3, or between EE1 and EE2. This follows from (66d) and (66e) and from (67) and (68), which show that interaction ratios $Y_{n3}/Y_{n3}^{(1)}$ and $Z_{n2}^{(3)}/Z_{n2}^{(1,3)}$ are each unity.

Two special cases of this general result may be of interest.

First, if $C_3 = 0$, the voltage gain H becomes that of a doubly damped C_1L_2 filter

$$H = \frac{1 + sC_1R_1}{1 + sC_1(R_1+R_2) + s^2C_1L_2} \tag{72}$$

better expresssed as

$$H = \frac{\left(1 + \dfrac{1}{Q_1}\dfrac{s}{\omega_{12}}\right)}{1 + \left(\dfrac{1}{Q_1} + \dfrac{1}{Q_2}\right)\left(\dfrac{s}{\omega_{12}}\right) + \left(\dfrac{s}{\omega_{12}}\right)^2} \tag{73}$$

where

$$\omega_{12} \equiv 1/\sqrt{C_1L_2} \tag{74}$$
$$Q_1 \equiv \frac{1}{R_1}\sqrt{\frac{L_2}{C_1}}, \tag{75a}$$
$$Q_2 \equiv \frac{1}{R_2}\sqrt{\frac{L_2}{C_1}}. \tag{75b}$$

Second, if C_3 is not zero but sufficiently small (as if it were the parasitic interwinding capacitance of inductor L_2), the general result may be approximated as

$$H = \frac{(1 + sC_1R_1)(1 + sC_3R_2 + s^2L_2C_3)}{[1 + sC_1(R_1+R_2) + s^2C_1L_2](1 + sC_3R_1)} \tag{76}$$

better expressed as shown in (77), at the bottom of the page, where

$$\omega_{23} \equiv 1/\sqrt{L_2C_3} \tag{78}$$

is the single new parameter introduced by a nonzero C_3.

$$H = \frac{\left(1 + \dfrac{1}{Q_1}\dfrac{s}{\omega_{12}}\right)\left[1 + \dfrac{\omega_{12}}{\omega_{23}}\dfrac{1}{Q_2}\left(\dfrac{s}{\omega_{23}}\right) + \left(\dfrac{s}{\omega_{23}}\right)^2\right]}{\left[1 + \left(\dfrac{1}{Q_1} + \dfrac{1}{Q_2}\right)\left(\dfrac{s}{\omega_{12}}\right) + \left(\dfrac{s}{\omega_{12}}\right)^2\right]\left(1 + \dfrac{\omega_{12}}{\omega_{23}}\dfrac{1}{Q_1}\dfrac{s}{\omega_{23}}\right)} \tag{77}$$

VII. Strategies for NEET Applications

If the NEET is to be a viable tool, it must be readily accessible to a designer-analyst in a useful form. Since the NEET is a complicated formula incorporating special symbols, and because it has many versions, it would be inefficient to assemble an encyclopedic collection of versions from which a potential user would have to select one version and remind oneself of the definitions of the symbols.

The alternative approach adopted here is to express the structure of the NEET in words, deferring the adoption of specific symbols as long as possible. It is convenient now to review statement of the NEET format, with use of the proposed new terminology and definitions, so that a designer-analyst can make an informed choice of which version to assemble with specific symbols.

The starting point is a linear system model and the identification of a specific transfer function to which the NEET is to be applied, such as the gain, or output impedance. The same model in the absence of N elements is designated the *ref* system model.

A. General NEET Theorem

$$\begin{bmatrix}\text{transfer function in}\\ \text{presence of } N \text{ EE's}\end{bmatrix} = \begin{bmatrix}\text{transfer function with}\\ N \text{ EE's in their } \textit{ref} \text{ states}\end{bmatrix} \times \begin{bmatrix}N\text{EET}\\ \text{correction factor}\end{bmatrix} \tag{79a}$$

$$\begin{bmatrix}N\text{EET}\\ \text{correction factor}\end{bmatrix} = \frac{\textit{Num}}{\textit{Denom}} \tag{79b}$$

$$\begin{aligned}\textit{Num} = 1 &+ \begin{bmatrix}\text{sum of EE}/(\textit{ndi dpi,}a)\\ \text{ratios 1 at a time}\end{bmatrix}\\ &+ \begin{bmatrix}\text{sum of EE}/(\textit{ndi dpi,}a)\\ \text{ratios 2 at a time}\end{bmatrix}\\ &+ \begin{bmatrix}\text{sum of EE}/(\textit{ndi dpi}, a)\\ \text{ratios 3 at a time}\end{bmatrix}\cdots\\ &+ \begin{bmatrix}\text{sum of EE}/(\textit{ndi dpi,}a)\\ \text{ratios } N \text{ at a time}\end{bmatrix}\end{aligned} \tag{79c}$$

$$\textit{Denom} = [\text{same as } \textit{Num} \text{ in terms of EE}/(\textit{si dpi,}a) \text{ ratios}]. \tag{79d}$$

An *si dpi,*a is a single-injection driving-point impedance or admittance seen by an extra element (EE), that is, the *dpi,*a seen by a test source substituted for that EE in the system model. An *ndi dpi,*a is a null double injection driving-point impedance or admittance seen by an EE, that is, the *dpi,*a seen by the same test source adjusted to null the transfer function output in the presence of the transfer function input signal. Examples of calculation of both *si* and *ndi dpi,*a's can be found in [2] and [3].

The statement of (79) is all a designer-analyst needs to have in mind in order to make the first round of decisions required for use of the NEET theorem. These decisions relate to how many, and which, elements are to be designated as EE's.

The NEET correction factor is a simultaneous bilinear representation of the EE/*dpi,*a ratios. Thus, it is a low-entropy expression in which the effect of each such ratio upon the system transfer function is exposed. Therefore, one criterion to determine which elements to designate as EE's is to choose the elements the effects of which it is wished to expose. One example is collector-base resistance in transistor models: the *ref* system model would give the first-order transfer function in the absence of these resistances, and the NEET correction factor would explicitly expose the modifications due to the collector-base resistances not being infinite.

Another consideration is the number of *dpi,*a's that need to be calculated for a particular number of EE's and a particular NEET version. The minimum number N of *dpi,*a's that need to be calculated for either the *Num* or *Denom* occur if the single EET is applied N times. As discussed in the Appendix, if all N EE's are incorporated simultaneously in the "basic" version of the NEET, the minimum number of different *dpi,*a's that appear in either the *Num* or *Denom* is $2^N - 1$. However, a choice of different equivalent *opref* products can introduce more *dpi,*a's that need to be calculated, with a maximum number of $N2^{N-1}$.

Still, to minimize the number of *dpi,*a's to be calculated is not a priority; on the contrary, a strong motivation for use of the NEET, rather than use of the single EET N times, is that the "divide and conquer" approach allows a small number of complicated calculations to be replaced by a larger number of simpler calculations, and this is usually an advantageous tradeoff.

A special case for application of the NEET arises when one wishes to expose the frequency response of the transfer function: as already seen in all three examples in the preceding text, choice of all the reactive elements as EE's leaves a purely resistive *ref* model, and all *dpi,*a's reduce to *dpr,*c's, whence the NEET correction factor is automatically in the form of a ratio of polynomials in frequency s.

Finally, regarding the choice of how many elements to designate as EE's, it is interesting to note that, in principle, one could take the NEET to the limit by removing all the elements from a circuit, leaving only a direct connection between the input and output, and then use the theorem to restore all the components. Unfortunately, this would not work: all the *dpi,*a's that one has to calculate would be either zero, infinite, or indeterminate, and the indeterminate ones are critical because only they contribute to a nontrivial final result. Calculation of these indeterminate impedances is more trouble than analyzing the circuit with a few elements left in place, and there is good reason to remove as few elements as possible from the circuit. The minimum number of *dpi,*a's needed for either the *Num* or *Denom* is $2^N - 1$, which increases rapidly with N. Thus, once the *ref* circuit has been reduced to the point where it is easy to calculate the required parameters, one should stop removing elements.

It will be noted that the first round of decisions regarding application of the NEET theorem is made without a particular NEET version in mind. The second round of decisions has two parts: first, choice of *ref* states for the EE's, and second, choice of equivalent *opref* products.

Choice of *ref* states will in many cases have been anticipated in the identification of which elements to designate as EE's.

The *ref* states, short or open, determine the *ref* system model and the *ref* transfer function. In the case where the EE's are all the reactances, choice of short *ref* for inductances and open *ref* for capacitances results in the *ref* transfer function being the zero-frequency value of the final transfer function. Other choices are possible—and necessary, if the zero-frequency transfer function is zero.

As an example, suppose that there are at least four EE's, of which EE3 has *ref* state open, and the others have *ref* states short. Once the *ref* states have been chosen, the basic form of the NEET correction factor of (79b) can be written down.

Starting with the 2EET of (20), repeated application of the NEET construction algorithm leads to the following versions of (79c) and (79d).

B. NEET Version for $N \geq 4$, EE3 Ref State Open

$$\begin{aligned}\textit{Num} = 1 &+ \left[\left(\frac{Z_1}{Z_{n1}} + \frac{Z_2}{Z_{n2}} + \frac{Y_3}{Y_{n3}}\right) + \frac{Z_4}{Z_{n4}} + \cdots\right] \\ &+ \left[\left\{\frac{Z_1}{Z_{n1}}\frac{Z_2}{Z_{n2}^{(1)}} + \frac{Z_1}{Z_{n1}}\frac{Y_3}{Y_{n3}^{(1)}} + \frac{Z_2}{Z_{n2}}\frac{Y_3}{Y_{n3}^{(2)}}\right\}\right. \\ &+ \left.\left(\frac{Z_1}{Z_{n1}}\frac{Z_4}{Z_{n4}^{(1)}} + \frac{Z_2}{Z_{n2}}\frac{Z_4}{Z_{n4}^{(2)}} + \frac{Y_3}{Y_{n3}}\frac{Z_4}{Z_{n4}^{(3)}}\right) + \cdots\right] \\ &+ \left[\left\langle\frac{Z_1}{Z_{n1}}\frac{Z_2}{Z_{n2}^{(1)}}\frac{Y_3}{Y_{n3}^{(1,2)}}\right\rangle + \left\{\frac{Z_1}{Z_{n1}}\frac{Z_2}{Z_{n2}^{(1)}}\frac{Z_4}{Z_{n4}^{(1,2)}}\right.\right. \\ &+ \left.\left.\frac{Z_1}{Z_{n1}}\frac{Y_3}{Y_{n3}^{(1)}}\frac{Z_4}{Z_{n4}^{(1,3)}} + \frac{Z_2}{Z_{n2}}\frac{Y_3}{Y_{n3}^{(2)}}\frac{Z_4}{Z_{n4}^{(2,3)}}\right\} + \cdots\right] \\ &+ \left[\left\langle\frac{Z_1}{Z_{n1}}\frac{Z_2}{Z_{n2}^{(1)}}\frac{Y_3}{Y_{n3}^{(1,2)}}\frac{Z_4}{Z_{n4}^{(1,2,3)}}\right\rangle + \cdots\right] + \cdots\end{aligned} \tag{80a}$$

$$\textit{Denom} = [\text{same as } \textit{Num} \text{ with sub } d \text{ instead of sub } n]. \tag{80b}$$

The three pairs of internal brackets reveal how the additional terms for EE4 originate from the 3EET according to the NEET construction algorithm, in a similar fashion in which the 2EET was extended to the 3EET in (30).

Equation (80) constitutes the most general version of the NEET that needs to be written down, since any EE can be represented as an impedance ratio or as an admittance ratio, depending, respectively, upon whether its *ref* value is short or open.

The version of (80) above is the "basic" version in that the *opref* superscripts in any *dpi*,a product accumulate in the same order as do the subscripts. However, by the equivalent *opref* product algorithm, the *opref* superscripts can accumulate in any order of the subscripts, that is, in any *dpi*,a product, any EE subscript can be assigned to a "first" *dpi*,a that has all the other EE's in their *ref* states (no *opref* superscript), any other EE subscript can be assigned to a "second" *dpi*,a that has the EE for the previous *dpi*,a in its *opref* state, and so on.

Thus, in (80a), the product in the basic form $Z_{n1}Z_{n2}^{(1)}Z_{n4}^{(1,2)}$ could be replaced by $Z_{n1}^{(2,4)}Z_{n2}Z_{n4}^{(2)}$ or by $Z_{n1}^{(4)}Z_{n2}^{(1,4)}Z_{n4}$, for a total of $3! = 6$ redundant forms. Or, the product in the basic form $Z_{n1}Z_{n2}^{(1)}Y_{n3}^{(1,2)}Z_{n4}^{(1,2,3)}$ could be replaced by $Z_{n1}^{(3)}Z_{n2}^{(1,3,4)}Y_{n3}Z_{n4}^{(1,3)}$, or by $Z_{n1}^{(2,3,4)}Z_{n2}^{(4)}Y_{n3}^{(2,4)}Z_{n4}$, for a total of $4! = 24$ redundant forms.

Another option is to invoke the redundancy relation involving the *opref* transfer function for any EEi, which is

$$\frac{Z_{di}}{Z_{ni}} = \frac{H_{\text{ref}}^{(i)}}{H_{\text{ref}}} = \frac{H_{\text{ref}}|_{Z_i=\infty}}{H_{\text{ref}}|_{Z_i=0}} \tag{81a}$$

or

$$\frac{Y_{di}}{Y_{ni}} = \frac{H_{\text{ref}}^{(i)}}{H_{\text{ref}}} = \frac{H_{\text{ref}}|_{Y_i=\infty}}{H_{\text{ref}}|_{Y_i=0}}. \tag{81b}$$

Thus, since a *Denom si dpi*,a is usually harder to calculate than the corresponding *Num ndi dpi*,a, it can be found indirectly from the corresponding *opref* transfer function $H_{\text{ref}}^{(i)}$, as was done in Example 2. This doesn't work if $H_{\text{ref}}^{(i)}$ is zero or infinite, however.

With such a plethora—even surfeit—of alternative *dpi*,a product redundancies, the important question is how to make a choice. As illustrated in Examples 2 and 3, an immediate criterion is which *dpi*,a's are easiest to calculate on any particular *ref* circuit. A second criterion, illustrated in Example 3, is that when an indeterminacy occurs, one merely tries a different equivalent *opref* product until a determinate result emerges.

There is a third criterion for choice of *dpi*,a equivalent *opref* product: different forms result in different low-entropy combinations of circuit elements, as illustrated in "Example 1 Revisited."

In general, this means that a consideration in the choice of *dpi*,a equivalent *opref* product is which low-entropy combination of circuit elements is desired in the result. In many, if not most, cases, the preferred result is not known until one has tried at least one version, but practice and experience, as in chess playing, enable a designer-analyst to see an ever-increasing number of steps ahead.

In summary, strategies for application of the NEET theorem to the determination of a transfer function in the form of (79) involve the following sequence of choices made according to the criteria discussed above.

1) Choose how many, and which, elements are to be designated as EE's.
2) Decide which EE's are to have *ref* states short, which open. The *ref* circuit is thus defined, and the *ref* transfer function can be determined.
3) Write the *Num* and the *Denom* of the NEET correction factor as in (80), using the NEET construction algorithm, with impedance ratios for EE's having short *ref* states, admittance ratios for those having open *ref* states. If certain interaction ratios, as in (37), are unity, the *Num* or *Denom* may be exactly factorable.
4) Substitute any *dpi*,a product with a redundant form using the equivalent opref product algorithm. Such substitutions can be done independently and differently for each product, and can be different in the *Num* and in the *Denom*.

VIII. Conclusion

The N Extra Element Theorem is an alternative means for analysis of a linear system model. Its principal distinction from conventional loop or node analysis, in which the system equations are solved simultaneously, is that a simpler *ref* system model in the absence of N designated "extra" elements is solved first, and the N EE's are then restored via a correction factor upon the result for the *ref* model. Parameters in the correction factor are various *dpi*,a's seen by the EE's, all calculated upon the *ref* model. Thus, no calculation is performed upon a model containing any of the designated EE's, and the final result is obtained by assembly of sequentially obtained results. This "divide and conquer" approach is potentially easier and/or shorter than the conventional approach.

The NEET is applicable to any transfer function of any linear system model, and of course is not limited to electrical systems. Any immittance can be designated as an Extra Element, and so can any dependent generator (see the Appendix, or [2]).

When applied to a self-immittance, the single EET is equivalent to Blackman's theorem [15]; an example is worked in [16], in which a second EE is incorporated by use of the single EET twice in succession.

The EET can also be used in "nested" fashion: that is, the EET can be used to find the driving point immittances for use in another EET correction factor. This is a key step in the NEET proof (see the Appendix), and is illustrated for a circuit example in [17].

The approach taken in this paper has been to develop the NEET theorem in an intuitive manner made possible by a rederivation and restatement of the 2EET in terms of condensed notation and definitions proposed in Sections II–IV. In Sections V and VI, the "basic" version of the 3EET, for all *ref* states short, is established by use of the NEET construction algorithm. Other versions result from application of the equivalent *opref* product algorithm, and reciprocity equalities between interaction ratios are verified by application of the 3EET to a ladder network in which the three inductances are designated as EE's. Also in Section VI, two other examples illustrate use of the 3EET with *ref* states other than all short, and how to use the equivalent *opref* products to avoid indeterminacies.

Section VII contains the salient features of the preceding sections in a format that suggests strategies for selecting a version of the NEET suitable for particular applications.

A less obvious, but equally (if not the most) valuable, feature of the NEET approach is that the result is derived in a low-entropy form, in contrast to the conventionally obtained high entropy form of a ratio of sums of products of various system elements. The NEET is in fact an extension of Bode's Bilinear Theorem, and exposes explicitly the contributions to the result of the elements designated as "extra" in simultaneous bilinear forms.

If the aspect of the transfer function of interest is its frequency response, the choice of all the reactive elements as EE's immediately sets up the NEET correction factor as a ratio of polynomials in complex frequency s.

In this special case, each *si dpr*,c forms a time constant with the associated capacitance or inductance, and the *Denom* polynomial, whose roots are the system poles, contains s coefficients that are the sum of the time constants, the sum of products of time constants 2 at a time, and so on, as in the technique developed by Cochrun and Grabel [8].

From the broader perspective of the NEET, it is seen that exactly the same process applies to the *Num* except that the time constants are formed from the *ndi dpr*,c's.

Hence the zeros, as well as the poles, and consequently the entire transfer function, can be found by assembly of small, separate, simple, low-entropy calculations upon a purely resistive reference circuit with all reactances absent.

Further, whether or not certain interaction ratios are unity determines whether or not the *Num* or *Denom* may be exactly factorable.

Superficially, the NEET may appear more complicated and harder to apply than the conventional method, since the formula itself has many components and many forms; this is to be expected since the formula represents a solution to a generalized problem. However, this is a benefit, rather than a penalty, because it is the effort put in by the analyst in making selections from the multiple choices that leads to the emergence of a low-entropy result of the desired form. That is, the format exposes how the designated EE's influence the result.

Nevertheless, in order to realize these benefits, it is desirable to employ condensed notation and terminology such as those introduced in this paper, although of course many other schemes are possible.

Appendix
A NEET Proof

A proof by induction of the NEET is presented. The algorithm is to postulate the "basic" version of the NEET, remove the Nth EE by setting it to its *ref* value, then to reinstate the Nth EE by use of the single EET and to show that the result is the same as the NEET first postulated. The process is a generalization of that by which the 2EET was derived from the single EET in [3].

The postulated form of the NEET, with all *ref* states short, is

$$H = H_{\text{ref}} \frac{\textit{Num}}{\textit{Denom}} \tag{A.1a}$$

$$\textit{Num} = 1 + \sum_{i=1}^{N} \frac{Z_i}{Z_{ni}} + \sum_{i=1}^{k-1}\sum_{k=2}^{N} \frac{Z_i}{Z_{ni}} \frac{Z_k}{Z_{nk}^{(i)}} + \sum_{i=1}^{k-1}\sum_{k=2}^{m-1}\sum_{m=3}^{N} \frac{Z_i}{Z_{ni}} \frac{Z_k}{Z_{nk}^{(i)}} \frac{Z_m}{Z_{nm}^{(i,k)}} + \cdots \tag{A.1b}$$

$$\textit{Denom} = [\text{same as } \textit{Num} \text{ with sub } d \text{ instead of sub } n]. \tag{A.1c}$$

The Nth EE is removed by setting it to its short *ref* state, which is accomplished by merely replacing N by $N-1$ in (A.1). Quantities relating to the $(N-1)$EET will be identified by the argument $[N-1]$:

$$H[N-1] = H_{\text{ref}} \frac{\textit{Num}[N-1]}{\textit{Denom}[N-1]}. \tag{A.2a}$$

As usual, the denominator has identical form to the numerator, so only the numerator needs to be considered:

$$\textit{Num}[N-1] = 1 + \sum_{i=1}^{N-1} \frac{Z_i}{Z_{ni}} + \sum_{i=1}^{k-1}\sum_{k=2}^{N-1} \frac{Z_i}{Z_{ni}}\frac{Z_k}{Z_{nk}^{(i)}} + \sum_{i=1}^{k-1}\sum_{k=2}^{m-1}\sum_{m=3}^{N-1} \frac{Z_i}{Z_{ni}}\frac{Z_k}{Z_{nk}^{(i)}}\frac{Z_m}{Z_{nm}^{(i,k)}} + \cdots. \tag{A.2b}$$

The next step is to restore the Nth EE by multiplication of (A.2a) by the single EET correction factor for the impedance Z_N:

$$H = H_{\text{ref}} \frac{\textit{Num}[N-1]}{\textit{Denom}[N-1]} \times \frac{1 + \frac{Z_N}{Z_{nN:Z_1\cdots Z_{N-1}}}}{[\text{same as } \textit{Num} \text{ with sub } d \text{ instead of } n]}. \tag{A.3}$$

Here, $Z_{nN:Z_1\cdots Z_{N-1}}$ is the *ndi dpi* seen by Z_N with EE's $Z_1 \cdots Z_{N-1}$ already in place.

The remaining task is to express $Z_{nN:Z_1\cdots Z_{N-1}}$ in terms of Z_{nN}, the *ndi dpi* seen by Z_N with $Z_1 \cdots Z_{N-1}$ in their (short) *ref* states. This is done by treating this *ndi dpi* as another transfer function of the system model, just as is H itself, and using the single EET in a "nested" fashion [inside the EET correction factor in (A.3)]. Thus,

$$Z_{nN:Z_1\cdots Z_{N-1}} = Z_{nN} \frac{\textit{Numz}_n[N-1]}{\textit{Denz}_n[N-1]} \tag{A.4}$$

and insertion into (A.3) gives

$$H = H_{\text{ref}} \frac{\textit{Num}[N-1] + \textit{Denz}_n[N-1] \frac{Z_N}{Z_{nN}} \frac{\textit{Num}[N-1]}{\textit{Numz}_n[N-1]}}{[\text{same as } \textit{Num} \text{ with sub } d \text{ instead of } n]}. \tag{A.5}$$

To find $\textit{Numz}_n[N-1]$ and $\textit{Denz}_n[N-1]$, we recall that a self-impedance is a transfer function whose "input" current produces an "output" voltage at the same port, and enables a special case of the EET correction factor [2], equivalent to Blackman's theorem [15], in which an *si dpi* is determined with the "input" open (zero), and an *ndi dpi* can be determined with "input" short instead of "output" nulled, since these two conditions are the same when the "output" is at the same port as the "input."

In the present context, $Z_{nN:Z_1\cdots Z_{N-1}}$ is the self-impedance seen by the Nth EE, and so the *dpi*'s for $Z1 \cdots Z_{N-1}$ are to be determined either with "input" short, which is the same as $Z_N = 0$, its *ref* value, or with "input" open, which is the same as $Z_N = \infty$, its *opref* value.

In $\textit{Numz}_n[N-1]$, the *dpi*'s for $Z_1 \cdots Z_{N-1}$ are to be determined with $Z_N = 0$, and the resulting conditions are identical to those for the *dpi*'s for $\textit{Num}[N-1]$. Hence, $\textit{Numz}_n[N-1]$ is the same as $\textit{Num}[N-1]$ given by (A.2b), and the numerator of (A.5) reduces to

$$\textit{Num} = \textit{Num}[N-1] + \textit{Denz}_n[N-1] \frac{Z_N}{Z_{nN}}. \tag{A.6}$$

In $\textit{Denz}_n[N-1]$, the *dpi*'s for $Z_1 \cdots Z_{N-1}$ are to be determined with $Z_N = \infty$, and the resulting conditions are the same as those for the *dpi*'s for $\textit{Num}[N-1]$ except that Z_N is in its *opref* state. Hence, $\textit{Denz}_n[N-1]$ is given by (A.2b) with all *dpi*'s having Z_N in its *opref* state:

$$\begin{aligned}\textit{Denz}_n[N-1] &= 1 + \sum_{i=1}^{N-1} \frac{Z_i}{Z_{ni}^{(N)}} + \sum_{i=1}^{k-1}\sum_{k=2}^{N-1} \frac{Z_i}{Z_{ni}^{(N)}}\frac{Z_k}{Z_{nk}^{(i,N)}} \\ &\quad + \sum_{i=1}^{k-1}\sum_{k=2}^{m-1}\sum_{m=3}^{N-1} \frac{Z_i}{Z_{ni}^{(N)}}\frac{Z_k}{Z_{nk}^{(i,N)}}\frac{Z_m}{Z_{nm}^{(i,k,N)}} + \cdots. \end{aligned}\tag{A.7}$$

After substitution of (A.2b) and (A.7), (A.6) is to be arranged into the form of (A.1b). The first step is

$$\begin{aligned}\textit{Num} &= 1 + \sum_{i=1}^{N-1} \frac{Z_i}{Z_{ni}} + \frac{Z_N}{Z_{nN}} + \sum_{i=1}^{k-1}\sum_{k=2}^{N-1} \frac{Z_i}{Z_{ni}}\frac{Z_k}{Z_{nk}^{(i)}} \\ &\quad + \left(\sum_{i=1}^{N-1} \frac{Z_i}{Z_{ni}^{(N)}}\right)\frac{Z_N}{Z_{nN}} + \sum_{i=1}^{k-1}\sum_{k=2}^{m-1}\sum_{m=3}^{N-1} \frac{Z_i}{Z_{ni}}\frac{Z_k}{Z_{nk}^{(i)}}\frac{Z_m}{Z_{nm}^{(i,k)}} \\ &\quad + \left(\sum_{i=1}^{k-1}\sum_{k=2}^{N-1} \frac{Z_i}{Z_{ni}^{(N)}}\frac{Z_k}{Z_{nk}^{(i,N)}}\right)\frac{Z_N}{Z_{nN}} + \cdots \end{aligned}\tag{A.8}$$

in which, in each line, the first term comes from $\textit{Num}[N-1]$ and the second term comes from $\textit{Denz}_n[N-1]$. The remaining several steps are to massage the second term in each line so that it can be combined with the first term. First, the 2EET redundancy relation $Z_{ni}Z_{nk}^{(i)} = Z_{ni}^{(k)}Z_{nk}$ is used to shift the *opref* superscript (N) to Z_{nN} from the other *dpi*'s:

$$\begin{aligned} Z_{ni}^{(N)}Z_{nN} &= Z_{ni}Z_{nN}^{(i)} \\ Z_{ni}^{(N)}Z_{nk}^{(i,N)}Z_{nN} &= Z_{ni}Z_{nk}^{(i,N)}Z_{nN}^{(i)} = Z_{ni}Z_{nk}^{(i)}Z_{nN}^{(i,k)} \cdots. \end{aligned}\tag{A.9}$$

Equation (A.8) then becomes

$$\begin{aligned}\textit{Num} &= 1 + \left(\sum_{i=1}^{N-1} \frac{Z_i}{Z_{ni}}\right) + \frac{Z_N}{Z_{nN}} + \left[\sum_{i=1}^{k-1}\sum_{k=2}^{N-1} \frac{Z_i}{Z_{ni}}\frac{Z_k}{Z_{nk}^{(i)}}\right] \\ &\quad + \left(\sum_{i=1}^{N-1} \frac{Z_i}{Z_{ni}}\right)\frac{Z_N}{Z_{nN}^{(i)}} + \left\{\sum_{i=1}^{k-1}\sum_{k=2}^{m-1}\sum_{m=3}^{N-1} \frac{Z_i}{Z_{ni}}\frac{Z_k}{Z_{nk}^{(i)}}\frac{Z_m}{Z_{nm}^{(i,k)}}\right\} \\ &\quad + \left[\sum_{i=1}^{k-1}\sum_{k=2}^{N-1} \frac{Z_i}{Z_{ni}}\frac{Z_k}{Z_{nk}^{(i)}}\right]\frac{Z_N}{Z_{nN}^{(i,k)}} + \cdots. \end{aligned}\tag{A.10}$$

This intermediate result verifies the NEET construction algorithm, introduced in Section V, for incorporating another EE; corresponding terms are identified by enclosure in a particular bracket shape.

The next step toward combining the two terms in each line of (A.10) is to identify each Z_N ratio as the result of a summation with a single index value of N:

$$\begin{aligned} \mathit{Num} = 1 &+ \sum_{i=1}^{N-1} \frac{Z_i}{Z_{ni}} + \sum_{i=N} \frac{Z_i}{Z_{ni}} + \sum_{i=1}^{k-1}\sum_{k=2}^{N-1} \frac{Z_i}{Z_{ni}} \frac{Z_k}{Z_{nk}^{(i)}} \\ &+ \sum_{i=1}^{k-1}\sum_{k=N} \frac{Z_i}{Z_{ni}} \frac{Z_k}{Z_{nk}^{(i)}} + \sum_{i=1}^{k-1}\sum_{k=2}^{m-1}\sum_{m=3}^{N-1} \frac{Z_i}{Z_{ni}} \frac{Z_k}{Z_{nk}^{(i)}} \frac{Z_m}{Z_{nm}^{(i,k)}} \\ &+ \sum_{i=1}^{k-1}\sum_{k=2}^{m-1}\sum_{m=N} \frac{Z_i}{Z_{ni}} \frac{Z_k}{Z_{nk}^{(i)}} \frac{Z_m}{Z_{nm}^{(i,k)}} + \cdots. \end{aligned} \tag{A.11}$$

In (A.11), the quantities to be summed in each line are the same, so the two terms can be combined into the first term simply by extending the summation limit from $N-1$ to N, upon which (A.1b) is restored.

This concludes the proof of the version postulated for the NEET in (A.1).

Proofs of other postulated versions can be conducted by a similar sequence of steps. For instance, use of the 2EET redundancy relation $Z_{ni}Z_{nk}^{(i)} = Z_{ni}^{(k)}Z_{nk}$ directly in (A.1b) leads to a version in which the order of the *opref* superscripts is reversed:

$$\begin{aligned} \mathit{Num} = 1 &+ \sum_{i=1}^{N} \frac{Z_i}{Z_{ni}} + \sum_{i=1}^{k-1}\sum_{k=2}^{N} \frac{Z_i}{Z_{ni}^{(k)}} \frac{Z_k}{Z_{nk}} \\ &+ \sum_{i=1}^{k-1}\sum_{k=2}^{m-1}\sum_{m=3}^{N} \frac{Z_i}{Z_{ni}^{(k,m)}} \frac{Z_k}{Z_{nk}^{(m)}} \frac{Z_m}{Z_{nm}} + \cdots. \end{aligned} \tag{A.12}$$

In retracing the steps of the proof for this version, the 2EET redundancy relations are no longer needed later, and a slightly different, but equally useful, algorithm emerges for the addition of one more EE.

It can also be seen that the steps in the proof are the same if one or more impedance ratios are replaced by admittance ratios. The same is true if an EE is a transimpedance or transadmittance, or a dimensionless current or voltage ratio. Thus, as shown in [3], the NEET includes dependent generators as EE's, and the corresponding *dpi*,a's become similarly dimensioned transfer functions of the *ref* model.

Hence, the NEET proof presented above can be considered general.

Three useful results regarding the number of parameters in the NEET can be derived from the binomial expansion of $(a+b)^N$, where $a = b = 1$. The expansion is

$$(1+1)^N = \sum_{k=0}^{N} \frac{N!}{k!(N-k)!} = 2^N \tag{A.13}$$

where $N!/[k!(N-k)!]$ is the number of combinations of N objects taken k at a time. This can be written

$$\begin{aligned} 1 &+ \begin{bmatrix} \text{no. of combinations of} \\ N \text{ objects 1 at a time} \end{bmatrix} + \begin{bmatrix} \text{no. of combinations of} \\ N \text{ objects 2 at a time} \end{bmatrix} \\ &\cdots + \begin{bmatrix} \text{no. of combinations of} \\ N \text{ objects } N \text{ at a time} \end{bmatrix} = 2^N. \end{aligned} \tag{A.14}$$

With regard to the NEET, the *ref* transfer function H_{ref} is determined by the *ref* model with all EE's in their *ref* states. A minimum of one H_{ref} is needed in one version of the NEET, which in (A.1) is the H_{ref} for all EE *ref* states short. However, other *ref* models have 1, 2, up to N EE's with open *ref* states, for a total number of combinations equal to 2^N as in (A.14). Hence, the maximum number of H_{ref}'s for N EE's is 2^N.

The *Num* (or *Denom*) of the NEET correction factor, as in (A.1) or explicitly in (80), is of the same form as (A.14) in which the "sum of products" corresponds to "number of combinations." Therefore, the total number of terms in either the *Num* or *Denom* is 2^N. However, each successive product term contains a minimum of one *dpi*,a that does not appear in a previous product term (see the NEET construction algorithm in Section V). This applies regardless of whether impedance or admittance ratios are present. Hence, the minimum number of different *dpi*,a's that appear in either the *Num* or the *Denom* is the same as the total number of terms less the leading 1, namely, $2^N - 1$.

On the other hand, many more different *dpi*,a's can be defined: each EE has a *dpi*,a that can have 1, 2, up to $N-1$ other EE's in their *opref* states, for a total which is again given by (A.14), namely 2^{N-1}. Since there are N EE's, the maximum number of *dpi*,a's in either the *Num* or *Denom* is $N2^{N-1}$.

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