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A two-dimensional sink in a density-stratified porous medium

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A solution is offered for the flow induced by a two-dimensional line sink in a saturated, density-stratified porous medium. It is found that fluid is selectively withdrawn from a thin layer at the elevation of the line sink and not from the entire medium. The velocity distributions predicted by the theory are checked by experiments in a Hele-Shaw cell and good agreement found.

1. Introduction

Investigations by Yih (1958), Kao (1965) and Koh (1966), have shown that a constant-strength sink situated in a density-stratified fluid withdraws fluid from a relatively thin layer at the level of the sink and not from the entire fluid. The phenomenon can easily be explained by the fact that a stable vertical density gradient inhibits vertical motion by providing a restoring force on any fluid particle with a tendency to move vertically. The problem is generally referred to in the literature as the selective withdrawal problem, for the obvious reason that the fluid is only withdrawn from about the level of the sink.

Koh (1964), in an unpublished manuscript, extended his boundary-layer-type theory for viscous fluids (Koh 1966) to a similar problem in a saturated density-stratified porous medium. He found no difficulty in solving the axially symmetric point sink problem but could not resolve an apparent difficulty associated with the boundary conditions at infinity in the two-dimensional sink case. The problem as Koh saw it was that the density anomaly which he determined had finite magnitude (but opposite sign) at large distances above and below the line of the sink. This would mean that in starting a sink from rest an infinite amount of salt would have to be transferred from above the sink to below before a steady-state flow was reached. This paper originated as an attempt to resolve this difficulty by attacking the problem in a somewhat more formal fashion using singular perturbation theory. It is found that Koh's solution is in fact a correct one and that the difficulty Koh found is merely one of interpretation.

2. Equations of motion

A two-dimensional sink of strength Q units of volume per unit time per unit length is supposed to be embedded in a porous medium. The medium is of intrinsic permeability k and is saturated with a viscous fluid of constant viscosity μ. The viscous fluid is linearly stratified so that the density, ρ, decreases with increasing elevation (see figure 1 for the flow configuration).

Two-dimensional Cartesian co-ordinate axes are chosen with the y-axis directed vertically upwards. The origin of the co-ordinate system is placed at the location of the two-dimensional sink. The equations governing the steady-state motion of a viscous fluid in a porous medium are Darcy's law,

$$\nabla p + \rho g \mathbf{j} + \mu \mathbf{q}/k = 0, \tag{1}$$

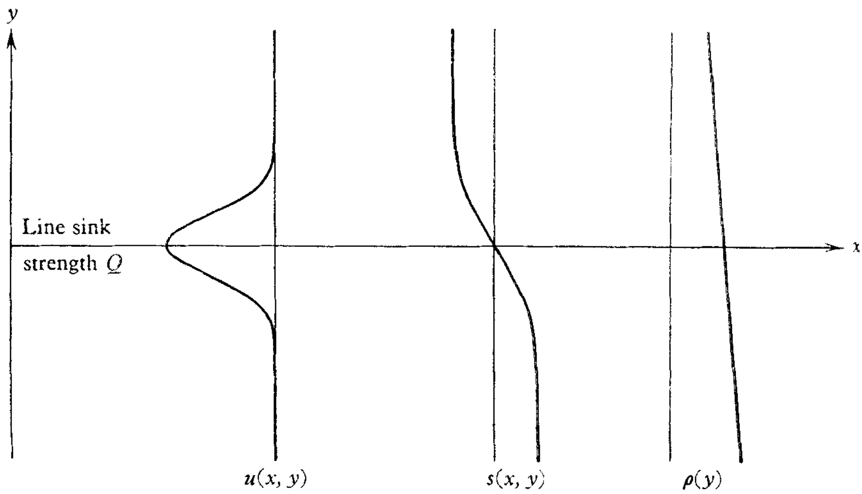


FIGURE 1. Flow configuration, line sink of strength Q centred at the origin.

where p is the fluid pressure, g the gravitational acceleration and $\mathbf{j}$ a unit vector directed in the positive y-direction and $\mathbf{q} = (u, v)$ is the seepage velocity vector (i.e. flux/pore area).

If the density-stratification is caused by a dissolved salt then the mass conservation equation for steady motion is

$$\sigma \nabla . (\rho \mathbf{q}) = D\nabla^2 C - \rho Q \delta(x)\delta(y), \tag{2}$$

where σ is the porosity of the medium and C the concentration (mass per unit volume) of dissolved salt and $\delta(x)$ is the Dirac delta function so that $\delta(x)\delta(y)$ corresponds to a unit source at the origin. D is the diffusivity of the salt in the saturated porous medium and may be assumed constant provided that the diffusive Péclet number

$$Ud/D_0 < O(10),$$

where U is a representative velocity, d the mean particle diameter and D_0 the molecular diffusivity of the salt (see List & Brooks 1967).

The salt conservation equation is

$$\sigma \nabla . (C\mathbf{q}) = D\nabla^2 C - CQ\delta(x)\delta(y). \tag{3}$$

When the salt concentration is low it is possible to write an equation of state

$$\rho/\rho_0 = 1 + \beta(C - C_0)/C_0, \tag{4}$$

where ρ_0 and C_0 are the density of fluid and concentration of the salt in the fluid at the level of the sink, and $\beta = 0{\cdot}03$.

Now if equation (3) is multiplied by β/C_0 and subtracted from equation (2) divided by ρ_0 then we get

$$\sigma(1-\beta)\nabla.\mathbf{q} = D(C_0/\rho_0-\beta)\nabla^2(C/C_0)-(1-\beta)Q\delta(x)\delta(y),$$

so that using (4) and dividing by $(1-\beta)$ gives the equation of continuity

$$\sigma\nabla.\mathbf{q} = \alpha D\nabla^2(\rho/\rho_0)-Q\delta(x)\delta(y), \tag{5}$$

where $\alpha = (C_0-\beta\rho_0)/(\rho_0(1-\beta)) \simeq 0{\cdot}4$, for usual values of C_0 and ρ_0.

Equation (5) is now multiplied by ρ and subtracted from (2) to give

$$\sigma\mathbf{q}.\nabla\rho = D\nabla^2C-\alpha\rho D\nabla^2(\rho/\rho_0),$$

and using (4) again yields the mass conservation equation in the form

$$\sigma\mathbf{q}.\nabla(\rho/\rho_0) = (1-\alpha\beta C/C_0)D\nabla^2(\rho/\rho_0). \tag{6}$$

To solve the problem it is now assumed that the density distribution for a steady state can be represented as a perturbation on a basic linear profile, that is

$$\rho/\rho_0 = 1-\epsilon y+s(x,y). \tag{7}$$

If it is further assumed that conditions are such that $\alpha\beta C/C_0 \ll 1$ then we can write (1), (5) and (6) as

$$\nabla P+sg\mathbf{j}+\mu\mathbf{q}/\rho_0 k = 0, \tag{8}$$

$$\sigma\nabla.\mathbf{q} = \alpha D\nabla^2 s-Q\delta(x)\delta(y), \tag{9}$$

$$-\epsilon v+\mathbf{q}.\nabla s = (D/\sigma)\nabla^2 s, \tag{10}$$

where

$$P = \frac{p}{\rho_0}+gy-\epsilon gy^2/2. \tag{11}$$

These equations are non-dimensionalized by choosing ϵ^{-1} as a characteristic length, and $\epsilon Q/\sigma$ as a characteristic velocity. The equations then become in non-dimensional variables

$$\nabla P/g+s\mathbf{j}+Q\mathbf{q}/\theta^2 D = 0, \tag{12}$$

$$\nabla.\mathbf{q} = (\alpha D/Q)\nabla^2 s-\delta(x)\delta(y), \tag{13}$$

$$-v+\mathbf{q}.\nabla s = (D/Q)\nabla^2 s, \tag{14}$$

where

$$\theta^2 = \rho_0 gk\sigma/\epsilon\mu D \tag{15}$$

can be interpreted as a Rayleigh number.

3. Existence of a boundary layer

The solution of this non-linear system of equations for a two-dimensional system appears to be difficult at first sight and in order to make some advance we consider the existence of a boundary-layer-type solution as suggested by Koh (1964). This will give some measure of the relative order of magnitude of the terms in (13) and (14) at locations away from the sink.

Now suppose that $L = \epsilon l$ (where l is a dimensional horizontal length) is a representative non-dimensional horizontal length and δ a representative non-dimensional vertical length. Then, for the moment ignoring the right-hand term, (13) states that if the flow occurs in layer of thickness δ then $u \sim \delta^{-1}$, $v \sim L^{-1}$ so that the curl of (12) implies that $s \sim QL/\theta^2 D\delta^2$, and therefore the terms of (14) are of the following order:

$$v \sim 1/L, \tag{16a}$$

$$\mathbf{q}.\nabla s \sim Q/\theta^2 D\delta^3, \tag{16b}$$

$$(D/Q)\nabla^2 s \sim L/\theta^2\delta^4. \tag{16c}$$

Suppose that we assume that the diffusive transport term $(D/Q)\nabla^2 s$ can be neglected, that is, that the terms (16a) and (16b) are of a comparably larger order of magnitude than (16c). The condition for this to be satisfied is easily seen to be

$$\delta \sim (QL/\theta^2 D)^{\frac{1}{3}},$$

$$L < (Q/D)^2/\theta.$$

Thus close to the sink we should expect an inertial boundary layer where horizontal velocities behave like $L^{-\frac{1}{3}}$.

Alternatively, suppose that we assume that (16a) and (16c) are comparable and larger in magnitude than the advection term (16b). The condition for this to be so is seen to be

$$\left.\begin{aligned} \delta &\sim (L/\theta)^{\frac{1}{2}}, \\ L &> (Q/D)^2/\theta. \end{aligned}\right\} \tag{17}$$

We should therefore expect a diffusive boundary layer further from the sink where horizontal velocities behave like $L^{-\frac{1}{2}}$.

It will be recalled that we have neglected the term $(\alpha D/Q)\nabla^2 s$ in both cases discussed above. It can be readily shown that in the inertial boundary layer this term can be neglected if $L < (Q/\alpha D)$ and it can be neglected in the diffusive boundary layer provided $L < \theta/\alpha^2$.

In the work to follow we shall, in general, be concerned with the diffusive boundary layer region $(Q/D)^2/\theta < L < \theta^2/\alpha$, so that we shall ignore the salt-transfer term $\alpha D\nabla^2 s$ contributing to the rate of dilatation.

4. Solution of the problem

The order-of-magnitude analysis above suggests that it will be worth while to scale the y co-ordinate by a factor η which will behave like $\theta^{-\frac{1}{2}}$ and then look for solutions of the form

$$s(x,y) = s_0(x,\bar{y})f_0(\eta) + s_1(x,\bar{y})f_1(\eta) + \ldots, \tag{18}$$

$$u(x,y) = u_0(x,\bar{y})g_0(\eta) + u_1(x,\bar{y})g_1(\eta) + \ldots, \tag{19}$$

$$v(x,y) = v_0(x,\bar{y})h_0(\eta) + v_1(x,\bar{y})h_1(\eta) + \ldots, \tag{20}$$

where

$$\bar{y} = y/\eta. \tag{21}$$

When these substitutions are made into the equations

$$\frac{Q}{\theta^2 D}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right) = \frac{\partial s}{\partial x}, \tag{22}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = -\delta(x)\,\delta(y), \tag{23}$$

$$-v+u\frac{\partial s}{\partial x}+v\frac{\partial s}{\partial y} = \frac{D}{Q}\nabla^2 s, \tag{24}$$

we are led to

$$\left.\begin{aligned} &\frac{\partial u_0}{\partial \bar{y}}-\frac{\partial s_0}{\partial x} = 0,\\ &v_0+\frac{\partial^2 s_0}{\partial \bar{y}^2} = 0,\\ &\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial \bar{y}} = -\delta(x)\,\delta(y),\end{aligned}\right\} \tag{25}$$

$$\left.\begin{aligned} &\frac{\partial u_1}{\partial \bar{y}}-\frac{\partial s_1}{\partial x} = 0,\\ &v_1+\frac{\partial^2 s_1}{\partial \bar{y}^2} = u_0\frac{\partial s_0}{\partial x}+v_0\frac{\partial s_0}{\partial \bar{y}},\\ &\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial \bar{y}} = 0,\end{aligned}\right\} \tag{26}$$

where

$$\left.\begin{aligned} &f_0(\eta) = Q\eta^2/D, &&f_1(\eta) = Q\eta^3/D,\\ &g_0(\eta) = 1/\eta, &&g_1(\eta) = 1,\\ &h_0(\eta) = 1, &&h_1(\eta) = \eta,\end{aligned}\right\} \tag{27}$$

and

$$\eta = \theta^{-\frac{1}{2}}.$$

Some discussion of the boundary conditions to be adopted is now in order. It is obvious from symmetry that there can be no flow across the vertical and horizontal axes. The boundary conditions at infinity are unknown and all that can be said is that the velocities must be finite and that the velocity distribution integrated over a control surface enclosing the sink must be constant. Suppose we choose a control volume with vertical and horizontal sides; it can be imagined that fluid and salt could be supplied in various ways through these surfaces. However, if there is to be a unique solution to the problem of a sink in a stratified fluid there must be some natural distribution of salt and fluid transport across the vertical and horizontal surfaces; the essential problem is that these distributions are unknown. The best we can do is make some assumptions which seem physically realistic and determine a solution which satisfies the integral condition above. In the absence of knowledge about the uniqueness of solutions of the equations under such integral boundary conditions there is no guarantee that the solution so determined is the natural one.

Thus, in the light of the boundary-layer discussion above it seems reasonable to assume that there is no flow across horizontal boundary surfaces of an infinitely large control volume and that half the flow to the sink must pass through any vertical surface; this can be written

$$\int_{-\infty}^{\infty} u\,dy = -\tfrac{1}{2}, \quad \text{all } x > 0.$$

In terms of the scaled co-ordinates, this condition becomes

$$\int_{-\infty}^{\infty} u_0\,d\bar{y} = -\tfrac{1}{2},$$

and can be obtained by partially integrating the third of equations (25), which it thereupon replaces.

If it is further assumed that the density anomaly does not contribute to the salt flux at infinity, then

$$\frac{\partial s_0}{\partial \bar{y}}(x, \pm\infty) = 0.$$

Now we let $u_0 = \partial\phi/\partial x$, $s_0 = \partial\phi/\partial\bar{y}$, $v_0 = -\partial^3\phi/\partial\bar{y}^3$ and we can then state the zeroth-order problem as

$$\frac{\partial^4\phi}{\partial\bar{y}^4} - \frac{\partial^2\phi}{\partial x^2} = 0,$$

$$\frac{\partial\phi}{\partial\bar{y}}(x,0) = \frac{\partial^3\phi}{\partial\bar{y}^3}(x,0) = 0,$$

$$\frac{\partial^2\phi}{\partial\bar{y}^2}(x, \pm\infty) = 0, \quad \frac{\partial\phi}{\partial x}(0,y) = 0,$$

$$\int_{-\infty}^{\infty} \frac{\partial\phi}{\partial x}\,d\bar{y} = -\tfrac{1}{2}.$$

It should be noted that the condition $\partial\phi(x, \pm\infty)/\partial x = 0$ is implicit in the last condition. The problem is very easily solved in terms of the similarity variable $\zeta = \bar{y}/x^{\frac{1}{4}}$ and the solution for $x > 0$ is

$$\phi(x,\bar{y}) = -x^{\frac{1}{2}}[(4\pi)^{-\frac{1}{2}}\exp(-\tfrac{1}{4}\zeta^2) + \tfrac{1}{4}\zeta\,\mathrm{erf}\,\tfrac{1}{2}\zeta],$$

so that

$$s_0(x,\bar{y}) = -\tfrac{1}{4}\,\mathrm{erf}\,(\tfrac{1}{4}\zeta),$$

$$u_0(x,\bar{y}) = -(16\pi x)^{-\frac{1}{2}}\exp(-\tfrac{1}{4}\zeta^2),$$

$$v_0(x,\bar{y}) = -(\zeta/8x\sqrt{\pi})\exp(-\tfrac{1}{4}\zeta^2).$$

(It is perhaps of interest to note that if x is regarded as a time-like variable there is a near analogy of this formal problem to the Uflyand impulsive loading on an Euler–Bernoulli beam.)

Now, it so happens that the term $u_0\partial s_0/\partial x + v_0\partial s_0/\partial\bar{y}$ in (26) is identically zero so that the solutions (18), (19) and (20) become

$$s(x,y) = -(Q/4D\theta)\,\mathrm{erf}\,(\tfrac{1}{2}\zeta) + O(\eta^4), \tag{28}$$

$$u(x,y) = -(\theta/16\pi x)^{\frac{1}{2}}\exp(-\tfrac{1}{4}\zeta^2) + O(\eta), \tag{29}$$

$$v(x,y) = -(\zeta/8x\sqrt{\pi})\exp(-\tfrac{1}{4}\zeta^2) + O(\eta^2). \tag{30}$$

The outer solution obtained by letting θ tend to infinity in equations (22), (23) and (24) gives $s = f(y)$ and matching with the inner solution gives

$$s(x,y) = -(Q/4D\theta)\operatorname{sgn} y.$$

Therefore (28), (29) and (30) are uniformly valid solutions to the order shown.

The fact that $u_0 \partial s_0/\partial x + v_0 \partial s_0/\partial \bar{y} \equiv 0$ means that s_0 is a function of the zeroth-order streamfunction ψ_0 defined by $u_0 = \partial\psi_0/\partial\bar{y}$, $v_0 = -\partial\psi_0/\partial x$ and, in fact, for this case $\psi_0 = s_0$.

This suggests that the non-linear term in the full equations (22), (23) and (24) may be negligibly small in comparison with the other terms. Therefore we ignore the term $u\,\partial s/\partial x + v\,\partial s/\partial y$ and eliminate s and v to find that $u(x,y)$ satisfies the equation

$$\nabla^4 u - \theta^2 \partial^2 u/\partial x^2 = -(\nabla^2 - \theta^2)\,\delta'(x)\,\delta(y). \tag{31}$$

Now if we let $u = (\nabla^2 - \theta^2)\,U$ we see that U satisfies

$$\nabla^4 U - \theta^2 \partial^2 U/\partial x^2 = -\delta'(x)\,\delta(y),$$

and this equation is easily solved using two-dimensional Fourier transforms (see Carrier, Krook & Pearson 1966) to give

$$U(x,y) = +(1/2\pi\theta)\sinh(\tfrac{1}{2}\theta x)\,K_0(\tfrac{1}{2}r\theta),$$

where $r^2 = x^2 + y^2$ and $K_0(\frac{1}{2}r\theta)$ is the modified Bessel function of zeroth order. We therefore find that

$$u(x,y) = -\frac{\theta}{4\pi}\left\{\sinh\left(\frac{\theta x}{2}\right)K_0\left(\frac{r\theta}{2}\right) + \frac{x}{r}\cosh\left(\frac{\theta x}{2}\right)K_1\left(\frac{r\theta}{2}\right)\right\}, \tag{32}$$

and, since $\qquad (x/r)\,K_1(\tfrac{1}{2}r\theta) = -(2/\theta)\,\partial K_0(\tfrac{1}{2}r\theta)/\partial x,$

and $\qquad \displaystyle\int_{-\infty}^{\infty} K_0\left(\frac{r\theta}{2}\right)dy = \frac{2\pi}{\theta}\exp(-\tfrac{1}{2}\theta x),$

we also have that $\qquad \displaystyle\int_{-\infty}^{\infty} u(x,y)\,dy = -\tfrac{1}{2}, \quad \text{all } x \neq 0.$

Since, in general, $\theta x > 10^2$, an asymptotic expansion is in order and

$$u(x,y) \sim -\left(\frac{\theta}{64\pi r}\right)^{\frac{1}{2}} \exp(-\tfrac{1}{2}(r-x)\,\theta)\left\{\frac{x}{r} + 1 + \frac{1}{4r\theta}\left(\frac{3x}{r} - 1\right) + O((r\theta)^{-2})\right\}.$$

In particular, when $y/x < 1$ it is possible to write

$$u/u_{\max} \sim \exp(-\theta y^2/4x) + O(y/x)^2,$$

which is just the boundary-layer solution given in (29).

It can also be shown that $s(x,y)$ satisfies the equation

$$\nabla^4 s - \theta^2 \partial^2 s/\partial x^2 = (Q/D)\,\delta(x)\,\delta'(y), \tag{33}$$

which leads to a two-dimensional singular integral when Fourier transforms are used; this difficulty arises because $s(x, \pm\infty)$ is not zero. The integral will have some value in a generalized sense but the writer has been unable to determine it. It seems possible that the $u(x,y)$ given in (32) above could be part of an exact solution to the non-linear equations but verification of this depends on solving

(33). The solutions determined above are quite different from those for the viscous flow problem solved by Koh (1966). Here, there is no return flow and, as a consequence, the density anomaly s does not vanish at $y = \pm\infty$. This means that when a sink is started in a linearly stratified fluid there must be an initial discontinuity in the stratification or otherwise a stable flow can never be reached, for otherwise an extremely large amount of salt must be transferred from above the line of the sink to below the line to make up the finite density anomaly at infinity. For all practical purposes however, this is not necessarily so because the total magnitude of this discontinuity has a value $Q/2D\theta^2$ which is extremely small since in general θ^2 is of $O(10^{10})$. Since this term is very small in comparison with ϵy there is no difficulty in doing an experiment with a continuous stratification because the local variations in density within the medium must be of this order. A series of experiments has therefore been done to verify the velocity profiles predicted by (29).

5. Experimental investigation

The problems associated with doing an experiment such as selective withdrawal in a porous medium are quite serious. It is difficult to establish a linear density-stratification using dissolved salt (Elder (1967) has recently done some studies with thermal stratification), and determining the flow velocities within the medium is not easy. It is fortunate therefore that the analogy between flows in a porous medium and flow in a Hele–Shaw cell exists. Wooding (1960) has shown that it is possible to use the Hele–Shaw cell for diffusive problems such as that under consideration here provided that if δ is the smallest length scale associated with the flow and U is the velocity scale then

$$\left.\begin{aligned} h/\delta &\ll 1, \\ \rho_0 U h^2/\mu\delta &\ll 1, \\ Uh^2/D_0\delta &\ll 1, \end{aligned}\right\} \tag{34}$$

where h is the spacing of the plates in the Hele–Shaw cell. The first condition is rather an obvious one if the flow is to be successfully modelled in a cell. The second condition is the requirement that the advective terms can be neglected in the viscous flow equations which apply within the cell. The third can be interpreted as the condition that the time required to modify significantly the density field is small compared with the time required for molecular diffusion to erase spanwise variations in the density profile. These conditions will be shown to have been barely satisfied for the experiments carried out. It should be noted that the flow in the Hele–Shaw cell is equivalent to a porous medium of unit porosity.

Experimental procedure

A Hele–Shaw cell was constructed of two sheets of $\frac{1}{2}$ in. thick optically flat plate glass approximately 100 cm long and 60 cm wide clamped 0·108 cm apart. One end of the cell was sealed with plastic tape except for the inclusion of a stainless capillary tube at mid-elevation. The entire cell was suspended in a vertical position in a tank 15 cm wide and slightly longer and deeper than the cell. The capillary tube forming the sink was led out of a sealed gland and connected to a power-driven hypodermic syringe which provided a constant withdrawal rate.

The outer tank had a hooded inlet in the base so that the tank, and hence the Hele–Shaw cell, could be filled from the bottom. A linearly stratified cell and tank were constructed by running layers of successively increasing specific gravity in at the bottom and then allowing the tank to stand for 12–18 h to smooth out the density discontinuities. The actual stratification in the tank was measured with a calibrated conductivity probe.

When a uniform density gradient had been established the power-driven hypodermic syringe was started and fluid withdrawn from between the walls of the Hele–Shaw cell. The resulting steady velocity profile was measured by timing successive photographs of traces of dye particles dropped between the plates of the Hele–Shaw cell.

Some difficulty was encountered in getting a dye particle of a suitable solubility, intensity and specific gravity so that a sufficiently contrasting dye trace would remain behind the dropped particle. A trial-and-error process finally produced an adequate particle made by combining zinc dust and methylene blue powder in the proportion of 1 to 2 by volume, pelletizing the mixture, crushing the pellets and sieving out suitably sized particles.

The time series photographs of the dye traces were projected onto squared paper and successive images aligned by fiduciary marks on the tank. Each image was then traced onto the squared paper. The velocity at any elevation was then determined by plotting the dye displacement against time, and the slope of the resulting straight line, after a suitable scaling, gave the velocity.

The technique is amazingly accurate when it is considered that errors in projector alignment, curve tracing and data read-off are included.

6. Comparison with theory

A comparison between experiment and theory is facilitated by the fact that for the experiments conducted (see table 1 for experimental parameters) $\theta^2 \sim 10^{10}$ and this will be true also for most physical situations. Consequently, the boundary-layer result should be quite accurate. In dimensional co-ordinates, equation (29) becomes when $y = 0$

$$u_{\max}/\epsilon Q = -(\theta/16\pi\epsilon x)^{\frac{1}{2}}.$$

Run	$\epsilon \times 10^3$ (cm^{-1})	Q (cm^2/sec)	ρ_0 (gm/cm^3)	$\mu \times 10^2$ (gm/cm sec)	$D \times 10^5$ (cm^2/sec)	$k \times 10^3$ (cm^2)	$\theta^2 \times 10^{-10}$
H 21	0·444	0·0118	1·011	0·988	1·25	0·971	1·75
H 22	0·444	0·0059	1·010	0·960	1·25	0·971	1·81
I 21	0·938	0·0059	1·021	0·972	1·25	0·971	0·854
I 22	0·938	0·0118	1·021	0·990	1·25	0·971	0·837
J 21	0·0898	0·0059	1·000	0·963	1·25	0·971	8·82
J 22	0·0934	0·00295	1·001	0·972	1·25	0·971	8·90
K 21	1·000	0·0295	1·027	0·981	1·25	0·971	0·798

TABLE 1. Experimental parameters

Experiment H 21		
Station	x (cm)	$u_{max} \times 10^2$ (cm/sec)
A	14·6	0·378
B	25·6	0·239
C	38·5	0·223
D	50·5	0·189
E	61·4	0·190
F	76·3	0·166
G	87·5	0·164

Experiment H 22		
Station	x (cm)	$u_{max} \times 10^2$ (cm/sec)
A	11·4	0·286
B	21·0	0·184
C	27·6	0·141
D	43·3	0·132
E	57·5	0·116
F	70·5	0·103
G	82·3	0·0916

Experiment I 21		
Station	x (cm)	$u_{max} \times 10^2$ (cm/sec)
A	28·1	0·210
B	43·4	0·165
C	58·5	0·143
D	70·9	0·132
E	79·6	0·131
F	90·6	0·120

Experiment I 22		
Station	x (cm)	$u_{max} \times 10^2$ (cm/sec)
A	40·0	0·342
B	58·0	0·250
C	70·0	0·216
D	74·3	0·212
E	86·0	0·207
F	90·7	0·210

Experiment J 21		
Station	x (cm)	$u_{max} \times 10^2$ (cm/sec)
A	23·0	0·095
B	36·6	0·0768
C	50·2	0·0623
D	66·8	0·0515
E	72·8	0·0475
F	81·3	0·0475

Experiment J 22		
Station	x (cm)	$u_{max} \times 10^2$ (cm/sec)
A	6·7	0·168
B	20·3	0·078
C	34·3	0·0612
D	42·0	0·048
E	50·8	0·043
F	76·6	0·0345
G	83·3	0·0340
H	91·9	0·0334

Experiment K 21		
Station	x (cm)	$u_{max} \times 10^2$ (cm/sec)
A	27·4	0·521
B	43·4	0·460
C	50·0	0·450
D	58·8	0·430
E	67·1	0·423
F	80·0	0·384
G	93·6	0·360
H	100·0	0·316

TABLE 2. Experimental results

A measure of the accuracy of the theory is given by figure 2, where u_{max} is plotted against x (see table 2) on logarithmic graph paper and lines of slope $-\frac{1}{2}$ are drawn through the experimental values. The agreement is seen to be remarkable except

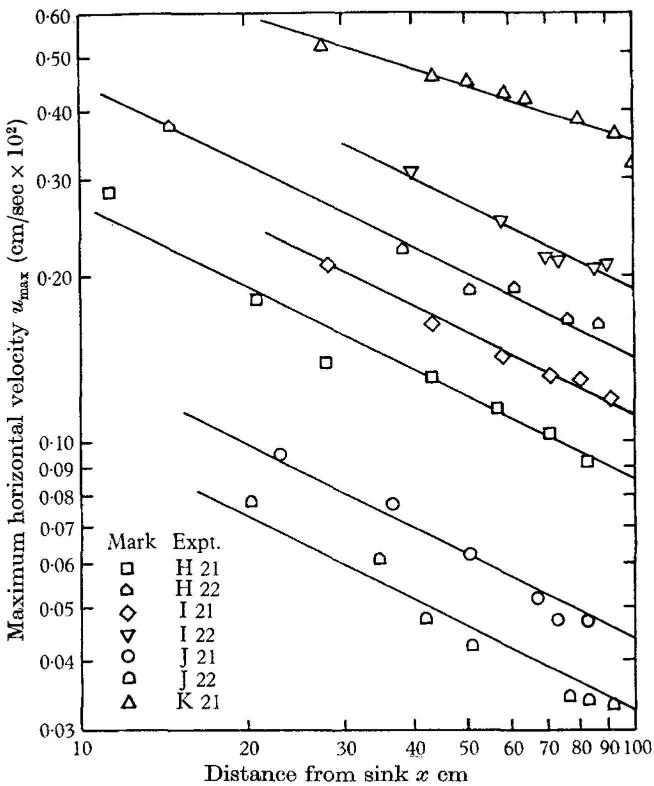


FIGURE 2. Graph of maximum velocity versus distance from the sink.

for experiment K 21, which is seen to have a slope of $-\frac{1}{3}$, and this will be explained later. This graph also gives further information. The intercept with the line $x = 100$ has the value $(\epsilon Q^2\theta/1600\pi)^{\frac{1}{2}}$ and can also be compared with the theoretical prediction. The results are given in table 3 below.

The error is seen to be not too large and the agreement in fact could be regarded as quite good.

Experiment		H21	H22	I21	I22	J21	J22	K21
$(\epsilon Q^2\theta/16\pi)^{\frac{1}{2}} \times 10^2$	theory	1·274	0·670	0·774	1·538	0·434	0·216	3·92
	expt.	1·43	0·860	1·225	1·91	0·441	0·328	3·53
Ratio		1·12	1·28	1·58	1·24	1·02	1·52	0·90

TABLE 3. Comparison of experimental and theoretically predicted values of $(\epsilon Q^2\theta/16\pi)^{\frac{1}{2}}$.
Mean ratio 1·23

Velocity distributions

The velocity profiles are self-similar according to the theory. Consequently if the theory is worthwhile it should be possible to collapse all the velocity profiles for a given experiment onto a single curve given by (29) written in the form (dimensional co-ordinates)

$$\frac{u}{u_{max}} = \exp(-\xi^2),$$

where

$$\xi^2 = \theta \epsilon y^2/4x.$$

The experimental data are plotted on figure 3 as u/u_{max} against ξ. As can be seen there it is necessary to introduce a scaling factor ξ_0 in some of the experiments

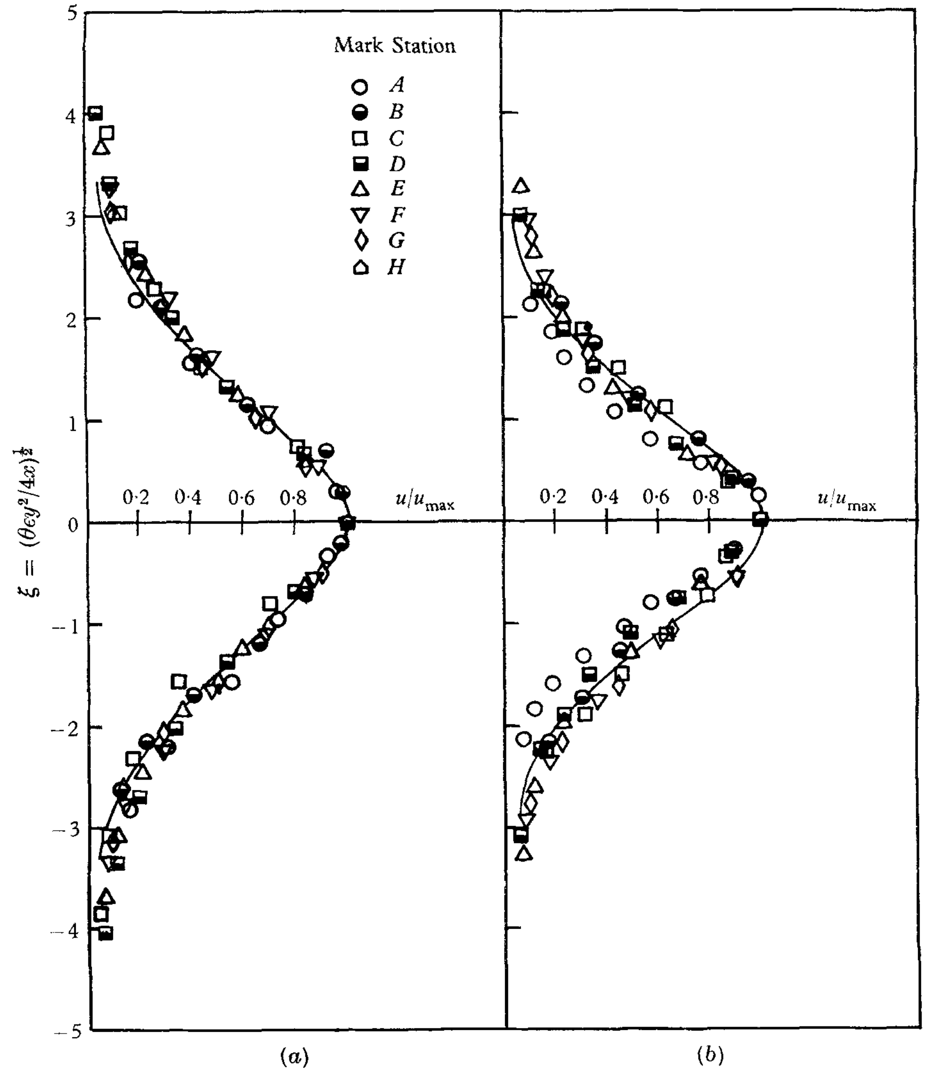
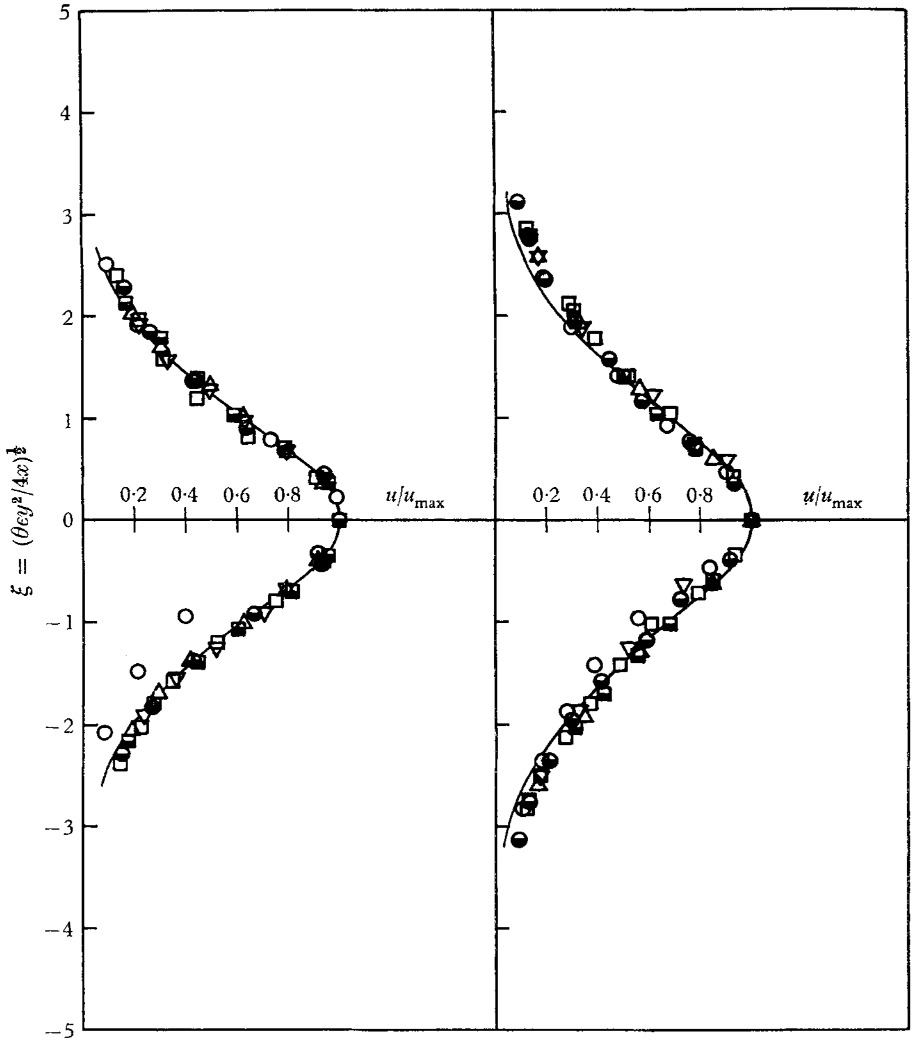


FIGURE 3. Experimentally determined velocity distributions compared with the theoretical prediction from equation (29), the full line is $u/u_{max} = \exp[-(\xi/\xi_0)^2]$. (*a*) Experiment H21, $\xi_0 = 1{\cdot}75$; (*b*) experiment H22, $\xi_0 = 1{\cdot}57$; (*c*) experiment I21, $\xi_0 = 1{\cdot}50$; (*d*) experiment I22, $\xi_0 = 1{\cdot}68$; (*e*) experiment J21; (*f*) experiment K21, $\xi_0 = 2{\cdot}39$.

in order to get the theoretical curve to fit. In general the results are seen to be very good except for the J-series experiments (experiment J 22 is similar to J 21 and has therefore been omitted). The reason for this discrepancy is unknown.



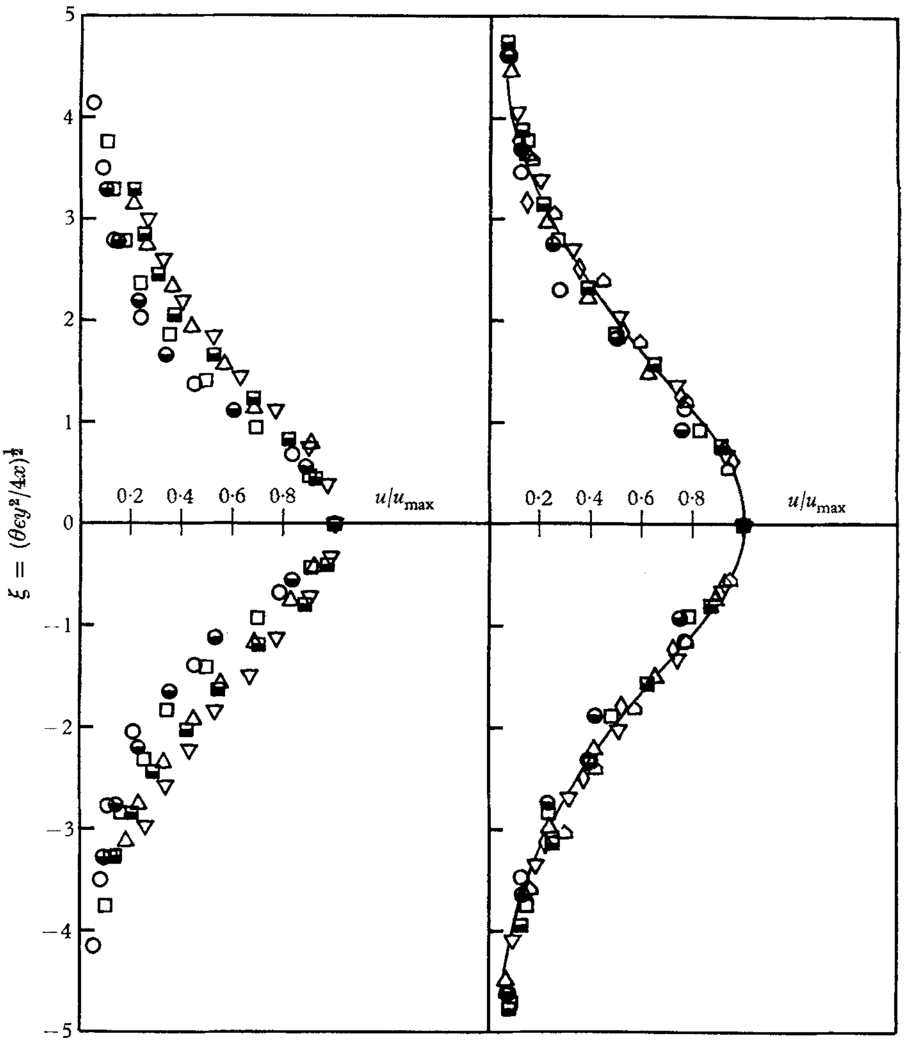
(*c*) (*d*)

FIGURE 3 (*c*) and (*d*). For legend see facing page.

It was not possible to measure the density anomaly within the cell. However, since the total change is $Q/2D\theta^2$ it can be seen from table 1 that this is of $O(10^{-8})$, which will be extremely small even in comparison with the variation in ϵy over 1 cm.

Validity of the Hele–Shaw analogy

The first two conditions of (34) for the validity of the Hele–Shaw analogy are well satisfied. The third condition $Uh^2/D_0\delta$ is found to be approximately 0·3 for the stations closest to the sink (given in table 2), and rapidly assumes much smaller values as the distance from the sink is increased. For experiment J 21, where best agreement of centreline velocities is found, all the conditions are well satisfied. It is interesting to note that experiment K 21 is by far the worst placed



(*e*) (*f*)

FIGURE 3 (*e*) and (*f*). For legend see page 540.

as $Uh^2/D_0\delta \simeq 0{\cdot}5$ for station C. This means that the Hele–Shaw analogy is not really valid for this experiment, a conclusion that is well borne out by the experimental result, where it is found that u_{max} decays like $x^{-\frac{1}{3}}$. This, incidentally, is the solution Koh (1966) found for viscous flows, which is not too surprising.

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