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Propagation in a non local reaction diffusion equation with spatial and genetic trait structure

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Abstract

We study existence and uniqueness of traveling fronts, and asymptotic speed of propagation for a non local reaction diffusion equation with spatial and genetic trait structure.

1 Introduction

In this article, we study bounded non-negative solutions of reaction-diffusion equations with non-local interactions of the type:

$$u_t - \Delta u + \alpha g(y) u = \left(1 - \int_{\mathbb{R}^N} K(z) u(t,x,z)\,\mathrm{d}z\right) u, \quad (t,x,y) \in \mathbb{R}^+ \times \mathbb{R}^m \times \mathbb{R}^N, \tag{1.1}$$

where $\mathbb{R}^+ = (0,+\infty), m \geq 1, N \geq 1$, Δ is the Laplacian operator in (x,y) variables, α is a positive constant, K and g are given non-negative functions.

This equation (1.1) arises in population dynamic models, see, e.g., [12], [2] and equations (2) and (3) in [3]. It describes a population which is structured by a set of quantitative genetic traits denoted $y \in \mathbb{R}^N$ and depends on the spatial location $x \in \mathbb{R}^m$. This population is subject to migration, mutations, growth, selection and intraspecific competition. The term $\Delta_x u$ accounts for migration by random dispersal through Brownian motion. For simplification, we assume here that mutations also involve a random dispersion in the y-variables, hence the Laplacian term $\Delta_y u$ with respect to y in the equation. Note that to simplify notations, we have taken the same diffusion coefficient, 1, both for the spatial diffusion and the diffusion in the trait space. The results remain unchanged if instead of Δu in the equation above we have $d_x\Delta_x u + d_y\Delta_y u$ with d_x and d_y positive constants. Next, local selection involves a fitness function represented here by the term $\alpha g(y)$. The effective growth rate is thus given by $u - \alpha g(y)u$. We assume that at every point in space the selection favors the trait $y = 0$ which translates into condition (1.4) below. In this context, α can be interpreted as an intensity of genetic pressure.

Lastly, at every point x in space and time t, each individual is subject to competition with all the individuals at the same location but with all possible values of the trait. The intensity of the competition can furthermore depend on the genetic traits of the competitors through a kernel $K = K(y)$. Let $u = u(t,x,y)$ denote the density of this postulation depending on time t, location

x and trait y. These various effects combine into equation (1.1) for u. When the intraspecific competition does not distinguish between the genetic features of competitors, the equation reads:

$$u_t - \Delta u + \alpha g(y) u = \left(1 - k \int_{\mathbb{R}^N} u(t,x,z)\ \mathrm{d}z\right) u, \quad (t,x,y) \in \mathbb{R}^+ \times \mathbb{R}^m \times \mathbb{R}^N, \tag{1.2}$$

where $k > 0$ is a constant. This is a particular case of (1.1) above.

This paper is about nonnegative bounded solutions of the reaction-diffusion equation with nonlocal interaction (1.1). We study the long time behavior of solutions and the traveling front solutions of (1.1).

If not otherwise stated, we always assume the kernel K to satisfy

$$K \not\equiv 0,\ 0 \le K(z) \le \kappa e^{\kappa |z|}, \quad \forall\, z \in \mathbb{R}^N \text{ with some fixed } \kappa > 0, \tag{1.3}$$

and g to be a Hölder continuous function satisfying

$$g(0) = 0,\ 0 < g(y) \le \kappa e^{\kappa|y|} \text{ in } \mathbb{R}^N \setminus \{0\}, \text{ and } \lim_{|y| \to +\infty} g(y) = +\infty. \tag{1.4}$$

With this assumption on g, the term $\alpha g(y) u$ expresses the preference that the most favorable trait is $y = 0$. Increasing the value of $\alpha > 0$ creates a tendency of the solution to decrease for all values $y \neq 0$. When $|y|$ is sufficiently large, it offsets the reproduction term in the equation in the sense that $1 - \alpha g(y) < 0$, and thus, the effective birth rate is negative for large $|y|$.

We will show that there is a constant $\bar{\alpha}$, which will be uniquely determined in Proposition 2.3, so that if $\alpha > \bar{\alpha}$, the solution $u(t,x,y) \to 0$ uniformly as $t \to +\infty$. This means that too large a genetic pressure always leads to extinction whatever the initial datum is.

Our main results concern the case $\alpha < \bar{\alpha}$. The first one describes the planar traveling wave solutions of (1.1). These are solutions of the type $u(x \cdot e - ct, y)$, where $c \in \mathbb{R}$ is a constant, $e \in \mathbb{S}^{m-1}$, $u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ solves

$$-cu_s(s,y) - \Delta u(s,y) + \alpha g(y) u(s,y) = \left(1 - \int_{\mathbb{R}^N} u(s,z) K(z)\ \mathrm{d}z\right) u(s,y), \tag{1.5}$$

with $s \in \mathbb{R}$, $y \in \mathbb{R}^N$ and such that

$$\lim_{s \to +\infty} u(s,\cdot) \equiv 0 \quad \text{and} \quad \liminf_{s \to -\infty} u(s,\cdot) > 0. \tag{1.6}$$

We also consider the stationary solution:

$$-\Delta v(x,y) + \alpha g(y) v(x,y) = \left(1 - \int_{\mathbb{R}^N} v(x,z) K(z)\ \mathrm{d}z\right) v(x,y), \quad (x,y) \in \mathbb{R}^m \times \mathbb{R}^N. \tag{1.7}$$

The results in the next theorem characterize these stationary solutions as well as the traveling wave solutions.

Theorem 1.1. *Assume that $0 < \alpha < \bar{\alpha}$. There exists a positive number c^* such that*

- *There exists a unique positive bounded stationary solution $v(x,y)$ of* (1.1)*, that is a solution of* (1.7)*. Moreover, this stationary solution is independent of x. We denote it $v = V(y)$.*

- *If* $0 \le c < c^*$, *there exists a unique positive bounded solution of* (1.5). *Moreover, the solution is independent of* s. *Therefore, it is equal to* $V(y)$.

- *For all* $c \ge c^*$, *there exists a unique nonnegative bounded solution* u *of* (1.5) *such that* (1.6) *holds. Moreover, in this case,* $\lim_{s\to-\infty} u(s,y) = V(y)$ *uniformly in* y.

We will see in the proof that c^* is explicitly given in terms of the ground state energy of the operator $-\Delta + \alpha g(y)$. Theorem 1.1 provides Liouville type results. The first part asserts the uniqueness of the stationary solution. The second one states that there are no non trivial traveling wave solutions of speed less than c^*. The third part says that there is a unique traveling wave for all speeds faster than or equal to c^*. The proof of Theorem 1.1 is obtained by a direct combination of Theorem 2.11, Theorem 2.13 and Theorem 3.4 in the main body of the paper.

Note that when $\alpha > \bar{\alpha}$ we can still make sense of the previous result by considering $c^* = +\infty$. If $\alpha < \bar{\alpha}$, then c^* is a finite positive number.

The second main result concerns the asymptotic speed of propagation of general non negative solutions of (1.1). Recall that $V(y)$ is the unique positive bounded stationary solution of (1.1) in Theorem 1.1.

Theorem 1.2. *We assume* $0 < \alpha < \bar{\alpha}$. *Let* u *be a solution of* (1.1) *so that* $u(0,x,y) = u_0(x,y)$ *is smooth, nonnegative, compactly supported and* $u_0 \not\equiv 0$. *Assume* (1.3), (1.4) *and also that* K *is bounded below in a neighborhood of the origin (condition* (3.4) *below). Then, there is invasion by* $V(y)$ *which means that at every point* $(x,y) \in \mathbb{R}^m \times \mathbb{R}^N$ *one has* $\lim_{t\to\infty} u(t,x,y) = V(y)$. *Furthermore, the asymptotic speed of propagation is equal to* c^* *in the sense that*

$$\lim_{t\to+\infty}\Big(\sup_{|x|\ge ct,\ y\in\mathbb{R}^N} u(t,x,y)\Big) = 0 \quad \textit{for all } c > c^*.$$

and

$$\lim_{t\to+\infty}\Big(\sup_{|x|\le ct,\ y\in\mathbb{R}^N} |u(t,x,y) - V(y)|\Big) = 0 \quad \textit{for all } 0 \le c < c^*.$$

There is a large literature devoted to local reaction-diffusion equations. When the competition term is replaced by a local one, (1.1) reduces to

$$u_t - \Delta u + \alpha g(y) u = f(u). \tag{1.8}$$

In [3], H. Berestycki and G. Chapuisat study this local equation. They establish the existence and characterization of traveling fronts, asymptotic speed of propagation and other related properties when f is a nonlinearity of either Fisher-KPP type or bistable type. The methods of [3] rely essentially on the maximum principle and comparison principles for parabolic equations. Therefore, they fall short for non-local equations as the one of interest here.

Several works address the questions of existence of traveling wave solutions and asymptotic speed of propagation for reaction-diffusion equations with nonlocal reaction terms related to (1.1). Using a topological degree argument and a priori estimates, M. Alfaro, J. Coville and G. Raoul [1] prove the existence of traveling waves for the equation (1.1) with K more general than here in that it also depends on y, i.e. $K = K(y,z)$, but with further restrictions on the growth of K. In particular, they assume that $k_1 \le K(y,z) \le k_2$ for all y, z, where k_1, k_2 are two positive constants. E. Bouin and V. Calvez [8] (see also [9]), constructed traveling wave solutions of equations for bounded traits with Neumann boundary conditions, where the space-diffusivity depends on the trait and the competition kernel is $K \equiv 1$. However, both of these works do not prove uniqueness results for the traveling waves and the asymptotic profile is not specified. E. Bouin and S. Mirrahimi [10] derive certain asymptotic speeds of propagation, and asymptotic behavior of either u or the average of u in the trait y, for equations with bounded traits and Neumann boundary conditions by a Hamilton-Jacobi approach. Very recently, we learned from O. Turanova [19] that she has generalized these results to equations of trait dependent space-diffusivity as those in [8]. Lastly, N. Berestycki, C. Mouhot and G. Raoul [7] establish a propagation law in $t^{3/2}$ for the model of [8] for toads invasion. The paper [9] provides a heuristic analysis and numerical computations for this model.

In Section 2.4 we consider some variations of the model (1.1) and extend our existence and uniqueness results. We analyze in particular the case in which the trait space is bounded, and also the case in which the diffusion in trait is fractional.

Another related nonlocal Fisher-KPP equation arises in ecology with a convolution term. This equation is of the form

$$u_t - \Delta u = u(1 - \phi * u) \tag{1.9}$$

where the nonlocal competition is given by a convolution with a kernel ϕ. The papers [4, 11, 15, 16, 20] and other works mentioned therein study the steady states, traveling waves and asymptotic speeds of propagation for (1.9).

The paper is organized as follows. In Section 2 we prove Theorem 1.1 on the existence and uniqueness of traveling fronts of (1.1). In Section 3 we show Theorem 1.2 on the asymptotic speed of propagation.

An important ingredient in Section 3 is a uniform pointwise bound for the solutions. One main difference between (1.8) and (1.1) (as well as (1.9)) is that in general we do not have comparison principles for solutions of (1.1) (nor (1.5)). Thus, many arguments used for the classical Fisher-KPP equation or for (1.8) as in [3] in general do not apply.

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2 Existence and uniqueness of traveling fronts

In this section, we will study existence and uniqueness of planar traveling fronts of (1.1), which are solutions of (1.5). This is actually equivalent to the case when the spatial dimension $m = 1$. Let us abuse the notations a little: we replace the variable s by x in the equation (1.5). Therefore, in this section, $x \in \mathbb{R}$ (not $\mathbb{R}^m$), and we study solutions $u(x,y) : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ of

$$-cu_x(x,y) - \Delta u(x,y) + \alpha g(y) u(x,y) = \left(1 - \int_{\mathbb{R}^N} u(x,z) K(z) \, \mathrm{d}z\right) u(x,y) \tag{2.1}$$

such that

$$\lim_{x \to +\infty} u(x,\cdot) \equiv 0 \quad \text{and} \quad \liminf_{x \to -\infty} u(x,\cdot) > 0. \tag{2.2}$$

One important observation is that when the solution u of (2.1) has the special structure $u(x,y) = v(x)\psi(y)$, where ψ is an eigenfunction of the left hand side of the equation, then the function v satisfies a classical KPP-Fisher reaction diffusion equation in x. The main difficulty in this section is to show that all traveling wave solutions u must have this separated variables structure.

2.1 A spectral lemma and asymptotic profiles

To start with, for g satisfying (1.4), we define the Hilbert space

$$\mathcal{H}(\mathbb{R}^N) = \{v \in H^1(\mathbb{R}^N) : \sqrt{g} v \in L^2(\mathbb{R}^N)\},$$

with its associated inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^N} \nabla u \nabla v + guv \, \mathrm{d}y.$$

We denote its norm as

$$\|v\|_{\mathcal{H}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\nabla v|^2 + g v^2 \, \mathrm{d}y\right)^{\frac{1}{2}}.$$

Since g is bounded from below by a positive constant in the complement B^c of the unit ball in $\mathbb{R}^N$, it is easily seen that $\mathcal{H}(\mathbb{R}^N) \hookrightarrow H^1(\mathbb{R}^N)$ with a continuous injection.

The following lemma is elementary. We include its proof here for completeness.

Lemma 2.1. *The embedding $\mathcal{H}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ is compact.*

Proof. Let $\{v_n\}$ be a bounded sequence in $\mathcal{H}(\mathbb{R}^N)$. By the assumption (1.4), $\forall\, \varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\|v_n\|_{L^2(B^c_{R_\varepsilon})} < \varepsilon \quad \text{for all } n.$$

Using the Rellich-Kondrachov theorem, there exists a subsequence $\{v_{k_n}\}$ such that

$$\limsup_{n,m \to \infty} \|v_{k_n} - v_{k_m}\|_{L^2(B_{R_\varepsilon})} = 0.$$

It follows that

$$\limsup_{n,m \to \infty} \|v_{k_n} - v_{k_m}\|_{L^2(\mathbb{R}^N)} < \varepsilon.$$

Finally, we can use a standard diagonal argument to extract a subsequence $\{v_{k_n}\}$ satisfying

$$\limsup_{n,m \to \infty} \|v_{k_n} - v_{k_m}\|_{L^2(\mathbb{R}^N)} = 0.$$

This finishes the proof. $\square$

Let $\mathcal{L}$ be the linear operator:

$$\mathcal{L}f := -\Delta_y f + \alpha g f,$$

where Δ_y denotes the Laplacian operator in the variable y only.

Lemma 2.2. *The spectrum of $\mathcal{L}$ consists only of eigenvalues. All its eigenvalues are positive and we can write them in a monotone increasing sequence $\{0 < \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots\}$ so that $\lambda_i \to \infty$ as $i \to \infty$. The first eigenvalue λ_0 is simple and corresponds to a positive eigenfunction ψ_0. All the other eigenfunctions ψ_i change signs. The eigenfunctions $\{\psi_i\}$ form an orthonormal basis of $L^2(\mathbb{R}^N)$.*

Proof. It follows from Riesz representation theorem that for each $f \in L^2(\mathbb{R}^N)$, there exists a unique function $u \in \mathcal{H}(\mathbb{R}^N)$ solving

$$\mathcal{L}u = f \tag{2.3}$$

in the sense that

$$\int_{\mathbb{R}^N} \nabla u \nabla v + guv = \int_{\mathbb{R}^N} fv \quad \text{for all } v \in \mathcal{H}(\mathbb{R}^N).$$

We write

$$u = \mathcal{L}^{-1} f.$$

Then $\mathcal{L}^{-1}$ is the operator which maps the right hand side f to the solution u in (2.3). This operator is naturally bounded from $L^2(\mathbb{R}^N)$ to $\mathcal{H}(\mathbb{R}^N)$.

Since the embedding

$$\mathcal{H}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N) \quad \text{is compact},$$

we have that

$$\mathcal{L}^{-1} : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N) \quad \text{is compact}.$$

Then the conclusion follows immediately from the standard spectral theorem for compact symmetric operators on Hilbert spaces. Since the eigenfunctions are mutually orthogonal and $\psi_0 > 0$, all the other eigenfunctions will change signs. $\square$

Since the first eigenvalue of $\mathcal{L}$ has monotonic and continuous dependence on α, we have

Lemma 2.3. *There exists some $\bar{\alpha}$ such that $\lambda_0(\alpha) < 1$ when $\alpha < \bar{\alpha}$, $\lambda_0(\bar{\alpha}) = 1$ and $\lambda_0(\alpha) > 1$ when $\alpha > \bar{\alpha}$.*

Proof. See Proposition 1 and Corollary 2 in [3] for the detailed proof. $\square$

Also, we have the following estimates for the first eigenfunction.

Proposition 2.4. *For every $\gamma > 0$ there exists $C > 0$ such that*

$$0 \leq \psi_0(y) \leq Ce^{-\gamma|y|} \quad \textit{and} \quad |\nabla \psi_0(y)| \leq Ce^{-\gamma|y|}.$$

Proof. The function ψ_0 satisfies the equation

$$\mathcal{L}\psi_0 = \lambda_0 \psi_0.$$

Then the bound for ψ_0 follows from Lemma 2.2 in [6]. Consequently, it follows from the gradient estimates for the Laplace operator that

$$|\nabla\psi_0(y)| \le 2 \sup_{|z-y|=1} |\psi_0(z)| + \sup_{|z-y|\le 1} (\alpha g(z) + \lambda_0)|\psi_0(z)|.$$

Therefore, the bound for $|\nabla\psi_0|$ follows from the bound for ψ_0. □

Moreover, we have

Proposition 2.5. *Let u be a nonnegative bounded solution of* (2.1) *such that $u \not\equiv 0$. Then for every $\gamma > 0$ there exists $C > 0$ such that for all $(x, y) \in \mathbb{R} \times \mathbb{R}^N$,*

$$0 < u(x, y) \le Ce^{-\gamma|y|} \quad \textit{and} \quad |\nabla u(x, y)| \le Ce^{-\gamma|y|}.$$

Proof. Let $\varphi_0(y) = e^{-\gamma|y|} + \varepsilon e^{\gamma|y|}$ for some $\varepsilon \in (0, 1)$. Then for $|y| \ge r_0 := (N-1)\gamma^{-1}$, we have

$$\Delta_y\varphi_0(y) = (\gamma^2 - (N-1)\gamma|y|^{-1})e^{-\gamma|y|} + \varepsilon(\gamma^2 + (N-1)\gamma|y|^{-1})e^{\gamma|y|} \le 2\gamma^2\varphi_0(y).$$

Let $w(x, y) = C(1 + \varepsilon x \arctan x)(e^{-\gamma|y|} + \varepsilon e^{\gamma|y|})$ for some $C > 0$. Then

$$\begin{aligned} & - cw_x - \Delta w + \alpha g w - w \\ &= C\Big(\varepsilon\big(- c(\arctan x + \frac{x}{1+x^2}) - \frac{2}{(1+x^2)^2}\big) \\ &\qquad + (\alpha g(y) - \varphi_0(y)^{-1}\Delta_y\varphi_0(y) - 1)(1 + \varepsilon x \arctan x)\Big)(e^{-\gamma|y|} + \varepsilon e^{\gamma|y|}) \\ &\ge C\left(-3|c| - 2 + \alpha g(y) - 2\gamma^2 - 1\right)(e^{-\gamma|y|} + \varepsilon e^{\gamma|y|}) \\ &\ge 0 \end{aligned} \tag{2.4}$$

for $(x, y) \in \Omega := \mathbb{R} \times \{y : |y| \ge R_0\}$, where $R_0 \ge r_0$ is chosen such that $g(y) \ge \alpha^{-1}(2\gamma^2 + 3 + 3|c|)$ for all $|y| \ge R_0$. It follows from (2.1) that

$$-cu_x - \Delta u + \alpha g u \le u \quad \text{in } \Omega.$$

Since u is a bounded function, we can choose C large so that $Ce^{-\gamma R_0} \ge \sup_{x,y} u(x, y)$. Meanwhile, $\alpha g > 3$ in Ω. From the maximum principle we infer that

$$u \le w \quad \text{in } \Omega.$$

By sending $\varepsilon \to 0$, we have

$$0 \le u(x, y) \le Ce^{-\gamma|y|} \quad \text{in } \mathbb{R} \times \mathbb{R}^N.$$

Consequently, by the gradient estimates we have

$$|\nabla u(x, y)| \le Ce^{-\gamma|y|} \quad \text{in } \mathbb{R} \times \mathbb{R}^N.$$

Meanwhile, it follows from strong maximum principle that $u > 0$ in $\mathbb{R} \times \mathbb{R}^N$. □

Let us consider the steady states of (2.1), i.e., the nonnegative bounded solutions of

$$-\Delta_y V(y) + \alpha g(y)V(y) = \left(1 - \int_{\mathbb{R}^N} V(z)K(z)\,\mathrm{d}z\right) V(y). \tag{2.5}$$

From Proposition 2.5 and Lemma 2.2 we see that $1 - \int_{\mathbb{R}^N} V(y)K(y)\,\mathrm{d}y$ is the first eigenvalue of $\mathcal{L}$ and $V \in \mathcal{H}(\mathbb{R}^N)$ is an eigenfunction. Thus, when $\alpha \geq \bar{\alpha}$ for $\bar{\alpha}$ be the one in Lemma 2.3, every nonnegative bounded solution of (2.5) has to be identically zero. When $\alpha \in (0, \bar{\alpha})$ and $V \not\equiv 0$, then

$$V = \mu\psi_0, \tag{2.6}$$

where

$$\mu = (1 - \lambda_0)\left(\int_{\mathbb{R}^N} \psi_0(y)K(y)\,\mathrm{d}y\right)^{-1}, \tag{2.7}$$

$\lambda_0 \in (0, 1)$ is the first eigenvalue and ψ_0 is the first eigenfunction in Lemma 2.2.

In general it is not clear whether the asymptotic profiles of traveling wave solutions to nonlocal equations are solutions of the steady equations like (2.5). See, e.g., [4]. However, we will show in Theorem 2.11 that it is the case for (2.1). That is, the solution of (2.1) and (2.2) will satisfy

$$\lim_{x\to-\infty} u(x, \cdot) = V.$$

2.2 Reduction to the classical Fisher-KPP equation

Let u be a nonnegative bounded solution of (2.1) such that $u \not\equiv 0$. Then $u > 0$ everywhere. Let $b : \mathbb{R} \to \mathbb{R}$ be its integral in y:

$$b(x) = \int_{\mathbb{R}^N} u(x, z)K(z)\,\mathrm{d}z.$$

Since $u > 0$ and $K \not\equiv 0$, we have $b(x) > 0$ for all $x \in \mathbb{R}$. For the rest of this section it is convenient to forget the relationship between u and b. We will only take into consideration that u solves the linear equation

$$-cu_x - \Delta u + \alpha g u = (1 - b(x))u \tag{2.8}$$

where b is some positive bounded function. Because of Lemma 2.2 and Proposition 2.5, we can write u as

$$u(x, y) = \sum_{i=0}^{\infty} v_i(x)\psi_i(y), \tag{2.9}$$

where

$$v_i(x) = \int_{\mathbb{R}} u(x, z)\psi_i(z)\,\mathrm{d}z \in L^\infty(\mathbb{R}).$$

By Proposition 2.5 again, it is easy to verify that the equation (2.8) splits into a sequence of equations for each v_i:

$$-c\partial_x v_i - \partial_{xx} v_i + \lambda_i v_i = (1 - b(x))v_i. \tag{2.10}$$

Lemma 2.6. *If $\lambda_i > 1$, then every bounded solution of* (2.10) *has to be identically zero.*

Proof. Let

$$w(x) = e^{\frac{-c-\sqrt{c^2+4(\lambda_i-1)}}{2}x} + e^{\frac{-c+\sqrt{c^2+4(\lambda_i-1)}}{2}x}$$

Since $\lambda_i - 1 > 0$, then $w(x) \to +\infty$ as $x \to \pm\infty$. Moreover

$$-c\partial_x w - \partial_{xx} w + (\lambda_i - 1)w = 0.$$

Thus, for every $\varepsilon > 0$, we have

$$-c\partial_x(\varepsilon w - v_i) - \partial_{xx}(\varepsilon w - v_i) + (\lambda_i + b(x) - 1)(\varepsilon w - v_i) = b\varepsilon w.$$

Since v_i is bounded and $b > 0$ in $\mathbb{R}$, the maximum principle yields that $v_i \leq \varepsilon w$. Similarly, $-v_i \leq \varepsilon w$. By sending $\varepsilon \to 0$, we have $v_i \equiv 0$. $\square$

The previous lemma tells us that there can be only finitely many terms in the expression for u:

$$u(x, y) = \sum_{i=0}^{J} v_i(x)\psi_i(y), \tag{2.11}$$

where J is a positive integer. Moreover, for each $i = 1, \ldots, J$, we have $\lambda_i \leq 1$. We suppose that $v_i \not\equiv 0$ for all $i = 0, 1, \ldots, J$. Note that $v_0(x) > 0$ for all $x \in \mathbb{R}$.

Lemma 2.7. *For each $i = 1, \cdots, J$, we have*

$$w_i := \frac{v_i}{v_0} \in L^\infty(\mathbb{R}).$$

Proof. Suppose that for some $k \in \{1, \ldots, J\}$, w_k is not bounded in $\mathbb{R}$. Then there exists a sequence $\{x_j\}$, $|x_j| \to \infty$, such that $|w_k(x_j)| \to \infty$. Then there exist $\ell \in \{1, \ldots, J\}$, and a subsequence of $\{x_j\}$ which will be still denoted as $\{x_j\}$, such that

$$|v_l(x_j)| = \max_{1 \leq i \leq J} |v_i(x_j)| \quad \text{for all } j, \quad \text{and thus } |w_\ell(x_j)| \to \infty.$$

We may also assume that all $\{v_l(x_j)\}$ have the same sign. From (2.11) we get

$$\frac{u(x_j, y)}{v_\ell(x_j)} = \frac{1}{w_\ell(x_j)}\psi_0(y) + \sum_{i=1, i\neq\ell}^{J} \frac{v_i(x_j)}{v_\ell(x_j)}\psi_i(y) + \psi_\ell(y).$$

Subject to taking a subsequence of $\{x_j\}$, we have

$$\lim_{j\to\infty} \frac{u(x_j, y)}{v_\ell(x_j)} = \sum_{i=1, i\neq\ell}^{J} \tau_i\psi_i(y) + \psi_\ell(y),$$

where each $\tau_i \in \mathbb{R}$. This is a contradiction since the right-hand side changes signs (since it is orthogonal to ψ_0) but the left-hand side does not change sign. $\square$

From now on, let $c \geq c^* = 2\sqrt{1 - \lambda_0}$. Then $c > 2\sqrt{1 - \lambda_i}$ for all $i = 1, \ldots, J$. In this case, we will have a lower bound of $|v_i|$ near $+\infty$.

Lemma 2.8. *There exists some $x_0 > 0$ such that $v_i(x)$ does not change signs in $[x_0, +\infty)$. Moreover,*

$$\liminf_{x\to+\infty} |v_i(x)|e^{\gamma_i x} > 0.$$

where

$$\gamma_i = \frac{c - \sqrt{c^2 - 4(1-\lambda_i)}}{2} \geq 0.$$

Proof. We argue by contradiction. Suppose there exist sequences $x_n \to +\infty$, $\varepsilon_n \to 0$ with

$$|v_i(x_n)| < \varepsilon_n e^{-\gamma_i x_n}.$$

Without loss of generality, we can assume $v_i(x_n) \geq 0$, since otherwise we can consider $-v_i$ instead. Let $\tilde{\gamma}_i = \frac{c+\sqrt{c^2-4(1-\lambda_i)}}{2} > \gamma_i \geq 0$. Note that in the following argument, $\tilde{\gamma}_i$ is not needed unless $\gamma_i = 0$.

We claim the following

$$v_i(x) < \varepsilon_n(e^{-\gamma_i x} + e^{-\tilde{\gamma}_i x}) \quad \forall\, x \in (-\infty, x_n]. \tag{2.12}$$

Indeed, suppose that there exists some $x' \in (-\infty, x_n)$ such that $v_i(x') \geq \varepsilon_n(e^{-\gamma_i x'} + e^{-\tilde{\gamma}_i x'})$. Since v_i is bounded and $w(x) \to +\infty$ as $x \to -\infty$, there exists some constant $C \geq 1$ such that $w(x) = C\varepsilon_n(e^{-\gamma_i x} + e^{-\tilde{\gamma}_i x})$ touches v_i from above in $(-\infty, x_n]$ at some point $\bar{x} \in (-\infty, x_n)$. Since

$$-cw_x - w_{xx} = (1-\lambda_i)w,$$

we have

$$-c(w - v_i)_x - (w - v_i)_{xx} + (\lambda_i + b - 1)(w - v_i) = bw.$$

But this is impossible if we evaluate the above equation at $\bar{x}$ since $b(x) > 0$ in $\mathbb{R}$.

Now we can let $n \to \infty$ in (2.12) to obtain $v_i \leq 0$ in $\mathbb{R}$. By applying the same arguments to $-v_i$, we obtain $v_i \equiv 0$, which is a contradiction. $\square$

Under the extra assumption $b(x) \to 0$ as $x \to +\infty$, we will have an upper bound of $|v_i|$ near $+\infty$.

Lemma 2.9. *Suppose $b(x) \to 0$ as $x \to +\infty$. For all $\delta > 0$ we have*

$$\limsup_{x\to+\infty} |v_i(x)|e^{(\gamma_i - \delta)x} < +\infty.$$

Proof. Suppose $v_i(x) > 0$ and $0 \leq b(x) \leq \delta^2$ for $x \in [x_0, +\infty)$. Let $\tilde{v}_i = e^{cx/2} v_i$. Then

$$\partial_{xx}\tilde{v}_i = (\lambda_i + b - 1 + c^2/4)\tilde{v}_i.$$

Let w_i be the solution of

$$\begin{cases} \partial_{xx} w_i = (\lambda_i + \delta^2 - 1 + c^2/4) w_i, \\ w_i(x_0) = \tilde{v}_i(x_0),\ \partial_x w_i(x_0) = \partial_x \tilde{v}_i(x_0) + 1. \end{cases}$$

Then $w_i > \tilde{v}_i$ near x_0. We claim that $w_i > \tilde{v}_i$ for all $x \in [x_0, +\infty)$. If not, let $x_1 \in [x_0, +\infty)$ be the smallest value such that $w_i(x_1) = \tilde{v}_i(x_1)$. Thus, $\partial_x w_i(x_1) \le \partial_x \tilde{v}_i(x_1)$. Then we have

$$\int_{x_0}^{x_1} (b - \delta^2) w_i \tilde{v}_i = \int_{x_0}^{x_1} w_i \partial_{xx} \tilde{v}_i - \tilde{v}_i \partial_{xx} w_i = (w_i \partial_x \tilde{v}_i - \tilde{v}_i \partial_x w_i)|_{x_0}^{x_1} > 0.$$

This is a contradiction since $b(x) \le \delta^2$ for $x \in [x_0, +\infty)$. Hence

$$\tilde{v}_i \le w_i \le C e^{\frac{\sqrt{c^2+4\lambda_i-4+4\delta^2}}{2} x} \le C e^{(\frac{\sqrt{c^2+4\lambda_i-4}}{2}+\delta)x} \le C e^{(\frac{c}{2}-\gamma_i+\delta)x}.$$

By the definition of $\tilde{v}_i$, we have

$$v_i(x) \le C e^{-(\gamma_i - \delta)x}.$$

This finishes the proof. $\square$

By combining the above three lemmas, we will conclude that $J = 0$ in the expansion (2.11) if $u(x, \cdot) \to 0$ as $x \to +\infty$.

Lemma 2.10. *Let u be a nonnegative bounded solution of* (2.1) *with $u \not\equiv 0$. Suppose in addition that for each $y \in \mathbb{R}^N$, $u(x, y) \to 0$ as $x \to +\infty$. Then the only non zero term in* (2.11) *is the one with $i = 0$.*

Proof. By Proposition 2.5 and dominated convergence theorem, we have

$$b(x) = \int_{\mathbb{R}^N} u(x, y) K(y)\, \mathrm{d}y \to 0 \quad \text{as } x \to +\infty.$$

Therefore, by Lemma 2.8 and Lemma 2.9 we have for every $i = 1, \ldots, J$,

$$\lim_{x \to +\infty} \frac{|v_i(x)|}{|v_0(x)|} = +\infty,$$

since $\gamma_i < \gamma_0$ if $\lambda_0 < \lambda_i$. This is in contradiction with Lemma 2.7. $\square$

Theorem 2.11. *Let $\alpha \in (0, \bar{\alpha})$. If $c \ge c^* = 2\sqrt{1-\lambda_0}$ then there exists a unique nonnegative bounded solution of* (2.1) *satisfying* (2.2)*. Moreover, $\lim_{x \to -\infty} u(x, y) = V(y)$, and the convergence at both $-\infty$ and $+\infty$ is uniform in y.*

Proof. After Lemma 2.10, we reduce the problem to functions u of the form

$$u(x, y) = v_0(x) \psi_0(y).$$

From (2.7) we see that $\int_{\mathbb{R}^N} \psi_0(y) K(y)\, \mathrm{d}y = (1 - \lambda_0)\mu^{-1}$, and thus $v = \mu^{-1} v_0$ satisfies

$$\begin{cases} -c\partial_x v - \partial_{xx} v = (1 - \lambda_0) v (1 - v), \\ \liminf_{x \to -\infty} v(x) > 0, \quad \lim_{x \to +\infty} v(x) = 0. \end{cases} \tag{2.13}$$

Now once we show $\lim_{x \to -\infty} v(x) = 1$, Theorem 2.11 will follow from the results on the existence and uniqueness for solutions of the classical Fisher-KPP model [17].

Let

$$m = \liminf_{x \to -\infty} v(x) > 0, \ M = \limsup_{x \to -\infty} v(x) < +\infty.$$

For $x_1 < x_2 < 0$, we integrate the first equation in (2.13) to obtain

$$c(v(x_1) - v(x_2)) + \partial_x v(x_1) - \partial_x v(x_2) = \int_{x_1}^{x_2} -c\partial_x v - \partial_{xx} v = \int_{x_1}^{x_2} (1-\lambda_0) v(1-v).$$

It follows from Proposition 2.5 that both v and $\partial_x v$ are bounded functions. Thus, the left-hand side of the above equation is bounded. Therefore, we must have $m \leq 1 \leq M$.

Therefore, we only need to show $m = M$. Suppose to the contrary that $m < M$. There exist two sequences $x_n \to -\infty$ and $z_n \to -\infty$ satisfying $z_{n+1} < x_{n+1} < z_n < x_n$ for all n, such that

$$m = \lim_{n \to \infty} v(x_n), \quad M = \lim_{n \to \infty} v(z_n).$$

Since $m < M$, there exist another two sequences $\{\tilde{x}_n\}$ and $\{\tilde{z}_n\}$ satisfying $x_{n+1} < \tilde{z}_n < x_n$, $z_{n+1} < \tilde{x}_{n+1} < z_n$, such that each $\tilde{z}_n$ is the maximum point of v in (x_{n+1}, x_n) and each $\tilde{x}_n$ is the minimum point of v in (z_{n+1}, z_n). Consequently,

$$m = \lim_{n \to \infty} v(\tilde{x}_n), \quad M = \lim_{n \to \infty} v(\tilde{z}_n).$$

By evaluating the first equation in (2.13) at $\tilde{x}_n$ and $\tilde{z}_n$, and sending $n \to \infty$, we obtain

$$M(1-M) \geq 0 \quad \text{and} \quad m(1-m) \leq 0.$$

This contradicts $m < M$. □

2.3 Non-existence of traveling fronts

In this subsection, we are going to show that when $\alpha \in (0, \bar{\alpha})$, i.e., $\lambda_0 < 1$, every bounded positive solution of (2.1) for $c < c^*$ has to be the steady solution V in (2.6).

Lemma 2.12. *Let $c \in [0, c^*)$. Then*

$$\inf_{x \in \mathbb{R}} v_0(x) > 0.$$

Proof. We argue by contradiction. Suppose that there exists a sequence $\{x_k\}$, $|x_k| \to \infty$, along which

$$v_0(x_k) \to 0.$$

Since $c < c^* = 2\sqrt{1-\lambda_0}$, we can choose $\delta > 0$ so as to have $c < 2\sqrt{1-\lambda_0-\delta}$. Let $-\frac{c}{2} + i\frac{\pi}{2L}$ be the complex root of $X^2 + cX + 1 - \lambda_0 - \delta = 0$ with $L > 0$. We first claim that

$$\lim_{k \to \infty} \sup_{|x - x_k| \leq L} v_0(x) = 0. \tag{2.14}$$

Indeed, we consider the translations of v_0 and b:

$$v_0^{(k)}(x) = v_0(x + x_k), \quad b^{(k)}(x) = b(x + x_k).$$

It follows from Proposition 2.5 that all v_0, v_0', b and b' are bounded. After extraction of a subsequence, $v_0^{(k)}$ and $b^{(k)}$ converge to $\bar{v}$ and $\bar{b}$, respectively, locally uniformly. The limits satisfy

$$-c\partial_x \bar{v} - \partial_{xx}\bar{v} + (\lambda_0 + \bar{b}(x) - 1)\bar{v} = 0.$$

Moreover, $\bar{v} \geq 0$ in $\mathbb{R}$ and $\bar{v}(0) = 0$. The strong maximum principle then shows $\bar{v} \equiv 0$. Thus $v_0^{(k)}$ converges uniformly on $[-L, L]$ to 0, which finishes the proof of the above claim.

Let

$$\phi_k(x) = e^{-c(x-x_k)/2} \cos\left(\frac{\pi}{2L}(x - x_k)\right),$$

which satisfies

$$-c\partial_x \phi_k - \partial_{xx}\phi_k + (\lambda_0 + \delta - 1)\phi_k = 0.$$

There exists $\varepsilon > 0$ such that $\varepsilon\phi_k$ touches v_0 from below at some point $\bar{x}_k \in (x_k - L, x_k + L)$. Then by evaluating the equation

$$-c\partial_x(v_0 - \varepsilon\phi_k) - \partial_{xx}(v_0 - \varepsilon\phi_k) + (\lambda_0 - 1)(v_0 - \varepsilon\phi_k) = -bv_0 + \delta\varepsilon\phi_k \quad \text{at } \bar{x}_k,$$

we have $b(\bar{x}_k) \geq \delta$. By Proposition 2.5 we get the existence of $R > 0$ (independent of k) such that

$$\int_{|y|>R} u(\bar{x}_k, y)K(y)\,\mathrm{d}y \leq \frac{\delta}{2} \quad \text{for all } k.$$

Thus by (1.3),

$$\kappa e^{\kappa R} \int_{|y|\leq R} u(\bar{x}_k, y)\,\mathrm{d}y \geq \int_{|y|\leq R} u(\bar{x}_k, y)K(y)\,\mathrm{d}y \geq \frac{\delta}{2} \quad \text{for all } k.$$

It follows that

$$\sup_{|x-x_k|\leq L} v_0(x) \geq v_0(\bar{x}_k) \geq \int_{|y|\leq R} u(\bar{x}_k, y)\psi_0(y)\,\mathrm{d}y \geq \min_{|y|\leq R} \psi_0(y) \frac{\delta e^{-\kappa R}}{2\kappa} \quad \text{for all } k.$$

This contradicts (2.14). □

Theorem 2.13. *Let* $\alpha \in (0, \bar{\alpha})$, $c \in [0, c^*)$ *and* u *be a nonnegative bounded solution of* (2.1) *with* $u \not\equiv 0$. *Then* $u \equiv V$.

Proof. If $u \not\equiv 0$, then $u > 0$. We decompose u as in (2.9). By Lemma 2.6, we have (2.11) holds for some J. Moreover, from Lemma 2.7 we know that $w_i = v_i/v_0$ is a bounded function for every $i = 1, \ldots, J$.

From (2.10) we get

$$\partial_{xx}w_i + c\partial_x w_i + \frac{2\partial_x v_0}{v_0}\partial_x w_i = (\lambda_i - \lambda_0)w_i. \tag{2.15}$$

This implies that w_i cannot have a positive local maximum, and w_i cannot have a negative local minimum. This reduces to the following structures of w_i. The function w_i must be either monotone (increasing or decreasing) or have only one local extrema. In the later case it would be either one nonnegative local minimum and monotone on each side, or one nonpositive local maximum and monotone on each side.

In any case, the function w_i must have limits as $x \to \pm\infty$. If both of these two limits are zero, we can easily conclude from the structure of w_i that w_i is identically zero, which is what we want.

Assume, to the contrary, that w_i converges monotonically to a positive number as $x \to +\infty$ (otherwise consider $-w_i$ instead). Since, in addition, w_i is bounded, there exists a sequence $\{x_k\} \to +\infty, x_{k+1}-x_k \geq 1$, such that $\partial_x w_i$ does not change signs on $[x_1, +\infty)$ and $\partial_x w_i(x_k) \to 0$. By Lemma 2.12 we may assume that $c + |2\partial_x v_0/v_0| \leq C$ in $\mathbb{R}$ for some positive constant C independent of k. Then by integrating (2.15) from x_k to x_{k+1}, we have

$$\begin{aligned}\int_{x_k}^{x_{k+1}} (\lambda_i - \lambda_0) w_i(x) \, \mathrm{d}x &\leq \partial_x w_i(x_{k+1}) - \partial_x w_i(x_k) + C \int_{x_k}^{x_{k+1}} |\partial_x w_i(x)| \, \mathrm{d}x \\ &= \partial_x w_i(x_{k+1}) - \partial_x w_i(x_k) + C \Big| \int_{x_k}^{x_{k+1}} \partial_x w_i(x) \, \mathrm{d}x \Big| \\ &= \partial_x w_i(x_{k+1}) - \partial_x w_i(x_k) + C |w_i(x_{k+1}) - w_i(x_k)| \\ &\to 0 \quad \text{as } k \to \infty.\end{aligned}$$

This is in contradiction with the assumption that w_i converges monotonically to a positive number. Similarly, we can show that $w_i(x)$ converges to 0 as $x \to -\infty$. Thus, we conclude $w_i \equiv 0$ for every $i = 1, \ldots, J$. It follows that $u(x, y) = v_0(x)\psi_0(y)$, $b(x) = v_0(x) \int_{\mathbb{R}^N} \psi_0(y) K(y) \, \mathrm{d}y = \mu^{-1}(1 - \lambda_0) v_0(x)$ where μ is the one in (2.7), and v_0 satisfies a classical Fisher-KPP equation

$$-c\partial_x v_0 - \partial_{xx} v_0 = \mu^{-1}(1 - \lambda_0)(\mu - v_0) v_0.$$

Since $c < 2\sqrt{1 - \lambda_0}$, we have $v_0 \equiv \mu$, and thus, $u \equiv V$. □

We remark that in (2.1), if $K = K(x, z)$ for $(x, z) \in \mathbb{R} \times \mathbb{R}^N$ and it satisfies (1.3) uniformly in $x \in \mathbb{R}$, then our proof still implies that the solution u of (2.1) has the separated structure $u(x, y) = v_0(x)\psi_0(y)$, where v_0 satisfies

$$-c\partial_x v_0 - \partial_{xx} v_0 = \big(1 - \lambda_0 - a(x) v_0\big) v_0$$

with

$$a(x) = \int_{\mathbb{R}^N} \psi_0(z) K(x, z) \, \mathrm{d}z.$$

2.4 Variations of the model

Our proofs of existence and uniqueness for traveling fronts also apply to other models. The first example would be those with bounded traits and Neumann boundary conditions:

$$\begin{aligned} cu_x - \Delta u + a(y) u &= \left(1 - \int_\Omega u(x, z) K(z) \, \mathrm{d}z\right) u, \quad (x, y) \in \mathbb{R} \times \Omega, \\ \frac{\partial u}{\partial \nu}(x, y) &= 0, \quad (x, y) \in \mathbb{R} \times \partial\Omega, \end{aligned} \tag{2.16}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, a, K are nonnegative bounded functions and the first eigenvalue λ_0 of the Neumann problem

$$
\begin{aligned}
-\Delta_y v + av = \lambda_0 v, \quad y \in \Omega,\\
\frac{\partial u}{\partial \nu}(y) = 0, \quad y \in \partial\Omega,
\end{aligned}
$$

satisfies $0 < \lambda_0 < 1$. The application of our proofs to (2.16) is quite straightforward. Therefore, we omit the details for (2.16) and will focus on the second example below.

The mutations in (1.1) in the space of trait y may be modelled by a diffusion process other than Brownian motions. Indeed, it would make sense to think of mutations as a jump process in the trait variable. In the case of a simple α-stable process, this leads us to models like

$$
u_t - \Delta_x u + (-\Delta_y)^\sigma u + \alpha g(y) u = \left(1 - \int_{\mathbb{R}^N} u(t,x,z)K(z)\,\mathrm{d}z\right)u, \quad (t,x,y) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^N, \tag{2.17}
$$

where $\sigma \in (0,1)$ and $(-\Delta_y)^\sigma$ is the fractional Laplacian operator in y. Since the heat kernel of the fractional Laplacian is of polynomial decay, we assume the K, g in (2.17) to satisfy

$$
K \not\equiv 0,\ 0 \le K(y) \le C_0|y|^{\kappa_1} \quad \forall\, y \in \mathbb{R}^N \text{ with some fixed } \kappa_1 \in [0, 2\sigma),\ C_0 > 0, \tag{2.18}
$$

and g is a Hölder continuous function satisfying

$$
g(0) = 0,\ 0 < g \le C_0|y|^{\kappa_1} \text{ in } \mathbb{R}^N \setminus \{0\}, \text{ and } \lim_{|y| \to +\infty} g(y) = +\infty. \tag{2.19}
$$

The traveling wave solutions of (2.17), which are solutions of the type $u(x - ct, y)$, where $c \in \mathbb{R}$ is a constant, $u : \mathbb{R}^{N+1} \to \mathbb{R}$ satisfies

$$
- cu_x - \Delta_x u + (-\Delta_y)^\sigma u + \alpha g(y) u = \left(1 - \int_{\mathbb{R}^N} u(x,z)K(z)\,\mathrm{d}z\right)u \tag{2.20}
$$

such that (2.2) holds. In addition, we require the traveling wave solutions u has finite energy in the sense that $\|(-\Delta_y)^{\sigma/2}u(x,\cdot)\|^2_{L^2(\mathbb{R}^N)}$ and $\|\nabla_x u(x,\cdot)\|^2_{L^2(\mathbb{R}^N)}$ are locally integrable in x.

To prove existence and uniqueness of traveling waves to (2.20), we only need propositions which are corresponding to Proposition 2.4 and Proposition 2.5. We start with the analysis for the principal eigenvalue of the linear operator

$$
\mathcal{L}_\sigma u(y) = (-\Delta)^\sigma u(y) + \alpha g(y) \quad \text{in } \mathbb{R}^N.
$$

Denote $H^\sigma(\mathbb{R}^N)$ be the standard fractional Sobolev space, and denote

$$
\mathcal{H}^\sigma(\mathbb{R}^N) = \{u \in H^\sigma(\mathbb{R}^N) : \sqrt{g}u \in L^2(\mathbb{R}^N)\}
$$

with norm

$$
\|u\|_{\mathcal{H}^\sigma(\mathbb{R}^N)} = \left(\int |(-\Delta)^{\sigma/2}u|^2 + gu^2\right)^{1/2}.
$$

As before, the embedding $\mathcal{H}^\sigma(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ is compact, and thus, Lemma 2.2 holds for $\mathcal{L}_\sigma$ as well. Let

$$
\lambda = \inf\{\|u\|^2_{\mathcal{H}^\sigma(\mathbb{R}^N)} : u \in \mathcal{H}^\sigma(\mathbb{R}^N), \|u\|_{L^2(\mathbb{R}^N)} = 1\},
$$

and for $R > 0$,

$$\lambda_R = \inf\{\|u\|^2_{\mathcal{H}^\sigma(\mathbb{R}^N)} : u \in \mathcal{H}^\sigma(\mathbb{R}^N), \|u\|_{L^2(\mathbb{R}^N)} = 1,\ u \equiv 0 \text{ in } \mathbb{R}^N \setminus B_R\}.$$

Note that λ is achieved by some positive function $\varphi \in \mathcal{H}^\sigma(\mathbb{R}^N)$ satisfying

$$(-\Delta)^\sigma \varphi + \alpha g \varphi = \lambda \varphi \quad \text{in } \mathbb{R}^N \tag{2.21}$$

and it is the principal eigenvalue for $\mathcal{L}_\sigma$. Also, λ_R is achieved by some nonnegative function $0 \not\equiv \varphi_R \in \mathcal{H}^\sigma$ satisfying

$$\begin{aligned}(-\Delta)^\sigma \varphi_R + \alpha g \varphi_R = \lambda_R \varphi_R \quad &\text{in } B_R,\\ \varphi_R = 0 \quad &\text{in } \mathbb{R}^N \setminus B_R.\end{aligned}$$

Then we will have λ_R converging to λ.

Lemma 2.14. *There holds*

$$\lim_{R\to\infty} \lambda_R = \lambda.$$

Proof. First of all, we know that λ_R is non-increasing in R. Let

$$\lambda_0 = \lim_{R\to\infty} \lambda_R.$$

Let R_0 be such that

$$\alpha g > \lambda_0 + 1 \quad \text{in } \mathbb{R}^N \setminus B_{R_0}.$$

Fixed φ_R such that $\|\varphi_R\|_{L^2(\mathbb{R}^N)} = 1$. We know that $\varphi_R > 0$ in B_R. Then for $R > 4R_0$, we have

$$\max_{B_{2R_0}} \varphi_R \le M,$$

where M is independent of R.

We claim that for all $R(> 4R_0)$ sufficiently large such that $\lambda_R < \lambda_0 + 1$, there holds

$$\varphi_R \le M \quad \text{in } \mathbb{R}^N.$$

Indeed, suppose there is a large R with $\max_{\mathbb{R}^N}(\varphi_R - M) > 0$. Then the maximum is achieved at some point $\bar{y} \in B_R \setminus B_{2R_0}$. Thus, we have

$$(-\Delta)^\sigma (\varphi_{R_0} - M)(\bar{y}) > 0.$$

This implies

$$0 > (\lambda_R - \alpha g(\bar{x}))\varphi_R(\bar{y}) = (-\Delta)^\sigma \varphi_R(\bar{y}) > 0,$$

which is a contradiction.

Therefore, φ_R is uniformly bounded. By the Hölder estimates (see, e.g., Proposition 2.9 in [18]), subject to a subsequence, φ_R converges locally uniformly to a bounded nonnegative continuous function φ. Since φ_R is also bounded in $\mathcal{H}^\sigma(\mathbb{R}^N)$, we have $\varphi \in \mathcal{H}^\sigma(\mathbb{R}^N)$ satisfies $\|\varphi\|_{L^2(\mathbb{R}^N)} = 1$, and is a solution of

$$(-\Delta)^\sigma \varphi + \alpha g \varphi = \lambda_0 \varphi.$$

Hence, φ is positive in $\mathbb{R}^N$, and therefore, $\lambda = \lambda_0$. $\square$

The first eigenfunction in (2.21) decays at polynomial rates. This follows by very classical methods. It is essentially the same decay as the Bessel potential or the fractional heat kernel. See for example the appendix in [13].

Proposition 2.15. *Suppose $\varphi \in \mathcal{H}^\sigma(\mathbb{R}^N)$ is a nonnegative solution of* (2.21)*, then*

$$\varphi(y) \le \frac{C}{|y|^{N+2\sigma}} \quad \textit{in } \mathbb{R}^N.$$

Proof. Let R_0 be such that

$$\alpha g > \lambda + 1 \quad \text{in } \mathbb{R}^N \setminus B_{R_0}.$$

Let G be a Green function satisfying

$$((-\Delta_y)^\sigma + 1)G = \delta_0.$$

We know that G is positive, radial, strictly decreasing in $|y|$, and satisfies

$$G(y) \le \frac{C}{|y|^{N+2\sigma}} \quad \text{for } |y| \ge 1,$$

where C is a positive constant depending only on N and σ. Let $\eta(y) = |y|^\sigma$. Then there exists $R_1 \in [R_0, +\infty)$ such that

$$(-\Delta)^\sigma \eta(y) + \eta(y) \ge 0 \quad \text{for all } |y| \ge R_1.$$

We know from the proof of Lemma 2.14 that φ is a bounded function. Therefore, we can choose M large enough to have

$$MG(R_1) \ge \varphi(y) \quad \text{for all } |y| \le R_1.$$

For every $\varepsilon \in (0,1)$, we claim

$$\varphi \le MG + \varepsilon\eta \quad \text{in } \mathbb{R}^N.$$

Suppose the contrary: $\max_{\mathbb{R}^N}(\varphi - MG - \varepsilon\eta) > 0$ is achieved at some point $\bar{y} \in \mathbb{R}^N \setminus \overline{B}_{R_1}$. Then

$$\begin{aligned} 0 &< (-\Delta)^\sigma(\varphi - MG - \varepsilon\eta)(\bar{y}) + (\varphi - MG - \varepsilon\eta)(\bar{y}) \\ &= (\lambda + 1 - \alpha g(\bar{y}))\varphi(\bar{y}) - \varepsilon\Big((-\Delta)^\sigma\eta(\bar{y}) + \eta(\bar{y})\Big) \\ &\le 0, \end{aligned}$$

which is a contradiction. Therefore, by sending $\varepsilon \to 0$, we have

$$\varphi(y) \le MG(y) \le \frac{C}{|y|^{N+2\sigma}} \quad \text{in } \mathbb{R}^N.$$

This finishes the proof. □

Proposition 2.16. *If u is a nonnegative bounded solution of the traveling wave equation* (2.20)*, then we have*

$$u(x,y) \le \frac{C}{|y|^{N+2\sigma}} \quad \textit{in } \mathbb{R}^{N+1}.$$

Proof. Let η and G be as the one in the proof of Proposition 2.15. For every $\varepsilon \in (0,1)$, we have

$$|\frac{(-\Delta)^\sigma (G+\varepsilon\eta)(y)}{(G+\varepsilon\eta)(y)}| \le \frac{G+\varepsilon|(-\Delta)^\sigma \eta(y)|}{G(y)+\varepsilon\eta(y)} \le 1 + \frac{|(-\Delta)^\sigma \eta(y)|}{\eta(y)} \le c_0 \quad \text{in } |y| \ge 1,$$

for some positive constant c_0. Let $R_0 > 1$ be such that

$$\alpha g(y) > 3|c| + c_0 + 3 \quad \text{in } |y| \ge R_0.$$

Since u is a bounded function, we can choose M large enough so that

$$MG(R_0) \ge u(x,y) \quad \text{in } |y| \le R_0.$$

Let $w(x,y) = M(1+\varepsilon x \arctan x)(G(y)+\varepsilon\eta(y))$. We claim

$$u(x,y) \le w(x,y) \quad \text{in } \mathbb{R}^{N+1}.$$

If not, then $\max_{\mathbb{R}^{N+1}}(u-w) > 0$ is achieved at some point $(\bar{x},\bar{y})$. It follows that $|\bar{y}| > R_0$. Therefore,

$$0 < -c(u-w)_x - \Delta_x(u-w) + (-\Delta_y)^\sigma(u-w) + \alpha g(u-w) + u - w$$

On the other hand, we have

$$\begin{aligned}
&- cw_x - \Delta_x w + (-\Delta_y)^\sigma w + \alpha g w - w \\
&= M\Big(\varepsilon\big(-c(\arctan x + \frac{x}{1+x^2}) - \frac{2}{(1+x^2)^2}\big) \\
&\qquad + (\alpha g(y) - \frac{(-\Delta)^\sigma (G+\varepsilon\eta)(y)}{(G+\varepsilon\eta)(y)} - 1)(1+\varepsilon x \arctan x)\Big)(G(y)+\varepsilon\eta(y)) \\
&\ge C\left(-3|c| - 2 + \alpha g(y) - c_0 - 1\right)\left(G(y)+\varepsilon\eta(y)\right) \\
&> 0.
\end{aligned}$$

Therefore, at $(\bar{x},\bar{y})$, we have

$$-c(u-w)_x - \Delta_x(u-w) + (-\Delta_y)^\sigma(u-w) + \alpha g(u-w) + u - w < 0,$$

which is a contradiction. Thus, our claim holds. By sending $\varepsilon \to 0$, we derive

$$u(x,y) \le \frac{C}{|y|^{N+2\sigma}} \quad \text{in } \mathbb{R}^{N+1}.$$

This completes the proof. □

From Proposition 2.16 we get the decomposition (2.9). Then for those traveling wave solutions with finite energy, we can split (2.20) into a sequence of equation as in (2.10). Owing to the assumptions on g and K, the terms gu and Ku are decaying faster than $|y|^{-N}$. Now we can conclude from the proof of Theorem 1.1 that for $0 < \alpha < \tilde{\alpha}$, where $\tilde{\alpha}$ is uniquely determined by $\lambda(\tilde{\alpha}) = 1$ in (2.21), we have

Theorem 2.17. *There exists a positive number c^* so that*

- *If $0 \le c < c^*$, there exists only one positive bounded solution u of* (2.20) *with finite energy. Moreover, the solution is constant in x.*
- *If $c \ge c^*$, there exists a unique non negative bounded solution u of* (2.20) *with finite energy such that* (2.2) *holds.*

3 Asymptotic speed of propagation

We consider the Cauchy problem (1.1) with $u(0,x,y) = u_0(x,y)$, where u_0 is smooth, with compact support in $\mathbb{R}^{m+N}$, $u_0 \geq 0$, and $u_0 \not\equiv 0$. Let C_0 be a positive constant such that

$$0 \leq u_0 \leq C_0\psi_0 \quad \text{in } \mathbb{R}^{m+N}, \tag{3.1}$$

where ψ_0 is the first eigenfunction in Lemma 2.2. The function $e^{(1-\lambda_0)t}\psi_0$ is a solution of the linear equation

$$\partial_t\psi - \Delta\psi + \alpha g\psi = \psi, \tag{3.2}$$

where λ_0 is the first eigenvalue in Lemma 2.2. By the comparison principle, standard parabolic equation estimates and fixed point arguments, there exists a unique solution u of (1.1) such that $u(0,x,y) = u_0(x,y)$ for all time $0 < t < \infty$, u is smooth in $(0,+\infty) \times \mathbb{R}^m \times \mathbb{R}^N$ and satisfies

$$0 \leq u(t,x,y) \leq C_0 e^{(1-\lambda_0)t}\psi_0(y) \quad \text{for all } (t,x,y) \in (0,+\infty) \times \mathbb{R}^m \times \mathbb{R}^N, \tag{3.3}$$

where C_0 is the constant in (3.1).

We are interested in the long time behavior of the solution u as $t \to \infty$. In this section, we are going to prove Theorem 1.2 on the asymptotic speed of propagation, which is the main result of this section. To prove Theorem 1.2, we proceed in two steps. We first prove the weaker version in Theorem 3.5 below. It consists in showing that for large time, for every y, the solution $u(t,x,y)$ is bounded from below by a positive constant on the sets $\{|x| \leq ct\}$ when $c < c^*$. Then, in Theorem 3.6 we make use of the decomposition as in Section 2.2 to obtain the more precise behavior that u converges to $V(y)$ on these sets. This yields Theorem 1.2.

Similar spreading rates for solutions of the local equation (1.8) were obtained in [3]. As usual, the bound (3.16) in Theorem 3.5 for $c > c^*$ follows immediately from comparing the solution of (1.1) and the solution of the linear equation (3.2).

However, because of the lack of general comparison principles, the proof of the bound (3.15) in Theorem 3.5 for $c < c^*$ is quite different from that in [3]. In this step, we shall adapt some compactness arguments used by Hamel and Ryzhik in [16]. The general idea is the following. If $u(t,x,y_0)$ is small for $|x| < c^*t$ and some point y_0, then $\int_{\mathbb{R}^N} u(t,x,y)K(y)\,\mathrm{d}y$ will be small. Hence, the behavior of u should be similar to that of the solution of the linear equation (3.2), which, however, is not small for $|x| < c^*t$.

To employ the compactness arguments, we first need to establish a uniform upper bound estimate for u, which, unlike (3.3), will be independent of the time t.

3.1 A priori estimates

To obtain the uniform upper bound of u, in addition to (1.3), we assume

$$K(y) \geq K_1 \text{ for } |y| \leq R_0 + 2, \tag{3.4}$$

where K_1 is a positive constant, and R_0 is chosen such that

$$\alpha g(y) \geq 1 \quad \text{for all } |y| \geq R_0. \tag{3.5}$$

As an intermediate step, we show the following auxiliary uniform estimate.

Lemma 3.1. *There exists a positive constant M_1 depending only on C_0, K_1, α and g such that*

$$\int_{B_1} u(t,x,y+s)\ \mathrm{d}s \le M_1 \quad \textit{for all } (t,x,y) \in (0,+\infty)\times\mathbb{R}^m\times\mathbb{R}^N,$$

where B_1 is the unit ball centered at the origin in $\mathbb{R}^N$.

Proof. Let

$$v(t,x,y) = \int_{B_1} u(t,x,y+s)\ \mathrm{d}s = \int_{B_1(y)} u(t,x,s)\ \mathrm{d}s,$$

where $B_1(y)$ is the ball in $\mathbb{R}^N$ with radius 1 and center y. Then

$$v_t - \Delta v + \alpha \int_{B_1(y)} g(s)u(t,x,s)\ \mathrm{d}s = v\left(1 - \int_{\mathbb{R}^N} u(t,x,z)K(z)\ \mathrm{d}z\right). \tag{3.6}$$

By (3.3), we have

$$0 \le v(t,x,y) \le |B_1| C_0 e^{(1-\lambda_0)t}\|\psi_0\|_{L^\infty(\mathbb{R}^N)}.$$

Let

$$M_1 = \max(1/K_1, |B_1|C_0 e^{1-\lambda_0}\|\psi_0\|_{L^\infty(\mathbb{R}^N)}) + 1.$$

We are going to show

$$v(t,x,y) < M_1 \quad \text{for all } (t,x,y) \in (0,+\infty)\times\mathbb{R}^m\times\mathbb{R}^N.$$

Suppose there is t_0 such that

$$\|v(t_0,\cdot)\|_{L^\infty(\mathbb{R}^{m+N})} = M_1, \quad \text{and} \quad \|v(t,\cdot)\|_{L^\infty(\mathbb{R}^{m+N})} < M_1 \quad \text{for } t < t_0.$$

Then $t_0 \ge 1$, and there exists a sequence $\{(x_n, y_n)\}$ such that $v(t_0, x_n, y_n) \to M_1$ as $n \to \infty$. From (3.3) we infer that $\{y_n\}$ is a bounded sequence. We define the translations (in x)

$$u_n(t,x,y) = u(t,x+x_n,y) \quad \text{and} \quad v_n(t,x,y) = v(t,x+x_n,y),$$

which also satisfy (1.1) and (3.6), respectively. By (3.3), parabolic equation estimates and dominated convergence theorem, up to a subsequence, $y_n \to y_\infty$, $\{u_n\}$ converges locally uniformly to u_∞ which satisfies (1.1), and $\{v_n\}$ converges locally uniformly to v_∞ which satisfies the equation (3.6) associated with u_∞. Moreover,

$$0 \le v_\infty(t,x,y) \le M_1 \quad \text{in } (0,t_0)\times\mathbb{R}^m\times\mathbb{R}^N, \quad v_\infty(t_0,0,y_\infty) = M_1,$$

and thus

$$\partial_t v_\infty(t_0,0,y_\infty) \ge 0, \quad \Delta v_\infty(t_0,0,y_\infty) \le 0.$$

This implies

$$\alpha \int_{B_1(y_\infty)} g(s)u_\infty(t_0,0,s)\ \mathrm{d}s \le v_\infty(t_0,0,y_\infty)\left(1 - \int_{\mathbb{R}^N} u_\infty(t_0,0,z)K(z)\ \mathrm{d}z\right).$$

Hence,

$$\alpha \int_{B_1(y_\infty)} g(s)u_\infty(t_0,0,s)\ \mathrm{d}s < v_\infty(t_0,0,y_\infty) = \int_{B_1(y_\infty)} u_\infty(t_0,0,s)\ \mathrm{d}s$$

and

$$\int_{\mathbb{R}^N} u_\infty(t_0,0,z)K(z)\ \mathrm{d}z \leq 1.$$

Thus, by the choice of R_0 in (3.5), we have

$$|y_\infty| \leq R_0 + 1.$$

From the assumption (3.4), we derive

$$K_1M_1 = K_1v_\infty(t_0,0,y_\infty) = K_1\int_{B_1(y_\infty)} u_\infty(t_0,0,z)\ \mathrm{d}z \leq \int_{B_1(y_\infty)} u_\infty(t_0,0,z)K(z)\ \mathrm{d}z \leq 1.$$

This contradicts the choice of M_1. Thus, we proved that no such t_0 exists, from which the lemma follows. ☐

We can now derive a uniform bound on u independently of the time t.

Lemma 3.2. *There exists a positive constant M_2 depending only on C_0, K_1, α and g such that*

$$0 \leq u(t,x,y) \leq M_2 \quad \textit{in } (0,+\infty)\times\mathbb{R}^m\times\mathbb{R}^N.$$

Proof. Let M_2 be a sufficiently large constant to be fixed in the proof. Suppose there exists $t_0 > 0$ such that

$$\|u(t_0,\cdot)\|_{L^\infty(\mathbb{R}^{m+N})} = M_2 \quad \text{ and } \|u(t,\cdot)\|_{L^\infty(\mathbb{R}^{m+N})} < M_2 \quad \text{ for } t < t_0.$$

By (3.3) we can choose M_2 large enough so that $t_0 \geq \sqrt{2R_0}$, where R_0 is the constant in (3.5). There exists a sequence $\{(x_n,y_n)\}$ for which $u(t_0,x_n,y_n) \to M_2$ as $n\to\infty$. We reason as in the proof of the preceding lemma. From (3.3) we infer that $\{y_n\}$ is a bounded sequence. As before, we define the translation (in x)

$$u_n(t,x,y) = u(t,x+x_n,y),$$

which also satisfies (1.1). By (3.3), parabolic equation estimates and dominated convergence theorem, up to a subsequence, $y_n \to y_\infty$, and $\{u_n\}$ converges locally uniformly to u_∞ which satisfies (1.1). Moreover,

$$0 \leq u_\infty(t,x,y) \leq M_2 \quad \text{ in } (0,t_0)\times\mathbb{R}^m\times\mathbb{R}^N, \quad u_\infty(t_0,0,y_\infty) = M_2,$$

and thus

$$\partial_t u_\infty(t_0,0,y_\infty) \geq 0, \quad \Delta u_\infty(t_0,0,y_\infty) \leq 0.$$

This implies

$$\alpha g(y_\infty)M_2 \leq (1 - \int_{\mathbb{R}^N} u_\infty(t_0,0,z)K(z)\ \mathrm{d}z)M_2.$$

Thus,

$$|y_\infty| \le R_0 \quad \text{and} \quad \int_{\mathbb{R}^N} u_\infty(t_0, 0, z)K(z)\, \mathrm{d}z \le 1.$$

Let $\Omega = (t_0 - \sqrt{2R_0}, t_0] \times \{x \in \mathbb{R}^m : |x| \le 2R_0\} \times \{y \in \mathbb{R}^N : |y| \le 2R_0\}$. The limit u_∞ satisfies

$$\partial_t u_\infty - \Delta u_\infty + \alpha g(y) u_\infty \le u_\infty \quad \text{in } \Omega.$$

By the local maximum principle in Lemma A.1, we have

$$M_2 = u_\infty(t_0, 0, y_\infty) \le C \int_\Omega u_\infty(t, x, y)\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}y,$$

where $C > 0$ depends only on α and g. By Lemma 3.1, we have

$$\int_\Omega u_\infty(t, x, y)\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}y \le \tilde{C} M_1.$$

where $\tilde{C} > 0$ depends only on R_0. Thus $M_2 \le C\tilde{C}M_1$. This is a contradiction if we choose M_2 large enough. Hence, we proved that no such t_0 exists, from which the lemma follows. $\square$

As a consequence we can show the uniformly exponential decay in y of u, independently of the time t.

Lemma 3.3. *For every $\gamma > 0$ there exists a positive constant M depending only on C_0, K_1, α, g and γ such that*

$$0 \le u(t, x, y) \le M e^{-\gamma|y|} \quad \textit{in } (0, +\infty) \times \mathbb{R}^m \times \mathbb{R}^N.$$

Proof. Let $w(x, y) = M(1 + \varepsilon|x|^2)(e^{-\gamma|y|} + \varepsilon e^{\gamma|y|})$ for some $M > 0$ and $\varepsilon \in (0, 1)$. A direct computation shows that

$$-\Delta w + \alpha g w - w \ge 0$$

for $(x, y) \in \Omega := \mathbb{R}^m \times \{y : |y| \ge R_0\}$, where $R_0 \ge (N-1)\gamma$ is chosen so that $g(y) \ge \alpha^{-1}(\gamma^2 + 2m + 2)$ for all $|y| \ge R_0$. By Proposition 2.4, we can choose M large so that $Me^{-\gamma y} \ge C_0\psi_0(y) \ge u_0(x, y)$ for $(x, y) \in \Omega$, and $Me^{-\gamma R_0} \ge M_2$, where M_2 is the one in Lemma 3.2. Since

$$u_t - \Delta u + \alpha g(y) u \le u, \tag{3.7}$$

it follows from the comparison principle that

$$u \le w \quad \text{in} \quad \text{in } (0, +\infty) \times \mathbb{R}^{m+N}.$$

The conclusion follows by sending $\varepsilon \to 0$. $\square$

3.2 Uniqueness of stationary solutions

In this section, we consider nonnegative bounded stationary solutions $u = u(x, y)$ of (1.1), that is, solutions of the equation:

$$-\Delta u + \alpha g(y) u = \left(1 - \int_{\mathbb{R}^N} K(z) u(x, z)\, \mathrm{d}z\right) u, \quad (x, y) \in \mathbb{R}^m \times \mathbb{R}^N. \tag{3.8}$$

By using the techniques in Section 2, we are able to show the uniqueness of the stationary solution.

Theorem 3.4. *For* $0 < \alpha < \bar{\alpha}$, *if* $u \not\equiv 0$ *is a non-negative bounded solution of* (3.8), *then* $u(x,y) \equiv V(y)$, *where* V *is defined in* (2.6). *When* $\alpha > \bar{\alpha}$, *the unique non-negative solution of* (3.8) *is identically zero.*

Proof. First of all, it follows from the same proof as for Lemma 3.3 that for every $\gamma > 0$ there exists a positive constant M such that

$$0 \le u(x,y) \le Me^{-\gamma|y|} \quad \text{in } \mathbb{R}^m \times \mathbb{R}^N. \tag{3.9}$$

Therefore, as in Section 2, we can write u as

$$u(x,y) = \sum_{i=0}^{\infty} v_i(x)\psi_i(y), \tag{3.10}$$

where the ψ_i are from the orthonormal basis in Lemma 2.2 and $v_i(x) = \int_{\mathbb{R}} u(x,z)\psi_i(z)\,\mathrm{d}z \in L^\infty(\mathbb{R}^m)$. Moreover, the equation (3.8) splits into a sequence of equations for each v_i:

$$-\Delta v_i = (1 - \lambda_i - b(x))v_i, \tag{3.11}$$

where $b(x) = \int_{\mathbb{R}^N} u(x,z)K(z)\,\mathrm{d}z$.

Suppose first that $\alpha < \bar{\alpha}$. By arguments similar to the proof of Lemma 2.12 we get

$$\inf_{x\in\mathbb{R}^m} v_0(x) > 0. \tag{3.12}$$

Indeed, suppose there exists a sequence $\{x_k\}$, $|x_k| \to \infty$, for which $v_0(x_k) \to 0$. Then, by exactly the same proof as for (2.14), for every $L > 0$, we have

$$\lim_{k\to\infty} \sup_{|x-x_k|\le L} v_0(x) = 0.$$

Since $\alpha < \bar{\alpha}$, we have $\lambda_0 < 1$. Choose $\delta > 0$ such that $\lambda_0 < 1 - \delta$. Let $\beta > 0$ and $\varphi_0(x)$ be the first eigenvalue and first eigenfunction of the Dirichlet problem in the unit ball of $\mathbb{R}^m$ with the normalization of unit L^∞ norm. That is,

$$\begin{cases} -\Delta_x\varphi_0 = \beta\varphi_0 \quad \text{on } B_1 := \{x \in \mathbb{R}^m : |x| < 1\}, \\ \varphi_0 > 0 \quad \text{in } B_1, \quad \varphi_0 = 0 \quad \text{on } \partial B_1, \quad \|\varphi_0\|_{L^\infty(B_1)} = 1. \end{cases} \tag{3.13}$$

Let $L = \sqrt{\frac{\beta}{1-\lambda_0-\delta}}$ and $\phi_k(x) = \varphi_0((x-x_k)/L)$, then it satisfies

$$-\Delta\phi_k + (\lambda_0 + \delta - 1)\phi_k = 0.$$

There exists $\varepsilon > 0$ such that $\varepsilon\phi_k$ touches v_0 from below at some point $\bar{x}_k \in \{x : |x - x_k| < L\}$. Then by evaluating the equation

$$-\Delta(v_0 - \varepsilon\phi_k) + (\lambda_0 - 1)(v_0 - \varepsilon\phi_k) = -bv_0 + \delta\varepsilon\phi_k \quad \text{at } \bar{x}_k,$$

we have $b(\bar{x}_k) \ge \delta$. The rest is identical to the proof of Lemma 2.12. This proves (3.12).

Let $i \geq 1$ be fixed. We set

$$w_i := \frac{v_i}{v_0}.$$

From (3.11) we see that w_i satisfies

$$-\Delta w_i - 2\frac{\nabla v_0}{v_0} \cdot \nabla w_i \;=\; (\lambda_0 - \lambda_i) w_i. \tag{3.14}$$

We know from (3.12) that w_i is bounded in $\mathbb{R}^m$ and the above equation has bounded coefficients. Suppose w_i is not identically zero. We can assume that it is positive somewhere. Then

$$0 < \sup_{\mathbb{R}^m} w_i < \infty.$$

If this supremum is reached at a point $\bar{x} \in \mathbb{R}^m$, then, since $\lambda_0 < \lambda_i$, we get $\Delta w_i(\bar{x}) > 0$ from (3.14), which is absurd. So let us assume that for some sequence $x_j \in \mathbb{R}^m$, with $|x_j| \to \infty$, we have

$$\lim_{j\to\infty} w_i(x_j) = \sup_{\mathbb{R}^m} w > 0.$$

Let us now set $w_{i,j}(x) := w_i(x_j + x)$ and $v_{0,j}(x) := v_0(x_j + x)$. From (3.14) we get

$$-\Delta w_{i,j} - 2\frac{\nabla v_{0,j}}{v_{0,j}} \cdot \nabla w_{i,j} \;=\; (\lambda_0 - \lambda_i) w_{i,j}.$$

Since v_0 is bounded from below and satisfies (3.11), by elliptic regularity estimates, we can strike out a subsequence, which is still denoted by j, such that

$$v_{0,j} \longrightarrow v_{0,\infty}, \quad w_{i,j} \longrightarrow w_{i,\infty}, \quad \inf_{\mathbb{R}^m} v_{0,\infty} > 0.$$

Moreover, we have $w_{i,\infty} \leq \sup_{\mathbb{R}^m} w_i$, $w_{i,\infty}(0) = \sup_{\mathbb{R}^m} w_i$ whence $w_{i,\infty}(0) = \sup_{\mathbb{R}^m} w_{i,\infty}$ and $w_{i,\infty}$ satisfies the equation

$$-\Delta w_{i,\infty} - 2\frac{\nabla v_{0,\infty}}{v_{0,\infty}} \cdot \nabla w_{i,\infty} \;=\; (\lambda_0 - \lambda_i) w_{i,\infty}.$$

We reach a contradiction by analyzing this equation at 0.

This proves $v_i \equiv 0$ for all $i \geq 1$. Therefore, every nonnegative bounded solution of (3.8) satisfies

$$u(x, y) = v_0(x)\psi_0(y).$$

Hence, $b(x) = v_0(x) \int_{\mathbb{R}^N} \psi_0(y) K(y) \, \mathrm{d}y = \mu^{-1}(1 - \lambda_0) v_0(x)$ where μ is the one in (2.7), and v_0 satisfies a classical Fisher-KPP equation

$$-\Delta v_0 = \mu^{-1}(1 - \lambda_0)(\mu - v_0) v_0.$$

We have $v_0 \equiv \mu$, and thus, $u \equiv V$. We remark that this translation and compactness proof can also be used to prove Theorem 2.13.

Suppose now that $\alpha > \bar{\alpha}$. We want to show that $v_i \equiv 0$ for all $i \geq 0$. We can do the above translation and compactness arguments for (3.11) directly, since $1 - \lambda_i - b(x) \leq 1 - \lambda_0 < 0$.

Suppose $v_i \not\equiv 0$ for some i. We can assume that it is positive somewhere, and then $0 < \sup_{\mathbb{R}^m} v_i < \infty$. By the equation (3.11), this positive supremum cannot be achieved at any point. So there exists some sequence $x_j \in \mathbb{R}^m$, with $|x_j| \to \infty$, for which

$$\lim_{j\to\infty} v_i(x_j) = \sup_{\mathbb{R}^m} v_i > 0.$$

Then we do a translation $v_{i,j}(x) = v_i(x_j + x)$, $b_j(x) = b(x_j + x)$ and $u_j(x, y) = u(x_j + x, y)$. By elliptic regularity estimates, after extraction of a subsequence, we can assume that $v_{i,j}$ and b_j are locally uniformly convergent to $v_{i,\infty}$ and b_∞, respectively. These limits satisfy

$$-\Delta v_{i,\infty} = (1 - \lambda_i - b_\infty(x))v_{i,\infty}.$$

Moreover, $v_{i,\infty} \le \sup_{\mathbb{R}^m} v_i$ in $\mathbb{R}^m$, $v_{i,\infty}(0) = \sup_{\mathbb{R}^m} v_i$. We reach a contradiction by evaluating the above equation at 0. Therefore, in the case of $\alpha > \bar{\alpha}$, every bounded nonnegative solution of (3.8) has to be identically zero. $\square$

3.3 Asymptotic speed of propagation

In this section we prove the following long time behavior properties for solutions of the Cauchy problem (1.1) with compactly supported nonnegative initial data. This is a weaker version of the results stated in Theorem 1.2 that we require as a first step in proving the stronger version.

Theorem 3.5. *Assume conditions* (1.3), (3.4) *and* (1.4). *Consider the solution* u *of* (1.1) *with* $u(\cdot, 0) = u_0$ *smooth, having compact support in* $\mathbb{R}^{m+N}$, $u_0 \ge 0$, *and* $u_0 \not\equiv 0$.

(i): if $\alpha > \bar{\alpha}$, *then* $u(t, x, y) \to 0$ *exponentially in* t, *uniformly in* (x, y).

(ii): if $0 < \alpha < \bar{\alpha}$, *then, for every* $y \in \mathbb{R}^N$,

$$\liminf_{t\to+\infty} \Big(\min_{|x|\le ct} u(t, x, y) \Big) > 0 \quad \textit{for all } 0 \le c < c^*, \tag{3.15}$$

and

$$\lim_{t\to+\infty} \Big(\sup_{|x|\ge ct, y\in\mathbb{R}^N} u(t, x, y) \Big) = 0 \quad \textit{for all } c > c^*. \tag{3.16}$$

Proof. The conclusion in (i) immediately follows from (3.3) since $\lambda_0 > 1$ when $\alpha > \bar{\alpha}$.

To prove (3.16) we shall use the exponential solutions $\psi_e(t, x, y) = M_3 e^{-\frac{c^*}{2}(x\cdot e - c^* t)} V(y)$, $e \in \mathbb{S}^{m-1}$, which are solutions of (3.2). From (3.7) and by the comparison principle, if we choose $M_3 > 0$ large, we have

$$u \le \psi_e, \quad \text{for all} \quad e \in \mathbb{S}^{m-1}.$$

By minimizing over e (for each x) we derive:

$$u(t, x, y) \le M_3 e^{-\frac{c^*}{2}(c - c^*)t} V(y), \quad \text{for all} \quad |x| \ge ct, y \in \mathbb{R}^N.$$

Therefore, for $c > c^*$,

$$\lim_{t\to\infty} \left(\sup_{|x|\ge ct, y\in\mathbb{R}^N} u(t, x, y) \right) = 0.$$

To prove (3.15), we use some compactness arguments as in [16]. We argue by contradiction. Suppose that there are $c < c^*$, $y_0 \in \mathbb{R}^N$ and a sequence $\{(t_n, x_n)\}$ such that

$$\begin{cases} |x_n| \le ct_n \quad \text{for all } n \in \mathbb{N}, \\ t_n \to \infty \quad \text{and } u(t_n, x_n, y_0) \to 0 \quad \text{as } n \to \infty. \end{cases}$$

We may assume that $c_n := |x_n|/t_n \to c_\infty \in [0, c]$ as $n \to \infty$. We let $e_n = x_n/|x_n| \in \mathbb{S}^{m-1}$ (if $x_n = 0$, we let e_n be the north pole of $\mathbb{S}^{m-1}$) and assume $e_n \to e_\infty$ as $n \to \infty$.

For each n and $(t, x) \in (-t_n, +\infty) \times \mathbb{R}^m$, we define the translation of u in (t, x)

$$u_n(t, x, y) = u(t + t_n, x + x_n, y).$$

By Lemma 3.3, standard parabolic equation estimates, and dominated convergence theorem, there exists a subsequence of $\{u_n\}$, which we still denote by $\{u_n\}$, such that u_n is locally uniformly convergent to U satisfying

$$\partial_t U - \Delta U + \alpha g U = \left(1 - \int_{\mathbb{R}^N} U(t, x, z) K(z)\, \mathrm{d}z\right) U \quad \text{in } \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^N. \tag{3.17}$$

Moreover,

$$U(0, 0, y_0) = 0, \quad U \ge 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^N.$$

By the strong maximum principle, $U \equiv 0$ in $(-\infty, 0] \times \mathbb{R}^{m+N}$. Consequently, by the comparison principle, we also have $U \equiv 0$ in $[0, +\infty) \times \mathbb{R}^{m+N}$, and thus,

$$U \equiv 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^{m+N}.$$

Let

$$v_n(t, x, y) = u_n(t, x + c_n t e_n, y) = u(t + t_n, x + c_n(t + t_n)e_n, y).$$

Then

$$\partial_t v_n - \Delta v_n - c_n e_n \cdot \nabla_x v_n + \alpha g v_n = \left(1 - \int_{\mathbb{R}^N} v_n(t, x, z) K(z)\, \mathrm{d}z\right) v_n, \tag{3.18}$$

Since c_n is bounded, $\{v_n\}$ also converges locally uniformly to 0 in $\mathbb{R} \times \mathbb{R}^{m+N}$. By Lemma 3.3, we see that $\int_{\mathbb{R}^N} v_n(t, x, y) K(y)\, \mathrm{d}y$ converges to 0 locally uniformly as well.

Since $c < c^* = 2\sqrt{1 - \lambda_0}$, we can choose $\delta > 0$ so as to have

$$|c_n| \le c < 2\sqrt{1 - \lambda_0 - 2\delta}.$$

Now let us use the property that λ_0 is the limit of the principal eigenvalue λ^R of the Dirichlet problem in $B_R \subset \mathbb{R}^N$ as $R \to \infty$ (see [5] for more details). That is:

$$\begin{cases} -\Delta_y \psi^R + \alpha g \psi^R = \lambda^R \psi^R \quad \text{on } B_R := \{y \in \mathbb{R}^N : |y| < R\}, \\ \psi^R > 0 \quad \text{in } B_R, \quad \psi^R = 0 \quad \text{on } \partial B_R, \quad \|\psi^R\|_{L^\infty} = 1. \end{cases}$$

More precisely, $\lambda^R > \lambda_0$ and $\lambda^R \to \lambda_0$ as $R \to \infty$. We can choose R large enough to have $\lambda_0 < \lambda^R < \lambda_0 + \delta$ and $|y_0| \le R/2$. Then $c < 2\sqrt{1 - \lambda^R - \delta}$.

Let $\beta > 0$ and $\varphi_0(x)$ be the elements in (3.13). Let $L = \sqrt{\frac{4\beta}{4(1-\delta-\lambda^R)-c^2}}$ and $\varphi_L(x) = \varphi_0(x/L)$. Then

$$\begin{cases} -\Delta_x \varphi_L = (1-\delta-\lambda^R - \frac{c^2}{4})\varphi_L \quad \text{on } B_L := \{x \in \mathbb{R}^m : |x| < L\}, \\ \varphi_L > 0 \quad \text{in } B_L, \quad \varphi_L = 0 \quad \text{on } \partial B_L, \quad \|\varphi_L\|_{L^\infty(B_L)} = 1. \end{cases}$$

Define

$$w_n = \begin{cases} e^{-c_n(x\cdot e_n + L)/2}\varphi_L(x)\psi^R(y) \quad \text{when } (x,y) \in S_{L,R} = \{(x,y) : |x| < L, |y| < R\}, \\ 0 \quad \text{elsewhere}. \end{cases}$$

It is easy to check that for $(x,y) \in S_{L,R}$, we have

$$-c_n e_n \cdot \nabla_x w_n - \Delta w_n + \alpha g w_n = \left(\frac{c_n^2 - c^2}{4} + 1 - \delta \right) w_n \le (1-\delta) w_n.$$

Since $u(1,\cdot,\cdot)$ is continuous and positive in $\mathbb{R}^{m+N}$, there exists $\eta > 0$ such that

$$u(1,x,y) \ge \eta > 0 \quad \text{for all } |x| \le L + c + 1, |y| \le R + 1.$$

Then

$$v_n(-t_n + 1, x, y) = u(1, x + c_n e_n, y) \ge \eta \quad \text{for all } |x| \le L + 1, |y| \le R + 1. \tag{3.19}$$

Since $\int_{\mathbb{R}^N} v_n(t,x,y)K(y)\,\mathrm{d}y \to 0$ locally uniformly as $n \to \infty$, we define, for $n > J$ (large),

$$t_n^* = \inf\{t \in [-t_n + 1, 0] : 0 \le \int_{\mathbb{R}^N} v_n(s,x,y)K(y)\,\mathrm{d}y \le \delta \text{ in } [t,0] \times \{x : |x| \le L+1\}\}.$$

We may assume $t_n^* < 0$. By continuity, we have

$$0 \le \int_{\mathbb{R}^N} v_n(t,x,y)K(y)\,\mathrm{d}y \le \delta \text{ in } [t_n^*, 0] \times \{x : |x| \le L+1\}, \tag{3.20}$$

and

$$\text{if} \quad t_n^* > -t_n + 1 \quad \text{then} \quad \max_{|x| \le L+1} \int_{\mathbb{R}^N} v_n(t_n^*, x, y)K(y)\,\mathrm{d}y = \delta. \tag{3.21}$$

We claim that there exists some $\rho > 0$ such that

$$\min_{|x| \le L, |y| \le R} v_n(t_n^*, \cdot, \cdot) \ge \rho \quad \text{for all} \quad n > J. \tag{3.22}$$

Let us postpone the proof of this claim, and use it to prove (3.15). By (3.18) and (3.20) we have,

$$\partial_t v_n - \Delta v_n - c_n e_n \cdot \nabla_x v_n + \alpha g v_n \ge (1-\delta)\, v_n \quad \text{in } [t_n^*, 0] \times S_{L,R}.$$

By the comparison principle, we have

$$v_n(t,x,y) \ge \rho w_n(x,y) \quad \text{in } [t_n^*, 0] \times S_{L,R}.$$

Since $|y_0| \le R/2$, we have

$$u(t_n, x_n, y_0) = v_n(0,0,y_0) \ge \rho w_n(0, y_0) = \rho e^{-c_n L/2} \varphi_L(0) \psi^R(y_0) \ge \rho e^{-cL/2} \varphi_L(0) \psi^R(y_0),$$

which is in contradiction with $u(t_n, x_n, y_0) \to 0$ as $n \to \infty$.

So it only remains to show (3.22).

If (3.22) fails, after extraction of a subsequence, there exists a sequence $\{(x_n, y_n)\}$ in $\overline{S}_{L,R}$ such that

$$v_n(t_n^*, x_n, y_n) \to 0 \quad \text{and} \quad (x_n, y_n) \to (\bar{x}, \bar{y}) \in \overline{S}_{L,R}.$$

Define

$$V_n(t,x,y) = v_n(t + t_n^*, x, y) \quad \text{for all } (t,x,y) \in (-t_n - t_n^*, +\infty) \times \mathbb{R}^{m+N},$$

which satisfies (3.18) as well. Notice that $-t_n - t_n^* \le -1$. As before, up to extracting a subsequence, V_n converges locally uniformly to V_∞ which is a bounded solution of

$$\partial_t V_\infty - \Delta V_\infty - c_\infty e_\infty \cdot \nabla_x V_\infty + \alpha g V_\infty = \left(1 - \int_{\mathbb{R}^N} V_\infty(t,x,z) K(z)\, \mathrm{d}z\right) V_\infty$$

in $(-1, +\infty) \times \mathbb{R}^{m+N}$. Moreover,

$$V_\infty(t,x,y) \ge 0 \quad \text{for all } (t,x,y) \in (-1, +\infty) \times \mathbb{R}^{m+N} \quad \text{and } V_\infty(0, \bar{x}, \bar{y}) = 0.$$

By the strong maximum principle, we have $V_\infty \equiv 0$ in $(-1, 0] \times \mathbb{R}^{m+N}$, and consequently, by the comparison principle, $V_\infty \equiv 0$ in $[0, +\infty) \times \mathbb{R}^{m+N}$. Hence V_n converges locally uniformly to 0 in $(-1, +\infty) \times \mathbb{R}^{m+N}$. By Lemma 3.3, $\int_{\mathbb{R}^N} V_n(t,x,y) K(y)\, \mathrm{d}y$ also converges to 0 locally uniformly. Hence $v_n(t_n^*, \cdot, \cdot) \to 0$ and $\int_{\mathbb{R}^N} v_n(t_n^*, \cdot, y) K(y)\, \mathrm{d}y \to 0$ locally uniformly. This contradicts (3.19) and (3.21). The proof is thereby complete. ☐

3.4 Asymptotic speed of propagation to $V(y)$

To complete the proof of Theorem 1.2, it remains to prove the following sharper statement.

Theorem 3.6. *Let $u(t,x,y)$ be as in Theorem 3.5. Assume $0 < \alpha < \bar{\alpha}$. Then, for every $0 \le c < c^*$, we have*

$$\lim_{t \to \infty} \sup_{|x| \le ct} |u(t,x,y) - V(y)| = 0 \quad \textit{uniformly in } y,$$

where V is the unique solution of (2.5) *given by* (2.6).

To prove Theorem 3.6, we will use the same decomposition as in Section 2.2. We know from Lemma 3.3 that u can be written as

$$u(t,x,y) = \sum_{i=0}^{\infty} v_i(t,x) \psi_i(y), \tag{3.23}$$

where ψ_i are those in Lemma 2.2. Then for each i, v_i solves

$$\partial_t v_i - \Delta v_i + \lambda_i v_i = (1 - b(t,x)) v_i, \tag{3.24}$$

where λ_i are those in Lemma 2.2, and

$$b(t,x) := \int_{\mathbb{R}^N} K(z)u(t,x,z)\ \mathrm{d}z.$$

We start with a lemma.

Lemma 3.7. *For each j for which $\lambda_j > 1$ the function $v_j(t,x)$ converges exponentially to 0 as $t \to \infty$, uniformly in x.*

Proof. From the equation (3.24), the function v_j satisfies

$$\partial_t v_i - \Delta v_i + \gamma v_i = 0,$$

where $\gamma(t,x) \geq \gamma_0 > 0$ for all t, x. Lemma 3.3 and the comparison principle yield

$$|v_j(t,x)| \leq Me^{-\gamma_0 t} \tag{3.25}$$

for some constant $M > 0$. □

Lemma 3.8. *Let $J \geq 1$ be an integer such that $\lambda_{J+1} > 1$. Let z_J be defined by*

$$z_J = \sum_{i=J+1}^{\infty} v_i(t,x)\psi_i(y).$$

Then, $z_J(t,x,y)$ converges to 0 as $t \to \infty$, uniformly in x and locally uniformly in y.

Proof. To start with, we have a bound for all $t > 0, x \in \mathbb{R}^m$,

$$\sum_{i=0}^{\infty} v_j^2(t,x) = \int_{\mathbb{R}^N} u^2(t,x,y)\ \mathrm{d}y \leq C.$$

for some constant $C > 0$. Therefore,

$$\|z_J(t,x,\cdot)\|_{L^2(\mathbb{R}^N)} \leq C.$$

We claim that

$$z_J(t,x,\cdot) \rightharpoonup 0 \quad \text{weakly in } L^2(\mathbb{R}^N) \quad \text{as } t \to \infty.$$

Indeed, for every $\varphi \in L^2(\mathbb{R}^N)$, for every $\varepsilon > 0$, there exist an integer $\ell > 0$ and $\mu_i \in \mathbb{R}$ such that

$$\|\varphi - \sum_{i=0}^{\ell} \mu_i\psi_i\|_{L^2(\mathbb{R}^N)} \leq \varepsilon.$$

Therefore,

$$\begin{aligned}
&|\int_{\mathbb{R}^n} z_J(t,x,y)\varphi(y)\ \mathrm{d}y| \\
&\leq \sum_{i=0}^{\ell} |\int_{\mathbb{R}^n} z_J(t,x,y)\mu_i\psi_i\ \mathrm{d}y| + |\int_{\mathbb{R}^n} z_J(t,x,y)(\varphi - \sum_{i=0}^{\ell}\mu_i\psi_i)\ \mathrm{d}y| \\
&\leq \sum_{i=J+1}^{\ell} |\mu_i v_i(t,x)| + C\varepsilon \\
&\leq 2C\varepsilon \quad \text{for all} \quad t \geq T,
\end{aligned}$$

where T is sufficiently large but independent of x, and we used (3.25) in the last inequality. This proves the claim.

Let $R > 0$. Since $z_J = u - \sum_{i=0}^{J} v_i\psi_i$ is Lipschitz continuous uniformly in (t, x) and locally uniformly in y for $t > 1$, for every $\varepsilon > 0$, there exists $r > 0$ such that for all $t > 1, x \in \mathbb{R}^m$, $|y| \leq R$, the oscillation of z_J in the ball $B_r(y)$ satisfies the bound:

$$\operatorname*{osc}_{B_r(y)} z_J(t, x, \cdot) \leq \varepsilon.$$

By the weak convergence, there exists $T = T(y) > 0$ such that for all $x \in \mathbb{R}^m$ and all $t > T$,

$$|\frac{1}{|B_r(y)|}\int_{B_r(y)} z_J(t, x, z)\ \mathrm{d}z| \leq \varepsilon.$$

Consequently, we have for $|y| \leq R$,

$$|z_J(t, x, z)| \leq 2\varepsilon \quad \text{for all}\ \ z \in B_r(y),\ \ \text{all}\ \ t > T = T(y),\ \ \text{and all}\ \ x \in \mathbb{R}^m.$$

Therefore, we can conclude that

$$z_J(t, x, y) \to 0 \quad \text{as}\ t \to \infty \quad \text{uniformly in}\ x\ \text{and locally uniformly in}\ y.$$

□

Therefore, to prove Theorem 3.6, we only have to deal with the finite sum $u - z_J$. We are going to prove that the finite number of functions $v_1(t, x), \ldots, v_J(t, x)$ converge to 0.

Let $w_i := v_i/v_0$. Using the equation (3.24) we derive an equation for w_i:

$$\partial_t w_i - \Delta w_i - \frac{2\nabla v_0}{v_0} \cdot \nabla w_i + (\lambda_i - \lambda_0)w_i = 0. \tag{3.26}$$

Owing to Theorem 3.5, we know that $\liminf_{t\to+\infty} v_0(t, x) > 0$ locally uniformly in x. Actually, we have a stronger information on the limit. Let $0 < \gamma < c^*$. For any $A > 0$, there exists $\rho > 0$ such that:

$$u(t, x, y) \geq \rho, \quad \text{for all} \quad t \geq 1, |x| \leq \gamma t, |y| \leq A. \tag{3.27}$$

Together with (3.4), this implies the existence of $\delta = \delta(\gamma) > 0$ such that

$$v_0(t, x) \geq \delta, \quad \text{for all} \quad t \geq 1, |x| \leq \gamma t. \tag{3.28}$$

Proposition 3.9. *Let $0 \leq c < c^*$. For each $j = 1, \ldots, J$, we have*

$$\lim_{t\to+\infty} \sup_{|x|\leq ct} |v_j(t, x)| = 0.$$

Proof. We argue by contradiction. Suppose there are $0 \leq c < c^*$, $\eta > 0$, and a sequence $\{(t_n, x_n)\}$ with

$$\begin{cases} |x_n| \leq ct_n \quad \text{for all}\ n \in \mathbb{N}, \\ t_n \to \infty \quad \text{and}\ |v_j(t_n, x_n)| \geq \eta \quad \text{as}\ n \to \infty. \end{cases}$$

Choose γ such that $c < \gamma < c^*$. We translate the functions v_0, w_j in time and space to define

$$V_n(t,x) := v_0(t_n + t, x + x_n), \quad \text{and} \quad W_n(t,x) := w_j(t_n + t, x + x_n)$$

for $t \geq -t_n$ and $x \in \mathbb{R}^m$. We also have the equation:

$$\partial_t W_n - \Delta W_n - \frac{2\nabla V_n}{V_n} \cdot \nabla W_n + (\lambda_j - \lambda_0) W_n = 0. \tag{3.29}$$

From (3.28), we observe that V_n is bounded from below by $\delta > 0$ on the larger and larger set $\Omega_n := \{(t,x) : |x| \leq (\gamma - c)t_n + \gamma t\}$ as $n \to \infty$. Therefore, W_n and the coefficient of the gradient term $\frac{2\nabla V_n}{V_n}$ are bounded on Ω_n as $n \to \infty$. By parabolic estimates, up to striking out a subsequence, we obtain the convergence of V_n and W_n locally uniformly to V_∞ and W. The limit functions satisfy an equation defined for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^m$:

$$\partial_t W - \Delta W - \frac{2\nabla V_\infty}{V_\infty} \cdot \nabla W + (\lambda_j - \lambda_0) W = 0, \quad t \in \mathbb{R}, x \in \mathbb{R}^m. \tag{3.30}$$

Furthermore, we know that

$$|W(0,0)| > 0, \quad \text{and} \quad V_\infty(t,x) \geq \delta \quad \forall\, t \in \mathbb{R}, x \in \mathbb{R}^m.$$

Therefore, the equation (3.30) has bounded coefficients. Moreover, W is bounded since V_∞ is bounded from below, and W is a time-global solution (i.e. defined for all t). Denote $M := \sup_{\mathbb{R}\times\mathbb{R}^m} |W(t,x)|$. The function

$$e^{(\lambda_0 - \lambda_j)t}$$

is a solution of (3.30). From the comparison principle applied to (3.30) we get:

$$|W(t,x)| \leq M e^{(\lambda_0 - \lambda_j)(t-\tau)} \quad \text{for all} \quad t \geq \tau.$$

Letting $\tau \to -\infty$, we get $W(t,x) = 0$ for all t and all x. This is in contradiction with the value of W at $(0,0)$. The proof of the proposition is thereby complete. $\square$

Proof of Theorem 3.6. We are going to first derive the following limit:

$$\sup_{|x| \leq ct} |v_0(t,x) - \mu| \to 0 \quad \text{as } t \to \infty, \tag{3.31}$$

where μ is given in (2.7). Letting

$$\tilde{b}(t,x) = -\int_{\mathbb{R}^N} K(z)[u(t,x,z) - v_0(t,x)\psi_0(z)]\,\mathrm{d}z,$$

we have the following equation:

$$\partial_t v_0 - \Delta v_0 = (\tilde{b}(t,x) + 1 - \lambda_0 - \mu^{-1}(1 - \lambda_0) v_0) v_0,$$

The proof of (3.31) is similar to the proof of Proposition 3.9. Suppose there are $0 \leq c < c^*$, $\eta > 0$, and a sequence $\{(t_k, x_k)\}$ such that

$$\begin{cases} |x_k| \leq c t_k \quad \text{for all } n \in \mathbb{N}, \\ t_k \to \infty \quad \text{and } |v_0(t_k, x_k) - \mu| \geq \eta \quad \text{as } n \to \infty. \end{cases}$$

Choose γ such that $c < \gamma < c^*$. From Lemma 3.8, Proposition 3.9, Lemma 3.3 and the dominated convergence theorem, we infer that

$$\lim_{t\to\infty} \sup_{|x|\le\gamma t} \tilde{b}(t,x) = 0.$$

Define

$$V_k(t,x) = v_0(t_k + t, x + x_k), \quad B_k(t,x) = \tilde{b}(t_k + t, x + x_k),$$

which satisfies

$$\partial_t V_k - \Delta V_k = (B_k(t,x) + 1 - \lambda_0 - \mu^{-1}(1-\lambda_0)V_k)V_k.$$

From (3.28), we observe that $V_k \ge \delta$ on the larger and larger set $\{(t,x) : |x| \le (\gamma - c)t_k + \gamma t\}$ as $k \to \infty$. Moreover, $B_k(t,x) \to 0$ locally uniformly as $k \to \infty$. From the upper bound of v_0 we can strike out a subsequence such that V_k converges locally uniformly to a bounded function V_∞, which satisfies $V_\infty \ge \delta$ in $\mathbb{R} \times \mathbb{R}^m$, and

$$\partial_t V_\infty - \Delta V_\infty = (1 - \lambda_0 - \mu^{-1}(1-\lambda_0)V_\infty)V_\infty \quad \text{for all } (t,x) \in \mathbb{R} \times \mathbb{R}^m. \tag{3.32}$$

We claim that

$$V_\infty \equiv \mu \quad \text{in } \mathbb{R} \times \mathbb{R}^m.$$

The proof of the claim is as follows. Let $M = \sup_{\mathbb{R}\times\mathbb{R}^m} V_\infty$ and $(t_n, x_n) \in \mathbb{R} \times \mathbb{R}^m$ be such that $V_\infty(t_n, x_n) \to M$ as $n \to \infty$. Let $W_n(t,x) = V_\infty(t + t_n, x + x_n)$. Then subject to a subsequence, W_k converges locally uniformly to W_∞, which also satisfies (3.32). Moreover, $W_\infty(0,0) = M = \sup_{\mathbb{R}\times\mathbb{R}^m} W_\infty$. By evaluating at $(0,0)$, we have $M \le \mu$, i.e., $\sup_{\mathbb{R}\times\mathbb{R}^m} V_\infty \le \mu$. Similarly, one can show that $\mu \le \inf_{\mathbb{R}\times\mathbb{R}^m} V_\infty$. Therefore, $V_\infty \equiv \mu$.

Hence, $v_0(t_k, x_k) = V_k(0,0) \to \mu$, which is a contradiction. This proves (3.31).

Once we have the limit (3.31), Lemma 3.8 and Proposition 3.9 yield

$$\lim_{t\to\infty} \sup_{|x|\le ct} |u(t,x,y) - V(y)| = 0 \quad \text{locally uniformly in} \quad y.$$

Since both u and V are uniformly exponentially decaying as $|y| \to \infty$, we can conclude that the above convergence is uniform in y. □

The proof of Theorem 1.2 is thereby complete. □

A Local maximum principle

In this appendix, we provide a short proof of the local maximum principle for heat equations, which was used in the proof of Lemma 3.2. The following statement and its proof are well-known, and we include them here for the purpose of completeness.

Lemma A.1. *Let $u \in C^{2,1}_{x,t}(Q_1)$ be a nonnegative solution of*

$$u_t - \Delta u + c(t,x)u \le 0 \quad \textit{in } Q_1 := (0,1] \times B_1,$$

where $c(t,x)$ is a bounded function in Q_1. Then there exists a positive constant C depending only on n and $\|c^-\|_{L^\infty(Q_1)}$ such that

$$u(t,x) \leq C \int_{Q_1} u \quad \textit{for all } (t,x) \in [15/16, 1] \times B_{1/4}.$$

Proof. Let $c_0 = \|c^-\|_{L^\infty(Q_1)}$ and $\tilde{u} = e^{-c_0 t} u$. Then

$$\tilde{u}_t - \Delta \tilde{u} \leq e^{-c_0 t}(-c_0 - c(t,x))u \leq 0 \quad \text{in } Q_1.$$

Let $\eta(t,x)$ be a smooth nonnegative cut-off function such that $\eta \equiv 1$ in $[3/4, 1] \times B_{1/2}$, $\eta \equiv 0$ in $([1/2, 1] \times B_{3/4})^c$ and $0 \leq \eta \leq 1$. Let $v = \eta \tilde{u}$. Then it satisfies that

$$v_t - \Delta v \leq (\eta_t - \Delta \eta)\tilde{u} - 2\nabla \eta \nabla \tilde{u} =: f(t,x).$$

Let

$$G(t,x) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$$

be the heat kernel. Therefore, we have

$$v(t,x) \leq \int_0^t \int_{\mathbb{R}^n} G(t-s, x-y) f(s,y) \, \mathrm{d}y \, \mathrm{d}s.$$

Then the conclusion follows from integration by parts and the observation that $f \equiv 0$ in $[3/4, 1] \times B_{1/2} \cup ([1/2, 1] \times B_{3/4})^c$. □

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