Step 2: To establish the validity of (4), note that if $n \geq 1$ and $k \geq 1$ then

$$\alpha(n, k) > \alpha(n-1, k). \tag{8}$$

In other words, the number of coefficients in a polynomial of degree n is greater than the number of coefficients in a polynomial of degree $n-1$. The condition $k \geq 1$ simply requires that the polynomials have at least one variable whereas the condition $n \geq 1$ is required so that the right side of (8) makes sense.

Any polynomial $p_n(\boldsymbol{x}, y)$ of degree n in the $k+1$ variables $(\boldsymbol{x}, y)$ may always be written uniquely as

$$p_n(\boldsymbol{x}, y) = q_n(\boldsymbol{x}) + q_{n-1}(\boldsymbol{x})\, y + q_{n-2}(\boldsymbol{x})\, y^2 + \cdots + q_0(\boldsymbol{x})\, y^n \tag{9}$$

where $q_i(\boldsymbol{x})$ is a polynomial of degree at most i. Therefore,

$$\alpha(n, k+1) = \alpha(n, k) + \alpha(n-1, k) + \cdots + \alpha(1, k) + 1 \tag{10}$$

or alternatively

$$\alpha(n, k+1) = [\alpha(n, k) + \cdots \alpha(l+1, k)] + \alpha(l, k+1). \tag{11}$$

Using (8), it follows from (11) that

$$\alpha(n, k+1) > [\alpha(n-l, k) + \cdots + \alpha(1, k)] + \alpha(l, k+1). \tag{12}$$

However, (10) implies that the term in brackets is equal to $\alpha(n-l, k+1) - 1$. Therefore,

$$\alpha(n, k+1) > \alpha(n-l, k+1) + \alpha(l, k+1) - 1. \tag{13}$$

Thus, (4) is true under the hypothesis of the theorem.

Step 3: Any reducible polynomial in $\mathcal{P}(n, k)$ is always contained within the finite union of the sets $\mathcal{P}_0, \mathcal{P}_c$, and $\mathcal{B}(l)$ for $l = 1, 2, \cdots, n-1$. Since each of these sets has measure zero in $R^{\alpha(n,k)}$, then does the set $\mathcal{B}$ and Theorem 2 follows.

III. Discussion

The result presented in Theorem 2 may be easily extended to other classes of polynomials. For example, it is straightforward to modify the proof of the theorem to show that the set of all reducible polynomials of degree n in k variables with complex coefficients corresponds to a set of measure zero in $R^{2\alpha(n,k)}$ provided $k > 1$ and $n > 1$.

Another class of polynomials which is often encountered consists of those which have a given degree in each variable. Specifically, let $\mathcal{Q}(\boldsymbol{n}, k)$ be the set of all polynomials which have degree n_i in x_i for $i = 1, \cdots, k$. A polynomial $p_{\boldsymbol{n}}(\boldsymbol{x})$ in $\mathcal{Q}(\boldsymbol{n}, k)$ is therefore of the form

$$p_{\boldsymbol{n}}(\boldsymbol{x}) = \sum_{l_1=0}^{n_1} \cdots \sum_{l_k=0}^{n_k} c(l_1, \cdots, l_k)\, x_1^{l_1} x_2^{l_2} \cdots x_k^{l_k}. \tag{14}$$

With $\beta(\boldsymbol{n}, k) = (n_1+1)(n_2+1)\cdots(n_k+1)$ the number of coefficients required to specify the polynomial $p_{\boldsymbol{n}}(\boldsymbol{x})$, it is easily shown that

$$\beta(\boldsymbol{l}, k) + \beta(\boldsymbol{n} - \boldsymbol{l}, k) \leq \beta(\boldsymbol{n}, k) \tag{15}$$

provided $k > 1$. Therefore, it follows in a style similar to that in the proof of Theorem 2 that the set of all reducible polynomials in $\mathcal{Q}(\boldsymbol{n}, k)$ corresponds to a set of measure zero in $R^{\beta(\boldsymbol{n},k)}$.

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On the Construction of a Digital Transfer Function from Its Real Part on Unit Circle

P. P. VAIDYANATHAN AND S. K. MITRA

***Abstract*—It is shown in this correspondence that the system function $H(z)$ of a linear time invariant (LTI) causal digital filter with real impulse response coefficients can be obtained from the real part of its frequency response $H_R(e^{j\omega})$ given in the form of a rational trigonomentric function, using algebraic methods rather than complex contour integration techniques.**

I. Introduction

For a causal linear time invariant (LTI) continuous time system with real impulse response, the transfer function $H_a(s)$ or any other network function (like the driving point impedance) can be calculated from a component of the function such as real part of the function or from the magnitude or phase response of the function [1]. Even though relations exist in the form of integrals (like the Hilbert Transform [2]), there are also available more convenient algebraic methods for this purpose [3]. For discrete time systems such as digital filters, one can still find relationships involving complex contour integrals, such as discussed in [4]. But there does not seem to be available an algebraic method, as for continuous time systems. In this correspondence, we develop an algebraic procedure for retrieving the system function $H(z)$ of a causal, stable LTI digital filter from the real part of its frequency response $H_R(e^{j\omega})$, specified as a ratio of two trigonomentric functions.

Consider a causal, stable LTI system described by

$$H(z) = A(z)/B(z) \tag{1}$$

with real impulse response sequence. Let the real part of the system function $H(z)$, evaluated on the unit circle be given by

$$H_R(e^{j\omega}) = N(e^{j\omega})/D(e^{j\omega}) \tag{2}$$

where $N(e^{j\omega})$ and $D(e^{j\omega})$ are trigonomentric functions of the radian frequency ω. It is shown in [4] that

$$H(z) = \frac{1}{2\pi j} \oint H_R(v) \frac{(z+v)\, dv}{(z-v)\, v}, \qquad |z| > 1 \tag{3}$$

Thus $H(z)$ can be evaluated everywhere in its region of convergence $|z| > 1$, using (3), where the contour of integration is the unit circle.

In what follows, we develop an algebraic method for calculation of $H(z)$ for $|z| > 1$ from $H_R(e^{j\omega})$. The process involves two steps: the first one is to identify the poles of $H(z)$; second is to identify the numerator polynomial and determine the scale factor.

II. Calculation of Poles of $H(z)$

We note that

$$H_R(e^{j\omega}) = \tfrac{1}{2}\, [H(z) + H(z^{-1})]\big|_{z=e^{j\omega}} \tag{4}$$

and hence

$$D(e^{j\omega}) = B(z)\, B(z^{-1})\big|_{z=e^{j\omega}} \tag{5}$$

as the coefficients of $B(z)$ are real. Thus $D(z)$ has zeroes occurring in mirror image pairs, and the zeros that fall inside the unit circle are precisely the poles of $H(z)$ from stability and causality requirements. As $H_R(e^{j\omega})$ is usually given in terms of sine and cosine functions, we make the following substitution in $D(e^{j\omega})$ in order to get $B(z)\, B(z^{-1})$:

$$\cos(n\omega) \leftarrow \frac{z^n + z^{-n}}{2} \qquad \sin(n\omega) \leftarrow \frac{z^n - z^{-n}}{2j}. \tag{6}$$

Once $D(z) = B(z)\, B(z^{-1})$ is thus obtained, we solve for its roots and take those roots inside the unit circle as the poles of $H(z)$. Thus $B(z)$ in

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The authors are with the Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106.

(1) is determined to within a scale factor. Note that (6) is analogous to the analytic continuation methods for continuous time systems, where ω is replaced by s/j to get $H_a(s)$ from $H_a(j\omega)$.

III. Calculation of the Numerator Polynomial $A(z)$ of $H(z)$

An examination of well-known methods adopted for a similar problem in continuous time systems leads us to look for a decomposition of $A(z)$ and $B(z)$

$$A(z) = m_1(z) + n_1(z); \qquad B(z) = m_2(z) + n_2(z) \tag{7a}$$

so that

$$H(z) = \frac{m_1 + n_1}{m_2 + n_2} \tag{7b}$$

in such a way that

$$m_i(z) = m_i(z^{-1}) \qquad n_i(z) = -n_i(z^{-1}), \qquad i = 1, 2. \tag{8}$$

The motivation behind such a decomposition is that, in view of (4), $H_R(e^{j\omega})$ becomes

$$H_R(e^{j\omega}) = \left.\frac{m_1m_2 - n_1n_2}{m_2^2 - n_2^2}\right|_{z=e^{j\omega}}. \tag{9}$$

Now, comparing the form (7) with (2), we find that

$$k(m_1m_2 - n_1n_2) = N(z) \tag{10}$$

(where k is a suitable scale factor whose determination is trivial, as shown in the example later).

Thus given $H_R(e^{j\omega})$, if we make the substitution (6) in $N(e^{j\omega})$ we get a function of z that can be equated to $m_1(z)m_2(z) - n_1(z)n_2(z)$ to within a scale factor. Now, since the poles of $H(z)$ are already found, m_2 and n_2 are known and therefore, by equating like powers of z in (8) we can determine $A(z) = m_1(z) + n_1(z)$.

A suitable way of defining $m_i(z)$ and $n_i(z)$ so that (10) is satisfied is:

$$m_1(z) = \frac{A(z) + A(z^{-1})}{2} \qquad n_1(z) = \frac{A(z) - A(z^{-1})}{2}$$

$$m_2(z) = \frac{B(z) + B(z^{-1})}{2} \qquad n_2(z) = \frac{B(z) - B(z^{-1})}{2} \tag{11}$$

Once we find $m_1 + n_1$ in the above manner, $H(z)$ is determined to within a constant factor. Since $h(n)$ is real, $H(z)$ evaluated at $z = 1$ is purely real and must equal $H_R(e^{j\omega})$ as $\omega = 0$ and this fact is made use of to determine the constant factor.

IV. An Example

Let

$$H_R(e^{j\omega}) = \frac{1 + \cos\omega + \cos 2\omega}{17 - 8\cos 2\omega} = \frac{N(e^{j\omega})}{D(e^{j\omega})}.$$

1) $D(z) = 17 - 8(z^2 + z^{-2})/2 = 17 - 4z^2 - 4z^{-2}$ and $D(z)$ has zeros at $z = 1/2, 2, -1/2, -2$. Thus we take $B(z) = 1 - \frac{1}{4}z^{-2}$ to be the denominator of $H(z)$. From here, we find

$$m_2(z) = -\tfrac{1}{8}z^2 + 1 - \tfrac{1}{8}z^{-2} \qquad n_2(z) = \tfrac{1}{8}z^2 - \tfrac{1}{8}z^{-2}.$$

2) Next, we find $N(z)$ from $N(e^{j\omega}) = 1 + \cos\omega + \cos 2\omega$ by substituting (6)

$$N(z) = 1 + \frac{z^2}{2} + \frac{z}{2} + \frac{z^{-1}}{2} + \frac{z^{-2}}{2}.$$

3) We assume $A(z) = a_0 + a_1z^{-1} + a_2z^{-2}$ since 2ω is the largest multiple of ω appearing. Then we write $m_1(z)$ and $n_1(z)$ as

$$m_1(z) = \frac{a_2}{2}(z^2 + z^{-2}) + \frac{a_1}{2}(z + z^{-1}) + a_0$$

$$n_1(z) = \frac{a_2}{2}(z^{-2} - z^2) + \frac{a_1}{2}(z^{-1} - z)$$

and find $m_1m_2 - n_1n_2$ to be

$$m_1m_2 - n_1n_2 = \left(\frac{a_2}{2} - \frac{a_0}{8}\right)(z^2 + z^{-2}) + a_1\left(\frac{7}{16} - \frac{1}{16}\right)(z + z^{-1}) + \left(a_0 - \frac{a_2}{4}\right).$$

Then we equate this to $N(z)$ to get a_0, a_1, and a_2 yielding $H(z) = (1 + z^{-1} + z^{-2})/12(1 - \frac{1}{4}z^{-2})$ where the scale factor is determined as described earlier.

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Circuit Diagnosis and Thevenin Equivalents

P. E. GRAY

***Abstract*—Roytman and Swamy have used the properties of orthonormal excitations to establish one method of circuit diagnosis. In this letter the method is extended to determine the driving-point or Thevenin equivalent impedance for each node pair.**

I. Introduction

The Roytman and Swamy [1] method of circuit diagnosis is concerned with the general, linear, time-invariant network of known topology exhibiting $(n + 1)$ nodes. Using orthonormal excitations this method results in the voltage measurement which become the coefficients of the inverse of the node admittance matrix $[Y]^{-1}$. The method proceeds to compute matrix $[Y]$ from which the values of individual circuit elements are calculated. Adaptations of the method to test-fixture development shows the research to be eminently practical.

The following discussion briefly summarizes the method and shows the resulting coefficients of $[U_n^{(k)}]$ to be related to the driving-point or Thevenin equivalent impedance at each node pair.

II. Calculation of Equivalent Impedance

The Roytman-Swamy method can be summarized

$$[U_n^{(k)}] = [Y]^{-1}\,[I] \tag{1}$$

where

- $[Y]^{-1}$ is the inverse of the node admittance matrix.
- $[I]$ is the unity matrix of orthonormal current excitations.
- $[U_n^{(k)}]$ is the matrix of node voltages of the kth node when only the nth node is excited by unity current.

$[Y]$ is then calculated and the values of the circuit elements determined.

Extending the method to calculate equivalent impedance at any node pair (p,m) requires consideration of the coefficients of $[U_n^{(k)}]$. That is when m is the reference node the corresponding equivalent impedance at node pair $(p, n + 1)$ is $U_p^{(p)}\ \Omega$. When m is not the reference node the corresponding equivalent impedance becomes

$$U_p^{(p)} - U_m^{(p)} + U_m^{(m)} - U_p^{(m)}. \tag{2}$$

This relationship can be derived using the Scott [2] concept that a

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The author is with the Electrical Engineering Department, University of Wisconsin, Platteville, WI 53818.

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