PHYSICAL REVIEW B VOLUME 40, NUMBER 1 1 JULY 1989

Dimensionality expansion for the dirty-boson problem

Peter B. Weichman and Kihong Kim
Condensed Matter Physics 114-36, California Institute of Technology, Pasadena, California 91125
(Received 19 December 1988)

We apply the double-dimensionality expansion of Dorogovstev to derive renormalization-group recursion relations for a Bose fluid in a random external potential. We find a nontrivial zero-temperature fixed point. The onset of mean-field behavior for dimensions $d > d_c = 4$ is unconventional, yielding discontinuous exponents, consistent with previous scaling arguments. Including positive temperatures, we give a clear picture of various crossover regimes, depending on the strength of the disorder and the boson density. This answers previous questions about the behavior of ^{4}He adsorbed in Vycor glass.

Over the past decade, enormous effort has been invested in trying to understand random-electron systems at zero temperature. In contrast, until recently, the corresponding boson problem has remained essentially unaddressed. This is in spite of the many experimental realizations of such systems, for example, ^{4}He adsorbed in various porous random media.[1,2] Of particular interest is the nature of the insulator to superfluid *onset* transition as the boson density n is increased through some critical density n_c at $T=0$, and also how this onset transition affects the finite temperature λ transitions at fixed densities $n > n_c$.

The system that has received the most attention is ^{4}He adsorbed in porous Vycor glass,[1,3,4] ironically, for the reason that it primarily displays the behavior characteristic of a pure nonrandom Bose fluid.[3] In fact, for very low coverages, $n - n_c \ll a^{-3}$, where a is the range of interactions, a crossover to *ideal* Bose-gas critical behavior is observed.

The reasons for the apparent invisibility of disorder in Vycor were explained qualitatively in Ref. 4 on the basis of scaling arguments, and the process of spinodal decomposition by which Vycor is made. However, a true quantitative understanding of the nature of the onset transition was still lacking. In this Rapid Communication we fill this gap by analyzing a model of bosons in a random external potential using the double-dimensionality expansion of Dorogovstev,[5,6] deriving lowest-order renormalization-group recursion relations. The resulting fixed-point structure clearly elucidates the relation between onset at $T=0$ and scaling near T_λ. In particular, for very weak disorder, there is a *range* of coverages over which the pure crossover to ideal gas behavior should be observed. Only at very low coverages is the $T=0$ disorder-dominated onset regime encountered, and should deviations from pure behavior become visible. The Vycor experiments[1] have probably not yet entered this regime.

Once inside the random onset region, various predictions can be made.[7-9] For example, the temperature can be treated within a finite-size scaling formalism, and this allows the prediction of various exponents, such as that which gives T_λ as a function of $n - n_c$. The scaling forms also predict universal *shapes* for constant density profiles when properly normalized and plotted versus T/T_λ. The lack of universal shape in the Vycor data is further evidence that random onset has not yet been observed.

The work of Refs. 7 and 8 has come a long way toward understanding the nature of the zero-temperature onset transition. What is still lacking is a quantitative understanding of the transition in higher dimensions. In particular, one would like to have a dimensionality expansion, analogous to the ϵ expansion for classical spin systems, about the upper critical dimension d_c above which mean-field theory is valid.

We will apply the Dorogovstev ideas[5,6] to the coherent state path integral formulation of the Bose-gas Hamiltonian.[10] When dealing with quenched disorder it is convenient to use the well-known replica trick[11] to derive an effective Lagrangian in which the random external potential has been integrated out. The final form with which we work is

$$L_{\rm eff}^{(p)} = \sum_{i=1}^{p} \int d^d r \int_0^\beta d\tau [\psi_i^*(\mathbf{r},\tau)\partial\psi_i(\mathbf{r},\tau)/\partial\tau + |\nabla\psi_i(\mathbf{r},\tau)|^2 + r|\psi_i(\mathbf{r},\tau)|^2 + v|\psi_i(\mathbf{r},\tau)|^4] - \sum_{i,j=1}^{p} \int d^d r \int_0^\beta d\tau \int_0^\beta d\tau' g |\psi_i(\mathbf{r},\tau)|^2 |\psi_j(\mathbf{r},\tau')|^2, \quad (1)$$

where $2v$ is the on-site soft-core repulsion, $-r=\mu$ is the chemical potential, $\beta=(k_B T)^{-1}$, and the randomness has been taken as Gaussian, δ-function correlated, with amplitude $2g$. Units have been chosen so that $\hbar = 2m = 1$, and an underlying spatial lattice with spacing $a_0 \approx a$ is assumed. Equivalently, a momentum space cutoff $k_\Lambda \sim \pi/a_0$ is imposed. The classical complex field $\psi_i(\mathbf{r},\tau)$ replaces the usual Bose-field operator, and the quantum-mechanical nature of the system is embodied in the extra imaginary time variable, τ. The *linear* time derivative $\psi_i^* \partial\psi_i/\partial\tau$ is characteristic of the Bose fluid.[12] The indices i,j label the p identical replicas, with the formal limit $p \rightarrow 0$ to be taken at the end. For ease of later comparison to O(n) spin models, it is convenient to generalize ψ_i to an m-component complex vector, with helium corresponding to $m=1$. One expects the correspondence $n=2m$.

What makes $L_{\rm eff}^{(p)}$ more difficult to analyze than the more standard classical spin models[11] is the fact that the interreplica coupling, proportional to g in Eq. (1), although local in space, is infinite ranged in time. Boyanovsky and Cardy,[6] extending earlier work of Dorogovstev,[5] have solved this problem for the spin-$\frac{1}{2}$ Ising version[12] of Eq. (1). They used field-theoretic techniques to generate a double expansion in the variables ϵ_d, the number of "temporal" dimensions along which the interreplica coupling has infinite range, and $\epsilon=4-D$, where $D=d+\epsilon_d$ is the total dimensionality. The actual physical situation corresponds to $\epsilon_d=1$. In this Rapid Communication we will carry out the analogous calculation to first order in ϵ and ϵ_d for the Bose gas using standard momentum shell renormalization-group techniques.

The first step in the momentum shell renormalization-group analysis involves integrating out the components of the fields ψ_i with wave numbers $\mathbf{k}$ in a shell $k_\Lambda/b<k<k_\Lambda$, and frequencies ω in a shell $\omega_\Lambda/b^z<\omega<\omega_\Lambda$, where $b=e^l>1$ is the rescaling parameter, eventually to be taken infinitesimally close to 1, and z is the dynamical exponent, to be specified below. The assumption of a frequency cutoff ω_Λ (or temporal lattice spacing $\tau_0\sim\pi/\omega_\Lambda$) seems difficult to justify, discrete time being a somewhat unnatural concept, but we have found no way of obtaining sensible answers without it. However, to order ϵ and ϵ_d, the fixed-point properties are independent of ω_Λ (and of k_Λ), and in the pure, nonrandom case where one can, in fact, calculate sensibly with $\omega_\Lambda=\infty$, the answers obtained are the same as for finite ω_Λ. The underlying problem seems to be in properly accounting for the analyticity properties of the boson Green's functions: Answers depend on whether frequency contour integrals are closed in the upper or lower half plane. In the more standard cases in which $L_{\rm eff}^{(p)}$ is even in ω, the upper and lower half planes are identical, and these problems do not arise. We can only speculate that a more careful calculation, to higher order in ϵ and ϵ_d, may provide some mechanism for an effective frequency cutoff, thereby eliminating the need for putting it in by hand.

The question arises at this point of how to analytically continue the frequency sums away from $\epsilon_d=1$. We do this simply by replacing the term $\psi^*\partial\psi/\partial\tau$ by $\sum_{\mu=1}^{\epsilon_d}\psi^*\partial\psi/\partial\tau_\mu$, but with the further restriction that ψ contain only those frequency components for which $\omega_\mu\geq 0$ for all μ or $\omega_\mu\leq 0$ for all μ.[13] To first order in ϵ_d it is sufficient to evaluate all frequency sums at $\epsilon_d=0$ so that only $\omega=0$ contributes. Hence, the lowest-order recursion relations require the existence of an analytic continuation, but are completely insensitive to its form. This, presumably, is why numerical values for the exponents obtained to $O(\epsilon_d)$ converge so poorly at $\epsilon_d=1$.

The second step in the renormalization process involves rescaling frequency and momentum to restore the original cutoffs k_Λ and ω_Λ, and rescaling the fields ψ_i to preserve the coefficients of $|\nabla\psi_i|^2$ and $\psi_i^*\partial\psi_i/\partial\tau$ in (1). This determines the choice of z, and ensures the existence of a well-defined fixed-point Lagrangian. When the resulting renormalized Lagrangian is restored to the original form (1), but with renormalized values of r, v, and g (we now take $\beta=\infty$), the resulting recursion relations, to lowest nontrivial order in ϵ and ϵ_d, are

$$d\tilde{r}/dl=2\tilde{r}+2(m+1)\tilde{v}/(1+\tilde{r})-2\tilde{g}/(1+\tilde{r})+O(\tilde{v}^2,\tilde{g}^2,\tilde{g}\tilde{v}), \tag{2a}$$

$$d\tilde{v}/dl=(\epsilon-\epsilon_d)\tilde{v}-2(m+4)\tilde{v}^2+12\tilde{v}\tilde{g}+O(\tilde{v}^3,\dots), \tag{2b}$$

$$d\tilde{g}/dl=(\epsilon+\epsilon_d)\tilde{g}+8\tilde{g}^2-4(m+1)\tilde{g}\tilde{v}+O(\tilde{v}^3,\dots), \tag{2c}$$

$$z=2+2\tilde{g}+O(\tilde{v}^2,\dots), \tag{2d}$$

where $\tilde{r}=r/k_\Lambda^2$, $\tilde{v}=K_dv$, $\tilde{g}=K_dg$, and $K_d=2/(4\pi)^{d/2}\Gamma(\frac{1}{2}d)$ is $(2\pi)^{-d}$ times the area of the unit sphere in d dimensions. These equations yield a nontrivial random fixed point, R_0, at $\tilde{v}^*=(\epsilon+5\epsilon_d)/4(2m-1)$, $\tilde{g}^*=[(2-m)\epsilon+3(m+2)\epsilon_d]/8(2m-1)$, and $\tilde{r}^*=-[3m\epsilon+(7m+4)\epsilon_d]/8(2m-1)$ with $z=2+2\tilde{g}^*$ and correlation-length exponent $\nu^{-1}=2-[3m\epsilon+7(m+4)\epsilon_d]/4(2m-1)$. As in Ref. 5, the eigenvalues associated with small deviations of $\tilde{v}$ and $\tilde{g}$ from their fixed-point values are complex, with negative real part, and hence, can give rise to oscillatory corrections to scaling. The crucial term, without which the fixed point would not exist, is the $-4(m+1)\tilde{g}\tilde{v}$ term in (2c). It is this term which disappears when $\omega_\Lambda=\infty$.

In Fig. 1 we plot the flows in the critical hyperspace, defined by (2b) and (2c). For $d<4(\epsilon+\epsilon_d>0)$, the Gaussian fixed-point G_0 at $\tilde{g}=\tilde{v}=0$, is unstable to the random fixed point, and the flows are plotted in Fig. 1(a). For $d\gtrsim 4(\epsilon+\epsilon_d\lesssim 0)$ *both* fixed points are stable, and a separatrix S divides the basins of attraction. This is shown in Fig. 1(b). Which of the two that governs the critical behavior depends on the strength of the randomness. As d increases further, the separatrix moves upwards, and for sufficiently large d it intersects the random fixed point which then becomes unstable. In all cases (except $\epsilon=\epsilon_d=0$) there are two distinct fixed points with different eigenvalues.

We see then that the transition to mean-field theory as d increases through $d_c=4$ is quite unconventional. The exponents change *discontinuously* to their mean-field values, whereas the conventional mechanism, involving the coalescence of the two fixed points, yields *continuous* exponents. In fact, it can be seen on very general grounds that precisely this kind of behavior must occur. A recent theorem of Chayes *et al.*[14] states that in any d-dimensional system with spatially uncorrelated disorder one has the bound $\nu\geq 2/d$. The mean-field value is $\nu_{\rm MF}=\frac{1}{2}$, which immediately implies $d_c\geq 4$. However, one also has the generalized Josephson hyperscaling relation[7,8] $\zeta=(d+z-2)\nu$ for the superfluid density exponent ζ, valid for $d<d_c$. The mean-field value is $\zeta_{\rm MF}=1$. However, if $d_c\geq 4$ and $z>0$ (in fact, scaling arguments yield $z=d$),[7,8] this relation implies $\zeta>1$ for d approaching d_c from below, i.e., a discontinuous change in ζ must occur as d passes through d_c. For the metal-insulator transition in noninteracting Fermi systems, this problem is apparently avoided by having $d_c=\infty$.[15] For classical systems, one has $z=0$ so that one can have $\nu=\frac{1}{2}$ and $\zeta=1$ in $d_c=4$ without contradiction.

The present method gives the dynamical exponent z as a nontrivial expansion in ϵ and ϵ_d. As mentioned, various

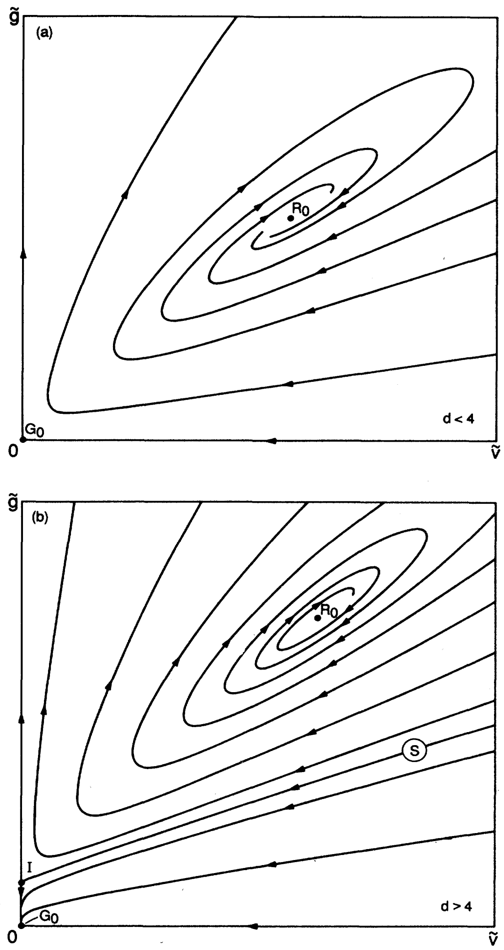

FIG. 1. Renormalization-group flows in the $T=0$ critical hyperplane for (a) $d<4(\epsilon/\epsilon_d=-0.5)$ and (b) $d>4(\epsilon/\epsilon_d=-2)$. The onset of mean-field theory occurs by way of a separatrix S, which separates the basins of attraction for the zero-temperature Gaussian G_0, and random R_0 fixed points.

scaling arguments predict the exact equality $z=d$ for $\epsilon_d=1$.[7,8] It is not at all clear how such a precise resummation of the series might come about. A better understanding would be obtained if the scaling arguments could be carried out for general ϵ_d; however, they seem rather special to $\epsilon_d=1$, and we have found no way to generalize them.

To end, we discuss the behavior for $T>0$. For $\beta<\infty$, a gap $2\pi/\beta$, opens up between the Matsubara frequencies. It is therefore convenient to change the renormalization procedure and integrate out only the momentum space shell $k_\Lambda/b<k<k_\Lambda$, allowing the frequency cutoff to renormalize. This circumvents the problem of counting the number of discrete frequencies in a thin shell, especially in the classical regime where the gap between frequencies is very large. The flow equation for β becomes

$$d\beta/dl=-z\beta\,, \tag{3}$$

which therefore flows toward $\beta=0$. The general β recursion relations are rather messy, and we will not display them here. However, the onset of classical behavior as $\beta\to 0$ is of interest. Various technical problems arise, and it becomes necessary to change the rescaling procedure for small β in order to obtain a well-defined fixed point.[16] Thus, when β reaches 1 we keep it fixed and allow the coefficient of $\psi^*\partial\psi/\partial\tau$ to vary instead, yielding the plausible classical result $z=0$. This coefficient then diverges as $l\to\infty$ and suppresses all but the $\omega=0$ Matsubara frequency. The recursion relations, in this limit, reduce precisely to the usual ϵ_d-independent classical spin random-bond recursion relations, with the identification $n=2m$.[11] These same recursion relations can be derived simply by setting $\epsilon_d=0$ in Eq. (2). For $m<2$ they possess an $O(\tilde\epsilon=4-d)$ random fixed point R at $\tilde v^*=\tilde\epsilon/4(2m-1)$ and $\tilde g^*=(2-m)\tilde\epsilon/8(2m-1)$. In fact, for $d=3(\tilde\epsilon=1)$ and $n=2$, the best estimates yield a negative specific-heat exponent $\alpha<0$, and hence, by the Harris criterion[11] randomness should be irrelevant for $m\gtrsim 1$. The $O(\tilde\epsilon)$ results therefore give a misleading picture of the flows for $m=1$. Qualitatively correct flows can be obtained by taking $m\gtrsim 2$ in the $O(\tilde\epsilon)$ recursion relations, which yields only a pure fixed point [at $\tilde g^*=0$ and $\tilde v^*=\tilde\epsilon/2(m+4)$] which is stable against disorder. Of course, *quantitative* estimates (for exponents, etc.) can only be obtained by going to higher order in $\tilde\epsilon$.

In Fig. 2 we show a schematic plot of the flows in the three-dimensional critical hyperspace defined by the variables $\tilde v$, $\tilde g$, and T. The picture is only schematic due to the different renormalization procedures used in different regions of parameter space. On this diagram various possible behaviors are shown, depending on the relative sizes of the starting parameters $\tilde v_0$, $\tilde g_0$, and T_λ. Since the pictured flows lie in the critical hyperspace, T_λ is in fact the physi-

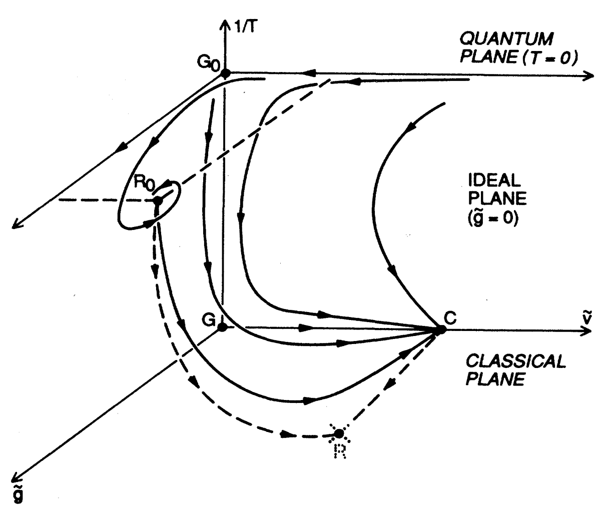

FIG. 2. Schematic plot of renormalization-group flows for $d<4$, including finite temperatures. Nonrandom flows remain in the $\tilde g=0$ plane, and crossover to ideal gas behavior involves flows that pass close to G_0, G, and finally C. The random onset regime involves flows which pass close to R_0 before collapsing into the classical plane. Depending on the sign of the specific-heat exponent, α, a classical random fixed point exists ($\alpha>0$, dashed line) or does not exist ($\alpha<0$, solid lines). The latter case probably holds for helium.

cal transition temperature, while $\tilde{v}_0$ and $\tilde{g}_0$ are fixed by the atomic properties of ^{4}He and by the random medium, respectively.

Suppose $\tilde{g}_0=0$: Then one is in the pure limit, a case that has been treated in great detail elsewhere.[3] When $T_\lambda^{(d-2)/2}\tilde{v}_0 \ll 1$, one explores the crossover to ideal gas behavior. This is described by flows which, due to the smallness of T_λ, closely approach the $T=0$ Gaussian fixed point G_0 before collapsing down into the neighborhood of the classical Gaussian fixed point G, from which they cross over to the critical fixed point C.[3,17] Now suppose that there exists a range of T_λ for which $T_\lambda^{(d-2)/2}\tilde{v}_0 \ll 1 \ll T_\lambda^{(4-d)/2}/\tilde{g}_0$.[4] Within this range of T_λ, the onset fixed point R_0 will play essentially no role. The flows will be dominated by pure crossover, being pulled down into the classical plane where randomness is irrelevant before R_0 has a chance to act. This range of T_λ corresponds to the range of coverages, alluded to in the Introduction, over which pure weakly interacting Bose-gas behavior is seen.[3] Only for $T_\lambda^{(4-d)/2}/\tilde{g}_0 \lesssim 1$ do the flows spend enough time near the $T=0$ plane to be attracted towards R_0. For $T_\lambda^{(4-d)/2}/\tilde{g}_0 \ll 1$ the flows are dominated completely by a direct crossover from R_0 to C. The scaling form that results from this crossover (essentially finite-size scaling in $1/T$) predicts, as mentioned earlier, constant density profiles with a universal shape.[7,8] This shape is determined by the asymptotic trajectory that connects R_0 to C, and could, in principle, be calculated within the ϵ,ϵ_d formalism. Although the Vycor data[1] does not enter this true region of random onset, this regime should be much more accessible in materials that are more strongly random.[18]

We are indebted to M. C. Cross, D. S. Fisher, and M. P. A. Fisher for useful discussions. Financial support from the Weingart Foundation through Caltech and the National Science Foundation through Grant No. DMR-8412543 is gratefully acknowledged. One of us (P.B.W.) enjoyed the hospitality of the Aspen Center for Physics during the early stages of this work.

[1] B. C. Crooker, B. Hebral, E. N. Smith, Y. Takano, and J. D. Reppy, Phys. Rev. Lett. **51**, 666 (1983); J. D. Reppy, Physica B **126**, 335 (1984).

[2] M. H. W. Chan, K. I. Blum, S. Q. Murphy, G. K. S. Wong, and J. D. Reppy, Phys. Rev. Lett. **61**, 1950 (1988).

[3] M. Rasolt, M. J. Stephen, M. E. Fisher, and P. B. Weichman, Phys. Rev. Lett. **53**, 798 (1984); P. B. Weichman, M. Rasolt, M. E. Fisher, and M. J. Stephen, Phys. Rev. B **33**, 4632 (1986).

[4] P. B. Weichman and M. E. Fisher, Phys. Rev. B **34**, 7652 (1986).

[5] S. N. Dorogovstev, Phys. Lett. **76A**, 169 (1980).

[6] D. Boyanovsky and J. L. Cardy, Phys. Rev. B **26**, 154 (1982); **27**, 6971(E) (1983); see also I. D. Lawrie and V. V. Prudnikov, J. Phys. C **17**, 1655 (1983).

[7] D. S. Fisher and M. P. A. Fisher, Phys. Rev. Lett. **61**, 1847 (1988).

[8] M. P. A. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher (unpublished).

[9] M. Ma, B. I. Halperin, and P. A. Lee, Phys. Rev. B **34**, 3136 (1986). The scaling analysis in this paper is correct, but the final model from which exponents are derived lacks the crucial $i\omega$ term emphasized here. This negates many of their quantitative conclusions. See Refs. 7 and 8 for details.

[10] For a fairly transparent derivation, see A. Casher, D. Lurié, and M. Revzen, J. Math. Phys. **9**, 1312 (1968); see also J. W. Negele and H. Orland, *Quantum Many Particle Systems* (Addison-Wesly, Reading, MA, 1988).

[11] See, e.g., A. Aharony, Phys. Rev. B **12**, 1038 (1975); for a review, see A. Aharony, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, London, 1976), Vol. 6, Chap. 6.

[12] A second derivative term arises in the spin-$\frac{1}{2}$ Ising model in a transverse field [A. P. Young, J. Phys. C **8**, L309 (1975)], and in granular superconductor models with particle-hole symmetry (Ref. 8). In these two cases time and space are symmetric and the d-dimensional quantum critical behavior is just that of the corresponding $(d+1)$-dimensional classical system. In general, however, the simple "add-a-dimension" rule fails [J. A. Hertz, Phys. Rev. B **14**, 1165 (1976)].

[13] This ensures that sums of the form $S[f]=\sum_{\{\omega_\mu\}} f(\sum_\mu \omega_\mu)$, where $\omega_\mu=2\pi n_\mu/\beta$, converge properly if f decreases sufficiently rapidly at $\pm\infty$, and yields the correct result when $\epsilon_d=0$ or 1. Analytic continuation is achieved by Laplace transforming the sum to yield

$$S[f]=\int_0^\infty dt\,[\tilde{f}_+(t)+\tilde{f}_-(t)]/[1-\exp(-2\pi t/\beta)]^{\epsilon_d}-f(0)\,,$$

where

$$f(\omega)=\int_0^\infty dt\,\tilde{f}_\pm(t)\exp(\mp\omega t)\ \text{ for } \pm\omega>0\,.$$

[14] J. T. Chayes, L. Chayes, D. S. Fisher, and T. Spencer, Phys. Rev. Lett. **57**, 2999 (1986).

[15] See P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985), and references therein.

[16] See D. S. Fisher and P. C. Hohenberg, Phys. Rev. B **37**, 4936 (1988); see also Weichman *et al.*, Ref. 3, Sec. VII for a detailed treatment of the nonrandom recursion relations.

[17] P. B. Weichman, Phys. Rev. B **38**, 8739 (1988).

[18] Preliminary data using carbon black show more promising behavior: K. I. Blum, J. S. Souris, and J. D. Reppy (unpublished).