D-brane probes of G_2 holonomy manifolds

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We describe how mirror symmetry of three-dimensional $\mathcal{N}=1$ supersymmetric gauge theories can be used to determine the theory on the world volume of a D2-brane probe of manifolds with G_2 holonomy.

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I. INTRODUCTION

The purpose of this paper is twofold: to present new, dual pairs of strongly interacting three-dimensional gauge theories, and to describe new, singular geometries which appear in realistic M-theory compactifications. In recent years, much progress has been made in understanding dualities between quantum field theories in various dimensions. In three space-time dimensions, a particularly intriguing duality, known as mirror symmetry, relates theories in the Coulomb phase with theories in the Higgs phase [1]. While the original mirror symmetry conjecture required extended supersymmetry, here we present mirror pairs with only minimal supersymmetry. It is to be hoped that such dualities may find application in condensed matter systems.

A somewhat grander, and seemingly unrelated, endeavor is the search for the vacuum of M-theory which correctly describes the universe in which we live. To reduce from the eleven space-time dimensions of M-theory to the four of the real world, it is natural to invoke the Kaluza-Klein idea and ask that seven of the dimensions are compact. Of particular phenomenological interest are seven-dimensional manifolds X of G_2 holonomy. Compactifications on such manifolds result in theories with minimal low-energy supersymmetry, and allow for natural solutions to problems which plague more conventional unified theories, for example the doublet-triplet splitting problem [2]. However, in order to incorporate both non-Abelian gauge fields, and charged chiral fermions, the manifold X must be singular at certain points. Until recently, little was known about the types of degenerations of manifolds with G_2 holonomy. Moreover, even for known examples of G_2 singularities, the dynamics of M-theory in these backgrounds was unclear.

The past year has seen much progress. The physics of M-theory on certain singularities is now well established [3–6] and may be applied to realistic model building. An important role in this development has been played by D-branes and, in this paper, we continue this line of investigation. We employ our new-found mirror pairs of three-dimensional gauge theories in order to describe the types of singularities that may appear in G_2 manifolds, providing

more flexibility in phenomenological constructions. The link between the three-dimensional gauge theory and the G_2 geometry is provided by the concept of the D-brane probe. This has proven to be one of the most important ideas amongst recent ideas in string theory, and here we extend this enterprise to D2-brane probes of non-compact manifolds of G_2 holonomy.

Here we include a single example of our technique: the G_2 manifold with topology $\mathbb{R}^3 \times \mathbb{CP}^2$, which is the simplest example to result in chiral fermions [5]. Many further examples will be discussed elsewhere [7].

II. MIRROR SYMMETRY

Let us firstly describe the mirror pairs that will be our tool in understanding the probe theories. Our mirror pairs have $\mathcal{N}=1$ supersymmetry, and are derived by deforming $\mathcal{N}=4$ mirror pairs using both field theory and string theory techniques. Of course with such little supersymmetry ($\mathcal{N}=1$ means two supercharges) we have little control over the strong coupling dynamics and must be wary of any conjectured duality. Our only savior is parity symmetry which may be used to prohibit the lifting of certain vacuum moduli spaces [8,9]. We hope that the success of our mirror pairs in describing manifolds of G_2 holonomy goes some way towards convincing the reader of their utility.

The mirror pairs preserve only $\mathcal{N}=1$ supersymmetry, and are given by

Theory A: $U(1)^r$ with k scalars and

N hypermultiplets

Theory B: $U(1)^{N-r}$ with $(3N-k)$ scalars and

N hypermultiplets.

The Abelian vector multiplets contain only a photon and a Majorana spinor, while the scalar multiplets, which we shall denote as Φ , contain a single real scalar and a Majorana fermion. In contrast, the hypermultiplets fill out representations of the $\mathcal{N}=4$ algebra: they each contain four Majorana fermions and two complex scalars, q and $\tilde{q}$. We write the superfield as a doublet, $W=(Q, \tilde{Q}^\dagger)^T$. The chiral multiplets Q and $\tilde{Q}$ carry conjugate charges under the gauge group. For

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Theory A, we denote the charge of the hypermultiplets as R_i^a , while for Theory B it is $\hat{R}_i^p$; $i=1, \dots, N$; $a=1, \dots, r$; $p=1, \dots, N-r$. Each of these matrices are assumed to be of maximal rank, and mirror symmetry requires

$$\sum_{i=1}^N R_i^a \hat{R}_i^p = 0 \quad \forall a, p. \quad (1)$$

In $\mathcal{N}=1$ theories, there are no holomorphic luxuries and interactions are determined in terms of a *real* superpotential, f . For Theory A, this superpotential has the cubic form associated with the $\mathcal{N}=4$ theories, and is determined by a triplet of $k \times N$ matrices T_c , $c=1,2,3$

$$f = \sum_{i=1}^N \sum_{c=1}^3 \sum_{\alpha=1}^k W_i^\dagger \tau^c W_i \cdot T_{c,i}^\alpha \Phi_\alpha \quad (2)$$

where $\alpha=1, \dots, k$ and τ^c are the three Pauli matrices. A similar coupling exists for Theory B, now with the triplet of $(3N-k) \times N$ coupling matrices $\hat{T}_c$, satisfying

$$\sum_{i=1}^N \sum_{c=1}^3 T_{c,i}^\alpha \hat{T}_{c,i}^p = 0 \quad \forall \alpha, p. \quad (3)$$

Further details of these theories, together with the methods used to derive them, will be presented elsewhere [7]. Here let us restrict ourselves to a few comments. The Coulomb branch of Theory A has dimension $(N+r)$, which coincides with the dimension of the Higgs branch of Theory B. (The converse also holds.) Mass and FI parameters may be added to both theories, partially lifting some branches of vacua, and the mirror map for these deformations is known.

III. D-BRANES AND G_2 HOLONOMY MANIFOLDS

The field theories of the previous section arise on the world volume of D2-brane probes in various string backgrounds. This provides the link between the gauge theories and G_2 holonomy manifolds. Moreover, mirror symmetry also has a natural interpretation on a D2-brane probe: it may be realized as an “M-theory flip” [10]:

In order to explain this duality, let us consider a membrane probe on a (singular) space X with G_2 holonomy. We can perform a reduction from M-theory to IIA string theory in two different ways. First, we can reduce on a circle transverse to both the manifold X and to the membrane. This takes us to IIA theory on X . The space X is reproduced as a Higgs

branch in the D2-brane world-volume theory, in a way reminiscent to the hyper-Kähler quotient construction. By analogy, we call it the G_2 quotient construction.

On the other hand, we can reduce on an $\mathbf{S}^1$ contained within X . There are many ways to choose the $\mathbf{S}^1$, but a particularly simple choice is to require $X/U(1) \cong \mathbb{R}^6$. If such a quotient exists it gives, after reduction to type IIA theory, a configuration of D6-branes in a (topologically) flat space-time. The positions of the D6-branes correspond to the fixed points of the circle action. In contrast to the previous reduction, the geometry of X is entirely encoded in the configuration of D6-branes, rather than in the geometry of space-time [5,11]. The M-theory geometry X is then reconstructed as the Coulomb branch of the world-volume theory on the D2-brane.

When the space X develops a conical singularity, the configuration of D6-branes also becomes singular. In particular, in some cases of interest it degenerates into a collection of flat D6-branes intersecting at the suitable angles [12]. Let us take $i=1, \dots, N$, flat D6-branes, each of which has spatial world-volume direction:

$$D6_i : 123[47]_{\theta_1^i}[58]_{\theta_2^i}[69]_{\theta_3^i}.$$

The D6-branes lie on a special Lagrangian locus if each rotation is contained in $SU(3)$ [12] or, more simply, if

$$\theta_1^i \pm \theta_2^i \pm \theta_3^i = 0 \quad \text{mod } 2\pi \quad \forall i \quad (4)$$

ensuring that $\mathcal{N}=1$ supersymmetry (four supercharges) is preserved on their common world volume. (For non-generic angles, more supersymmetry may be preserved. We will assume this is not the case.)

We probe this configuration with a D2-brane lying in the x^1 - x^2 plane. This breaks supersymmetry by a further half, resulting in a $d=2+1$ dimensional world-volume theory with $\mathcal{N}=1$ supersymmetry (two supercharges). For the singular case of intersecting, flat D6-branes, the theory on the D2-brane probe is simple to write down. The 2-2 strings give rise to the usual gauge field and seven scalars. Of these, there is one free $\mathcal{N}=1$ scalar multiplet parametrizing motion in the x^3 direction common to all D6-branes. Further fields arise from the 2-6 strings. These give rise to N hypermultiplets. Thus, we have the interacting $\mathcal{N}=1$ supersymmetric theory on the probe,

$$\text{Theory A: } U(1) \text{ with 6 scalar multiplets and } N \text{ hypermultiplets}$$

where each hypermultiplet has charge $+1$ under the gauge field. The couplings of the hypermultiplets to the scalar multiplets are determined by the geometry of the D6-branes: each hypermultiplet couples minimally to the three scalar fields orthogonal to the corresponding D6-brane. If we define the scalar fields $\phi_\alpha = x^{\alpha+3}$, $\alpha=1, \dots, 6$, then the superpotential is of the form (2) with the couplings determined by the D6-brane orientations:

$$T_{c,i}^\alpha = -\sin \theta_c^i \phi_\alpha \delta_{c,\alpha} + \cos \theta_c^i \phi_\alpha \delta_{c,\alpha-3} \quad c=1,2,3. \quad (5)$$

From the IIA space-time picture, we are lead to the natural conjecture that the Coulomb branch of this theory, parametrized by the six real scalars ϕ_a , together with the dual photon σ , is a seven-dimensional manifold X that admits a metric of G_2 holonomy. However, this description of X in terms of Coulomb branch variables is not overly useful. In particular, the isometries of X are lost in the reduction to IIA, and are only expected to be recovered as isometries of the Coulomb branch in the strong coupling limit. It would be desirable to have an algebraic description of X , in which the symmetries are manifest. This is exactly what the mirror theory provides for us.

Since Theory A is in the class of theories discussed above, we may simply write down the mirror theory whose Higgs branch is conjectured to give the G_2 manifold X :

Theory B: $U(1)^{N-1}$ with $3(N-2)$ scalar
and N hypermultiplets.

The gauge couplings are determined by the A_{N-1} quiver diagram, i.e. the i th gauge group acts on the i th hypermultiplet with charge $+1$, and the $(i+1)$ th hypermultiplet with charge -1 . All other hypermultiplets are neutral. The Yukawa terms are of the form (2), with the triplet of coupling matrices $\hat{T}$ determined by Eq. (3). The Higgs branch of this theory is parametrized by $w_i = (q_i, \tilde{q}_i^\dagger)^T$, the N doublets of complex scalars in the hypermultiplets. These are constrained by the $3(N-2)$ D terms, modulo $(N-1)$ $U(1)$ gauge quotients,

$$\sum_{i,c} \hat{T}_{c,i}^\rho w_i^\dagger \tau^c w_i = 0, \quad \rho = 1, \dots, 3(N-2). \quad (6)$$

This quotient construction yields a conical manifold, which is expected to admit a metric of G_2 holonomy. In some cases the conical singularity may be (partially) resolved by adding constants to the right-hand side of Eq. (6). This enlarges by two cycles and, in the IIA picture, corresponds to translating the D6-branes in the x^4 - x^9 directions. Note that when the Yukawa matrices $\hat{T}$ fall into suitable $SU(2)$ triplets, the above method coincides with the toric hyper-Kähler quotient construction, supplemented by a further quotient by a tri-holomorphic isometry to yield a manifold of dimension seven. This is the construction discussed by Acharya and Witten [6]. However, in general, the charges in Eq. (6) differ.

IV. AN EXAMPLE

Let us now examine the G_2 -quotient construction applied to a specific example. Our choice for consideration is the G_2 manifold X given by the cone over the flag manifold $SU(3)/U(1)^2$ [13,14]. This example was also discussed in detail by Atiyah and Witten [5]. They show that, with a suitable choice of M-theory circle, X can be reduced to three, flat, intersecting D6-branes in type IIA string theory. The symmetry of X (to be discussed below), together with the special Lagrangian condition (4) determines the angles of these three branes to be $\theta_c^1 = 0$, $\theta_c^2 = 2\pi/3$ and $\theta_c^3 = 4\pi/3$ for each c . The configuration is drawn in Fig. 1.

FIG. 1. Intersection of special Lagrangian D6-branes dual to M-theory on G_2 holonomy cone over $SU(3)/U(1)^2$ (a), and its non-singular deformation (b).

In order to make the symmetries of the configuration manifest, we define two triplets of scalars, $\vec{\phi}_1 = (x^7, x^8, x^9)^T$ and $\vec{\phi}_2 = (x^4, x^5, x^6)^T$ in terms of which the orientation of the i th D6-brane can be described by the set of linear equations:

$$\begin{aligned} D6_1: \quad & \vec{\phi}_1 = 0 \\ D6_2: \quad & \frac{1}{2}\vec{\phi}_1 + \frac{\sqrt{3}}{2}\vec{\phi}_2 = 0 \\ D6_3: \quad & -\frac{1}{2}\vec{\phi}_1 + \frac{\sqrt{3}}{2}\vec{\phi}_2 = 0. \end{aligned}$$

The original G_2 holonomy manifold X enjoyed an $SU(3)$ continuous isometry. It is not surprising that, upon taking the quotient to IIA string theory, this isometry is partially lost. In fact, the D6-brane background has only a $SU(2)$ symmetry, under which each $\vec{\phi}_a$ transforms as a triplet. The M-theory circle itself provides one further, hidden, $U(1)$ action. We therefore conclude that the reduction to IIA string theory has broken the isometry group to $SU(3) \rightarrow SU(2) \times U(1)$. Now, let us introduce a probe D2-brane in this background, and look at the $\mathcal{N}=1$ gauge theory on its world volume:

Theory A: $U(1)$ with 6 scalars and
3 hypermultiplets.

As described above, the 6 scalar multiplets combine into two triplets whose interactions with the hypermultiplets are of the form (2), where the interaction matrices are determined by Eq. (5). The Coulomb branch of this theory is parametrized by the six scalars, together with the dual photon. It has the $SU(2) \times U(1)$ isometry group, which is expected to be enhanced to the full $SU(3)$ only in the strong coupling, infrared limit.

Using the results described earlier, the mirror theory is the $\mathcal{N}=1$ gauge theory with matter content,

Theory B: $U(1)^2$ with 3 scalar and
3 hypermultiplets.

The charges of the three hypermultiplets under the $U(1)^2$ gauge group are $(+1, -1, 0)$ and $(0, +1, -1)$. The three sca-

lars couple through the usual superpotential (2), with interactions determined by Eqs. (3) and (5) to be $\hat{T}_{c,i}^\rho = \delta_c^\rho$ for all i . Let us examine the Higgs branch of this theory. The superpotential provides 3 real constraints on the 12 real scalar fields contained in the hypermultiplets. After dividing by the gauge group, we are left with a Higgs branch of dimension 7, as required. The constraints are

$$\sum_{i=1}^3 |q_i|^2 - |\tilde{q}_i|^2 = 0, \quad \sum_{i=1}^3 \tilde{q}_i q_i = 0. \quad (7)$$

Firstly notice that this space has a manifest $SU(3)$ symmetry, thus recovering the full isometry group of X . It is not difficult to further show that the space is indeed isomorphic to the cone over $SU(3)$, ensuring that the full Higgs branch is the flag manifold $SU(3)/U(1)^2$.

There is a single normalizable deformation of this space, which yields a smooth G_2 manifold:

$$X \cong \mathbb{R}^3 \times \mathbb{CP}^2. \quad (8)$$

In the D6-brane picture, the singularity is resolved by deforming the singular locus of flat, intersecting D6-branes into a smooth special Lagrangian curve $L \subset \mathbb{C}^3$:

$$L \cong \mathbb{R} \times \mathbb{S}^2 \cup \mathbb{R}^3. \quad (9)$$

In the present case this deformation involves only two out of the three D6-branes. To see this more explicitly, let us choose the first and second D6-branes, which deform to lie on the special Lagrangian curve:

$$\vec{\phi}_1 \cdot \vec{\phi}_2 = -|\vec{\phi}_1||\vec{\phi}_2|, \quad |\vec{\phi}_1| \cdot (3|\vec{\phi}_1|^2 - |\vec{\phi}_2|^2) = \rho. \quad (10)$$

This curve has a remarkable property: it creates a hole through which the remaining D6-brane can pass, see Fig. 1. Therefore, it suffices to deform only two of the three D6-branes in order to completely remove the conical singularity. Of course, one has three different ways to pick a pair of D6-branes, leading to three different resolutions of the space, meeting at a singular point. This is precisely the picture suggested in [5].

It is natural to ask how the probe theory responds to such a deformation. From the perspective of Theory A, one can show that there is essentially a unique deformation consistent with all the symmetries of the model; it is a Yukawa term coupling a pair of hypermultiplets [7]. Moreover, the locus of zeroes of the fermion mass matrix has the same topology as the locus (10).

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