Initial conditions for perturbations in $R+\epsilon R^2$ cosmology

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In a classical inflationary cosmology based on the $R+\epsilon R^2$ Lagrangian the parameters of the model (such as ϵ and the initial conditions for inflationary trajectories) are constrained by the observational requirement that any perturbations be delivered small to the present horizon volume. Previous calculations of the evolution of these perturbations (and, hence, of the parameter constraints enforced by their evolution) have assumed that the modes begin in their ground state. In this paper, following the procedure of Halliwell and Hawking, the Wheeler-DeWitt equation is derived for this model's inhomogeneous modes in perturbative superspace. Then, the two boundary-condition proposals of Hartle and Hawking ("no boundary") and Vilenkin ("tunneling from nothing") are implemented, verifying that both boundary conditions require the inhomogeneous modes to begin in their ground states.

I. INTRODUCTION

In Ref. 1, Mijić, Suen, and I explored a classical cosmological model based on the $R+\epsilon R^2$ Lagrangian. We showed that Robertson-Walker domains would inflate for a wide range of initial conditions, that this pure gravity inflation would smoothly shut itself down, and that the evolution of perturbations on the background could be used to constrain ϵ and the initial parameters of the model. In Ref. 2, we turned to the wave-function formalism and applied it to the same model to obtain distributions over initial conditions for the classical model. There, we derived approximately the general solution in minisuperspace to the Wheeler-DeWitt equation, we implemented the two boundary-condition proposals of Vilenkin[3] ("tunneling from nothing") and Hartle and Hawking[4] ("no boundary") to obtain specific solutions, and we compared the resulting distributions by restricting these wave functions to the initial edge of the Lorentzian semiclassical domain of inflationary trajectories.

In Ref. 1, we showed that the classical inflation tends to smooth out scalar and tensor perturbations. We thus could convert the observational bound (that perturbations presently reentering the horizon be small) into a lower bound on ϵ, $\epsilon > 10^{11} l_{\rm Pl}^2$. The only necessary input was the "usual" assumption that the inhomogeneous scalar and tensor modes begin in their ground states.[5] In this paper, I obtain the wave function for these inhomogeneous modes in the perturbative superspace approximation that the mode strengths are small (this on top of the approximations already made in Ref. 2 to determine the wave function in minisuperspace). I then apply the boundary-condition proposals to verify the ground-state assumption for both. This ground-state conclusion should not be surprising since in work on perturbations in a model of Einstein gravity plus a scalar field Vilenkin[6] has found his boundary condition to fix the inhomogeneous parts of the wave function precisely the same as they are fixed in Halliwell and Hawking.[7] There (Einstein gravity plus a scalar field) as here ($R+\epsilon R^2$ cosmology) both boundary conditions start perturbations off in the ground state; the proposals differ (to semiclassical order) only in the initial state of the expansion degree of freedom.

I thus conceive of this paper as an application of the perturbation analysis of Ref. 7—an application which supports and extends the work in Refs. 1 and 2, and verifies the intuition gleaned from the wave-function formalism applied to the scalar field model (Refs. 6 and 7). My notation will accordingly follow closely that of Refs. 1 and 2. All three papers, though, should be viewed in the larger contexts of work on higher-derivative gravity and the wave-function formalism.[8]

My approach here becomes straightforward after I exploit one strategic fact: Whitt[9] has exhibited a conformal transformation that expresses $R+\epsilon R^2$ as Einstein gravity plus a scalar field. This transformation, important to the calculation and insight of Ref. 1 and central to the method of Ref. 2, is no less key here. The potential for this "conformal-picture" scalar field (which, of course, carries the extra scalar degree of freedom present in the scalar curvature in higher derivative gravity) is zero for large values of the field (the inflationary regime) and approaches the "scalaron mass," $\sim 1/\sqrt{6\epsilon}$, in the linearized limit.

Once in the conformal picture, I can borrow (almost) wholesale the formalism of Halliwell and Hawking[7] to set up and analyze the wave function for the perturbations. In their paper, Halliwell and Hawking present the mode expansion in detail for the perturbed Friedmann model. My application of their work requires me simply to consider the effect of the special form of the potential for the $R+\epsilon R^2$ model. Halliwell and Hawking require that the perturbations match the Hartle-Hawking compact-manifold boundary condition at zero size for the Universe (in the Euclidean regime), leading to a wave function regular in the perturbations. Vilenkin[6] directly requires regularity in the wave function to enforce literally his "tunneling from nothing" proposal. As they show (and I shall show in the present context) this leads to the ground-state initial conditions. Wada[10] has analyzed the wave function tensor modes in some detail for a model of

Einstein gravity plus a cosmological constant that is nearly equivalent to the $R+\epsilon R^2$ conformal picture in the inflationary limit. His methods for solution of the perturbative superspace Wheeler-DeWitt equation will prove useful here. Other authors have variously employed the perturbative superspace approach to problems in quantum cosmology.[11]

The body of this paper is split into two sections: In Sec. II, I obtain the Wheeler-DeWitt equation including all the mode-strength variables out to quadratic perturbative order. In Sec. III, I solve the perturbed Wheeler-DeWitt equation for the inhomogeneous-mode parts of the wave function (for high mode number) and apply the boundary condition(s), verifying that these modes begin in the ground state.

II. THE WHEELER-DeWITT EQUATION WITH PERTURBATIONS

I study a model cosmology governed by the action[1,2]

$$S=\frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R+\epsilon R^2)+\frac{1}{8\pi G}\int d^3x\sqrt{h}\,K(1+2\epsilon R)\ , \tag{2.1}$$

which represents Einstein gravity with an additional quadratic gravitational term. Here, R is the scalar curvature, g is the determinant of the spacetime four-metric, h is the determinant of the induced spatial three-metric on the boundary of integration, and K is the trace of the extrinsic curvature. The sign conventions are those of Ref. 1 and I choose units in which $\hbar=c=1$ and $G=l_{\rm Pl}^2$. The parameter ϵ will then have dimensions of l^2. Under the Whitt conformal transformation,[9]

$$\tilde g_{\mu\nu}=e^{2\phi}g_{\mu\nu}\ ,\quad \phi\equiv\tfrac{1}{2}\ln(1+2\epsilon R)\ , \tag{2.2}$$

the action (2.1) can be reexpressed as Einstein gravity plus a scalar field:

$$S=\frac{1}{16\pi G}\int d^4x(-\tilde g)^{1/2}\tilde R-\int d^4x(-\tilde g)^{1/2}\times\left[\frac{3}{8\pi G}\phi_{,\mu}\phi_{,\nu}\tilde g^{\mu\nu}+\frac{1}{64\pi G\epsilon}(e^{-2\phi}-1)^2\right]+\frac{1}{8\pi G}\int d^3x(\tilde h)^{1/2}\tilde K\ . \tag{2.3}$$

Geometric quantities in conformal space are here denoted by a tilde. During the classical inflationary epoch, the ϵR^2 term will dominate the action (2.1). This corresponds to large ϕ, generating in (2.3) an effective cosmological constant, $1/(8\epsilon)$. The unperturbed Robertson-Walker line element is

$$ds^2=-dt^2+a^2(t)[d\chi^2+\sin^2\chi(d\theta^2+\sin^2\theta\,d\phi^2)]=-dt^2+a^2(t)(\Omega_{ij}dx^idx^j)\ , \tag{2.4}$$

where $0\le\chi\le\pi$, $0\le\theta\le\pi$, and $0\le\phi\le 2\pi$. With a convenient choice of variables in the conformal picture, this can be rewritten

$$d\tilde s^2=\sigma^2[-d\tau^2+\alpha^2(\tau)(\Omega_{ij}dx^idx^j)]\ , \tag{2.5}$$

where $\sigma^2=2G/3\pi$, $\alpha=a(1+2\epsilon R_0)^{1/2}\sigma^{-1}$, $\tau=t(1+2\epsilon R_0)^{1/2}\sigma^{-1}$, $\phi_0\equiv\frac{1}{2}\ln(1+2\epsilon R_0)$, and the 0 subscript denotes the homogeneous part. Now, the action (2.3) in this unperturbed model can be written

$$S_0=\frac{1}{2}\int d\tau\left\{-\alpha\left[\frac{d\alpha}{d\tau}\right]^2+\alpha^3\left[\frac{d\phi_0}{d\tau}\right]^2+\alpha\left[1-\left[\frac{\alpha}{\alpha_*}\right]^2(1-e^{-2\phi_0})^2\right]\right\}\ , \tag{2.6}$$

where $\alpha_*^2\equiv 36\pi\epsilon/G$. This action, and its corresponding Wheeler-DeWitt equation, has been studied in Ref. 2, where the 0 subscript distinction on the homogeneous variables was omitted. To include the effect of perturbations, I now explore the full action (2.3) in the manner of Halliwell and Hawking.[7] In the conformal picture, the 3+1 split is written

$$d\tilde s^2=-(\tilde N^2-\tilde N_i\tilde N^i)d\tau^2+2\tilde N_idx^id\tau+\tilde h_{ij}dx^idx^j\ . \tag{2.7}$$

This metric may then be expressed as a general expansion around the unperturbed metric (2.5):

$$\tilde N=\sigma\left[1+6^{-1/2}\sum_{nlm}g_{nlm}Q^n{}_{lm}\right]\ , \tag{2.8a}$$

$$\tilde N_i=\sigma\alpha(\tau)\sum_{nlm}[6^{-1/2}k_{nlm}(P_i)^n{}_{lm}+2^{1/2}j_{nlm}(S_i)^n{}_{lm}]\ , \tag{2.8b}$$

$$\phi=\phi_o(\tau)+2^{1/2}\pi\sum_{nlm}f_{nlm}Q^n{}_{lm}\ , \tag{2.8c}$$

and

$$\tilde h_{ij}=\sigma^2\alpha^2(\tau)(\Omega_{ij}+\epsilon_{ij})\ , \tag{2.8d}$$

where

$$\epsilon_{ij}=\sum_{nlm}[6^{1/2}a_{nlm}\tfrac{1}{3}\Omega_{ij}Q^n{}_{lm}+6^{1/2}b_{nlm}(P_{ij})^n{}_{lm}+2^{1/2}c^{(o,e)}{}_{nlm}(S_{ij}{}^{(o,e)})^n{}_{lm}+2d^{(o,e)}{}_{nlm}(G_{ij}{}^{(o,e)})^n{}_{lm}]\ . \tag{2.8e}$$

The coefficients a_n, b_n, c_n, d_n, f_n, g_n, j_n, and k_n are all perturbatively small functions of time. I will henceforth follow the convention of denoting all the indices n,l,m and the odd-even parity designators o,e by the single index n. The Q^n are hyperspherical scalar harmonics; $P_i{}^n$ and $S_i{}^n$ are hyperspherical vector harmonics of the scalar and vector types; $P_{ij}{}^n$, $S_{ij}{}^n$, and $G_{ij}{}^n$ are hyperspherical tensor harmonics of the scalar, vector, and tensor types, respectively. All are defined and displayed (together with some of their most useful properties) in Ref. 7. The three-metric $\tilde h_{ij}\sigma^{-2}\alpha^{-2}$ will be used to raise and lower spatial indices.

To simplify the calculation I introduce the gauge choice, $a_n=b_n=c_n=j_n=0$. Then, following the procedure of Ref. 7, I can expand the action in the conformal picture (2.3) out to quadratic order in the perturbations. The only wrinkle concerns the potential term

$$-\int d^4x(-\tilde{g})^{1/2}\frac{(e^{-2\phi}-1)^2}{64\pi G\epsilon} . \tag{2.9}$$

The simplifying assumption I wish to make is to hold the homogeneous part of the scalar field, ϕ_0 in Eq. (2.8c), large, corresponding to the strongly inflationary regime. Indeed, in Ref. 2, the wave function is derived only up to first order in $e^{-2\phi_0}$ [first order in $1/(\epsilon R_0)$]. If I keep only terms to this order here and assume additionally that the perturbation mode strengths are small, I can rewrite the potential term as

$$-\int d^4x(-\tilde{g})^{1/2}\frac{1-2e^{-2\phi_0}}{64\pi G\epsilon} . \tag{2.10}$$

Note that the only remaining coupling of the perturbations to the potential will come from perturbations of $(-\tilde{g})^{1/2}$. The rest of the perturbed action is straightforward, if tedious, to obtain.

I should stress that this method of analysis consigns the wave function to three successive approximations: First to small inhomogeneous mode strengths, then to first order in $1/(\epsilon R_0)$, and finally (below) to a first-order Wentzel-Kramers-Brillouin (WKB) approximation. The latter two approximations already severely restrict the realm of validity in minisuperspace, and the wave functions here must then be held near the unperturbed wave functions of Ref. 2. But, these approximations are valid on the quantum-classical boundary at the initial edge of inflationary trajectories in the Lorentzian domain of minisuperspace. I will thus stay with the interpretation of Ref. 2 and consider the wave function to give the amplitude for branching to a classical trajectory in the expansion degree of freedom on this boundary (other degrees of freedom may remain in the highly quantum regime long after this branching).

The calculation sketched above gives for the action to quadratic perturbative order

$$S=S_0+\sum_n S_n , \tag{2.11a}$$

where the unperturbed action is now

$$S_0=\frac{1}{2}\int d\tau\left\{-\alpha\dot{\alpha}^2+\alpha^3\dot{\phi}_0{}^2 +\alpha\left[1-\left[\frac{\alpha}{\alpha_*}\right]^2(1-2e^{-2\phi_0})\right]\right\} \tag{2.11b}$$

(the overdot denotes $d/d\tau$) and

$$S_n=\int d\tau L_n , \tag{2.11c}$$

where

$$L_n=\frac{1}{2}\alpha^3\left\{\left\{\left[\dot{d}_n+4\frac{\dot{\alpha}}{\alpha}d_n\right]^2-d_n^2\left[10\left[\frac{\dot{\alpha}}{\alpha}\right]^2+6\dot{\phi}_0^2\right]-\frac{d_n^2}{\alpha^2}\left[n^2+1-6\left[\frac{\alpha}{\alpha_*}\right]^2\right]\right\} + \left[(\dot{f}_n-g_n\dot{\phi}_0)^2-(n^2-1)\frac{f_n^2}{\alpha^2}\right]+\left[\frac{2}{3}\left[\frac{\dot{\alpha}}{\alpha}\right]\frac{k_ng_n}{\alpha}-\left[\frac{\dot{\alpha}}{\alpha}\right]^2g_n^2-\frac{k_n^2}{3\alpha^2(n^2-1)}-\frac{2}{\alpha}k_nf_n\dot{\phi}_0\right]\right\} . \tag{2.11d}$$

These equations (2.11) are now the $R+\epsilon R^2$ version of Eqs. (B1)–(B5) of Ref. 7 with my choice of gauge and notational conventions. At this point, it is possible to achieve a vast simplification to the tensor-mode parts of these equations by choosing a new expansion variable in the manner of Wada:[10]

$$\bar{\alpha}\equiv\alpha\prod_n e^{-2d_n^2} . \tag{2.12}$$

Then the action (2.11) can be reexpressed as

$$S=S_0+\sum_n S_n , \tag{2.13a}$$

where

$$S_0=\frac{1}{2}\int d\tau\left\{-\bar{\alpha}\dot{\bar{\alpha}}^2+\bar{\alpha}^3\dot{\phi}_0^2 +\bar{\alpha}\left[1-\left[\frac{\bar{\alpha}}{\alpha_*}\right]^2(1-2e^{-2\phi_0})\right]\right\} \tag{2.13b}$$

and

$$S_n=\int d\tau L_n , \tag{2.13c}$$

where

$$L_n=\frac{1}{2}\bar{\alpha}^3\left\{\left[\dot{d}_n^2-\frac{n^2-1}{\bar{\alpha}^2}d_n^2\right] + \left[(\dot{f}_n-g_n\dot{\phi}_0)^2-(n^2-1)\frac{f_n^2}{\bar{\alpha}^2}\right] + \left[\frac{2}{3}\left[\frac{\dot{\bar{\alpha}}}{\bar{\alpha}}\right]\frac{k_ng_n}{\bar{\alpha}}-\left[\frac{\dot{\bar{\alpha}}}{\bar{\alpha}}\right]^2g_n^2 -\frac{k_n^2}{3\bar{\alpha}^2(n^2-1)}-\frac{2}{\bar{\alpha}}k_nf_n\dot{\phi}_0\right]\right\} . \tag{2.13d}$$

Now, $\partial L_n/\partial g_n=0$ and $\partial L_n/\partial k_n=0$ provide the constraints

$$k_n=3(n^2-1)\bar{\alpha}\left[\frac{\frac{1}{3}\frac{\dot{\bar{\alpha}}}{\bar{\alpha}}\dot{f}_n\dot{\phi}_0-f_n(\dot{\phi}_0)^3+f_n\dot{\phi}_0\left[\frac{\dot{\bar{\alpha}}}{\bar{\alpha}}\right]^2}{\dot{\phi}_0^2+\frac{n^2-4}{3}\left[\frac{\dot{\bar{\alpha}}}{\bar{\alpha}}\right]^2}\right] \tag{2.14a}$$

and

$$g_n=\left[\frac{\left[(n^2-1)\frac{\dot{\bar{\alpha}}}{\bar{\alpha}}f_n+\dot{f}_n\right]\dot{\phi}_0}{\dot{\phi}_0^2+\frac{n^2-4}{3}\left[\frac{\dot{\bar{\alpha}}}{\bar{\alpha}}\right]^2}\right] . \tag{2.14b}$$

The only perturbative degrees of freedom are the scalar modes, carried by the f_n, and the tensor modes, carried by the d_n.

Now, the canonical momenta are

$$\pi_{\bar{\alpha}}=\frac{\partial L_n}{\partial\dot{\bar{\alpha}}}=-\bar{\alpha}\dot{\bar{\alpha}}+\sum_n(-\bar{\alpha}\dot{\bar{\alpha}}g_n^2+\tfrac{1}{3}k_ng_n\bar{\alpha}) , \tag{2.15a}$$

$$\pi_{\phi_0}=\frac{\partial L_n}{\partial\dot{\phi}_0}=\bar{\alpha}^3\dot{\phi}_0+\sum_n[-g_n\bar{\alpha}^3(\dot{f}_n-g_n\dot{\phi}_0)-\bar{\alpha}^2k_nf_n] , \tag{2.15b}$$

$$\pi_{d_n}=\frac{\partial L_n}{\partial\dot{d}_n}=\bar{\alpha}^3\dot{d}_n , \tag{2.15c}$$

and

$$\pi_{f_n}=\frac{\partial L_n}{\partial\dot{f}_n}=\bar{\alpha}^3(\dot{f}_n-g_n\dot{\phi}_0) . \tag{2.15d}$$

The Hamiltonian is obtained by the usual prescription, $H=\pi_x\dot{x}-L$:

$$\begin{aligned}H=\frac{1}{2}\Bigg\{&-\frac{\pi_{\bar{\alpha}}^2}{\bar{\alpha}}+\frac{\pi_{\phi_0}^2}{\bar{\alpha}^3}-\bar{\alpha}\left[1-\left[\frac{\bar{\alpha}}{\alpha_*}\right]^2(1-2e^{-2\phi_0})\right]\\&+\sum_n\left[\frac{\pi_{f_n}^2}{\bar{\alpha}^3}+\bar{\alpha}(n^2-1)f_n^2\right]\\&+\sum_n\left[\frac{\pi_{d_n}^2}{\bar{\alpha}^3}+\bar{\alpha}(n^2-1)d_n^2\right]\\&+\left[\sum_n g_n\frac{\pi_{\phi_0}\pi_{f_n}}{\bar{\alpha}^3}+\sum_n k_n\frac{f_n\pi_{\phi_0}}{\bar{\alpha}}\right]\Bigg\} .\end{aligned} \tag{2.16}$$

The constraint parts of this Hamiltonian (the last two sums), corresponding to the independent constants g_n, k_n must be individually satisfied by the wave function (since the wave function is independent of g_n, k_n). They will be trivially satisfied at the order of approximation used here because they are of quadratic order in the perturbations and π_{ϕ_0} is small. Factor-ordering worries left out (again to this order of approximation), canonical quantization, $\hat{\pi}_x=-i\partial/\partial x$, yields finally the Wheeler-DeWitt equation appropriate to this approximation and gauge,

$$\hat{H}\Psi(\bar{\alpha},\phi_0,\{d_n\},\{f_n\})=0 , \tag{2.17a}$$

where

$$\hat{H}=\hat{H}_0+\sum_n\hat{H}_n , \tag{2.17b}$$

$$\hat{H}_0=\frac{1}{2\bar{\alpha}}\left\{\frac{\partial^2}{\partial\bar{\alpha}^2}-\frac{1}{\bar{\alpha}^2}\frac{\partial^2}{\partial\phi^2}-\bar{\alpha}^2\left[1-\left[\frac{\bar{\alpha}}{\alpha_*}\right]^2(1-2e^{-2\phi_0})\right]\right\} , \tag{2.17c}$$

and

$$\hat{H}_n=\frac{1}{2\bar{\alpha}}\left[\left[-\frac{1}{\bar{\alpha}^2}\frac{\partial^2}{\partial f_n^2}+\bar{\alpha}^2(n^2-1)f_n^2\right]+\left[-\frac{1}{\bar{\alpha}^2}\frac{\partial^2}{\partial d_n^2}+\bar{\alpha}^2(n^2-1)d_n^2\right]\right] . \tag{2.17d}$$

III. SOLUTION OF THE WHEELER-DeWITT EQUATION

To solve the infinite-dimensional Wheeler-DeWitt equation (2.17) I follow Wada[10] and write the wave function as

$$\Psi=\exp(iS) , \tag{3.1}$$

and expand $S(\bar{\alpha},\phi_0,\{d_n\},\{f_n\})$ out to quadratic order in the perturbations,

$$S(\bar{\alpha},\phi_0,\{d_n\},\{f_n\})=S_0(\bar{\alpha},\phi_0)+\tfrac{1}{2}\sum_n S_n^d(\bar{\alpha})d_n^2+\tfrac{1}{2}\sum_n S_n^f(\bar{\alpha})f_n^2 . \tag{3.2}$$

Separating the order of perturbation, and keeping terms to semiclassical order, I obtain three equations:

$$-\left[\frac{\partial S_0}{\partial\bar{\alpha}}\right]^2+\frac{1}{\bar{\alpha}^2}\left[\frac{\partial S_0}{\partial\phi_0}\right]^2-\bar{\alpha}^2\left[1-\left[\frac{\bar{\alpha}}{\alpha_*}\right]^2(1-2e^{-2\phi_0})\right]=0 , \tag{3.3a}$$

$$-\left[\frac{\partial S_0}{\partial\bar{\alpha}}\right]\left[\frac{dS_n^d}{d\bar{\alpha}}\right]+\frac{(S_n^d)^2}{\bar{\alpha}^2}+\bar{\alpha}^2(n^2-1)=0 , \tag{3.3b}$$

and

$$-\left[\frac{\partial S_0}{\partial\bar{\alpha}}\right]\left[\frac{dS_n^f}{d\bar{\alpha}}\right]+\frac{(S_n^f)^2}{\bar{\alpha}^2}+\bar{\alpha}^2(n^2-1)=0 . \tag{3.3c}$$

Now, Eq. (3.3a) has been solved in Ref. 2 in the region of minisuperspace where the kinetic term in ϕ_0 is ignorable (near the Hartle-Hawking and Vilenkin wave functions). For Eqs. (3.3b) and (3.3c), I can make the adiabatic approximation for large mode number n. Writing $\partial S_0/\partial\bar{\alpha}=\pi_{\bar{\alpha}}=-\bar{\alpha}\dot{\bar{\alpha}}$ and assuming that $\bar{\alpha}$ is slowly varying, Eqs. (3.3b) and (3.3c) can be rewritten as

$$(S_n{}^d)^2+\bar{\alpha}^4(n^2-1)\approx 0 \qquad (3.3d)$$

and

$$(S_n{}^f)^2+\bar{\alpha}^4(n^2-1)\approx 0 \; . \qquad (3.3e)$$

These equations (3.3d) and (3.3e) are algebraically solvable to get

$$S_n{}^d\approx\pm i\sqrt{n^2-1}\bar{\alpha}^2 \text{ and } S_n{}^f\approx\pm i\sqrt{n^2-1}\bar{\alpha}^2 \; . \qquad (3.4)$$

Now, in the treatment by Halliwell and Hawking, they require that the mode strengths $d_n(\tau)$ and $f_n(\tau)$, when evaluated in the Euclidean domain for small scale factor, should vanish. That is, the Hartle-Hawking boundary condition requires that the contributions to the path integral are from those paths which start out at $d_n=f_n=0$. To see what this leads to in the wave function here, I briefly consider the scalar modes: In the semiclassical regime (whether inside or outside the barrier) I find [from Eq. (2.15c) and Eqs. (3.1)–(3.3)]:

$$\pi_{f_n}\approx\bar{\alpha}^3\dot{f}_n=S_n{}^f f_n=\pm i\sqrt{n^2-1}\bar{\alpha}^2 f_n \; . \qquad (3.5a)$$

In the Euclidean semiclassical regime (under the barrier), I retrieve the solution, $\bar{\alpha}\approx\alpha(\tau_E)=\alpha_*\sin(\tau_E/\alpha_*)$, where $\tau_E=i\tau$, from the discussion preceding Eq. (4.27) of Ref. 2. Then, the solution to Eq. (3.5a) is

$$f_n(\tau_E)=f_n(\tau_{E_0})\left[\frac{\tan(\tau_E/2\alpha_*)}{\tan(\tau_{E_0}/2\alpha_*)}\right]^{\pm\sqrt{n^2-1}} \; . \qquad (3.5b)$$

The solution which matches the boundary condition that $f_n(\tau_E=0)=0$ picks out the positive sign. As can be seen from Eqs. (3.1)–(3.4), this sign just corresponds to the wave-function mode regular in f_n. Now Vilenkin's boundary condition enforces this regularity directly on the wave function. Though the procedure is admittedly not rigorous (as Vilenkin[6] has pointed out), because f_n and d_n have been assumed small in deriving (3.4), this requirement demands the positive sign and both boundary conditions are seen to yield the same form for the perturbative parts of the wave function.

A final expression for the wave function may now be written down. I use the notation [based on Eqs. (3.1) and (3.2)]

$$\Psi(\bar{\alpha},\phi_0,\{f_n\},\{d_n\}) =\Psi_0(\bar{\alpha},\phi_0)\prod_n \Psi^n{}_{\text{scalar}}(\bar{\alpha},f_n)\Psi^n{}_{\text{tensor}}(\bar{\alpha},d_n) \; . \qquad (3.6a)$$

The homogeneous part of this wave function $\Psi_0=\exp(iS_0[\bar{\alpha},\phi_0])$ is given by Eq. (4.37) of Ref. 2 for Vilenkin's boundary condition and by Eq. (4.38) of Ref. 2 for Hartle and Hawking's. The wave functions for the inhomogeneous modes for large n in the adiabatic approximation, as inferred from (3.1), (3.2), and (3.4) with the + sign, can be written for both boundary conditions as

$$\Psi^n{}_{\text{scalar}}(\bar{\alpha},f_n)\approx e^{-\sqrt{n^2-1}\bar{\alpha}^2 f_n^2/2} \qquad (3.6b)$$

and

$$\Psi^n{}_{\text{tensor}}(\bar{\alpha},d_n)\approx e^{-\sqrt{n^2-1}\bar{\alpha}^2 d_n^2/2} \; . \qquad (3.6c)$$

Both of these wave functions have the ground-state form for harmonic oscillators of frequency $\omega_n^2\approx\bar{\alpha}^4(n^2-1)$. Indeed, from Eq. (2.17) they individually satisfy Schrödinger equations in the form for such an oscillator in the semiclassical approximation:

$$i\frac{\partial}{\partial\tau}\Psi^n{}_{\text{scalar}} =-\frac{i}{\bar{\alpha}}\left[\frac{\partial S_0}{\partial\bar{\alpha}}\right]\frac{\partial}{\partial\bar{\alpha}}\Psi^n{}_{\text{scalar}} =\frac{1}{2\bar{\alpha}}\left[-\frac{1}{\bar{\alpha}^2}\frac{\partial^2}{\partial f_n^2}+\bar{\alpha}^2(n^2-1)f_n^2\right]\Psi^n{}_{\text{scalar}}(\bar{\alpha},f_n) \qquad (3.7a)$$

and

$$i\frac{\partial}{\partial\tau}\Psi^n{}_{\text{tensor}} =-\frac{i}{\bar{\alpha}}\left[\frac{\partial S_0}{\partial\bar{\alpha}}\right]\frac{\partial}{\partial\bar{\alpha}}\Psi^n{}_{\text{tensor}} =\frac{1}{2\bar{\alpha}}\left[-\frac{1}{\bar{\alpha}^2}\frac{\partial^2}{\partial d_n^2}+\bar{\alpha}^2(n^2-1)d_n^2\right]\Psi^n{}_{\text{tensor}}(\bar{\alpha},d_n) \; . \qquad (3.7b)$$

Here time has been reintroduced in terms of the expansion of the classical background $\bar{\alpha}(\tau)$ in the Lorentzian semiclassical domain. That tensor modes should satisfy the same Schrödinger equation as scalar modes directly follows from the work of Ford and Parker,[12] who showed that odd- and even-parity gravitational perturbations are equivalent to massless minimally coupled scalar fields.

The modes remain in the ground state until the adiabatic approximation breaks down—until they cross out of the horizon.[13] This crossing was shown in Ref. 1 to occur during the inflationary epoch, where the approximation of large ϕ_0 still holds. The ground state wave function (3.6) at the outgoing horizon crossing is the starting point of the evolution calculations in Secs. IV and V of Ref. 1.

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