

Duality and Fibrations on G_2 Manifolds

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Abstract

We argue that G_2 manifolds for M-theory admitting string theory Calabi-Yau duals are fibered by coassociative submanifolds. Dual theories are constructed using the moduli space of M5-brane fibers as target space. Mirror symmetry and various string and M-theory dualities involving G_2 manifolds may be incorporated into this framework. To give some examples, we construct two non-compact manifolds with G_2 structures: one with a $K3$ fibration, and one with a torus fibration and a metric of G_2 holonomy. Kaluza-Klein reduction of the latter solution gives abelian BPS monopoles in $3 + 1$ dimensions.

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1 Introduction

One of the main achievements in string theory during the last decade was the discovery of string dualities and relations among them. A particularly rich and interesting example of string duality is *mirror symmetry* between pairs of Calabi-Yau manifolds. A geometric framework for understanding this duality was proposed in [30], and involves constructing the mirror manifold by dualizing a torus fibration. This construction arose from the correspondence among nonperturbative states of dual theories. M-theory has united the disparate string theories and promises to reveal the nature of string dualities. In M-theory, the analogue of a Calabi-Yau manifold is a manifold with G_2 holonomy, simply by the counting of dimensions: what was $10 = 4+6$ for string theory is $11 = 4+7$ in M-theory. According to this simple formula, seven-dimensional G_2 -holonomy manifolds are natural candidates for minimally supersymmetric (and phenomenologically interesting [3]) compactifications of M-theory to $3+1$ dimensions. If manifolds with G_2 holonomy are M-theory analogues of Calabi-Yau spaces, then what is the corresponding notion of mirror symmetry, and what is the geometry behind duality? Is there a fibration structure on G_2 manifolds relevant to this and possibly other string/M-theory dualities? These are the questions that one might naturally ask, and that we attempt to address in this paper.

We argue that, just as Calabi-Yau manifolds involved in mirror symmetry are fibered by special Lagrangian tori, in M-theory G_2 -holonomy manifolds which admit string theory duals are fibered by coassociative 4-manifolds. Specifically, M-Theory on a seven-manifold X , with G_2 holonomy, leads to an effective field theory in four dimensions with $\mathcal{N} = 1$ supersymmetry. The same is true for the heterotic string theory on a Calabi-Yau manifold, Y , with an appropriate choice of holomorphic bundle.¹ Similarly, Type-IIA string theory on a noncompact Calabi-Yau manifold, Z , with Ramond-Ramond fluxes turned on (or with a spacetime-filling brane) has $\mathcal{N} = 1$ supersymmetry if the fluxes satisfy certain first-order equations. Are there pairs² (X, Y) or (X, Z) which lead to equivalent theories? If so, how are the geometry and topology of X related to the choice of Y or Z , as well as to the bundle or Ramond-Ramond field data? Is there a constructive way of producing duals?

In this paper we address these questions from two points of view, then produce two manifolds which may serve as sources of further study in these directions: one a torus fibration with a G_2 metric constructed from Hitchin's method [18] in Sec. 4, and

¹One requires $E \rightarrow Y$ to obey $p_1(E) = p_1(TY)$ and $c_1(E) = 0$ so that there is no anomaly, *i.e.* the heterotic theory contains no fivebranes.

²Kachru and Vafa first found heterotic-Type-IIA (Y, Z) pairs in [19]; some (X, Z) pairs are studied in [4, 5].

one a K3 fibration with a G_2 structure (but neither closed nor co-closed three-form) constructed in Sec. 3. The two lines of reasoning are as follows. First, we study the moduli space of an M-theory five-brane wrapped around a coassociative (internal) four-cycle, C . By allowing the moduli of C to vary slowly in spacetime directions, one sees that the resulting theory on the spacetime soliton string is a conventional string theory with target space the M-brane moduli space. The moduli space is Y or Z , depending on C . This line of reasoning follows [15]. Second, as in [3], we use fiberwise duality of M-theory on K3 with heterotic strings on T^3 , as well as the “fact” that Calabi-Yau’s admit torus fibrations, to argue that heterotic string theory on a Calabi-Yau manifold should be dual to M-theory on a K3 fibration.

Remark 1 *Some duality conjectures involving G_2 manifolds have been proposed in [21] and [1]. Our arguments don’t involve pairs of G_2 manifolds per se, though do lead to relations. For example, if one takes a Calabi-Yau resulting from the moduli space of a coassociative fiber, one can look for a different G_2 manifold which has that Calabi-Yau as its Kaluza-Klein reduction. One would then expect that two G_2 manifolds related in this way would be mirror, in the sense of Shatashvili-Vafa [29]. The setting for Acharya’s arguments in [1] is string theory and duality of G_2 manifolds via dual torus fibrations. The motivation in [21] is more mathematical, where a fiberwise Fourier-Mukai transform leads to conjectured dual G_2 manifolds near limiting points in the moduli space of G_2 -holonomy metrics.*

Remark 2 *Results in Sec. 2 rely on physical arguments, and include some speculative mathematics. While the metrics of Secs. 3 and 4 are motivated by the physical reasoning, these sections are purely mathematical in nature, and can be read independently. Sec. 5 is a mixed bag.*

2 Fibrations from Brane Moduli Spaces

2.1 Fibrations from M-theory

The arguments of mirror symmetry as T-duality [30] arise from recognizing that string duality demands a correspondence among the nonperturbative states of the theory. The dual theory is then found as a sigma model on the moduli space of a relevant brane. We shall try to apply similar reasoning to M-theory on a compact G_2 manifold, X . Here, instead of D-branes we have M-theory five-branes.³ On a G_2 manifold we

³We recall that the field content of M-theory contains a three-form H with four-form field strength; it obeys $dH = \delta_D$, where D is the five-brane world-volume. This leads to a condition that the normal bundle have trivial Euler characteristic, which is true for the examples $(T^4, K3)$ in this paper.

can choose a five-brane whose world-volume is $\Sigma \times C$, with $\Sigma \subset \mathbb{R}^4$ a Riemann surface in flat space, and $C \subset X$ a coassociative four-cycle (this means $\Phi|_C = 0$, where Φ is the associative calibration three-form of the torsion-free G_2 structure on X). The five-brane is a string in the effective theory, the so-called *black string*, and we have chosen a supersymmetric brane, in the sense that the theory on the string worldsheet is a *two-dimensional* $\mathcal{N} = (2, 0)$ supersymmetric theory. Its moduli space equals the moduli space of the five-brane. Recall that similar reasoning led to the discovery of the heterotic string as a type-II soliton [15].

The question now arises: What is the five-brane moduli space? This is comprised of a choice of coassociative 4-cycle $C \subset X$, a choice of $\Sigma \subset \mathbb{R}^4$, and a point in the intermediate Jacobian [34]:

$$J_{C \times \Sigma} \equiv H^3(C \times \Sigma, \mathbb{R}) / H^3(C \times \Sigma, \mathbb{Z}) \quad (2.1)$$

In the world-volume action, only the self-dual three-form couples to the five-brane. We can write a self-dual three-form as a linear combination of $\alpha_+ \wedge \beta_+$ and $\alpha_- \wedge \beta_-$, where α_{\pm} are self-dual/anti-self-dual forms on Σ , and β_{\pm} are self-dual/anti-self-dual forms on C . There are $b_2^{\pm}(C)$ self-dual/anti-self-dual forms on C . Let us now turn to the moduli fields of the five-brane $\Sigma \times C$. There are two transverse directions for $\Sigma \subset \mathbb{R}^4$ (or four total, as $\dim G_{\mathbb{R}}(2, 4) = 4$). As for the number of deformations of a coassociative submanifold [23], this is equal to $b_2^+(C)$, and these moduli fields have left- and right-dependencies.

Now let's look at the effective theory on the black string worldsheet, Σ . The low-lying excitations are described by allowing the coassociative cycle C and gauge two-form on it to vary slowly over Σ . The effective string theory is therefore a supersymmetric sigma model with target space described by the $2b_2^+(C) + 2$ left-moving and $b_2^+(C) + b_2^-(C) + 2$ right-moving moduli fields found above.⁴ These fields live in a compactification defined by the moduli space and the integer lattice $H^2(C, \mathbb{Z})$. When $C \cong T^4$ we have $b_2^+ = b_2^- = 3$, and we find eight left-movers and eight right-movers, equal in number to the transverse oscillations of the Type-II string. When $C \cong K3$ we have $b_2^+ = 3, b_2^- = 19$, so we get eight left-movers and 24 right-movers, as in the heterotic string theory. We

⁴ The spectrum of fermions on the black string worldsheet Σ follows by supersymmetry. Since the five-brane breaks the six-dimensional Lorentz invariance down to: $\text{Spin}(1, 1)_{\Sigma} \times \text{Spin}(4)_C$ the fermions in the five-brane tensor multiplet transform as $(+, \mathbf{2}_+) \oplus (-, \mathbf{2}_-)$. On the other hand, the structure group of the five-brane normal bundle, N , becomes the R-symmetry group $\text{Spin}(4)_N \cong SU(2) \times SU(2)$ of the world-volume theory. The fermions transform in $(\mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{2})$ under this group [6]. Therefore, on the coassociative 4-manifold C we have a topologically twisted $\mathcal{N} = 4$ gauge theory, the so called Vafa-Witten theory [33]. The partition function of this theory counts the euler number of the moduli space of instantons on C . It would be interesting to investigate a further relation between this topological theory and M-theory on G_2 -holonomy manifolds. Mirror symmetry and the counting associative 3-cycles should play an important role in such a relation.

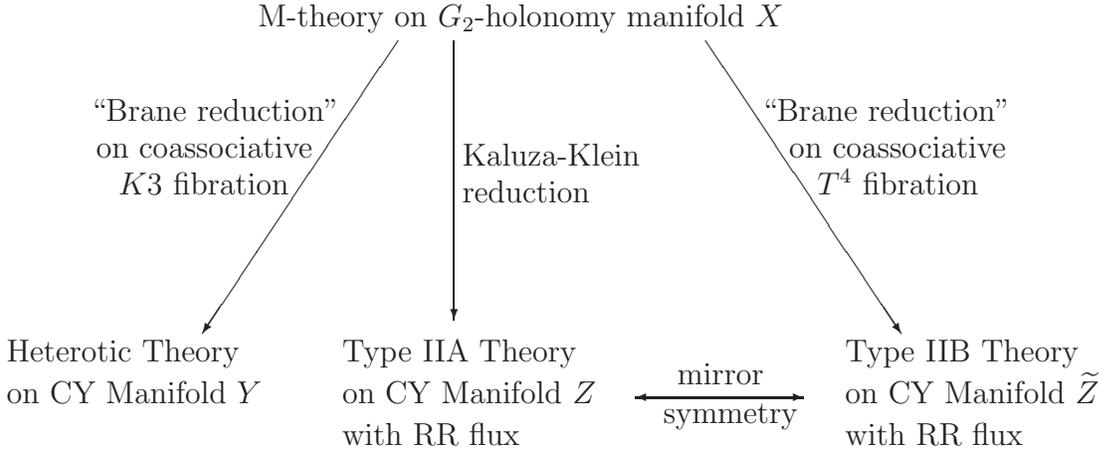


Figure 1: Various dualities between M-theory on manifold X with G_2 holonomy and string theory on Calabi-Yau manifolds.

expect these string theories to be dual to the original M-theory on X . As with SYZ, the geometric structures which we have found should emerge in some regime of large radius and small fibers.

In the following subsections we will consider torus and $K3$ fibrations in turn.

2.2 Torus Fibrations and Type-II/M-theory Duality

In the usual equivalence between M-theory and Type IIA string theory, one employs simple Kaluza-Klein reduction to the fields. In the reduction from M-theory on T^4 to IIA on a three-torus, the metric field of a four-torus gives a metric on the three-torus, a Ramond-Ramond gauge field, and a scalar. In our situation, if we take X to be fibered by $C \cong T^4$, then the Ramond-Ramond gauge field will be varying and generically produce a non-zero field strength. This leads to the generation of a superpotential, for an $\mathcal{N} = 1$ theory⁵ [13, 31, 22].

However, in the reasoning of Sec. 2.1, we have done something *different* to arrive at a Calabi-Yau manifold starting from X , via self-dual forms. What’s the relation? By analogy with the mirror symmetry argument of [30], we should be performing not the Kaluza-Klein reduction, but its *mirror*. Therefore, we should arrive at a Type-IIB theory on our Calabi-Yau three-fold, and that IIB theory should have a *dual* torus fibration — see Fig. 1. (Generalizations of the SYZ conjecture to G_2 manifolds with torus fibrations have been made in [1] and [21].)

⁵If the fibration of T^4 over T^3 is not changing, then the dilaton is constant and $X \cong CY \times S^1$, so we recover $\mathcal{N} = 2$ supersymmetry.

To show this is true, we isolate a vector v in the direction of the M-theory circle. Then, we use the G_2 structure to find a perfect pairing between $\mathbb{R}^4/\mathbb{R}v$ and the self-dual directions in \mathbb{R}^4 , where \mathbb{R}^4 is the tangent space to the coassociative fiber. But note that the G_2 three-form Φ_0 on \mathbb{R}^7 already gives an isomorphism $\Lambda_+^2 \mathbb{R}^4 \cong (\mathbb{R}^4)^\perp$, so we need only find, given a vector v , a pairing between vectors $w \in \mathbb{R}^4/\mathbb{R}v$ and vectors n normal to the four-plane. This pairing is simply

$$\Phi_0(v, w, n), \tag{2.2}$$

and is perfect.⁶

Let us call the Calabi-Yau formed from the moduli space of coassociative torus fibers the “brane reduction” of the G_2 manifold X , as opposed to the Kaluza-Klein reduction. What we have argued then is that *brane reduction is mirror to Kaluza-Klein reduction*.

Another piece of evidence for this correspondence can be found by studying the effect of deforming the C -field in M-theory. Such a deformation doesn’t affect the geometry of the Kaluza-Klein reduction, but it does alter the B -field, hence the complexified Kähler form. How does it affect the brane reduction? It should enter the equations for the two-form gauge field on the five-brane, hence change the self-duality condition. This alters the pairing between base and fiber directions for the brane reduction, hence changes the complex structure of the Calabi-Yau. We therefore see once again the mirror relation between brane and Kaluza-Klein reduction.

As described in Remark 1, one would naturally conjecture that the Kaluza-Klein lift of a brane-reduction would yield a mirror G_2 manifold, in the sense of Shatashvili-Vafa [29]. In fact, B. Acharya informs us that he has constructed G_2 orbifolds with T^4 and T^3 fibrations (and discrete torsion), and finds that dualizing along T^3 or along T^4 fibers produces G_2 manifolds with the same values of $b_2 + b_3$, as required by [29]. We thank him for informing us of these interesting examples, which suggest that mirror Calabi-Yau manifolds correspond to mirror G_2 manifolds.

2.3 $K3$ Fibrations and Heterotic/M-theory Duality

Consider heterotic string theory on a Calabi-Yau three-fold Y . Following [30], we view Y as a fibration by special Lagrangian tori over a 3-dimensional base ($= S^3$). Since heterotic string on T^3 is dual to M-theory on $K3$, we can apply this duality fiberwise,

⁶For example, if we write

$$\Phi_0 = e_{125} + e_{345} + e_{136} - e_{246} + e_{147} + e_{237} + e_{567} \tag{2.3}$$

and $v = e_1$, then the duality pairs $e_2 \leftrightarrow e_5$, as $\iota_v \Phi_0 = e_{25} + e_{36} + e_{47}$, etc.

and conclude that heterotic string on Y should be dual to M-theory on a G_2 -holonomy manifold X , which in turn can be viewed as a fibration by (possibly singular) $K3$ fibers.

Identifying BPS domain walls in M-theory and in heterotic string, we can obtain a relation between Betti numbers of X and Y , assuming that both manifolds are smooth [26]:

$$b_3(X) = 2h^{2,1}(Y) + 1 \tag{2.4}$$

Moreover Betti numbers of Calabi-Yau space Y must obey $h^{2,1} = h^{1,1}$. Finally, we could also look at the effective theory in M-theory on X and in heterotic theory on Y . The spectra of light particles should match. In particular, we should expect the matching of numbers of chiral/vector multiplets in dual descriptions.

In the simple case (when singularities in the $K3$ fiber can be resolved or deformed), both X and Y are smooth manifolds. It follows that the gauge group in the effective four-dimensional theory is abelian, typically of rank (notice an obvious mistake in [26]):

$$14 = \text{number of } \mathcal{N}=1 \text{ vector multiplets} \tag{2.5}$$

This is the rank of the gauge group in heterotic theory broken to a subgroup by Wilson lines, which can be continuously connected to a trivial Wilson line, *i.e.* we are on the main branch of the moduli space corresponding to the so-called *standard embedding*.

On the other hand, in M-theory on G_2 manifold X , vector fields come from KK modes of the C -field. So, there are $b_2(X)$ of them. Therefore, identifying $\mathcal{N} = 1$ vector multiplets in the low-energy effective theory, we seem to find a peculiar condition:

$$b_2(X) \leq 16 \tag{2.6}$$

for all G_2 -holonomy manifolds (with generically non-singular $K3$ fibers) that have heterotic duals. Of course, we assume that the heterotic dual is purely geometrical, *i.e.* there are no space-filling five-branes.

In general, one needs space-filling branes to cancel anomalies. For example, in F-theory on a Calabi-Yau four-fold these are D3-branes, needed to cancel the $\chi/24$ tadpole of the F-theory compactification. On the other hand, in heterotic string theory on a Calabi-Yau space Y these are five-branes wrapped on holomorphic curves inside Y . Via duality to M-theory these space-filling branes become certain singularities of the coassociative fibration (such that the whole G_2 manifold may still be non-singular).

Summarizing, we argued that the Calabi-Yau dual of M-theory on X is a heterotic compactification on the moduli space $\mathcal{M}_{\text{coassoc}}$ of coassociative cycles in the deformation class of C . There are a few important remarks in place here:

Remark 3 *Note that the metric on the torus part of the heterotic compactification is changing. This scenario is somewhat similar to the stringy cosmic string, in which the modulus of the compactification has spatial dependence. In that case, the equations of motion ensured holomorphicity of the total space. We would hope to find that the G_2 holonomy condition is related to the equations of motion for the stringy cosmic string in this generalized setting [10].*

Remark 4 *By analogy to the Calabi-Yau case, where a multiple of the holomorphic form becomes an integral form near the large radius limit,⁷ one anticipates that near some limit in G_2 moduli space, the self-dual forms can be represented by integral forms, and the left- and right-moving spaces compactify separately (i.e. the lattice is compatible with the left/right split). The limiting Calabi-Yau, then, would look like the quotient of $T^*\mathcal{M}_{\text{coassoc}}$ by a lattice. This is the G_2 analogue of the fact that there is no quantum correction to the complex structure of the special Lagrangian fibration near the large radius limit. Perhaps one could then use a Gauss-Manin connection to follow the Calabi-Yau manifold away from this limit.*

We now highlight some features of $\mathcal{M}_{\text{coassoc}}$ which make it a possible candidate for the base of a Calabi-Yau with torus fibration. The moduli space $\mathcal{M}_{\text{coassoc}}$ has a natural metric on it, given by the inner product of anti-self-dual forms:

$$g(V_1, V_2) = - \int_C \theta_1 \wedge \theta_2 \quad (2.7)$$

where θ_i are the anti-self-dual two-forms corresponding to the tangent vectors $V_i \in T\mathcal{M}_{\text{coassoc}}|_C$. We also have a correspondence between moduli (harmonic self-dual forms) and left-moving field strength directions (self-dual cohomology classes). This defines an almost complex structure on the left-moving target space. In addition, $\mathcal{M}_{\text{coassoc}}$ has a natural three-form Ω on it, defined as follows. Let V_1, V_2, V_3 be three vectors in $T\mathcal{M}_{\text{coassoc}}|_C$, and let v_1, v_2, v_3 be the corresponding normal vectors to C in X . We define

$$\Omega(V_1, V_2, V_3)|_C = \int_C \Phi(v_1, v_2, v_3) dV, \quad (2.8)$$

where Φ is the three-form defining the G_2 structure. When $b_2^+(C) = 3$, this is a top-dimensional form. We expect, by analogy with the special Lagrangian D-brane case, that this three-form gets complexified, to define a holomorphic three-form on the left-moving part of the five-brane moduli space. For more about geometric structures on $\mathcal{M}_{\text{coassoc}}$ see [21].

⁷This is pointed out in [25], for example.

2.4 Mirror Symmetry as Fourier-Mukai Transform

Using various dualities between string theory on Calabi-Yau manifolds and M-theory on G_2 -holonomy manifolds, we argued that such G_2 -holonomy manifolds should be fibered by coassociative tori or $K3$ surfaces. In this subsection, we come to the same conclusion using only string dualities and interpreting mirror symmetry for G_2 manifolds as a Fourier-Mukai transform on the coassociative fibers.

For concreteness, consider a $K3$ fibration, and let B be the base of this fibration:

$$\pi: X \rightarrow B. \quad (2.9)$$

Let us analyze in more detail the structure of this fibration. One natural question about this fibration by $K3$ surfaces is about the geometric meaning of the Fourier-Mukai transform acting on each fiber. By analogy with the SYZ conjecture [30], it is natural to expect that this transformation corresponds to a symmetry of the full quantum string theory on X , viz. to T-duality or mirror symmetry [29]. In order to follow the arguments of [30] in the G_2 case, let us go to Type-IIA theory on X .

We can take a D0-brane on X , with moduli space equal to X . Locally, we can identify the moduli with (p, q) , where q is the position of the D0-brane on B , and p is its position in a $K3$ fiber. On the 7-manifold X we can describe a D0-brane as a (non-holomorphic) skyscraper sheaf, \mathcal{E} , supported at (p, q) . Now since everything is going to happen in the fiber, we can think of \mathcal{E}_p as a (holomorphic) coherent sheaf on the $K3$ at q . Let $v(\mathcal{E}_p)$ be the corresponding Mukai vector:

$$v(\mathcal{E}_p) = \text{D-brane charge} = ch(\mathcal{E}_p) \sqrt{\hat{A}(K3)}. \quad (2.10)$$

In particular, for a D0-brane we have $v = (0, 0, -1)$.

Naively, one might expect that via Fourier-Mukai transform a D0-brane becomes a D4-brane wrapped on the entire $K3$ fiber. It should be also 1/2-BPS, so immediately we infer that $K3$ fibers should be volume minimizing, i.e. coassociative submanifolds inside X .

As the number of deformations of coassociative $K3$'s in X equals $b_2^+(K3) = 3$, it is natural to identify position on the base, $q \in B$, with the local coordinate on this moduli space. But this clearly cannot be the full story since the moduli space of the original D0-brane was 7-dimensional (a copy of X), and the same should be true for the dual D4-brane.

The solution is that after we make a Fourier-Mukai transform we obtain the Mukai vector

$$v_{dual} = (1, 0, 0). \quad (2.11)$$

This is not the right charge vector for a D4-brane on $K3$. Since $p_1(K3) = 48$, the latter would be $v = (1, 0, -1)$. So, after performing Fourier-Mukai transform we actually get a bound state of D4-brane and a D0-brane! It has the right charge vector, $v = (1, 0, 0)$, and the right dimension of the moduli space. In fact, according to Mukai, the real dimension of the moduli space of a sheaf \mathcal{E} with Mukai vector $v = (r, l, s)$ is

$$\dim(M(\mathcal{E})) = 2l^2 - 4rs + 4. \quad (2.12)$$

which is equal to 4 when $v = (1, 0, 0)$. So, the total dimensional of the moduli space of the dual D0/D4 bound state is indeed equal to 7, as expected, in complete analogy with the SYZ case. Note, that instead of $K3$ we could take T^4 as a coassociative fiber. In this case, the story is much easier: there is no induced D0-brane charge on the dual D4-brane, after we make four T-dualities along the T^4 . In this case again one has $b_3^-(T^4) = 3$ for the number of deformations of coassociative T^4 cycle, and $b_1(T^4) = 4$ for the number of moduli associated with flat connections. Hence, the total dimension of the moduli space of dual D4-brane is equal to 7, which is the right dimension to describe mirror G_2 manifold. This case was already studied in [2].

Rigidity of the Base ?

In both $K3$ and T^4 fibrations, we could take an appropriate D7-brane wrapped on the entire X , and conclude that B is itself a supersymmetric 3-cycle in X — an associative cycle. Indeed, dualizing the D7-brane along the fibers we find a D3-brane wrapped around the base B . Since the moduli spaces of these two D-branes should be the same, one might expect that both are rigid. In fact, the D7-brane does not have any geometric deformations. Furthermore, $\text{Hol}(X) = G_2$ implies $b_1(X) = 0$, which means that the space of flat $U(1)$ gauge connections on the D7-brane is also zero-dimensional. However, a complete answer to this question should involve a more careful analysis of the gauge bundle on the D7-brane, and it would interesting to study it further both from physics and mathematics points of view.

3 A $K3$ Fibration

3.1 Idea and Basic Set-Up

Imagine a G_2 manifold which is a $K3$ fibration over a base S^3 , with a discriminant locus Δ , which we assume to be a closed manifold of codimension two — a knot or link. If we consider the case of a non-satellite knot, then by Thurston's theorem there exists a hyperbolic metric on the complement $S^3 \setminus \Delta$. In this section, we use this reasoning to

look for a G_2 structure on a $K3$ fibration X over a non-compact hyperbolic manifold.⁸ For simplicity, we take the contractible hyperbolic space $B = SO(3, 1)/SO(3)$ for our base, with its hyperbolic metric g_B , left-invariant under the action of $SO(3, 1)$. Thus, $X = B \times K3$ as a differentiable manifold. We write

$$\pi : X \rightarrow B$$

for the projection to base. Note that at a point $p \in X$ the vertical vectors are defined as the kernel of π and span a sub-bundle $T_V X$ of TX , but there is no canonical notion of horizontal vectors until we have a *connection*, i.e. a choice of “horizontal” subbundle $T_H X$ of TX . Such a choice allows us to decompose TX as $TX = T_H X \oplus T_V X$, and we write P_H and P_V for the corresponding projection operators. We will discuss such a choice in section 3.2.

Choose over a point $b \in B$ a marking for the $K3$ fiber. Recall from Ref. [20] that the moduli space of Einstein metrics on a marked $K3$ manifold with unit volume is isomorphic to

$$\mathcal{M}_{K3} = SO(3, 19)/[SO(3) \times SO(19)].$$

(Quotienting on the left by $SO(3, 19; \mathbb{Z})$ removes the choice of marking.) Since we will wish to fiber X with Kähler-Einstein $K3$'s, we will employ a map from B to \mathcal{M}_{K3} . Let $\tau : B \rightarrow \mathcal{M}_{K3}$. The next section now shows that we can choose a family of Ricci-flat metrics for $K3$'s over B realizing the family defined by τ , and that we have a natural connection.

3.2 The fiberwise metric and connection

Let $RicMet$ be the space of unit-volume Ricci-flat metrics on a fixed differentiable $K3$ manifold. On $RicMet \times K3$ we have a natural fiberwise metric: at (p, q) we have the metric p defines at q . Then the group of diffeomorphisms $Diff$ acts on $RicMet$ on the right via pull-back, and on $K3$ on the left. The fiber product $RicMet \times_{Diff} K3$ inherits the fiberwise metric, and defines a universal family of Ricci-flat $K3$ manifolds over $\mathcal{M}_{K3} = RicMet/Diff$. This is universal in the sense that any other family of Ricci-flat $K3$ metrics can be mapped to the constructed family by a canonical family of diffeomorphisms, with the fiberwise metric defined by pull-back.

We now can assume that X has been chosen to realize the map τ , i.e. that $X \rightarrow B$ is a family of Ricci-flat $K3$'s such that the equivalence class of the metric on $\pi^{-1}(b)$ equals $\tau(b)$. We now construct a canonical connection on X . We need to define the horizontal sub-bundle of TX .

⁸This construction works for T^4 fibrations too, if in the following we simply replace $SO(3, 19)$ and its maximal compact subgroup by $SO(3, 3)$ with its corresponding subgroup.

We will define the horizontal part of a vector V in X at q , a point in the fiber over $b \in B$. Let $U \subset B$ be a neighborhood containing b such that $\pi^{-1}(U) \cong U \times K3$. Using a trivialization for the fiber bundle $X \rightarrow B$, we may assume we have a family of metrics on a fixed K3. Let t parametrize a path $\gamma(t)$ in the base through b such that $\dot{\gamma}(0) = \pi_* V$. Then g_t (the metric over $\gamma(t)$) defines a family of metrics, and we look for a family of diffeomorphisms $f_t : K3 \rightarrow K3$ such that

$$f_0 = \text{id}, \quad \frac{d}{dt}(f_t^* g_t) \perp \text{ gauge orbit of diffeos.} \quad (3.13)$$

Here perpendicularity is in the space of metrics, which is equipped with the natural metric on symmetric two-tensors. As we will show, this uniquely determines f_t . We therefore get a curve $\Gamma(t)$ passing through q defined by $\Gamma(t) = (\gamma(t), f_t(q))$, and we define the horizontal component of V to be

$$V_H \equiv \frac{d}{dt} \Gamma(t)|_{t=0}.$$

Lemma 5 *The conditions in (3.13) uniquely determine f_t .*

Proof. In fact we only need the first derivative of f_t at $t = 0$, which is defined by a vector field $\hat{\xi}$. Using the metric, we can equate this with a one form $\xi = \xi_\mu dx^\mu$, written in local coordinates x^μ . Let $\eta = \frac{d}{dt} g_t|_{t=0}$. Then

$$A_{\mu\nu} \equiv \frac{d}{dt}(f_t^* g_t)_{\mu\nu}|_{t=0} = \eta_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu.$$

Now perpendicularity of $A_{\mu\nu}$ to the gauge orbit under diffeomorphisms means

$$\nabla^\mu A_{\mu\nu} = 0. \quad (3.14)$$

Imposing this condition leads to the equation

$$\nabla^\mu \nabla_\mu \xi_\nu + \nabla^\mu \nabla_\nu \xi_\mu = -\nabla^\mu \eta_{\mu\nu} \equiv -B_\nu, \quad (3.15)$$

where we have defined the one-form $B_\nu = \nabla^\mu \eta_{\mu\nu}$. Now note

$$\nabla^\mu \nabla_\nu \xi_\mu = [\nabla^\mu, \nabla_\nu] \xi_\mu + \nabla_\nu \nabla^\mu \xi_\mu = R_\nu{}^\alpha \xi_\alpha + \nabla_\nu \nabla^\mu \xi_\mu = \nabla_\nu \nabla^\mu \xi_\mu,$$

where we have used Ricci flatness. Therefore, (3.15) becomes

$$\nabla^\mu \nabla_\mu \xi_\nu + \nabla_\nu \nabla^\mu \xi_\mu = -B_\nu,$$

or

$$d^\dagger d\xi + 2dd^\dagger \xi = B.$$

Using Hodge decomposition and the fact that $K3$ is simply-connected, we can write $\xi = dh + d^\dagger k$ for some function h and two-form k , and we may choose h and k perpendicular to the kernel of d and d^\dagger , respectively. Likewise, B has a decomposition as $dH + d^\dagger K$, with H and K chosen similarly. Therefore $d^\dagger dd^\dagger k + 2dd^\dagger dh = dH + d^\dagger K$, and we have

$$k = \frac{1}{dd^\dagger}K, \quad h = \frac{1}{2d^\dagger d}H.$$

This determines ξ , thus $\frac{d\xi}{dt}|_{t=0}$, uniquely. ■

The set of vectors V_H spans a sub-bundle $T_H \subset TX$. Further, we have a splitting $TX = T_H \oplus T_V$ defined on U , where T_V is the vertical sub-bundle. The definition of the splitting is independent of the trivialization, so we have shown:

Corollary 6 *The sub-bundle T_H is well-defined on all of X and defines a connection or splitting, $TX = T_H \oplus T_V$.*

3.3 Defining the Three-form, Φ

From the previous construction, we can now form a natural metric on X as follows. Let $p \in X$, $b = \pi(p)$ and $\mu = \tau(b)$, with g_K^μ the metric defined by $\mu \in \mathcal{M}_{K3}$. Let $U, V \in T_p X$. Then we can define

$$g(U, V) = g_B(\pi_* U, \pi_* V)|_b + g_K^\mu(P_V U, P_V V)|_p \quad (3.16)$$

(one could also multiply the two terms by positive functions of X), where P_V is the vertical projection. In particular, horizontal and vertical directions are declared to be perpendicular.

Note that both B and \mathcal{M}_{K3} are homogeneous spaces G/H . Then H acts on the tangent space of G/H , since the stabilizer of the transitive left G action at $[g]$ is $H_g \equiv gHg^{-1} \cong H$. This is obvious, since $(ghg^{-1})[g] = [gh] = [g]$, where $[g]$ denotes the coset gH .

Now let $\tilde{\tau} : SO(3, 1) \rightarrow SO(3, 19)$ be a group homomorphism such that

$$\tilde{\tau}(SO(3)) \subset SO(3) \times SO(19)$$

and

$$p_1 \circ \tilde{\tau} \text{ is an isomorphism,}$$

where p_1 is the projection to the first factor. Now $\tilde{\tau}$ induces a map

$$\tau : B \rightarrow \mathcal{M}_{K3},$$

equivariant in the following sense. Let $c \in SO(3)$, which acts on TB . Let $C = \tilde{\tau}(c) \in SO(3) \times SO(19)$ acting on $T\mathcal{M}_{K3}$. Then

$$\tau_* \circ c = C \circ \tau_*, \quad (3.17)$$

where τ_* is the push-forward of tangent vectors. Namely, we have equivariance under corresponding $SO(3)$ actions.⁹

We now try to construct a positive G_2 calibration Φ on X , so that Φ is a nowhere-vanishing, closed three-form which at every point $p \in X$ lies in the $GL(T_p X)$ orbit of the standard associative form Φ_0 (encoding the structure constants of multiplication on the imaginary octonians). Recall that the construction of Φ_0 involves writing \mathbb{R}^7 as $\text{Im}\mathbb{H} \oplus \mathbb{H}$, then identifying for each orthonormal basis element e_i in $\text{Im}\mathbb{H}$ a self-dual form α_i in $\Lambda^2\mathbb{H}$ which encodes multiplication (in \mathbb{H}) by e_i . For example, $\alpha_1 = (e_4 \wedge e_5 + e_6 \wedge e_7)$. Then $\Phi_0 = e_1 \wedge e_2 \wedge e_3 + \sum_{i=1}^3 e_i \wedge \alpha_i$. A G_2 form ϕ defines a metric g by $g(u, v)dV = \iota_u \phi \wedge \iota_v \phi \wedge \phi$, where $dV = \sqrt{\det g} e_1 \wedge \dots \wedge e_7$.

To construct such a Φ then, we must relate the tangential directions on B to self-dual forms on the fiber. (The metric on B allows us to equate tangent vectors and one-forms.) Recall that $TB|_{[g]} \cong \mathfrak{g}/\mathfrak{h}_g$, where $\mathfrak{g} = so(3, 1)$, $\mathfrak{h} = so(3)$, and we have defined $\mathfrak{h}_g = g\mathfrak{h}g^{-1} = \text{Ad}_g\mathfrak{h}$ (independent of the representative of the coset $[g]$). Our key construction will be the simple observation that

$$\mathfrak{g}/\mathfrak{h}_g \cong \mathfrak{h}_g^\perp \cong \mathfrak{h}_g \quad (3.18)$$

as vector spaces, where the first equivalence comes from the metric on \mathfrak{g} and the second comes from the (pseudo-)symplectic structure on $\mathfrak{g} = so(3, 1) \cong sl(2, \mathbb{C}) \cong \mathbb{C}^3$, by which \mathfrak{h} is a Lagrangian subspace.¹⁰ This allows us to associate to $V \in T_g B$ an element $C_V \in so(3)$. Next we shall get from C_V a self-dual form on the fiber. Recall now that \mathcal{M}_{K3} is a Grassmannian of positive, oriented three-planes in $\mathbb{R}^{3,19}$, which is interpreted as the plane of self-dual harmonic two-forms inside $H^2(K3)$. The first factor $SO(3) \subset SO(3) \times SO(19)$ acts on the three-plane. An element of $SO(3)$ singles out a direction defined by its zero eigenspace (the axis defining the rotation in three-space).

⁹This construction was meant to mimic the notion of holomorphicity for an elliptic fibration, which can be written as the equivariant property $\tau_* \circ j = J \circ \tau_*$, where j and J are complex structures on the base of the fibration and the moduli of elliptic curves, respectively. We thus think of a complex structure as the $\pi/4$ element of an $S^1 = SO(2)$ action. In our case, $SO(3)$ is the relevant group.

¹⁰Explicitly, at the identity coset, we identify $so(3, 1) = sl(2, \mathbb{C})$ with traceless, 2×2 complex matrices with the indefinite metric $\langle A, B \rangle = -\frac{1}{2}\text{Tr}(AB)$, then $so(3) = su(2)$ are the anti-hermitian ones, the map $\mathfrak{g}/\mathfrak{h} \rightarrow \mathfrak{h}^\perp$ sends $A \mapsto (A - A^\dagger)/2$. The involution from $\mathfrak{h}^\perp \leftrightarrow \mathfrak{h}$ is just $A \mapsto iA^\dagger$, or $\sqrt{-1}$ times the Cartan involution. Put physically, the projection to \mathfrak{h}^\perp eliminates rotational pieces of an infinitesimal Lorentz transformation, and then the correspondence $\mathfrak{h}^\perp \cong \mathfrak{h}$ associates to a pure boost in some direction a rotation in the same direction.

Therefore, from V at b we form C_V which maps by $p_1 \circ \tilde{\tau}$ to $so(3)$ inside the stabilizer of the three-plane of self-dual forms on $\pi^{-1}(b)$. Let θ_V be a generator of $\text{Ker}[p_1(\tilde{\tau}(C_V))]$, defined up to sign, and normalized so that $\int(\theta_V)^2/2 = |V|^2$. (The signs can be chosen consistent with the orientation.¹¹) This is our sought-after self-dual two-form, and we have found a map

$$\theta : T_b B \rightarrow H_+^2(\pi^{-1}(b)),$$

defined up to sign. In short, we have described another sequence of isomorphisms

$$\mathfrak{h}_g \cong so(3) \xrightarrow{p_1 \circ \tilde{\tau}} so(H_+^2(K3)) \cong H_+^2(K3). \quad (3.19)$$

θ is the composition of the isomorphisms in (3.18) and (3.19). Note that the forms in H_+^2 are harmonic, so no choice of representative class is necessary.

Now let $p \in X$ with $\pi(p) = [g] \in B$. Let $\{e_i\}$ be the pull-back of an orthonormal frame for $T_{[g]}^* B \cong \mathfrak{g}/\mathfrak{h}_g$, and let $\{\theta_i\}$ be the corresponding self-dual two-forms on $\pi^{-1}(b) \cong K3$. We construct

$$\Phi = Vol_B + \sum_i e_i \wedge \theta_i, \quad (3.20)$$

where Vol_B is the volume form on B and the pull-backs of the various forms to X are understood. More invariantly, we can write

$$\Phi(U, V, W) = Vol_B(\pi_* U, \pi_* V, \pi_* W) + (\theta_{\pi_* U}(P_V V, P_V W) + \text{cyclic}). \quad (3.21)$$

Note that as $\tilde{\tau}$ is an inclusion of groups, then since the e_i are orthogonal, the unit-norm self-dual forms θ_i are mutually orthogonal, and the metric defined from Φ agrees with (3.16).

3.4 Explicit formulas

The constructions above can be made explicit. Let $\eta_{\mu\nu}$ be a metric deformation of K3, perpendicular to diffeomorphisms.

Lemma 7 *η is locally volume-preserving.*

¹¹If we write an element C_V of $so(3)$ as $aX + bY + cZ$, then (a, b, c) defines the axis of rotation in three-space. After equating the three-plane \mathbb{R}^3 isometrically with $so(3)$, then θ_V is simply (a, b, c) . That is, $\text{End}(E) \cong E$ for oriented, three-dimensional metric vector spaces.

Proof. The globally volume-preserving deformation of the metric η induces a change of the Riemann tensor. Imposing the gauge condition (3.14) means $\nabla^\mu \eta_{\mu\alpha} = 0$. Working in Riemann normal coordinates and using the fact that the Ricci curvature is zero ($R^\mu{}_{\alpha\mu\beta} = 0$), one can compute that vanishing of the infinitesimal variation of the Ricci curvature is equivalent to

$$\nabla^\mu \nabla_\mu \eta_{\alpha\beta} + 2R_\alpha{}^\mu{}_\beta{}^\nu \eta_{\mu\nu} + \nabla_\alpha \nabla_\beta \eta^\mu{}_\mu = 0.$$

Multiplying by $g^{\alpha\beta}$ and summing, using the fact that the metric is covariantly constant and Ricci flat, gives

$$2\nabla^\mu \nabla_\mu \eta^\alpha{}_\alpha = 0.$$

Now $\Delta(\eta^\alpha{}_\alpha) = 0$ means that $(\eta^\alpha{}_\alpha)$ is a constant, C , but since η is globally volume-preserving, $\int C = C \cdot Vol = 0$, and so η is pointwise traceless. ■

Now let S^a , $a = 1, \dots, 3$, be an orthonormal set of self-dual forms. (In an orthonormal frame, the S^a are antisymmetric matrices obeying the algebra of the quaternions i, j, k , and η is traceless.) Then we can define the anti-self-dual forms A^a as follows:

$$(A^a)_{\mu\nu} = \frac{1}{2}(S^a{}_{\mu\sigma} \eta^\sigma{}_\nu + \eta_{\mu\sigma} S^{a\sigma}{}_\nu).$$

Conversely, if we have a trio of anti-self-dual forms A^a we can define a metric deformation

$$\eta_{\mu\nu} = - \sum_a A^a{}_{\mu\sigma} S^{a\sigma}{}_\nu.$$

These operations are indeed inverses of each other when η is traceless and symmetric.¹²

To define the self-dual two-form defined by a metric deformation, we can do the following. Fix at a point in K3 moduli space a four-plane W in $\mathbb{R}^{3,19} \cong H^2(K3)$ with signature $(3, 1)$ and containing $H^2_+(K3)$. Then choose $\tilde{\tau}$ to be an isomorphism $SO(3, 1) \cong SO(W)$. Then at each point $b \in B$ there is a unique anti-self-dual form β in W perpendicular to the S^a , so that $A^a = r^a \beta$ for all $a = 1, \dots, 3$. Now each tangent vector V on B defines a metric deformation η , which defines a trio of anti-self-dual forms A^a , which then define a single, self-dual form

$$\theta_V = \sum_a r^a S^a.$$

This is the isomorphism described in (3.18) and (3.19).

¹²This depends on some nice facts, including the following identity. Let A be a traceless, symmetric, four-by-four matrix acting on the quaternions \mathbb{R}^4 . Let I, J, K be matrices representing multiplication by i, j, k . Then $A = IAI + JAJ + KAK$.

3.5 $d\Phi \neq 0$

We have not been able to show that Φ is closed, and in fact this seems unlikely, despite the fact that the map τ was meant to mimick the (more successful) stringy cosmic string construction. However, we believe that this may lead to weak holonomy G_2 , for which $d\Phi = \lambda * \Phi$. Then, one would also hope that this construction can be modified to produce a torsion-free G_2 structure, hence a manifold with true G_2 holonomy. This is currently under investigation.

Also, it is worth mentioning that since our constructions are left-invariant under the transitive action of G on B , the behavior of Φ can be analyzed at a single point, e.g. the identity coset. Also, if $\tilde{\tau}$ maps the discrete subgroup $SO(3, 1; \mathbb{Z})$ to the subgroup $SO(3, 19; \mathbb{Z})$ then this entire construction will descend to the finite-volume quotient by this discrete group.

4 A Torus Fibration

4.1 Outline of Hitchin's Construction

Recently Hitchin has shown how certain functionals on differential forms in six dimensions generate metrics with G_2 and weak $SU(3)$ holonomy [17, 18]. Here, we outline his construction and use his result to construct new G_2 metrics. The main point is to consider the Hamiltonian flow of a volume functional on a symplectic space of stable three- and four-forms on a six-manifold. When a group acts on the six-fold, the invariant differential forms can restrict the infinite-dimensional variational problem to a finite-dimensional set of equations governing the evolution. Including the “time” direction, one is able to create a closed and co-closed G_2 three-form, thus a metric of G_2 holonomy.

The key element in the construction is the following:

Theorem 8 [18]: *Let M be a 6-manifold, $\mathcal{A} \in H^3(M, \mathbb{R})$ and $\mathcal{B} \in H^4(M, \mathbb{R})$ be fixed cohomology classes, and let $(\rho, \sigma) \in \mathcal{A} \times \mathcal{B}$ be stable forms of positive type which evolve via Hamiltonian flow of the functional:*

$$H = V(\rho) - 2V(\sigma). \tag{4.22}$$

Here, $V(\rho)$ and $V(\sigma)$ are suitable volume forms (which we define below), with ϕ their integrands: $V = \int_M \phi$. If for some $t = t_0$, ρ and σ satisfy the compatibility conditions $\omega \wedge \rho = 0$ and $\phi(\rho) = 2\phi(\sigma)$ (where $\sigma = \omega^2/2$) then the three-form

$$\Phi = dt \wedge \omega + \rho, \tag{4.23}$$

defines a G_2 structure on $X = M \times (a, b)$ for some interval (a, b) .

The converse is also true [18].

Stable forms are defined in an earlier paper by Hitchin:

Defintion 9 [17]: *Let M be a manifold of real dimension n , and $V = TM$. Then, the form $\rho \in \Lambda^p V^*$ is stable if it lies in an open orbit of the (natural) $GL(V)$ action on $\Lambda^p V^*$.*

In other words, this means that all forms in the neighborhood of ρ are $GL(V)$ -equivalent to ρ . This definition is useful because it allows one to define a volume. For example, a symplectic form ω is stable if and only if $\omega^{n/2} \neq 0$.

Relevant to our discussion are 3-forms and 4-forms on a 6-manifold M . If these forms are stable, we can define the corresponding volumes as follows. Let's start with a stable 4-form

$$\sigma \in \Lambda^4 V^* \cong \Lambda^2 V \otimes \Lambda^6 V^*.$$

Therefore, we find

$$\sigma^3 \in \Lambda^6 V \otimes (\Lambda^6 V^*)^3 \cong (\Lambda^6 V^*)^2$$

and

$$V(\sigma) = \int_M |\sigma^3|^{\frac{1}{2}}. \quad (4.24)$$

In order to define the volume $V(\rho)$ for a 3-form $\rho \in \Lambda^3 V^*$, one first defines a map

$$K_\rho: V \rightarrow V \otimes \Lambda^6 V^*,$$

such that for a vector $v \in V = TM$ it gives

$$K(v) = \iota(v)\rho \wedge \rho \in \Lambda^5 V^* \cong V \otimes \Lambda^6 V^*. \quad (4.25)$$

Hence, one can define

$$\text{tr}(K^2) \in (\Lambda^6 V^*)^2.$$

Since stable forms with stabilizer $SL(3, \mathbb{C})$ are characterised by $\text{tr}(K^2) < 0$, following [17], we define

$$V(\rho) = \int_M |\sqrt{-\text{tr}K^2}|. \quad (4.26)$$

The last fact used in the Hitchin's theorem is that there is a natural symplectic structure on the space

$$\mathcal{A} \times \mathcal{B} \cong \Omega_{exact}^3(M) \times \Omega_{exact}^4.$$

Explicitly, it can be written as

$$\omega((\rho_1, \sigma_1), (\rho_2, \sigma_2)) = \langle \rho_1, \sigma_2 \rangle - \langle \rho_2, \sigma_1 \rangle,$$

where, in general, for $\rho = d\beta \in \Omega_{exact}^p(M)$ and $\sigma = d\gamma \in \Omega_{exact}^{n-p+1}(M)$ one has a nondegenerate pairing

$$\langle \rho, \sigma \rangle = \int_M d\beta \wedge \gamma = (-1)^p \int_M \beta \wedge d\gamma. \quad (4.27)$$

Then, Hitchin shows that the first-order Hamiltonian flow equations in the theorem quoted above are equivalent to the closure and co-closure of the associative form $\Phi = dt \wedge \omega + \rho$:

$$d\Phi = 0, \quad d * \Phi = 0.$$

In order to construct the metric with G_2 holonomy from the form Φ we should take $v, w \in W$, where $W = TX$ is the seven-dimensional vector space and define a symmetric bilinear form on W with values in $\Lambda^7 W^*$ by

$$B_\Phi = -\frac{1}{6} \iota(v)\Phi \wedge \iota(w)\Phi \wedge \Phi. \quad (4.28)$$

This defines a linear map $K_\Phi : W \rightarrow W^* \otimes \Lambda^7 W^*$. Then the G_2 holonomy metric can be written as [17]

$$g_\Phi(v, w) = B_\Phi(v, w)(\det K_\Phi)^{-\frac{1}{9}}. \quad (4.29)$$

4.2 Equations for the Metric

Now let us consider an example of (non-compact) G_2 manifold, with principal orbits

$$M = S^3 \times T^3.$$

We think of S^3 as a group manifold $SU(2)$. The space M appears as one of the examples in the recent work of Cleyton and Swann [9], where they classified principal orbits of cohomogeneity-one G_2 manifolds under a compact, connected Lie group.

In order to construct differential forms ρ and σ , let us choose a basis of left-invariant one-forms on $SU(2)$:

$$\begin{aligned} \Sigma_1 &= \cos \psi d\theta + \sin \psi \sin \theta d\phi, \\ \Sigma_2 &= -\sin \psi d\theta + \cos \psi \sin \theta d\phi, \\ \Sigma_3 &= d\psi + \cos \theta d\phi. \end{aligned} \quad (4.30)$$

which enjoy the $su(2)$ algebra

$$d\Sigma_a = -\frac{1}{2}\epsilon_{abc}\Sigma_b \wedge \Sigma_c.$$

We also choose closed, but not exact one-forms α_i , which generate the $H^1(T^3)$ cohomology of the torus:

$$\alpha_{1,2,3} \in H^1(T^3) = \mathbb{R}^3.$$

Explicitly, if we define

$$T^3 = \mathbb{R}^3/\mathbb{Z}^3,$$

where \mathbb{R}^3 is parametrized by affine coordinates u_1 , u_2 , and u_3 , we can write

$$\alpha_i = du_i, \quad i = 1, 2, 3.$$

Now, we have to fix cohomology classes \mathcal{A} and \mathcal{B} in $H^3(M)$ and in $H^4(M)$, respectively. The cohomology groups are non-trivial:

$$\begin{aligned} H^3(M; \mathbb{R}) &= \mathbb{R} \oplus \mathbb{R}, \\ H^4(M; \mathbb{R}) &= \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}, \end{aligned}$$

so that our choice depends on five *real* parameters that we call m , n , k_1 , k_2 , and k_3 . As we will see in a moment, the construction depends only on the parameters m and n , which determine the class $\mathcal{A} \in H^3(M; \mathbb{R})$. Specifically, m corresponds to the class $[T^3]$ and n corresponds to the class $[S^3]$.

Now we can write the $SU(2)$ -invariant 3-form $\rho \in \mathcal{A}$ as

$$\rho = n\Sigma_1\Sigma_2\Sigma_3 - m\alpha_1\alpha_2\alpha_3 + x_1d(\Sigma_1\alpha_1) + x_2d(\Sigma_2\alpha_2) + x_3d(\Sigma_3\alpha_3).$$

Here, $x_i(t)$ are functions of the extra variable t that describe variation of the 3-form ρ within a given cohomology class (determined by m and n). The radial direction t is going to play the role of time variable for the Hamiltonian evolution. Clearly, the form ρ is colsed.

Similarly, we can write a natural 4-form:

$$\begin{aligned} \sigma &= k_1\Sigma_1\Sigma_2\Sigma_3\alpha_1 + k_2\Sigma_1\Sigma_2\Sigma_3\alpha_2 + k_3\Sigma_1\Sigma_2\Sigma_3\alpha_3 \\ &+ y_1\Sigma_2\alpha_2\Sigma_3\alpha_3 + y_2\Sigma_3\alpha_3\Sigma_1\alpha_1 + y_3\Sigma_1\alpha_1\Sigma_2\alpha_2. \end{aligned}$$

The first line in this expression is cohomologically non-trivial, whereas the second line contains three exact terms. Indeed,

$$\Sigma_2\alpha_2\Sigma_3\alpha_3 = d(\Sigma_1)\alpha_2\alpha_3 = d(\Sigma_1\alpha_2\alpha_3),$$

and similarly for other terms. Therefore, both ρ and σ are closed forms.

In order to see that parameters $k_{1,2,3}$ are irrelevant, let us evaluate the volume corresponding to the form σ :

$$V^2(\sigma) = y_1 y_2 y_3.$$

Since it does not depend on the choice of the cohomology class, in what follows we set $k_1 = k_2 = k_3 = 0$. Hence, the $SU(2)$ -invariant 4-form σ can be written as

$$\sigma = y_1 d(\Sigma_1 \alpha_2 \alpha_3) + y_2 d(\Sigma_2 \alpha_3 \alpha_1) + y_3 d(\Sigma_3 \alpha_1 \alpha_2).$$

Finally, we want to show that σ can be written as $\omega^2/2$ for some two-form ω , and that $\omega \wedge \rho = 0$. Explicitly, we can write

$$\omega = \sqrt{\frac{y_2 y_3}{y_1}} \Sigma_1 \alpha_1 + \sqrt{\frac{y_1 y_3}{y_2}} \Sigma_2 \alpha_2 + \sqrt{\frac{y_1 y_2}{y_3}} \Sigma_3 \alpha_3.$$

It is straightforward to check that this form ω satisfies the required properties, namely

$$\sigma = \frac{1}{2} \omega^2,$$

and

$$\omega \wedge \rho = 0.$$

The last thing we need to check before we proceed to the Hamiltonian flow is to make sure that $x_i(t)$ and $y_i(t)$ are conjugate coordinate and momenta. In other words, we need to show that there is a non-degenerate pairing between invariant 3-forms and 4-forms:

$$\langle \Sigma_2 \alpha_2 \Sigma_3 \alpha_3, d(\Sigma_1 \alpha_1) \rangle = \int_{S^3 \times T^3} \Sigma_2 \alpha_2 \Sigma_3 \alpha_3 \Sigma_1 \alpha_1 = \text{vol}(S^3) \text{vol}(T^3) \neq 0.$$

Therefore, just as in the model with $SU(2) \times SU(2)$ principal orbits [18, 7], it turns out that the symplectic form is a multiple of

$$dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3.$$

Using (4.26) we find a simple expression for $V(\rho)$:

$$V^2(\rho) = -m^2 n^2 - 4m x_1 x_2 x_3.$$

Since both $V(\rho)$ and $V(\sigma)$ must be real, we have two constraints:

$$y_1 y_2 y_3 > 0, \quad 4m x_1 x_2 x_3 < -m^2 n^2. \quad (4.31)$$

Provided these relations are satisfied, we can write the Hamiltonian

$$\begin{aligned} H &= V(\rho) - 2V(\sigma) \\ &= \sqrt{-m^2n^2 - 4mx_1x_2x_3} - 2\sqrt{y_1y_2y_3}, \end{aligned} \quad (4.32)$$

which is constrained by the hypothesis of Theorem 8 to be zero.

Now choosing $i \neq j \neq k \neq i$ among 1, 2, 3, the corresponding Hamiltonian flow equations read

$$\begin{aligned} \dot{y}_i &= \frac{2mx_jx_k}{\sqrt{-m^2n^2 - 4mx_ix_jx_k}}, \\ \dot{x}_i &= -\sqrt{\frac{y_jy_k}{y_i}}. \end{aligned} \quad (4.33)$$

A solution to these first-order differential equations define a G_2 structure on $(a, b) \times S^3 \times T^3$. Explicitly, the associative 3-form is given by

$$\begin{aligned} \Phi &= dt \wedge \left(\sqrt{\frac{y_2y_3}{y_1}} \Sigma_1 \alpha_1 + \sqrt{\frac{y_1y_3}{y_2}} \Sigma_2 \alpha_2 + \sqrt{\frac{y_1y_2}{y_3}} \Sigma_3 \alpha_3 \right) + \\ &+ n \Sigma_1 \Sigma_2 \Sigma_3 - m \alpha_1 \alpha_2 \alpha_3 + x_1 d(\Sigma_1 \alpha_1) + x_2 d(\Sigma_2 \alpha_2) + x_3 d(\Sigma_3 \alpha_3). \end{aligned} \quad (4.34)$$

It follows that for $n = 1$ the 3-sphere $B = SU(2)$ is an associative submanifold inside $X = (a, b) \times SU(2) \times T^3$, while the non-compact fiber $(a, b) \times T^3$ is coassociative.

Moreover, from the expression (4.34) for the associative 3-form Φ , it follows that the volume of S^3 and T^3 (measured with respect to the G_2 -holonomy metric (4.29) obtained from Φ) is bounded below:

$$\text{Vol}(S^3) \geq |n|, \quad (4.35)$$

$$\text{Vol}(T^3) \geq |m|. \quad (4.36)$$

In general, the G_2 -holonomy metric looks like

$$\begin{aligned} ds^2 &= \det(K_\Phi)^{-1/9} \left(\sqrt{y_1y_2y_3} dt^2 + \sqrt{\frac{y_2y_3}{y_1}} (x_2x_3 \Sigma_1^2 + mn \Sigma_1 \alpha_1 - mx_1 \alpha_1^2) + \right. \\ &+ \left. \sqrt{\frac{y_1y_3}{y_2}} (x_1x_3 \Sigma_2^2 + mn \Sigma_2 \alpha_2 - mx_2 \alpha_2^2) + \sqrt{\frac{y_1y_2}{y_3}} (x_1x_2 \Sigma_3^2 + mn \Sigma_3 \alpha_3 - mx_3 \alpha_3^2) \right) \end{aligned} \quad (4.37)$$

where

$$\det(K_\Phi) = -m^3 (y_1y_2y_3)^{3/2} \left(x_1x_2x_3 + \frac{1}{4} mn^2 \right)^3 = (y_1y_2y_3)^{9/2}, \quad (4.38)$$

where in the last equality we used the conservation of the Hamiltonian, $H = 0$. Then, the overall factor $\det(K_\Phi)^{-1/9}$ in the metric (4.37) cancels the coefficient in front of dt^2 , so that the resulting expression looks like

$$ds^2 = dt^2 + \frac{1}{y_1}(x_2x_3\Sigma_1^2 + mn\Sigma_1\alpha_1 - mx_1\alpha_1^2) + \frac{1}{y_2}(x_1x_3\Sigma_2^2 + mn\Sigma_2\alpha_2 - mx_2\alpha_2^2) + \frac{1}{y_3}(x_1x_2\Sigma_3^2 + mn\Sigma_3\alpha_3 - mx_3\alpha_3^2). \quad (4.39)$$

4.3 $SU(2)$ Symmetric Solution and Large Distance Asymptotics

Let us study various limits of the new G_2 metric, and try to understand the role of various parameters, m and n , in particular. It is instructive to look first at the simple case, where:

$$x_1 = x_2 = x_3, \quad y_1 = y_2 = y_3. \quad (4.40)$$

This set of extra conditions restricts us to a class of metrics with extra $SU(2)$ symmetry. As will be shown below, a study of much simpler $SU(2)$ -invariant metrics illustrates all important properties of the generic solutions to (4.33).

The extra conditions (4.40) lead to a consistent truncation of the first-order system (4.33), *cf.* [8] and [7]:

$$\begin{aligned} \dot{x} &= -\sqrt{y}, \\ \dot{y} &= \frac{2mx^2}{\sqrt{-m^2n^2 - 4mx^3}}. \end{aligned} \quad (4.41)$$

Without loss of generality, we can assume that m is positive. It implies that the values of $x(t)$ and $y(t)$ range in

$$\begin{aligned} -\infty < x(t) &\leq -\left(\frac{mn^2}{4}\right)^{1/3}, \\ 0 &\leq y(t) < +\infty. \end{aligned} \quad (4.42)$$

Since the system (4.41) is Hamiltonian, we have one obvious integral of motion, namely $H(t) = 0$ which we express in the form

$$-mx^3 = y^3 + \frac{1}{4}m^2n^2. \quad (4.43)$$

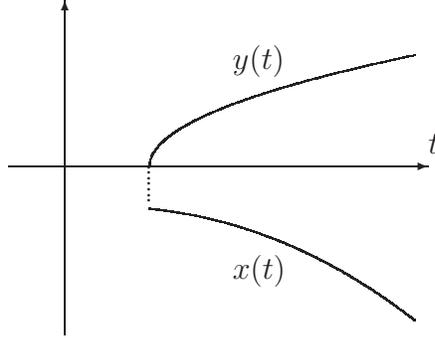


Figure 2: The behavior of functions $x(t)$ and $y(t)$.

This allows to reduce the system (4.41) to a single differential equation of a hypergeometric type,

$$\frac{dy}{dt} = \frac{m}{y^{3/2}} \left(\frac{1}{m} y^3 + \frac{1}{4} mn^2 \right)^{2/3}, \quad (4.44)$$

which is exactly solvable:

$$t = t_0 + \frac{1}{5} \left(\frac{2^7}{m^5 n^4} \right)^{1/3} y^{5/2} F \left(\left[\frac{5}{6}, \frac{2}{3} \right], \left[\frac{11}{6} \right], -\frac{4y^3}{m^2 n^2} \right). \quad (4.45)$$

This solution leads to a simple G_2 -holonomy metric:

$$ds^2 = dt^2 + \frac{1}{y} \sum_{j=1}^3 (x^2 \Sigma_j^2 + mn \Sigma_j \alpha_j - mx \alpha_j^2),$$

with the isometry group:

$$SU(2) \times SU(2) \times U(1)^3.$$

The behavior of $x(t)$ and $y(t)$ is sketched in Fig. 2. Next, let us analyze various limits of this metric.

Solution with $n = 0$ and Large Distance Asymptotics

In the special case $n = 0$, the solution takes a very simple polynomial form:

$$x = -\frac{m^{1/3}}{4} (t - t_0)^2, \quad y = \frac{m^{2/3}}{4} (t - t_0)^2. \quad (4.46)$$

The corresponding metric looks like (for simplicity, we put the integration constant $t_0 = 0$)

$$ds^2 = dt^2 + \frac{1}{4} t^2 (\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2) + m^{2/3} (\alpha_1^2 + \alpha_2^2 + \alpha_3^2). \quad (4.47)$$

A numerical factor $1/4$ in front of the Σ_i terms is crucial here for these terms to become the usual Einstein metric on the round 3-sphere. Hence, the above expression is nothing but the usual Ricci-flat metric on

$$\mathbb{R}^4 \times T^3. \tag{4.48}$$

The metric on the regular 3-torus in this solution depends on the value of the parameter m . Namely, an easy computation gives the asymptotic volume of the T^3 in this metric:

$$\text{Vol}(T^3)|_{t=\infty} = m, \tag{4.49}$$

which is in complete agreement with the bound (4.36). It is natural to expect that more general solutions, without the $SU(2)$ symmetry, exhibit a similar behavior. In the next subsection we will show that this is indeed the case; changing various parameters it is easy to modify the asymptotic shape of the 3-torus, but not the overall volume, which is determined only by m . The metric (4.47) is manifestly Ricci-flat for all values of the parameter m , so it is clearly a modulus.

Another remark is that (4.47) describes the asymptotic behavior of the metric with non-zero n at large distances. Indeed, if the absolute value of $y(t)$ (and, therefore, of $x(t)$ as well) grows as $t \rightarrow \infty$, the term with n^2 in the first order equation (4.44) can be neglected and one finds a simple ODE:

$$\frac{dy}{dt} = m^{1/3} \sqrt{y},$$

which leads to the *approximate* solution (4.46). Therefore, even for $SU(2)$ symmetric solutions with non-zero parameter n , the metric is asymptotic to $\mathbb{R}^4 \times T^3$ at large distances. Of course, for $n \neq 0$ the metric can no longer have global topology of $\mathbb{R}^4 \times T^3$ because the volume of the 3-sphere is bounded by $|n|$, *cf.* (4.35). Nevertheless, as we pointed out, the parameter n does not change the asymptotic behaviour of the metric.

4.4 $U(1)$ Symmetric Solution and Short Distance Asymptotics

We have studied the asymptotic behavior of the $SU(2)$ symmetric solution when the functions $x(t)$ and $y(t)$ approach one limit in the range of allowed values (4.42). This limit corresponds to large distance asymptotics of the G_2 -holonomy metric. Here, we discuss the other limit,

$$x \rightarrow -\left(\frac{mn^2}{4}\right)^{1/3}, \quad y \rightarrow 0,$$

as t approaches some value, say $t \rightarrow 0$.

In the special case $n = 0$, we have also seen that the metric is non-singular in this limit since the principal orbit $M = S^3 \times T^3$ degenerates into T^3 at $t = 0$ in such a way that the total space has topology (4.48):

$$X \cong \mathbb{R}^4 \times T^3$$

On the other hand, if $m \neq 0$ and $n \neq 0$ the constraints (4.35) and (4.36) prevent both S^3 and T^3 cycles from shrinking. In such cases one finds a rather exotic metric of the form:

$$ds^2 = dt^2 + t^{-2/5} \sum_{j=1}^3 (\Sigma_j + \alpha_j)^2 + \dots$$

where the dots stand for the terms vanishing in the limit $t \rightarrow 0$.

In order to find other G_2 -holonomy metrics with more regular behavior at $t = 0$, we have to allow some cycle to collapse and relax the $SU(2)$ symmetry condition. The natural step to consider between $SU(2)$ symmetry and no symmetry at all is when only a $U(1) \subset SU(2)$ symmetry group is preserved. This can be achieved, for example, by imposing the following conditions:

$$x_1 = x_2, \quad y_1 = y_2.$$

The corresponding expression for the metric (4.39) looks like:

$$\begin{aligned} ds^2 = & dt^2 + \frac{x_1 x_3}{y_1} (\Sigma_1^2 + \Sigma_2^2) + \frac{mn}{y_1} (\Sigma_1 \alpha_1 - \Sigma_2 \alpha_2) - \frac{m x_1}{y_1} (\alpha_1^2 + \alpha_2^2) + \\ & + \frac{1}{y_3} (x_1^2 \Sigma_3^2 + mn \Sigma_3 \alpha_3 - m x_3 \alpha_3^2). \end{aligned} \quad (4.50)$$

This leads to a rather simple system of only four first-order equations:

$$\begin{aligned} \dot{x}_1 = -\sqrt{y_3}, \quad \dot{y}_1 = \frac{m x_1 x_3}{y_1 \sqrt{y_3}}, \\ \dot{x}_3 = -\frac{y_1}{\sqrt{y_3}}, \quad \dot{y}_3 = \frac{m x_1^2}{y_1 \sqrt{y_3}}. \end{aligned} \quad (4.51)$$

In some special cases, this system has simple explicit solutions. For example, if $n = 0$ one finds

$$t = t_0 - x_1 \left(-\frac{B^2}{\beta m} \right)^{1/4} F \left(\left(\left[\frac{1}{2}, \frac{1}{4} \right], \left[\frac{3}{2} \right], -B x_1^2 / \beta \right), \quad (4.52)$$

where

$$\beta = x_3 y_3 - x_1 y_1 \quad (4.53)$$

and $B \leq 0$ is assumed to be non-zero. In the case $B = 0$ we recover the $SU(2)$ symmetric solution (4.46). Notice, β is an integral of motion. In the next subsection we explain this in more detail and find all the integrals of motion for the general first-order system (4.33).

However, trying to find a general solution to the reduced first-order system (4.51) amounts, essentially, to solving the general system (4.33). It can be written in terms of the Weierstrass function and will be studied in the next subsection. Here, let us consider approximate solutions at $t \rightarrow 0$. There are many possibilities corresponding to different assumptions about the vanishing of the functions x_i and y_i at $t = 0$. We consider just one such possibility corresponding to a solution, where the S^3 cycle degenerates into a two-sphere:

$$S^3 \rightarrow S^2, \quad t \rightarrow 0.$$

Of course, we also have to assume $n = 0$ in order to obey the condition (4.35). Specifically, this solution corresponds to $x_1(t)$ and $y_1(t)$ vanishing at $t = 0$, whereas $x_3(t)$ and $y_3(t)$ are assumed to take finite non-zero values at this point. Therefore, from the integral of motion (4.53) we get ($\beta < 0$)

$$y_3 \approx \frac{\beta}{x_3}, \tag{4.54}$$

and the conservation of the Hamiltonian (4.32) gives

$$y_1 \approx \sqrt{-\frac{m}{\beta}} x_1 x_3$$

For x_1 and x_3 we find a simple system of the first-order differential equations, which has a nice trigonometric solution:

$$\begin{aligned} x_1 &= -A \sin(\gamma t), & y_1 &= \frac{\sqrt{-m\beta} \sin(\gamma t)}{\gamma^2 A \cos^2(\gamma t)}, \\ x_3 &= \frac{\beta}{\gamma^2 A^2 \cos^2(\gamma t)}, & y_3 &= \gamma^2 A^2 \cos^2(\gamma t), \end{aligned} \tag{4.55}$$

with

$$\gamma^2 = \sqrt{\frac{m}{4|\beta|}}.$$

The approximation (4.54) is valid for $|\beta| \gg |x_1 y_1|$, which means

$$\tan^2(\gamma t) \ll 2$$

In particular, this includes the interesting range of small t , where the solution approximately behaves as

$$\begin{aligned} x_1 &\approx -A\gamma t, & y_1 &\approx \frac{\sqrt{-m\beta t}}{\gamma A}, \\ x_3 &\approx \frac{\beta}{\gamma^2 A^2}, & y_3 &\approx \gamma^2 A^2. \end{aligned} \quad (4.56)$$

The corresponding metric looks like

$$ds^2 \approx dt^2 + \sqrt{\frac{|\beta|}{m}}(\Sigma_1^2 + \Sigma_2^2) + \frac{mA^2}{2|\beta|}(\alpha_1^2 + \alpha_2^2) + t^2\Sigma_3^2 + \frac{4\beta^2}{A^4}\alpha_3^2. \quad (4.57)$$

It is natural to expect that for $\gamma t \gg 1$ this metric interpolates to the asymptotic metric (4.46). Then, we obtain a smooth manifold X with G_2 holonomy, such that

$$X \cong S^2 \times \mathbb{R}^2 \times T^3, \quad (4.58)$$

where the volume of the two-sphere is proportional to $\sqrt{|\beta|}$. In the next subsection we verify that our expectation is correct by constructing the explicit solution to the general first-order system (4.33).

4.5 General Solution

Given

$$H = \sqrt{-m^2n^2 - 4mx_1x_2x_3} - 2\sqrt{y_1y_2y_3} = V(\rho) - 2V(\sigma),$$

we recall that $H = 0$, i.e. $V(\rho) = 2V(\sigma)$, along our orbit. Then consider the Hamiltonian

$$\tilde{H} = V(\rho)^2 - 4V(\sigma)^2 = -m^2n^2 - 4mx_1x_2x_3 - 4y_1y_2y_3.$$

Since $d(V(\rho)^2 - 4V(\sigma)^2) = 2V(\rho)(dV(\rho) - 2dV(\sigma))$ along the orbit, the orbits are the same, though parametrized differently. Indeed,

$$\frac{dt}{d\tilde{t}} = 2V(\rho) = 4V(\sigma) = 4\sqrt{y_1y_2y_3}. \quad (4.59)$$

Now the action $(\vec{x}, \vec{y}) \mapsto (M \cdot \vec{x}, M^{-1} \cdot \vec{y})$, where M is a diagonal matrix with determinant one, is symplectic and leaves the Hamiltonian invariant. This symmetry group is two dimensional, and the corresponding conserved quantities are

$$x_1y_1 - x_3y_3 \quad \text{and} \quad x_2y_2 - x_3y_3.$$

Now choose $i \neq j \neq k \neq i$. The equations of motion are

$$\dot{x}_i = -4y_j y_k; \quad \dot{y}_i = 4m x_j x_k$$

where we now are using a dot over a variable to represent $\frac{d}{dt}$. Let's define

$$z_i = x_i y_i.$$

Then $\frac{d}{dt}(z_i - z_j) = 0$, as these are our conserved quantities. Therefore, we may write

$$\begin{aligned} z &\equiv z_1 \\ z_2 &= z + \alpha \\ z_3 &= z + \beta. \end{aligned} \tag{4.60}$$

Now define

$$X \equiv x_1 x_2 x_3 \quad Y \equiv y_1 y_2 y_3.$$

Since our Hamiltonian is zero on our orbit, we have

$$m^2 n^2 + 4mX + 4Y = 0. \tag{4.61}$$

Now compute

$$\dot{z} = \dot{x}_1 y_1 + x_1 \dot{y}_1 = -4Y + 4mX = -m^2 n^2 - 8Y.$$

But note

$$\dot{Y} = 4m(x_2 x_3 y_2 y_3 + x_1 x_3 y_1 y_3 + x_1 x_2 y_1 y_2),$$

so we see

$$\ddot{z} = -32m(3z^2 + 2z(\alpha + \beta) + \alpha\beta). \tag{4.62}$$

Assuming we can solve this equation, we can get

$$Y = -(\dot{z} + m^2 n^2)/8, \tag{4.63}$$

and we can find the individual x_i and y_i as follows: $x_1 = -4y_2 y_3 = -4Y/y_1 = -4(Y/z)x_1$, so we see

$$x_1 = -A_1 \exp[-4 \int (Y/z) d\tilde{t}], \tag{4.64}$$

with A_1 a (positive) constant. Then $y_1 = z/x_1$. The quantities x_2, y_2, x_3, y_3 are similarly calculated, with one of the integration constants fixed by the constraint (4.61). Finally, to connect with the form of the metric in (4.37) we must rewrite our answers in terms of t , which means solving (4.59):

$$\frac{dt}{d\tilde{t}} = 4\sqrt{Y}.$$

Equation for z

So let's try to solve the equation (4.62). First note that by a change of variables

$$u = -96mz - 32m(\alpha + \beta)$$

the equation takes the form

$$\ddot{u} = u^2 + D \tag{4.65}$$

where

$$D = (32m)^2(3\alpha\beta - (\alpha + \beta)^2) = -(32m)^2(\alpha^2 - \alpha\beta + \beta^2).$$

We can try to solve this equation with a solution satisfying a first-order equation $\dot{u} = f(u)$. Then $\ddot{u} = f'\dot{u} = ff'$, so

$$(f^2)' = 2(u^2 + D),$$

and $f^2 = \frac{2}{3}u^3 + Du + E$. Therefore, $f = \pm\sqrt{\frac{2}{3}u^3 + Du + E}$ and we see $\dot{u} = f(u)$ has the solution

$$\tilde{t} = C \pm \int \frac{du}{\sqrt{\frac{2}{3}u^3 + Du + E}},$$

which is the equation for the integral of an abelian differential on the elliptic curve

$$y^2 = \frac{2}{3}x^3 + Dx + E.$$

Therefore, to express u in terms of \tilde{t} we need to invert this (Abel-Jacobi) map. This is precisely what the Weierstrass function does!

To make this explicit, we'll want to put things in Weierstrass form. So let's make the change of variables $v = 6u$ and define

$$g_2 \equiv -D/3.$$

The equation for v becomes

$$\ddot{v} = 6v^2 - g_2/2 \tag{4.66}$$

with the solutions

$$\tilde{t} = C \pm \frac{dv}{\sqrt{4v^3 - g_2v - g_3}}$$

where C and g_3 are constants. To invert this, we simply use the Weierstrass function

$$v = \mathbf{p}_\tau(\tilde{t} + C),$$

where τ is the modular parameter determined by g_2 and the constant g_3 . Note that this is a two-parameter family of solutions (since the Weierstrass function is even, we don't gain anything by including the solution with $-\tilde{t} + C$ as the argument). To be sure, note that \mathbf{p} famously satisfies the differential equation

$$(\mathbf{p}')^2 = 4\mathbf{p}^3 - g_2\mathbf{p} - g_3.$$

Differentiating once more, we see

$$2\mathbf{p}'\mathbf{p}'' = 12\mathbf{p}^2\mathbf{p}' - g_2\mathbf{p}',$$

so

$$\mathbf{p}'' = 6\mathbf{p}^2 - g_2/2.$$

Since the lattice of the torus is rectangular when g_2 and g_3 are real, the Weierstrass function has complex conjugation symmetry, i.e. it's real.

For some special examples, when $g_2 = 8$, we have the simple solution

$$v = \csc^2(\tilde{t} + C),$$

and if $g_2 = 4/3$ we have the solution

$$v = \csc^2(\tilde{t} + C) - 1/3.$$

Asymptotics and Behavior Near the Poles

Note that the Weierstrass \mathbf{p} -function has a second-order pole. Below we argue that it corresponds to the asymptotic region of our solution, where the metric has the same asymptotic $\mathbb{R}^4 \times T^3$ behavior as in (4.48).

Specifically, near a pole we have

$$v \approx \tilde{t}^{-2}.$$

In order to verify that this gives an approximate solution to the eqn. (4.66), note that for large v the constant term $g_2/6$ can be neglected. Unwinding the definitions gives (omitting the subleading terms)

$$z \approx -\frac{u}{96m} \approx -\frac{v}{576m} \approx -\frac{1}{576m\tilde{t}^2}.$$

Therefore, $x_i(\tilde{t})$ and $y_i(\tilde{t})$ look like

$$x_i \sim y_i \sim \frac{1}{\tilde{t}},$$

and

$$Y \approx \frac{1}{2304m\tilde{t}^3}.$$

Using the relation (4.59) between t and \tilde{t} , we find the asymptotic solution in terms of the original variable t :

$$\begin{aligned} x_1 &= -A_1 t^2, & y_1 &= \frac{m}{16A_1} t^2, \\ x_2 &= -A_2 t^2, & y_2 &= \frac{m}{16A_2} t^2, \end{aligned} \tag{4.67}$$

$$x_3 = -A_3 t^2, \quad y_3 = \frac{m}{16A_3} t^2, \tag{4.68}$$

where we assume that the integration constants A_i are positive (to agree with our earlier conventions $x_i < 0$). As we pointed out earlier, one of the integration constants is not independent due to the constraint (4.61), which reads

$$64A_1 A_2 A_3 = m. \tag{4.69}$$

Hence, we can eliminate one of the constants A_i , say, A_3 :

$$A_3 = \frac{m}{64A_1 A_2}.$$

It is easy to check directly that (4.67) is an asymptotic solution to the original first order system (4.33). In the limit when x_i are large (equivalently, in the limit $n \rightarrow 0$), the equations look like

$$\dot{y}_i = \sqrt{-\frac{m x_j x_k}{x_i}}, \quad \dot{x}_i = -\sqrt{\frac{y_j y_k}{y_i}}. \tag{4.70}$$

Analysis, similar to what we have done in the $SU(2)$ -invariant case, shows that x_i and y_i have to grow as t^2 , and straightforward calculation leads to the solution (4.67).

Now, substituting (4.67) into (4.39), we find the (asymptotic) expression for the G_2 -holonomy metric:

$$ds^2 = dt^2 + \frac{1}{4} t^2 (\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2) + 16(A_1^2 \alpha_1^2 + A_2^2 \alpha_2^2 + A_3^2 \alpha_3^2). \tag{4.71}$$

Note, that the first terms describe the usual metric on \mathbb{R}^4 . The coefficient $1/4$ is crucial for this and comes as follows:

$$\frac{x_1 x_2}{t^2 y_3} = A_1 A_2 \left(\frac{m}{16A_3} \right)^{-1} = \frac{16A_1 A_2 A_3}{m} = \frac{1}{4}.$$

Here we used the identity (4.69) satisfied by A 's.

Therefore, the metric (4.71) describes the flat metric on

$$\mathbb{R}^4 \times T^3.$$

However, unlike the $SU(2)$ -symmetric metric (4.47), the solution here describes a 3-torus of arbitrary shape (determined by A_1 and A_2) with a fixed volume:

$$\text{Vol}(T^3) = \int_{T^3} \sqrt{g} = 16^{3/2} A_1 A_2 A_3 = m, \quad (4.72)$$

where we again used the important condition (4.69).

To summarize, we have demonstrated that the general solution is asymptotic to $\mathbb{R}^4 \times T^3$ where the Weierstrass function has a second-order pole. Moreover, the size of the 3-torus asymptotically saturates the bound (4.36).

At this point we shall remark on the interpretation of various parameters in the general solution described here. In total we have eight parameters: six integration constants for the first-order system (4.33) and the original parameters m and n . One of the integration constants (corresponding to the conservation of the Hamiltonian) is a constant in the definition of the radial variable t , and therefore does not play an interesting role. The remaining five independent constants have been denoted α , β , E , A_1 and A_2 . Two of them, A_1 and A_2 , affect the behavior of the metric at infinity. Namely, they describe the asymptotic form of the T^3 . Furthermore, the volume of the torus is determined by m . On the other hand, the parameter n along with the integration constants α , β , and E should be interpreted as ‘dynamical moduli’ since they don’t change behavior of the metric at infinity. In particular, the value of n determines the minimal volume of the 3-sphere (*cf.* (4.35)), and if $n = 0$ the value of β determines the volume of the two-sphere in (4.58).

5 Abelian BPS Monopoles from Torus Fibrations

The analysis in this section is motivated by the cosmic string solution [10], where compactification of Type-II string theory on a two-torus with varying modulus τ was considered. According to the equations of motion of the effective theory, if $\tau = \tau(z)$ depends only on two real directions of space-time (which can be combined in one complex coordinate z), it must be holomorphic. Then, at the points of space-time where $\tau \rightarrow \infty$, a real codimension two singularity — a cosmic string — is found. In this way, one views fibrations by special-holonomy fibers as supersymmetric topological defects in lower dimension.

Our solutions, constructed in the previous sections, do not have degenerate fibers. Therefore, one would not expect to find extreme concentration of energy via Kaluza-Klein reduction. Nevertheless, we will explain below that the dimensionally-reduced

G_2 Manifold X	Monopole Solution
$SU(2)$ Symmetry	Spherical Symmetry
$U(1)$ Symmetry	Axial Symmetry
No Extra Symmetry	No Rotational Symmetry

Table 1: Relation between extra symmetry of the G_2 -holonomy metric (4.39) and rotational symmetry of the corresponding monopole solution.

field configurations carry topological charge. So, they indeed represent stable solitons — monopoles, cosmic strings, or domain walls, depending on the configuration of energy density. The soliton obtained by Kaluza-Klein reduction is guaranteed to be BPS because the original metric admits a covariantly constant spinor.

Disk instanton corrections to the geometry near the cosmic string singularities should smoothen the metric, and the authors of [10] argued that a smooth, even compact, total space could result. This line of thinking was given credence by the explicit metrics near a degeneration found in Ooguri and Vafa in [24]. While our metrics involve the smooth part of a fibration, we hope that similar effects involving associative three-cycles will result in compact manifolds of G_2 holonomy, fibered over an S^3 base which includes the discriminant locus of a torus or $K3$ fibration (*cf.* [12]).

Since the base of a coassociative fibration is three-dimensional, the reduced solution could be interpreted as a monopole, after we supplement the metric on X with time direction τ :

$$\mathbb{R}_\tau \times X.$$

Of course, this space-time has the same holonomy as X , so the solution is guaranteed to be supersymmetric. Mainly interested in abelian monopoles, we shall focus on the torus fibrations¹³ found in Sec. 4.

The solutions we found can be classified according to their isometry. After Kaluza-Klein reduction to 3 + 1 dimension this translates to the rotational symmetry of the monopole solution [14]. Namely, the generic metric (4.39) is expected to give a (3 + 1)-dimensional monopole metric with no rotational symmetry. On the other hand, solutions with extra $SU(2)$ (resp. $U(1)$) symmetry lead to spherically (resp. axially) symmetric monopoles. We summarize this general pattern in Table 1.

In order to avoid possible confusion with a space-like coordinate t , we introduce a time-like variable τ and replace t by r , to emphasize that it plays a role of radial

¹³Kaluza-Klein reduction of certain G_2 holonomy metrics to non-abelian monopoles has been discussed recently in [14].

variable. Now, let us rewrite the eight-dimensional metric on $\mathbb{R}_\tau \times X$ in the new notation:

$$ds^2 = -d\tau^2 + dr^2 + \frac{1}{y_1}(x_2x_3\Sigma_1^2 + mn\Sigma_1\alpha_1 - mx_1\alpha_1^2) + \frac{1}{y_2}(x_1x_3\Sigma_2^2 + mn\Sigma_2\alpha_2 - mx_2\alpha_2^2) + \frac{1}{y_3}(x_1x_2\Sigma_3^2 + mn\Sigma_3\alpha_3 - mx_3\alpha_3^2), \quad (5.73)$$

where x_i and y_i should be understood as functions of the radial variable r :

$$\begin{aligned} x_i &\equiv x_i(r), \\ y_i &\equiv y_i(r). \end{aligned} \quad (5.74)$$

Since the G_2 -holonomy manifold X has principal orbits $SU(2) \times T^3$, it has four natural $U(1)$ isometries: three from the isometries of the 3-torus, and a $U(1) \subset SU(2)$. The latter is generated by shifts of the angular variable ϕ , *cf.* (4.30). In order to treat this latter $U(1)$ in the same way as the three directions of the T^3 , it is convenient to introduce

$$\alpha_4 \equiv d\phi.$$

Then, α_i , $i = 1, 2, 3, 4$ is a natural basis of one-forms on the 4-torus, $T^4 = T^3 \times U(1)$.

Now, we are ready to make a Kaluza-Klein reduction on the T^4 . We write the metric in the usual Scherk-Schwarz form [28]

$$ds^2 = ds_{1,3}^2 + h_{ij}(\alpha_i + A_i)(\alpha_j + A_j), \quad (5.75)$$

where $ds_{1,3}^2$ is the four-dimensional metric of the static gravitating monopole solution and A_i is the gauge connection for the i -th $U(1)$ gauge factor. The dilaton-like scalar fields h_{ij} have charge +1 under i -th $U(1)$ gauge factor, and -1 under the j -th $U(1)$. All these fields appear in the appropriate supermultiplets of the effective four-dimensional theory. Summarizing, after the Kaluza-Klein reduction we find the following spectrum of the effective supersymmetric theory in four dimensions:

4D Theory: Supergravity coupled to 4 vector and 10 matter multiplets.

Straightforward but technical calculations give the scalar field matrix corresponding to the general solution (4.39):

$$h = \begin{pmatrix} -\frac{mx_1}{y_1} & 0 & 0 & \frac{mn}{2y_1} \sin \psi \sin \theta \\ 0 & -\frac{mx_2}{y_2} & 0 & \frac{mn}{2y_2} \cos \psi \sin \theta \\ 0 & 0 & -\frac{mx_3}{y_3} & \frac{mn}{2y_3} \cos \theta \\ \frac{mn}{2y_1} \sin \psi \sin \theta & \frac{mn}{2y_2} \cos \psi \sin \theta & \frac{mn}{2y_3} \cos \theta & h_{44} \end{pmatrix}, \quad (5.76)$$

where

$$h_{44} = \frac{x_1 x_2}{y_3} \cos^2 \theta + \frac{x_3}{y_1 y_2} (x_1 y_1 \cos^2 \psi + x_2 y_2 \cos^2 \psi) \sin^2 \theta.$$

For a general solution, the gauge connections can be conveniently written as:

$$A_i = \sum_{k=1}^4 h_{ik}^{-1} \tilde{A}_k, \quad (5.77)$$

where

$$\begin{aligned} \tilde{A}_1 &= \frac{mn}{2y_1} \cos \psi d\theta, \\ \tilde{A}_2 &= -\frac{mn}{2y_2} \sin \psi d\theta, \\ \tilde{A}_3 &= \frac{mn}{2y_3} d\psi, \\ \tilde{A}_4 &= \frac{x_1 x_2}{y_3} \cos \theta d\psi + \left(\frac{x_2 x_3}{y_1} - \frac{x_1 x_3}{y_2} \right) \cos \psi \sin \psi \sin \theta d\theta. \end{aligned} \quad (5.78)$$

Finally, the metric in (3 + 1) dimensions looks like:

$$ds_{1,3}^2 = -d\tau^2 + dr^2 + \frac{x_1 x_2}{y_3} d\psi^2 + \left(\frac{x_2 x_3}{y_1} \cos^2 \psi + \frac{x_1 x_3}{y_2} \sin^2 \psi \right) d\theta^2 - \sum_{i,j=1}^4 A_i h_{ij}^{-1} A_j. \quad (5.79)$$

Evaluating the last term leads to a rather complicated form of the metric, which we write explicitly only in a few simple examples below.

5.1 Spherically Symmetric Monopoles

The above formulas considerably simplify in the case of the $SU(2)$ symmetric solution (4.47):

$$x_1 = x_2 = x_3, \quad y_1 = y_2 = y_3.$$

For example, the gauge connections (5.77) can be written explicitly:

$$\begin{aligned} A_1 &= -\left(\frac{n}{2x}\right) (\cos \psi d\theta - \cos \psi \sin \psi \sin \theta d\psi), \\ A_2 &= \left(\frac{n}{2x}\right) (\sin \psi d\theta + \cos \psi \cos \theta \sin \theta d\psi), \\ A_3 &= -\left(\frac{n}{2x}\right) \sin^2 \theta d\psi, \\ A_4 &= \cos \theta d\psi. \end{aligned} \quad (5.80)$$

Note, that A_1 , A_2 , and A_3 are all proportional to n , unlike A_4 .

Also, as we alluded to earlier, the $SU(2)$ symmetric solution automatically leads to the spherically symmetric monopole metric (in the Einstein frame):

$$ds_E^2 = \left(-\frac{mx}{y^{1/2}} \right) (-d\tau^2 + dr^2) + y^{3/2} d\Omega_2^2. \quad (5.81)$$

In the simple case $n = 0$, the metric on the G_2 -holonomy manifold X becomes the usual metric on $\mathbb{R}^3 \times T^3$. Specifically, the functions $x(r)$ and $y(r)$ take a simple polynomial form (4.46):

$$x = -\frac{m^{1/3}}{4} r^2, \quad y = \frac{m^{2/3}}{4} r^2. \quad (5.82)$$

After reduction to 3 + 1 dimensions, the scalar field matrix (5.76) turns out to be diagonal:

$$h = \text{diag}(m^{2/3}, m^{2/3}, m^{2/3}, \varphi), \quad \varphi = \frac{1}{4} r^2, \quad (5.83)$$

and the resulting metric is spherically symmetric:

$$ds_E^2 = \frac{1}{2} mr (-d\tau^2 + dr^2) + \frac{m}{8} r^3 d\Omega_2^2. \quad (5.84)$$

What is particularly nice about this solution is that the gauge fields A_1 , A_2 , and A_3 vanish in this background, so that we end up with a localized particle, magnetically charged under a single $U(1)$. Furthermore, A_4 resembles the gauge connection of the Pollard-Gross-Perry-Sorkin magnetic monopole [27]:

$$A_4 = \cos \theta d\psi. \quad (5.85)$$

In order to realize (5.83)–(5.85) as a solution in the Kaluza-Klein theory, it is convenient to combine the (3 + 1)-dimensional metric, the gauge field A_4 , and the “dilaton” φ into a five-dimensional metric. Then the equations for all of these fields follow from the five-dimensional supergravity action with the usual bosonic piece

$$S = -\frac{1}{16\pi\kappa_5} \int d^5x \sqrt{-g_5} R_5.$$

Unlike the usual Kaluza-Klein monopole [27], however, this solution represents a distribution of the magnetic charge in the entire three-dimensional space. It follows, for example, from (5.83) that the dilaton field φ has a uniform source in the \mathbb{R}^3 . The reason is that the size of the circle, parametrized by ϕ grows at large distances, *cf.* (5.82). In order to obtain a solution with localized source, one has to start with a metric (5.73), where the functions $x_i x_j / y_k$ are bounded at large r . This would work,

for example, if we had a Taub-NUT space instead of \mathbb{R}^4 in our special solution. It is easy to check, however, that $\text{TN}^4 \times T^3$ is not among the metrics of the form (4.39) that we consider here.

Another important observation is that $(3+1)$ -dimensional metric (5.79) always has the simple asymptotic behavior (5.84) of a Kaluza-Klein magnetic monopole. Indeed, even for a general solution without $SU(2)$ symmetry, the asymptotic form of the metric (4.71) describes a flat metric on $\mathbb{R}^4 \times T^3$, where the shape of T^3 can be different now. After reduction to $3+1$ dimensions this may change the asymptotic vevs of the scalar fields, but not the $(3+1)$ -dimensional metric (5.84).

5.2 Axially Symmetric Monopoles

Axially symmetric BPS monopoles follow from the $U(1)$ symmetric solution, just like spherically symmetric ones follow from solutions with additional $SU(2)$ isometry. The simplest way to see this is to put $x_1 = x_2$ and $y_1 = y_2$ in the general expressions for the scalar fields (5.76), $U(1)$ gauge connections (5.78), and monopole metric (5.79). For example,

$$A_4 = \frac{z \cos \theta}{z + \beta \sin^2 \theta} d\psi,$$

where, following the notation of section 4, we use $z = x_1 y_1$. Notice, that A_4 does not depend on the angular variable ψ , indicating the axial symmetry of the four-dimensional solution. The same is true for all the other fields. Thus, in the Einstein frame the metric reads

$$\begin{aligned} ds_E^2 &= \sqrt{\det h} \left[-d\tau^2 + dr^2 - \frac{y_1^2 y_3}{mz} \left(d\theta^2 + \frac{z \sin^2 \theta}{z + \beta \sin^2 \theta} d\psi^2 \right) \right] = \\ &= \frac{m \sqrt{z^2 + \beta z \sin^2 \theta}}{y_1 \sqrt{y_3}} \left(-d\tau^2 + dr^2 - \frac{y_1^2 y_3}{mz} d\theta^2 \right) - \frac{y_1 \sqrt{y_3} z \sin^2 \theta}{\sqrt{z^2 + \beta z \sin^2 \theta}} d\psi^2. \end{aligned} \quad (5.86)$$

As expected, the metric is manifestly axially symmetric. At large distances, $|z| \gg |\beta|$, the dependence on θ drops out and the metric takes the asymptotic, spherically symmetric form (5.84). This is also expected from the general no-hair theorem.

On the other hand, at small distances, the magnetic source is extended in one of the directions, thus breaking $SO(3) \cong SU(2)$ rotational symmetry down to $U(1)$ axial symmetry. Therefore, it is natural to interpret such metric as a magnetic monopole, which also carries some dipole charge given by β . Indeed, as $\beta \rightarrow 0$ the solution reduces to a spherically symmetric monopole. A further argument for this interpretation is that at large distances the field of a dipole falls off much faster than the field of a monopole, in agreement with the asymptotic behavior of the axially symmetric solution (5.86).

Depending on the internal structure of the dipole, one might say that it consists of two point-like sources connected by a finite string.

Let us consider a specific solution (4.57) studied in the previous section. As $z \rightarrow 0$, the corresponding four-dimensional metric looks like

$$ds_E^2 \approx \frac{m}{\sqrt{2}} \sin \theta \left(-d\tau^2 + dr^2 + \frac{2|\beta|}{m} d\theta^2 + 2\sqrt{\frac{|\beta|}{m}} r^2 d\psi^2 \right).$$

It has two cosmological singularities: one at $\theta = 0$, and another at $\theta = \pi$.

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