hep-th/0403225
HUTP-04/A014

Chern-Simons Gauge Theory and the AdS_3/CFT_2 Correspondence

Sergei Gukov*, Emil Martinec**, Gregory Moore† and Andrew Strominger*

Abstract

The bulk partition function of pure Chern-Simons theory on a three-manifold is a state in the space of conformal blocks of the dual boundary RCFT, and therefore transforms non-trivially under the boundary modular group. In contrast the bulk partition function of AdS_3 string theory is the modular-invariant partition function of the dual CFT on the boundary. This is a puzzle because AdS_3 string theory formally reduces to pure Chern-Simons theory at long distances. We study this puzzle in the context of massive Chern-Simons theory. We show that the puzzle is resolved in this context by the appearance of a chiral "spectator boson" in the boundary CFT which restores modular invariance. It couples to the conformal metric but not to the gauge field on the boundary. Consequently, we find a generalization of the standard Chern-Simons/RCFT correspondence involving "nonholomorphic conformal blocks" and nonrational boundary CFTs. These generalizations appear in the long-distance limit of AdS_3 string theory, where the role of the spectator boson is played by other degrees of freedom in the theory.

March 19, 2004

* *Jefferson Physical Laboratory, Harvard University, Cambridge, MA 02138*
** *University of Chicago, Chicago, IL 60637*
† *Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08855*

Contents

1. Introduction

One of the most beautiful examples of a holographic correspondence is the equivalence between three-dimensional Chern-Simons gauge theory and the chiral half of a rational conformal field theory [1]. (For reviews see [2,3,4]). We will refer to this as the CSW/RCFT correspondence. In recent years a more ambitious example of holography has been investigated, that of the AdS/CFT correspondence [5]. In this paper we discuss some aspects of the relation between these two holographic dualities.

We expect to find a relation between the AdS/CFT correspondence and the CSW/RCFT correspondence in the special case of superstring theories on spacetimes of the form $AdS_3 \times K_7$, where K_7 is a compact 7-manifold. The reason is that the low energy supergravity on AdS_3 typically contains gauge fields with Chern-Simons terms. This raises a puzzle when K_7 is a product of spheres, such as $K_7 = \mathbf{S}^3 \times (\mathbf{S}^1)^4$ or $K_7 = \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$, because in those cases the dual conformal field theory associated with the $\mathbf{S}^1$ factors is in general *not* a rational conformal field theory. The present paper resolves that puzzle.

In this paper we examine in some detail the holography of the massive abelian gauge theories that appear in the AdS/CFT examples we have just cited. At long distances these theories are dominated by the Chern-Simons terms. We will show that the partition function of these theories has a kind of factorization into "non-holomorphic conformal blocks," which transform in a finite-dimensional representation of the modular group. They are associated to a theory of a *nonchiral* boson, consisting of the usual chiral boson plus an antichiral "spectator", and have a continuously variable radius. We think this is an interesting extension of the standard holographic duality of CSW theory to the chiral half of a rational conformal field theory.

Let us describe our results in some more detail. In section 2 we review the well-studied example of a single massive abelian gauge field with action [6,7]

$$S = \int \frac{1}{2e^2} dA * dA - 2\pi i k A dA. \tag{1.1}$$

(Here it is in euclidean signature; our conventions are spelled out in the text below.) The partition function of the theory is a product of two factors; one factor is associated with a massive scalar field, and the other with a topological sector of the theory. We are mainly interested in the latter, although we shall see that the effects of the first term do not entirely disappear at long distances. The most natural way to study this theory — especially in the context of AdS_3 string theory — is to compute the path integral on a three-manifold Y as a function of the boundary conditions on the metric and gauge fields on $X := \partial Y$.

In this paper we focus on the quantization of the theory on a solid torus with Dirichlet boundary conditions for the gauge fields. We consider the limit of an infinite-volume torus (such as a quotient by $\mathbf{Z}$ of hyperbolic space). In this limit we can study the partition function by studying the groundstate of the gauge theory on T^2. We do this by solving explicitly for the Landau level wavefunctions in the quantization on the plane, and then projecting onto gauge invariant wavefunctions, taking proper account of the Gauss law. In this way we produce a finite dimensional space of wavefunctions, and the partition function on the torus is a linear combination of these wavefunctions.

In the above approach it turns out to be important to include both chiralities of the boson on the boundary, although only one of these couples to the gauge field -this being the usual chiral boson of CSW theory. Put differently, the partition function on the solid torus, in the limit of infinite volume, is equivalent to that of a nonchiral boson with Euclidean action:

$$\pi k \int d\phi * d\phi + 4\pi i k \int \bar{\partial}\phi \wedge A^{1,0} \tag{1.2}$$

where $\phi \sim \phi + 1$ and therefore the target space of the boson is a circle of radius $R^2 = k\alpha'$. (We have assumed $k > 0$). The fact that we can even speak of the radius shows that we must include both left- and right-movers. We refer to the left-moving part of ϕ, which is invisible to A, as a "spectator chiral boson." Note that the spectator does couple to the conformal metric on the boundary.

In section 3 we turn to the main model of interest here, namely the theory of two abelian gauge fields with off-diagonal Chern-Simons coupling. The action is:

$$S_a = \int \frac{1}{2e_A^2} dA * dA + \frac{1}{2e_B^2} dB * dB - 2\pi i k A dB \tag{1.3}$$

Our primary motivation is that this is the form of Lagrangian appearing in the low-energy supergravity theory on AdS_3 in the examples cited above, although as we discuss at the end of this introduction, there are other potential applications of our remarks.

The topological sector of the theory has two parameters, these are the integer k and the real number $\mu := |e_B/e_A|$. One might think that (1.3) is a trivial extension of (1.1) since one could introduce the change of variables

$$\begin{aligned} A &= \frac{1}{\sqrt{2\mu}}(A^{(+)} - A^{(-)}) \\ B &= \sqrt{\frac{\mu}{2}}(A^{(+)} + A^{(-)}) \end{aligned} \tag{1.4}$$

which gives two copies of (1.1), but with $e^2 \to |e_A e_B|$, and $k \to +\frac{1}{2}k$ for one term while $k \to -\frac{1}{2}k$ for the other. It turns out that we do not get a trivial extension of the previous theory, because of the quantization conditions on the periods of A and B. The dual theory is a theory of two bosons, ϕ^A, ϕ^B of period 1 with Euclidean action of the form $S_1 + S_2 + S_3$ where

$$S_1 = \frac{\pi k}{2} \int \mu d\phi^A * d\phi^A + \mu^{-1} d\phi^B * d\phi^B \tag{1.5}$$

shows the bosons have radius $R_A^2 = \frac{1}{2}k\mu\alpha'$ and $R_B^2 = \frac{1}{2}k\mu^{-1}\alpha'$ while

$$S_2 = i\pi k \int d\phi^A \wedge d\phi^B \tag{1.6}$$

shows there is a nontrivial B-field, for k odd, and finally

$$S_3 = 2\pi ik \int \left[(A^{(-)})^{0,1} \wedge \partial \phi^{(-)} - (A^{+})^{1,0} \wedge \bar{\partial} \phi^{(+)} \right] \tag{1.7}$$

gives the coupling to the gauge fields. In conformity with (1.4) we have defined

$$\begin{aligned} \phi^{(+)} &:= \frac{1}{\sqrt{2}}(\mu^{-1/2}\phi^B + \mu^{1/2}\phi^A) \\ \phi^{(-)} &:= \frac{1}{\sqrt{2}}(\mu^{-1/2}\phi^B - \mu^{1/2}\phi^A) \end{aligned} \tag{1.8}$$

Note that $\phi_L^{(-)} + \phi_R^{(+)}$ is a nonchiral scalar coupling to the gauge fields, but, for μ nonrational, it does not have a well-defined periodicity, as promised.

The radii satisfy[1]

$$\begin{aligned} \frac{R_A}{R_B} &= \mu \\ R_A R_B &= k \end{aligned} \tag{1.9}$$

Although the boundary conformal field theory is *not* rational (when μ is not rational), thanks to the quantization of $R_A R_B$, the partition function on the torus is always a linear combination:

$$Z = \sum_{\beta \in \Lambda^*/\Lambda} \zeta^{\beta} \Psi_{\beta}(A,B). \tag{1.10}$$

Here $\Psi_\beta(A,B)$ span a finite-dimensional space of states. They are proportional to Siegel-Narain theta functions (defined in appendix A) associated to the hyperbolic lattice $\Lambda = \sqrt{k} II^{1,1}$, and

$$\beta \in \Lambda^*/\Lambda \cong (\mathbf{Z}/k\mathbf{Z})^2. \tag{1.11}$$

The Ψ_β are also not holomorphic in τ, but do transform in a simple finite dimensional representation of the modular group. These higher level theta functions generalize the familiar holomorphic level k theta functions of RCFT. The case $k = 1$ is simply the modular invariant partition function of a single compact boson.

In section 4 we show that our considerations easily extend to the most general abelian Chern-Simons theory with gauge group $U(1)^d$ and action

$$\int \frac{1}{2e^2} \lambda^{-1}_{\alpha\beta} dA^{\alpha} * dA^{\beta} - 2\pi i K_{\alpha\beta} A^{\alpha} dA^{\beta} \tag{1.12}$$

[1] In what follows, $\alpha' = 2$ unless noted otherwise.

where $2K_{\alpha\beta}$ is an even integral nondegenerate symmetric matrix, and hence defines a lattice $\bar{\Lambda}$, while $\lambda^{\alpha\beta}$ is a positive definite symmetric matrix. The boundary theory, including the spectator chiralities, is a theory of d nonchiral bosons. The metric for the bosons is determined by $\lambda^{\alpha\beta}$ and $K_{\alpha\beta}$ while the B-field is

$$-2\pi i\int\sum_{\alpha<\beta}K_{\alpha\beta}d\phi^{\alpha}\wedge d\phi^{\beta} \tag{1.13}$$

Left plus right movers move in a target space $V_L\oplus V_R$, where $V\cong\mathbf{R}^d$. Using the data of both $\lambda^{\alpha\beta}$ and $K_{\alpha\beta}$ one constructs a projection matrix $P_\pm$ on V which is compatible with the projection into left and right movers. The bosons coupling to the gauge fields lie in $V_{L,-}\oplus V_{R,+}$. The "spectator chiralities" lie in $V_{L,+}\oplus V_{R,-}$.

Finally, the computations also generalize in a natural way from the torus to higher genus Riemann surfaces.

Now let us discuss the relation to the purely topological CSW theory. In the AB theory, the space of states spanned by Ψ_β in (1.10) is k^2-dimensional, in harmony with a standard analysis of the pure Chern-Simons theory associated to the $e_A^2,e_B^2\to\infty$ limit of (1.3) [1,8,9,10]. Indeed, the topological Hilbert space and the representation of the modular group are independent of μ (and independent of $\lambda^{\alpha\beta}$ in the higher rank case). Nevertheless, the path integral on the torus naturally introduces μ-dependence in the basis of wavefunctions, and is essential in writing the path integral of the massive Chern-Simons theory. The dependence of the topological field theory on μ is quite analogous to the dependence of the topological Hilbert spaces $\mathcal{H}(X)$ associated to a Riemann surface X on the complex structure of X. Because it is the fields $A_z^{(+)}$, $A_{\bar z}^{(-)}$ which couple to the currents, the holomorphic polarization is more natural when using the AdS/CFT correspondence. Indeed, the path integral on AdS_3 with no operator insertions is in the state (1.10) with $\zeta^\beta\sim\delta_{\beta,0}$.

Our work touches on a number of other closely related investigations. First it touches on an old problem in the CSW/RCFT correspondence. The chiral half of an RCFT is only part of the data needed to construct the conformal field theory, as stressed in [11,12]. Indeed, in general different CFT's can be made from gluing together the chiral parts using different automorphisms of the fusion rules [12,13]. [2] Thus, a vexing question has always

[2] There are important subtleties in this statement which have been investigated in [14,15,16]. However they do not affect the very simple models considered in this paper.

been: "How does one modify the CSW theory to incorporate both left- and right-movers?" The present paper provides the beginning of an answer to that question, at least in the case of abelian gauge theories.

The importance of including the kinetic terms $\sim \int F * F$ in studying the holography of abelian Chern-Simons theory was stressed by S. Carlip and Y. Kogan in their attempt to rewrite string theory as a topological membrane theory [17]. They did not explain the role of left- and right-movers in the way we are doing, but introduced this term to account for dependence on the boundary conformal structure. More recently, off-diagonal Chern-Simons terms have been discussed by Witten in [10]. In his discussion it is crucial that the theory with $k = 1$ is "trivial." What this means, in our context, is that there is only one wavefunction Ψ_β, and it transforms trivially under the modular group. Indeed, as we have noted, the level 1 Siegel-Narain theta function is simply the theta function appearing in the modular invariant partition function of a conformal field theory of both left- and right-movers.

The present computations might conceivably find a use in condensed matter physics, where classification of quantum hall states involves the study of general abelian Chern-Simons gauge theories [18,19,20,21,22,23,24]. Curiously, for related reasons, massive Chern-Simons theories with two gauge fields and opposite sign Chern-Simons terms have recently been recognized as being important in condensed matter physics with a view towards quantum computation. See, for example, [25].[3] Also in [10] Witten pointed out that the triviality of the AB theory for $k = 1$ has important consequences for the classification of quantum Hall states. In the simple case where we do not consider spin theories, the results of this paper, combined with the Nikulin embedding theorem [26] show that abelian Chern-Simons theories are classified by the signature of Λ modulo 24, together with the discriminant form of the lattice Λ, where Λ is the lattice determined by $-2K_{\alpha\beta}$.

Finally, one motivation for the present work was a project involving the AdS/CFT correspondence, so let us mention briefly here some implications for the AdS/CFT correspondence. (Further details are in [27].) The relevance of topological field theories to the AdS/CFT correspondence was first discussed in [28]. The authors of [29] discussed in some detail the singleton sector of supergravity theories in the AdS/CFT correspondence in a variety of dimensions. We will improve on [29] in two ways. First, we show that the Hamiltonian for the singleton is naturally chosen by using Dirichlet boundary conditions

[3] We thank Paul Fendley for pointing this out.

for the second order system in the Euclidean path integral. Second we show how one can discuss the radius of the singleton scalar.

We have studied here the simple free-field theory of abelian gauge fields. In the AdS/CFT context these gauge fields couple to other degrees of freedom in the low energy supergravity. Nevertheless, based on simple considerations of the decoupling of topological modes at long distance, we conjecture that the full partition function of the string theory on $AdS_3 \times K_7$ can still be written as

$$Z_{\text{string}} = \sum_{\beta} \zeta^{\beta}_{\text{string}} \Psi_{\beta}(A, B) \tag{1.14}$$

where $\zeta^{\beta}_{\text{string}}$ is A, B-independent, and $\Psi_{\beta}(A, B)$ are the same functions as in (1.10). That is, the dependence on the boundary values of the $U(1)$ gauge fields is given by a wavefunction in the topological Hilbert space determined by the free massive gauge theory. The essential difference from the massive gauge theory (which is *not* a holographic theory, since it does not contain gravity), is that $\zeta^{\beta}_{\text{string}}$ only depends on boundary data. In [27] the conjecture (1.14) is used to investigate the holographic correspondence for $AdS_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$.

2. Review of the standard case

The massive 3d gauge theory was analyzed in a classic paper [6,7] and the topological sector of the theory was understood in [30,1,31,8,9]. We review it here as preparation for the AB theory.

2.1. The classical theory

We are interested in studying abelian Chern-Simons gauge theory on a topologically non-trivial 3-manifold Y. In this section, we focus on the simplest example of such a theory, with $U(1)$ gauge group, whose action is (in Euclidean signature),

$$S_E = \int \frac{1}{2e^2} dA * dA - 2\pi i k A dA \tag{2.1}$$

Here, the gauge connection A is a section of a principal $U(1)$-bundle over Y, normalized so that dA has integral periods and large gauge transformations are $A \to A + \omega$ with ω a closed 1-form with integral periods. In order to obtain the partition function of the Euclidean theory, one has to integrate over the space of all gauge connections A (modulo the gauge equivalence) with the measure e^{-S}. Similarly, on the Lorentzian space-time with signature $(-,+,+)$ the action looks like

$$S = \int \frac{-1}{2e^2} dA * dA + 2\pi k A dA \tag{2.2}$$

and the measure is given by e^{iS}.

The coupling k is an integer if the Euclidean theory is to be well-defined on all 3-manifolds. If we use spin bounding 4-folds then we can take $k_{min} = \pm\frac{1}{2}$.

The coupling e^2 has dimensions of mass. Under a conformal rescaling $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ of the 3-dimensional metric the first term in the action scales as Ω^{-1}, while the second is invariant. Therefore, we expect that at long distances the topological term dominates. Note that in this sense the long-distance limit is the $e^2 \to \infty$ limit.

The equations of motion are

$$d * F - 4\pi k e^2 F = 0 \tag{2.3}$$

In the presence of a boundary we vary in a space of fields such that the two form

$$\delta A \wedge (*F - 2\pi k e^2 A) = 0 \tag{2.4}$$

vanishes when pulled back to the boundary.

2.2. Solutions to equations of motion

We are interested in formulating carefully the phase space of the theory. One way of formulating physical phase space is that it is the space of gauge inequivalent solutions of the equations of motion.

In the present theory, thanks to linearity the space of (not necessarily gauge inequivalent) solutions of the equations of motion is a product

$$S = S_f \times S_{nf} \tag{2.5}$$

where S_f is the space of flat solutions $F = 0$. These are the solutions of the topological sector. $A = A_f + A_{nf}$ where A_{nf} is orthogonal to the flat subspace in, say, the Hodge metric.

More generally, on $Y = X \times \mathbf{R}$, X compact we can take $A = A_f + A_{nf}$ where the nonflat component A_{nf} is defined by saying it is orthogonal to $\ker d$ in the Hodge metric. The space of solutions to the equations of motion is a *product*. When X is noncompact one needs to include boundary conditions, and the space of solutions might or might not be a product.

The main result of [6,7] is the "equivalence" of the massive gauge theory to a theory of a massive scalar field. In our context this means that we can identify the factor S_{nf} with the space of solutions of the massive scalar equation.

2.3. Hamiltonian Formalism

Let us work out the Hamiltonian formulation on a spacetime of the form $X \times \mathbf{R}$, with metric $-dt^2 + g_{ij}dx^i dx^j$ and orientation $dtdx^1dx^2$. The canonically conjugate momentum as a vector-density is ($\epsilon^{12} = +1$):

$$\Pi^i = \frac{1}{e^2}\sqrt{g}g^{ij}(\dot{A}_j - \partial_j A_0) + 2\pi k \epsilon^{ij} A_j \tag{2.6}$$

The action can be written as $S = \int dtL$ with

$$L = \int_X \Pi^i \dot{A}_i - H + \int_X A_0 \Big(\partial_i \Pi^i + 2\pi k \epsilon^{ij} \partial_i A_j \Big) \tag{2.7}$$

where we find a Hamiltonian

$$H = \int_X \frac{e^2}{2\sqrt{g}} g_{ij} E^i E^j + \frac{1}{2e^2} F \wedge *_2 F \tag{2.8}$$

where $*_2$ is the Hodge star on X and

$$E^i := \Pi^i - 2\pi k \epsilon^{ij} A_j \tag{2.9}$$

(We will also denote $E^i = \tilde{\Pi}^i$ below.) The Gauss law is:

$$\partial_i \Pi^i + 2\pi k \epsilon^{ij} \partial_i A_j = 0 \tag{2.10}$$

That is, $\nabla \cdot E + 4\pi k B = 0$.

2.4. Phase space and symplectic structure

There are two descriptions of the phase space, depending on how one works with Hamiltonian reduction.

One way to formulate physical phase space is as the space of gauge inequivalent solutions of the equations of motion. This point of view makes it obvious that the phase space is a product of the phase space for flat gauge fields and for nonflat gauge fields, $\mathcal{P} = \mathcal{P}_f \times \mathcal{P}_{nf}$ for the flat and nonflat parts of the theory.

Another way to formulate the theory "upstairs" in $A_0 = 0$ gauge is to take phase space to be the cotangent space with coordinates (Π^i, A_i) and symplectic form:

$$\Omega = \int_X \delta \Pi^i \wedge \delta A_i \tag{2.11}$$

where δ is exterior derivative on the infinite dimensional phase space. Notice that when (2.11) is restricted to the subspace of flat gauge fields, by (2.9) we get second class constraints and the phase space is the Chern-Simons symplectic form

$$\Omega_f = \int_X 2\pi k \delta A \wedge \delta A \tag{2.12}$$

This is gauge invariant on the subspace $F = 0$ and one may then perform Hamiltonian reduction.

It is instructive to consider the $e^2 \to \infty$ limit. Using (2.8), we see that if we restrict to finite energy field configurations then we must set $E^i = 0$. Then, by the Gauss law we must put $F = 0$. As we have said, restriction to this subspace imposes second class constraints and we are restricting to the flat factor in phase space.

2.5. Quantization in the Schrodinger representation

If we quantize on phase space and then impose the Gauss law we have wavefunctionals $\Psi(A_i)$, and we quantize using the symplectic form (2.11). Thus

$$\Pi^i = -i \frac{\delta}{\delta A_i} \tag{2.13}$$

Since we can split $A = A_f + A_{nf}$ and the Hamiltonian does not mix these, the Hilbert space of the theory is naturally thought of as a product

$$\mathcal{H} = \mathcal{H}_f \otimes \mathcal{H}_{nf} \tag{2.14}$$

where $\mathcal{H}_f$ is the space of wavefunctions of flat potentials.

The Gauss law is:

$$\Psi(A + \omega) = e^{-2\pi i k \int \omega \wedge A} \Psi(A) \tag{2.15}$$

This is valid also for large gauge transformations. [4] Here ω is a closed 1-form with integral periods. Note that this does not affect the A_{nf} variable.

[4] This requires explanation. The proper mathematical formulation involves regarding Ψ as a section of a line bundle over the space of gauge potentials $\mathcal{A}(X)$ on X. We then lift the group action, and find that a lift only exists when $c_1(P) = 0$. There is a canonical trivialization of the line in this case, as well as a canonical connection, and the wavefunction becomes a function. A similar discussion holds for the more subtle case of the M-theory C-field [32].

2.6. Euclidean Path integral on the solid torus

We will determine the Hamiltonian for the singletons by considering the Euclidean path integral of the theory on the solid torus, and then interpreting that path integral in terms of Hamiltonian evolution in the radial direction.

Since our action is second order in derivatives, when formulating the path integral on a handlebody Y we should specify all components of A_X on the boundary X. This is to be contrasted with the Chern-Simons path integral which is a phase space path integral, and in which we specify just one component of A_X on the boundary X.

Let us consider the Euclidean partition function of the theory on a solid torus with radius ρ denoted $Y_\rho \cong D\times\mathbf{S}^1$. We assume the torus has a metric that behaves asymptotically like $d\rho^2+\Omega^2(\rho)g_X$. The path integral defines a state $\Psi_{Y_\rho}(A)$ given by

$$\Psi_{Y_\rho}(A)=\int \frac{dA_Y}{\mathrm{vol}(\mathcal{G}(Y))}e^{-\int \frac{1}{2e^2}dA*dA+2\pi ik\int AdA} \tag{2.16}$$

where $\mathcal{G}(Y)$ is the gauge group on Y_ρ. We can understand the behavior for $\rho\to\infty$ just from the above understanding of the spectrum.

We can view the evolution to large ρ as evolution in a Euclidean time direction. The large ρ behavior projects onto the lowest energy states.

$$\lim_{\rho\to\infty}\Psi_{Y_\rho}(A)=e^{-\rho E_0}\Psi_0 \tag{2.17}$$

with Ψ_0 in the space of ground states on the torus with energy E_0. The insertion of local operators such as Wilson lines or other disturbances induces transitions between vectors within this space of ground states.

2.7. Quantization on the torus

We now consider quantization on $T^2\times\mathbf{R}$. Our wavefunction is $\Psi(A_f)\otimes\Psi(A_{nf})$. The spectrum of the nonflat sector is clear, and we take the unique groundstate wavefunction for this factor: It is the product of harmonic oscillator groundstates for the oscillators of the massive scalar of [6,7]. In this section we drop this factor so we can focus on the dependence on A_f.

To simplify matters, we work on a torus $X=T^2$ with $z=\sigma^1+\tau\sigma^2$ and metric $\Omega^2|dz|^2$, $\sigma^i\sim\sigma^i+1$. We fix the small gauge transformations by assuming A_f is constant.

In complex coordinates $A = A_z dz + A_{\bar z} d\bar z$ we have

$$A_z = \frac{A_2 - \bar\tau A_1}{\tau - \bar\tau} \qquad A_{\bar z} = -\frac{A_2 - \tau A_1}{\tau - \bar\tau}.$$

We further define the zero mode of the shifted momentum (2.9) as

$$\begin{aligned} \tilde\Pi^z &= \int d^2 z \big(-i\frac{\delta}{\delta A_z(z)} - 2\pi k \epsilon^{z\bar z} A_{\bar z}(\bar z)\big) = -i\left(\frac{\partial}{\partial A_z} - 4\pi k \mathrm{Im}\tau\ A_{\bar z}\right) \\ \tilde\Pi^{\bar z} &= -i\left(\frac{\partial}{\partial A_{\bar z}} + 4\pi k \mathrm{Im}\tau\ A_z\right) \end{aligned} \tag{2.18}$$

so that the Hamiltonian is:

$$H = \frac{e^2}{4\mathrm{Im}\tau}(\tilde\Pi^z\tilde\Pi^{\bar z} + \tilde\Pi^{\bar z}\tilde\Pi^z) \tag{2.19}$$

Note that these do not commute: $[\tilde\Pi^z, \tilde\Pi^{\bar z}] = -8\pi k \mathrm{Im}\tau$. The ground state energy density is $2\pi|k|e^2$ and is infinitely degenerate, as in the standard Landau-level problem. If $k > 0$ we have

$$\tilde\Pi^{\bar z}\Psi = 0 \Rightarrow \Psi = e^{-4\pi k \mathrm{Im}\tau\ A_z A_{\bar z}}\psi(A_z) \tag{2.20}$$

If $k < 0$ we have

$$\tilde\Pi^{z}\Psi = 0 \Rightarrow \Psi = e^{4\pi k \mathrm{Im}\tau\ A_z A_{\bar z}}\psi(A_{\bar z}) \tag{2.21}$$

Here ψ are arbitrary holomorphic functions. Indeed, if we take $\psi = \psi_\lambda$, where

$$\psi_\lambda(x) := e^{\lambda x} \tag{2.22}$$

then the set of wavefunctions $\{\Psi_\lambda | \lambda \in \mathbf{C}\}$ is an overcomplete set spanning the infinite-dimensional lowest Landau level.

The set of states spanned by (2.22) is infinite dimensional, but when we consider gauge invariant wavefunctions on the torus the lowest Landau level (LLL) becomes finite dimensional. We have already enforced the invariance under small gauge transformations by choosing our flat connections to be constants on the torus. We can impose the invariance under large gauge transformations by averaging over large gauge transformations. Given *any* wavefunction $\Psi(A)$ the average:

$$\bar\Psi(A) := \sum_{\omega \in \mathrm{Harm}^1_Z} \Psi(A+\omega) e^{2\pi i k \int \omega A} \tag{2.23}$$

where Harm^1_Z are the harmonic 1-forms with integral periods, transforms according to the Gauss law (2.15).

Now assume $k > 0$ and get the projected wavefunctions of the LLL:

$$\bar{\Psi}(A) := \mathcal{N} e^{-4\pi k \mathrm{Im}\tau\ A_z A_{\bar{z}}} \sum_{\omega \in \mathrm{Harm}^1_Z} e^{-4\pi k \mathrm{Im}\tau\ \omega_z \omega_{\bar{z}}} e^{-8\pi k \mathrm{Im}\tau\ \omega_{\bar{z}} A_z} \psi(A_z + \omega_z) \tag{2.24}$$

where $\mathcal{N}$ is a normalization constant, which might depend on τ.

Let us now consider the space of wavefunctions — as functions of A_z — that we obtain from (2.24) and (2.22). At first, one might think that the space is infinite dimensional since λ in (2.22) can be any complex number. However, using the Poisson summation formula we find that

$$\overline{\Psi_\lambda} = e^{-4\pi k \mathrm{Im}\tau\ (A_z A_{\bar{z}} + A_z^2) - \frac{\lambda^2}{16\pi k \mathrm{Im}\tau}} \sqrt{\frac{\mathrm{Im}\tau}{k}} \sum q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2} e^{-\frac{p_R}{R} 8\pi k \mathrm{Im}\tau\ A_z - \frac{p_L}{R}\lambda} \tag{2.25}$$

where $q = e^{2\pi i \tau}$, and

$$\begin{aligned} p_L &= (n/R + mR/2), p_R = (n/R - mR/2) \\ R^2 &= 2k \end{aligned} \tag{2.26}$$

We recognize that we have the soliton sum of the partition function of a scalar field with radius R. Since $R^2 = 2k$ is integral it is a rational conformal field theory, and the infinite sum can be split as a finite sum of terms of the form $f_i(A) g_i(\lambda)$. The sublattices $p_L = 0$ and $p_R = 0$ are of index $2k$ in the Narain lattice $(p_L; p_R)$. Indeed, after a little algebra we see that (2.25) can be written as:

$$\overline{\Psi_\lambda} = e^{-Q} \sqrt{\frac{\mathrm{Im}\tau}{k}} \sum_{0 \le \mu < 2k} \Theta_{-\mu,k}(-2i \mathrm{Im}\tau\ A_z, -\bar{\tau}) \Theta_{\mu,k}(\frac{-\lambda}{4\pi i k}, \tau). \tag{2.27}$$

where

$$Q = 4\pi k \mathrm{Im}\tau\ (A_z A_{\bar{z}} + A_z^2) + \frac{\lambda^2}{16\pi k \mathrm{Im}\tau} \tag{2.28}$$

The level k theta functions $\mu = -k+1, \dots, k$ are defined by

$$\Theta_{\mu,k}(\omega, \tau) = \sum_{n \in \mathbf{Z}} q^{k(n+\mu/(2k))^2} y^{(\mu + 2kn)} \tag{2.29}$$

where $y = \exp(2\pi i \omega)$. Equation (2.27) shows quite explicitly that the space of quantum states is in fact only finite dimensional. A basis for the vector space of states is

$$\psi_\mu = \mathcal{N} \sqrt{\frac{\mathrm{Im}\tau}{k}} e^{-4\pi k \mathrm{Im}\tau\ (A_z A_{\bar{z}} + A_z^2)} \Theta_{-\mu,k}(-2i \mathrm{Im}\tau\ A_z, -\bar{\tau}) \qquad 1 \le \mu \le 2k \tag{2.30}$$

Finally, we would like to determine the normalization $\mathcal{N}$. We do this following a trick in [9].

The flat gauge fields on the torus can be written $A = d\chi + A_z dz + A_{\bar z} d\bar z$. From the Gauss law $\Psi(A) = \psi(A_z, A_{\bar z})$. However there is a Jacobian for the change of variables from A to $\chi, A_z, A_{\bar z}$. Now

$$\int \frac{[dA]}{\mathrm{vol}\mathcal{G}} \Psi^*_\mu[A]\Psi_\nu[A] = \det{}'(d) \int_0^1 dA_1 \int_0^1 dA_2 \psi^*_\mu \psi_\nu \tag{2.31}$$

We can regularize $\det'(d) = \sqrt{\mathrm{Im}\tau}\,|\eta|^2$. We can also evaluate the inner product of the states (2.30):

$$\begin{aligned}\int_0^1 dA_1 \int_0^1 dA_2 \psi^*_\mu \psi_\nu &= \delta_{\mu,\nu} \frac{\mathrm{Im}\tau}{k} |\mathcal{N}|^2 \int_0^1 dA_1 \sum_n e^{-4\pi k \mathrm{Im}\tau\ (A_1 + n - \mu/2k)^2} \\ &= \frac{\sqrt{\mathrm{Im}\tau}}{2k^{3/2}} |\mathcal{N}|^2\end{aligned} \tag{2.32}$$

Normalizing the wavefunctions to one gives:

$$\psi_\mu = \frac{k^{3/4}}{\bar\eta} e^{-4\pi k \mathrm{Im}\tau\ (A_z A_{\bar z} + A_z^2)} \Theta_{-\mu,k}(-2i\mathrm{Im}\tau\ A_z, -\bar\tau) \tag{2.33}$$

Remarks:

1. The higher Landau levels are obtained by acting with $\tilde\Pi^z$ to give energy densities $2\pi k e^2 (2N+1)$, $N > 0$. Note that (2.24) is independent of e^2, and hence has a smooth limit as $e^2 \to \infty$. Moreover, the gap between Landau levels becomes infinite in this limit.
2. The dependence on A_z is that of the wavefunctions in the holomorphic polarization of the pure Chern-Simons theory. Equivalently, they are conformal blocks for the Gaussian model at $R^2 = 2k$, coupled to an external gauge field.

2.8. Holographic mapping to the Gaussian model

We now interpret the sum (2.24) in terms of conformal field theory. The first exponential factor in the sum in (2.24) is just the standard value of the Gaussian model action

$$k\pi \int d\phi * d\phi \tag{2.34}$$

evaluated on a soliton configuration $d\phi = \omega = n_1 d\sigma^1 + n_2 d\sigma^2$. In our conventions, we use a scalar of periodicity 1 and hence we get the Gaussian model on a circle with radius

$$R^2 = k\alpha', \tag{2.35}$$

with both left-movers and right-movers. Of course, we then recognize the Narain lattice in (2.26) with $\alpha' = 2$. Note, however, that the coupling of ϕ to A_z is chiral, and given by the Lagrangian

$$4\pi i k \int \bar{\partial}\phi \wedge A^{1,0} \tag{2.36}$$

For $k > 0$ we have holomorphic functions of A_z coupling to the rightmoving current $\bar{\partial}\phi$ and for $k < 0$ we have holomorphic functions of $A_{\bar{z}}$ coupling to the leftmoving current $\partial\phi$.

Remarks:

1. In [17] Carlip and Kogan discuss very closely related matters in their attempt to rewrite string theory as a topological membrane theory. The Landau levels are solved for in their eq. 3.4, which they are thinking of as the solutions of the full Schrodinger equation in the limit $e^2 \to \infty$. Their motivation was to introduce dependence on the conformal structure of the boundary into the wavefunctions. They intended to get left and right-moving degrees of freedom from the inner and outer radii of an annulus, as in [31,9].
2. We now propose a somewhat heterodox interpretation of the equation (2.24). We propose that the dual conformal field theory is a theory of both a left-moving and a right-moving boson with the radius (2.35). The fact that both chiralities are present is surprising since the canonical quantization of the pure Chern-Simons theory is well-known to lead to a single chiral boson. In particular, with appropriate boundary conditions the quantization on the disk gives a chiral boson degree of freedom on the boundary. One should distinguish between the *modes of A on the boundary* which, with proper boundary conditions are those of a chiral scalar and *the dual field theory variable ϕ*. Note that $\partial\phi$ *couples* to A, it is not one of the degrees of freedom of A. Moreover, only one chirality of ϕ couples to A. The other chirality is a "spectator" in the sense that in (2.24) only one chirality $\omega_{\bar{z}}$ couples to the external gauge field. Nevertheless, there are really two chiralities present in (2.24). Both chiralities couple to the conformal metric.

3. One way to make the point about the "reality" of the spectator chirality is to note that we have identified a definite radius, (2.35). In order to understand why this is surprising one must recall some standard points from RCFT. In RCFT the wave-functions (2.33) are the conformal blocks of a Gaussian model with "$U(1)$ level N current algebra." By definition, this is a holomorphic $U(1)$ current algebra extended by holomorphic currents

$$e^{\pm i\sqrt{2N}\phi(z)} \tag{2.37}$$

of conformal dimension N. There are $2N$ distinct representations of this algebra generated by $\exp[i\frac{r}{\sqrt{2N}}\phi(z)]$ for $r \sim r + 2N$. The conformal blocks of this theory on the torus are level N theta functions. This theory is dual to the pure Chern-Simons theory with action

$$2\pi i N \int AdA \tag{2.38}$$

in a normalization where dA has integer periods. In units $\alpha' = 2$ the Gaussian model with radius R has $U(1)$ level N current algebra whenever R^2 is rational. More precisely, if $R^2 = p/q$ is in lowest terms then $N = 2pq$ for p odd and $N = pq/2$ for p even. In the present section we have $R^2 = 2k$ and hence $N = k$, hence $2k$ topological states. Returning to the general case, for a given N there are several Gaussian models with the same chiral algebra, corresponding to the different ways of factoring N. The choice of a definite radius is a choice of how to combine left- and right-moving conformal blocks [12,13]. One cannot speak of the radius without introducing both left and right movers.

4. The role of the spectator chiralities is further clarified if one compares carefully the Euclidean and Lorentzian versions of holography. In the Lorentzian case we have an isomorphism of Hilbert spaces. As we have mentioned, quantization on $D \times \mathbf{R}$ yields the Hilbert space of a chiral boson, depending on which boundary condition we impose. Since we could impose either boundary condition, both chiralities are "present." Perhaps a good analogy is the light-cone gauge quantization of a massless scalar. Making one gauge choice one only sees one of two chiralities. In the Euclidean formulation, the path integral on the bulk manifold is equivalent to the path integral of some CFT on the boundary. Here we impose Dirichlet boundary conditions on the gauge field and compute a wavefunctional $\Psi(A)$ of the boundary value of A. It is here that we see the necessity of both chiralities in identifying $\Psi(A)$ with a conformal field theory partition function.

3. The AB theory

3.1. Action

Now let us consider the AB theory with action:

$$S_a = \int \frac{-1}{2e_A^2} dA * dA + \frac{-1}{2e_B^2} dB * dB + 2\pi k A dB \tag{3.1}$$

The gauge group is $U(1) \times U(1)$, and in particular large gauge transformations are $A \to A + \omega^A$ and $B \to B + \omega^B$ where the ω^A, ω^B are closed 1-forms with integral periods.

The above treatment is asymmetric in A, B. By using

$$\int AdB = \int BdA + \int d(BA) \tag{3.2}$$

we can convert to a theory with action:

$$S_s = \int \frac{-1}{2e_A^2} dA * dA + \frac{-1}{2e_B^2} dB * dB + \pi k (AdB + BdA) \tag{3.3}$$

which is manifestly symmetric under exchanging $A \leftrightarrow B$, $e_A \leftrightarrow e_B$. More generally, we can use (3.2) to formulate the action

$$S_x = \int \frac{-1}{2e_A^2} dA * dA + \frac{-1}{2e_B^2} dB * dB + \pi k \big[(1+x) AdB + (1-x) BdA\big]. \tag{3.4}$$

It is very useful to introduce $\mu := |e_B/e_A|$ and the linear combinations

$$\begin{aligned} A^{(+)} &:= \frac{1}{\sqrt{2}}\Big(\mu^{-1/2} B + \mu^{1/2} A\Big) \\ A^{(-)} &:= \frac{1}{\sqrt{2}}\Big(\mu^{-1/2} B - \mu^{1/2} A\Big) \end{aligned} \tag{3.5}$$

the inverse relation being (1.4). If (and only if) $x = 0$ in (3.4) we may use these fields to write the action as a sum of two "decoupled" theories:

$$S_s = \int \left[\frac{-1}{2|e_A e_B|} dA^{(+)} * dA^{(+)} + \pi k A^{(+)} dA^{(+)}\right] + \int \left[\frac{-1}{2|e_A e_B|} dA^{(-)} * dA^{(-)} - \pi k A^{(-)} dA^{(-)}\right] \tag{3.6}$$

One might conclude that the AB theory is merely two copies of the one-field case with opposite signs of k. *However, if μ is not rational then $A^{(+)}, A^{(-)}$ cannot be defined as connections on topologically nontrivial line bundles. They are not truly independent. In particular, to implement the Gauss law on wavefunctions we cannot simply take a product* *of wavefunctions for $A^{(+)}, A^{(-)}$ and implement the Gauss laws separately. This is what makes the AB theory an interesting and nontrivial extension of the one-field case.*

Another interesting new point is that the topological limit is $e_A^2 \to \infty, e_B^2 \to \infty$ holding μ fixed. Thus, the topological sector of the theory has a *continuous* parameter μ, in addition to the discrete parameter k. It is usually said that in the long distance limit the kinetic terms have no effect. As we shall see, this is not quite true. The ratio μ does affect the wavefunctions in the topological Hilbert space.

3.2. Equations of motion

The equations of motion are

$$\begin{aligned} d * dA^{(+)} &= 2\pi k |e_A e_B| dA^{(+)} \\ d * dA^{(-)} &= -2\pi k |e_A e_B| dA^{(+)} \end{aligned} \tag{3.7}$$

and therefore there are two propagating vector fields of $m^2 = (2\pi k e_A e_B)^2$.

The boundary conditions should be such that when pulled back we have

$$-\frac{1}{e_A^2}\delta A * dA - \frac{1}{e_B^2}\delta B * dB + \pi k((1+x)\delta B A + (1-x)\delta A B) = 0 \tag{3.8}$$

3.3. Hamiltonian formalism: Symmetric formulation $(x=0)$

The Hamiltonian formulation is easily deduced by combining (3.6) with section 2.3. We have

$$S_s = -H + \int \Pi_+^i \dot{A}_i^{(+)} + \Pi_-^i \dot{A}_i^{(-)} + A_0^+(\partial_i \Pi_+^i + \pi k \epsilon^{ij}\partial_i A_j^{(+)}) + A_0^-(\partial_i \Pi_-^i - \pi k \epsilon^{ij}\partial_i A_j^{(-)}) \tag{3.9}$$

where H is the Hamiltonian (two copies of the usual one) and

$$\begin{aligned} \Pi_+^i &= \frac{1}{e_A e_B}\sqrt{g} g^{ij}(\partial_0 A_i^{(+)} - \partial_i A_0^{(+)}) + \pi k \epsilon^{ij} A_j^+ \\ \Pi_-^i &= \frac{1}{e_A e_B}\sqrt{g} g^{ij}(\partial_0 A_i^{(-)} - \partial_i A_0^{(-)}) - \pi k \epsilon^{ij} A_j^+ \end{aligned} \tag{3.10}$$

The symplectic structure is

$$\Omega = \int_X \delta\Pi_+^i \delta A_i^{(+)} + \delta \Pi_-^i \delta A_i^{(-)} = \int_X \delta \Pi_A^i \delta A_i + \delta \Pi_B^i \delta B_i \tag{3.11}$$

The Gauss laws become:

$$\begin{aligned} \partial_i \Pi_B^i + \pi k \epsilon^{ij}\partial_i A_j &= 0 \\ \partial_i \Pi_A^i + \pi k \epsilon^{ij}\partial_i B_j &= 0 \end{aligned} \tag{3.12}$$

Imposing the second class constraints of restriction to flat gauge fields gives symplectic form

$$\Omega_f = \int_X 2\pi k \epsilon^{ij} \delta B_i \wedge \delta A_j \tag{3.13}$$

Quantum mechanically, working in "upstairs formalism" the Gauss laws become

$$\begin{aligned} \Psi_s(A + \omega_A, B) &= e^{-\pi i k \int \omega_A \wedge B} \Psi_s(A, B) \\ \Psi_s(A, B + \omega_B) &= e^{-\pi i k \int \omega_B \wedge A} \Psi_s(A, B) \end{aligned} \tag{3.14}$$

Thus, if we shift by both ω^A, ω^B then:

$$\Psi_s(A + \omega^A, B + \omega^B) = e^{-\pi i k \int \omega^A \omega^B + \omega^A B + \omega^B A} \Psi_s(A, B) \tag{3.15}$$

Note that this is only a consistent transformation law so long as ω^A, ω^B have integral periods and k is an integer.

Remark: Here we encounter a truly treacherous point. Since the action separates as in (3.6) one might have expected the Gauss law to be simply the product of that for the $A^{(+)}$ and the $A^{(-)}$ theory. That is, one might have expected that

$$\Psi_s(A^{(+)} + \omega^{(+)}, A^{(-)} + \omega^{(-)}) \stackrel{?}{=} e^{i\pi k \int (\omega^{(-)} A^{(-)} - \omega^{(+)} A^{(+)})} \Psi_s(A^{(+)}, A^{(-)}) \tag{3.16}$$

While this indeed agrees with (3.14) if $\omega^A = 0$ or if $\omega^B = 0$ *it does not agree with (3.15) !* We will discuss this subtlety more thoroughly in the general case in section 4.2 below.

3.4. Hamiltonian analysis

For completeness, in this subsection we give the formulation for an arbitrary value of x. The conjugate momenta are:

$$\begin{aligned} \Pi_A^i &= \tilde{\Pi}_A^i + \pi k (1 - x) \epsilon^{ij} B_j \\ \Pi_B^i &= \tilde{\Pi}_B^i + \pi k (1 + x) \epsilon^{ij} A_j \end{aligned} \tag{3.17}$$

where $\tilde{\Pi}_{A,B}^i$ is x-independent. Then

$$\begin{aligned} \int \Pi_A^i \dot{A}_i + \Pi_B^i \dot{B}_i - S = & \int H dt \\ & + \int \Pi_A^i \partial_i A_0 - \pi k (1 + x) A_0 \epsilon^{ij} \partial_i B_j \\ & + \int \Pi_B^i \partial_i B_0 - \pi k (1 - x) B_0 \epsilon^{ij} \partial_i B_j \end{aligned} \tag{3.18}$$

Note that *no integration by parts has been used at this point.*

The classical Gauss law expressed in terms of $\tilde{\Pi}$ is x-independent. On the other hand, the quantum Gauss law is

$$\begin{aligned}\Psi_x(A+\omega^A, B) &= e^{-i\pi(1+x)k\int \omega^A\wedge B}\Psi_x(A,B)\\ \Psi_x(A, B+\omega_B) &= e^{-i\pi(1-x)k\int \omega^B\wedge A}\Psi_x(A,B)\end{aligned} \tag{3.19}$$

The wavefunction and Hamiltonian depend on the choice of x. The general transformation between wavefunctions is

$$\Psi_x(A,B) = \Omega_x\Psi_s(A,B) = e^{-i\pi kx\int A\wedge B}\Psi_s(A,B) \tag{3.20}$$

where $x = 0$ is the symmetric formulation. The Hamiltonian is obtained from $H_x = \Omega_x H_s \Omega_x^{-1}$.

We henceforth set $x = 0$ but the formulae for general x can be obtained using (3.20).

3.5. Groundstates on T^2

The standard Hamiltonian analysis on $D\times\mathbf{R}$ yields a left- and right-moving chiral boson, once one chooses appropriate boundary conditions. However, as in the previous section, we focus on the Euclidean path integral on the solid torus, since the natural Dirichlet boundary conditions on the fields distinguishes a Hamiltonian for the singleton modes. Therefore, we use the same trick of considering the gauge invariant groundstate wavefunctions on T^2.

Again we have the factorization $\mathcal{H}_f\otimes\mathcal{H}_{nf}$ of the Hilbert space and we concentrate on the wavefunctions of flat gauge fields. We do this, and fix the small gauge transformations by taking our wavefunctions to be functions of the constant gauge potentials.

The Hamiltonian can be written as:

$$\begin{aligned}H_s = &-\frac{e_A^2}{2\mathrm{Im}\tau}\left(\frac{\partial}{\partial A_z} - 2\pi k\mathrm{Im}\tau\ B_{\bar z}\right)\left(\frac{\partial}{\partial A_{\bar z}} + 2\pi k\mathrm{Im}\tau\ B_z\right)\\ &-\frac{e_B^2}{2\mathrm{Im}\tau}\left(\frac{\partial}{\partial B_z} - 2\pi k\mathrm{Im}\tau\ A_{\bar z}\right)\left(\frac{\partial}{\partial B_{\bar z}} + 2\pi k\mathrm{Im}\tau\ A_z\right)\end{aligned} \tag{3.21}$$

and one can solve for the Landau levels. A trick for finding these is to write the Hamiltonian as a sum of two copies of (2.19), with opposite signs of k. From (2.20) and (2.21) we can write without further ado the wavefunctions in the lowest Landau level (assuming $k>0$):

$$\Psi_{\lambda,\bar\lambda} := e^{-2\pi k\mathrm{Im}\tau\ A_z^{(+)}A_{\bar z}^{(+)} - 2\pi k\mathrm{Im}\tau\ A_z^{(-)}A_{\bar z}^{(-)}}e^{\bar\lambda A_z^{(+)}+\lambda A_{\bar z}^{(-)}} \tag{3.22}$$

These have energy $2\pi k|e_A e_B|$ for all values of $\lambda, \bar{\lambda}$, (they are not related by complex conjugation), and (3.22) forms an overcomplete set for the LLL. Again, this space of states is infinite dimensional.

Let us now follow the procedure used in the one-field case. Averaging the wavefunctions (3.22) over the large gauge transformations for A, B to enforce the Gauss laws (3.14) gives a family of gauge invariant ground states parametrized by $\lambda, \bar{\lambda}$:

$$\begin{aligned} \overline{\Psi_{\lambda,\bar{\lambda}}} = & \sum_{\omega^A,\omega^B} \Psi_{\lambda,\bar{\lambda}}(A+\omega^A, B+\omega^B) \times \\ & \times e^{i\pi k\int \omega^A\omega^B + i\pi k\int \omega^A B + i\pi k\int \omega^B A} \end{aligned} \tag{3.23}$$

Applying this to the wavefunctions (3.22) we have the averaged sum:

$$\begin{aligned} \overline{\Psi_{\lambda,\bar{\lambda}}} = \Psi_{\lambda,\bar{\lambda}}(A,B) \sum e^{-2\pi k \mathrm{Im}\tau\ (\omega_z^{(+)}\omega_{\bar{z}}^{(+)} + \omega_z^{(-)}\omega_{\bar{z}}^{(-)})} e^{i\pi k\int \omega^A\wedge\omega^B} \\ e^{\bar{\lambda}\omega_z^{(+)} - 4\pi k \mathrm{Im}\tau\ A_z^{(+)}\omega_{\bar{z}}^{(+)} - 4\pi k \mathrm{Im}\tau\ A_{\bar{z}}^{(-)}\omega_z^{(-)} + \lambda\omega_{\bar{z}}^{(-)}} \end{aligned} \tag{3.24}$$

where $\omega^{\pm}$ are related to ω^A, ω^B by the same linear transformation as (3.5).

Our next move is to give an interpretation of the sum (3.24) as an instanton sum in the partition function of a Gaussian model on the torus. To begin, we write

$$\omega_z^{(+)}\omega_{\bar{z}}^{(+)} + \omega_z^{(-)}\omega_{\bar{z}}^{(-)} = \mu\omega_z^A\omega_{\bar{z}}^A + \mu^{-1}\omega_z^B\omega_{\bar{z}}^B \tag{3.25}$$

Therefore, we see from the quadratic terms in ω in (3.24) that we have *two Gaussian models*, one at radius $R_A^2 = \frac{1}{2}k\mu\alpha'$ and one at radius $R_B^2 = \frac{1}{2}k\mu^{-1}\alpha'$. Let us call these Gaussian fields ϕ^A, ϕ^B. They have periodicity 1 and both left- and right-movers, so $\omega^A = d\phi^A$ in an instanton configuration on the torus. The quadratic piece of the action is

$$S_1 = \frac{\pi k}{2}\int \mu d\phi^A * d\phi^A + \mu^{-1} d\phi^B * d\phi^B \tag{3.26}$$

There is evidentally a B-field:

$$S_2 = i\pi k\int d\phi^A \wedge d\phi^B \tag{3.27}$$

Now let us consider the coupling to the external gauge field. Let us form the linear combinations:

$$\begin{aligned} \phi^{(+)} &:= \frac{1}{\sqrt{2}}(\mu^{-1/2}\phi^B + \mu^{1/2}\phi^A) \\ \phi^{(-)} &:= \frac{1}{\sqrt{2}}(\mu^{-1/2}\phi^B - \mu^{1/2}\phi^A) \end{aligned} \tag{3.28}$$

and similarly for $\omega^{(\pm)}$. Comparing with (3.24) we see that the only couplings of the Gaussian fields are $A_{\bar{z}}^{(-)}$ couples to $\partial_z \phi^{(-)}$ while $A_z^{(+)}$ couples to $\partial_{\bar{z}} \phi^{(+)}$. To be more precise, the linear terms correspond to an action:

$$S_3 = 2\pi i k \int \left[(A^{(-)})^{0,1} \wedge \partial \phi^{(-)} - (A^{+})^{1,0} \wedge \bar{\partial} \phi^{(+)} \right] \tag{3.29}$$

Thus, the rightmoving part of ϕ^+ and the leftmoving part of ϕ^- couple to the external gauge fields, and correspondingly, one chirality of each of ϕ^A and ϕ^B "decouples" from the gauge fields, but not from the metric.

Note, that unless μ is *rational* the scalar fields $\phi^{(\pm)}$ do not individually have a discrete periodicity, that is, we cannot consider ϕ^+ to be a well-defined periodic scalar field on its own. The unusual and interesting point is that, nevertheless $\phi_L^- + \phi_R^+$ and $\phi_L^+ + \phi_R^-$ *are* very nearly well-defined periodic scalars.

Remarks:

1. Notice that (with $\alpha' = 2$) the radii satisfy (1.9). The second equation relating R_A and R_B is analogous to the T-duality relation. Standard T-duality is $R_A R_B = 2$ in units $\alpha' = 2$.
2. Note that the wavefunction (3.22) only depends on e_A, e_B through the ratio μ. Thus, if $e_A, e_B \to \infty$ holding μ fixed then the wavefunction has a smooth limit. This is the limit in which we expect the topological theory to dominate. The gap to the next Landau level is $\sim 2\pi k |e_A e_B|$.

3.6. Vector space of wavefunctions on T^2

At this point we could proceed with standard quantization of the CFT defined by (3.26) and (3.29). Let us stress that for generic μ this conformal field theory is *not* a rational conformal field theory. Nevertheless, as we will show momentarily, the space of wavefunctions (3.24) spanned by $\lambda, \bar{\lambda} \in \mathbf{C}$ is *finite dimensional* and defines an analog of the space of conformal blocks. Moreover, we will show that the partition function can be written as a finite sum of factorized terms in a fashion very reminiscent of RCFT.

In order to get at the spectrum we will take a shortcut and simply perform a Poisson resummation of the instanton sum (3.24). We rewrite the sum in terms of ω^A, ω^B. We write $\omega^A = n_1 d\sigma^1 + n_2 d\sigma^2$ and $\omega^B = \tilde{n}_1 d\sigma^1 + \tilde{n}_2 d\sigma^2$. Next we do a Poisson resummation on $n_2, \tilde{n}_2$ and convert the sum to a form where we recognize the Hamiltonian formalism (of the conformal field theory). After some algebra one arrives at the result:

$$
\begin{aligned}
\overline{\bar{\Psi}_{\lambda,\bar{\lambda}}} = & \frac{2\mathrm{Im}\tau}{k} e^{-Q} \sum q^{\frac{1}{2}(p_L^2+\tilde{p}_L^2)} \bar{q}^{\frac{1}{2}(p_R^2+\tilde{p}_R^2)} \\
& \exp[-4\pi\sqrt{k}\mathrm{Im}\tau\ A_z^{(+)}(p_R+\tilde{p}_R)/\sqrt{2} - 4\pi\sqrt{k}\mathrm{Im}\tau\ A_{\bar{z}}^{(-)}(p_L-\tilde{p}_L)/\sqrt{2} \\
& - \frac{\lambda}{\sqrt{k}}(p_R-\tilde{p}_R)/\sqrt{2} - \frac{\bar{\lambda}}{\sqrt{k}}(p_L+\tilde{p}_L)/\sqrt{2}]
\end{aligned}
\tag{3.30}
$$

where the prefactor e^{-Q} is determined by

$$
\begin{aligned}
Q = & 2\pi k\mathrm{Im}\tau\ [A_z^+ A_{\bar{z}}^+ + A_z^- A_{\bar{z}}^-] \\
& + 2\pi k\mathrm{Im}\tau \left((A_z^+)^2 + (A_{\bar{z}}^-)^2 \right) + \frac{1}{8\pi k\mathrm{Im}\tau}(\lambda^2 + \bar{\lambda}^2)
\end{aligned}
\tag{3.31}
$$

Now we have

$$
\begin{aligned}
\frac{p_L-\tilde{p}_L}{\sqrt{2}} &= \frac{1}{R}m_2 - \frac{R}{2k}\tilde{m}_2 \\
\frac{p_R+\tilde{p}_R}{\sqrt{2}} &= \frac{1}{R}m_2 + \frac{R}{2k}\tilde{m}_2 \\
\frac{p_L+\tilde{p}_L}{\sqrt{2}} &= \frac{1}{R}(m_2-k\tilde{n}_1) + \frac{R}{2k}(\tilde{m}_2-kn_1) \\
\frac{p_R-\tilde{p}_R}{\sqrt{2}} &= \frac{1}{R}(m_2-k\tilde{n}_1) - \frac{R}{2k}(\tilde{m}_2-kn_1)
\end{aligned}
\tag{3.32}
$$

where $R = \sqrt{2}R_A = \sqrt{2\mu k}$, $m_2, \tilde{m}_2, n_1, \tilde{n}_1 \in \mathbf{Z}$.

At this point we can recognize the following. The sum (3.30) is a sum over a signature $(2,2)$ Narain lattice. We can define two sublattices: Λ^A is the lattice of vectors "coupling only to A and not to λ." Thus, it is defined by $p_R - \tilde{p}_R = 0, p_L + \tilde{p}_L = 0$. Similarly, Λ^λ is the lattice of vectors $p_R + \tilde{p}_R = 0, p_L - \tilde{p}_L = 0$. The main observation is that these are each sublattices of signature $(1,1)$ and $\Lambda^A \oplus \Lambda^\lambda$ is of *finite index* in the full Narain lattice. The analog of the chiral splitting of RCFT is obtained by summing over the lattice vectors in $\Lambda^A \oplus \Lambda^\lambda$. This sum is a factorized product of a function of A and a function of λ. Then, the full sum is given by a sum of this factorized form over the coset representatives and takes the form

$$
\sum_{\beta\in\Lambda^*/\Lambda} \Psi_\beta(A)\Psi_{\bar{\beta}}(\lambda)
\tag{3.33}
$$

where the lattice Λ will be defined presently. In this way we have defined a factorization into "nonholomorphic conformal blocks."

Let us make all this explicit. Note that we may write

$$
\begin{aligned}
m_2 &= ka + \rho \\
\tilde{m}_2 &= kb + \tilde{\rho} \\
m_2 - k\tilde{n}_1 &= kc + \rho \\
\tilde{m}_2 - kn_1 &= kd + \tilde{\rho}
\end{aligned}
\tag{3.34}
$$

with $a, b, c, d \in \mathbf{Z}$ and $0 \le \rho, \tilde{\rho} \le k-1$ all uncorrelated. In this parametrization we may write

$$
\begin{aligned}
\left(\frac{p_L - \tilde{p}_L}{\sqrt{2}}; \frac{p_R + \tilde{p}_R}{\sqrt{2}}\right) &= a\sqrt{\frac{k}{2\mu}}e_0 - b\sqrt{\frac{\mu k}{2}}f_0 + \beta \\
&\equiv ae_1 - bf_1 + \beta \\
\left(\frac{p_L + \tilde{p}_L}{\sqrt{2}}; \frac{p_R - \tilde{p}_R}{\sqrt{2}}\right) &= c\sqrt{\frac{k}{2\mu}}e_0 + d\sqrt{\frac{\mu k}{2}}f_0 + \bar{\beta} \\
&= ce_1 - df_1 + \bar{\beta}
\end{aligned}
\tag{3.35}
$$

where $\beta = \rho/ke_1 - \tilde{\rho}/kf_1$ and $\bar{\beta} = \rho/ke_1 + \tilde{\rho}/kf_1$. Here $e_0 := (1;1), f_0 := (1;-1)$ generate the lattice $\sqrt{2}II^{1,1}$. The vectors e_1, f_1 generate a lattice $\Lambda = e_1\mathbf{Z} + f_1\mathbf{Z} \cong \sqrt{k}II^{1,1}$. Note that $\Lambda^* \cong \frac{1}{k}\Lambda$, and we may regard $\beta, \bar{\beta}$ as representatives of elements of the dual quotient group Λ^*/Λ.

Now, with any lattice of indefinite signature, but with a projection into definite signature subspaces one may form a Siegel-Narain theta function. The definition is reviewed in appendix A. We may write our analogs of "conformal blocks" in terms of Siegel-Narain theta functions for Λ. Specifically, we have

$$
\Psi_\beta(A) = \mathcal{N}\frac{2\mathrm{Im}\tau}{k}e^{-\pi k\int[A^{(+)}*A^{(+)}+A^{(-)}*A^{(-)}]}\Theta_\Lambda(\tau, 0, \beta; P; \xi(A))
\tag{3.36}
$$

$\mathcal{N}$ is a normalization constant and

$$
\Psi_{\bar{\beta}}(\lambda) = \Theta_\Lambda(\tau, 0, \bar{\beta}; P; \xi(\lambda))
\tag{3.37}
$$

where we have defined:

$$
\xi(A) = (\sqrt{k}2i\mathrm{Im}\tau\ A_{\bar{z}}^{(-)}; -\sqrt{k}2i\mathrm{Im}\tau\ A_z^{(+)})
\tag{3.38}
$$

$$
\xi(\lambda) = \left(-\frac{\bar{\lambda}}{2\pi i\sqrt{k}}; \frac{\lambda}{2\pi i\sqrt{k}}\right)
\tag{3.39}
$$

One can now compute that

$$\int_0^1 dA_1 dA_2 dB_1 dB_2 (\Psi_\beta(A,B))^* \Psi_{\beta'}(A,B) = \delta_{\beta,\beta'} \frac{2\mathrm{Im}\tau}{k^3} |\mathcal{N}|^2 \tag{3.40}$$

Taking into account the Jacobian factor $\mathrm{Im}\tau\ |\eta|^4$ for going from the wavefunctional $\Psi(A(z))$ to the wavefunction on harmonic 1-forms we finally get [5]

$$\Psi_\beta(A) = \frac{2k^{1/2}}{\eta\bar{\eta}} e^{-\pi k \int [A^{(+)} * A^{(+)} + A^{(-)} * A^{(-)}]} \Theta_\Lambda(\tau, 0, \beta; P; \xi(A)) \tag{3.41}$$

It is now straightforward to compute the representation of $\Psi_\beta(\tau, A)$ under the action of the modular group. Specifically, the matrix elements of the T- and S-transformations are given by

$$T_{\beta,\beta'} = e^{i\pi(\beta,\beta)} \delta_{\beta,\beta'} \tag{3.42}$$

and

$$S_{\beta,\beta'} = \frac{1}{k} e^{-2\pi i(\beta,\beta')} \tag{3.43}$$

where $\beta, \beta' \in \Lambda^*/\Lambda \cong (\mathbf{Z}/k\mathbf{Z})^2$ inherits a quadratic form from the hyperbolic inner product.

This is the same representation of $SL(2,\mathbf{Z})$ as that studied in [28], and for similar reasons. There is a natural action of the modular group on the irrep of the discrete Heisenberg group which is a central extension of $H^1(X;\mathbf{Z}/k\mathbf{Z}) \times H^1(X;\mathbf{Z}/k\mathbf{Z})$.

3.7. Comment on a clash of terminology

The term "level k $U(1)$ current algebra" is, regrettably, used in two very different ways in the context of the theories discussed in this paper. In [33,34] Kutasov and Seiberg, and Larsen and Martinec, use it to refer to the structure of conformal weights $h \sim p^2/k, \tilde{h} \sim \tilde{p}^2/k$ where $(p, \tilde{p})$ lie in a (Narain) lattice of charges. Unfortunately, the same terminology is used with a different meaning in a closely related context in rational conformal field theory. In the latter setting "level k $U(1)$ current algebra" is the chiral algebra of the RCFT one obtains for a Gaussian model on a rational square-radius, as described near (2.37) above. One of our motivations in this paper is to clarify the relation between the two uses of this term. We do this in the present section.

---

[5] Of course, we have made a choice of factorization of $\mathrm{Im}\tau\ |\eta|^4$. Our choice was to take the positive square root. This seems reasonable, and gives a nice representation of the modular group below, but should be better justified. It is certainly necessary to match to the topological theory.

Let us consider only the momentum coupling to $A^{(+)}, A^{(-)}$. Let us define the left and right "charges" by:

$$\begin{aligned} u_L &:= \xi(p_L - \tilde{p}_L) \\ u_R &:= \xi(p_R + \tilde{p}_R) \end{aligned} \tag{3.44}$$

where ξ is a real normalization constant to be determined below.

The set of charges (3.44) forms a lattice in $\mathbf{R}^{1,1}$ defined by

$$\Lambda := \{(u_L; u_R) | n, m, \tilde{n}, \tilde{m} \in \mathbf{Z}\} \subset \mathbf{R}^{1,1} \tag{3.45}$$

This lattice is generated by integral combinations of 2 vectors:

$$\begin{aligned} e_1 &= \xi \frac{1}{R_A} e_0 \\ f_1 &= \xi \frac{R_A}{k} f_0 \end{aligned} \tag{3.46}$$

where $e_0 := (1;1), f_0 := (1;-1)$ generate the lattice $\sqrt{2} II^{1,1}$. Thus, $e_1 \cdot f_1 = 2\xi^2/k$, while $e_1^2 = f_1^2 = 0$. The charge lattice is $e_1 \mathbf{Z} \oplus f_1 \mathbf{Z}$. So choosing

$$\xi = \sqrt{\frac{k}{2}} \tag{3.47}$$

we obtain a self-dual lattice.

In terms of these charges we can write the conformal weights of the states counted in (3.30) as:

$$\begin{aligned} h &= \tfrac{1}{2}(p_L^2 + \tilde{p}_L^2) \\ &= \frac{1}{4}\big(p_L + \tilde{p}_L\big)^2 + \frac{1}{2k} u_L^2 \end{aligned} \tag{3.48}$$

$$\begin{aligned} \tilde{h} &= \tfrac{1}{2}(p_R^2 + \tilde{p}_R^2) \\ &= \frac{1}{4}\big(p_R - \tilde{p}_R\big)^2 + \frac{1}{2k} u_R^2 \end{aligned} \tag{3.49}$$

Now, for fixed values of the "spectator charges" $(p_L + \tilde{p}_L; p_R - \tilde{p}_R)$ we recognize, after using (3.47) that the dependence of the conformal weight on $(u_L; u_R)$ is that of "level k $U(1)$ current algebra." Note especially that

$$\begin{aligned} \frac{1}{4}(p_L - \tilde{p}_L)^2 - \frac{1}{4}(p_R + \tilde{p}_R)^2 &= \frac{1}{4\xi^2}(u_L^2 - u_R^2) \\ &= -\frac{m_2 \tilde{m}_2}{k} = \frac{N}{k} \end{aligned} \tag{3.50}$$

where N can be any integer.

Remark. *The purely topological quantization.* In [10] Witten studied the off-diagonal Chern-Simons theory for the case that $k=1$ and concluded that the pure Chern-Simons theroy is "trivial." It is straightforward to analyze the purely topological theory on $D\times \mathbf{R}$ using the methods of [1][31][9]. One finds a left and a right-moving boson, but, we stress, *these are not the left- and right-moving components of a single boson of well-defined discrete periodicity.* One can compute $L_0-\bar{L}_0$ in this approach and one finds $L_0-\bar{L}_0=N/k$. Without further input it is difficult to decide whether we should allow all integers N, or whether one should project to $N=0$ mod k. The approach we are taking in this paper answers that question. We see that the integer N in (3.50) can be *any* integer.

4. General massive abelian Chern-Simons theories

Both in the theory of the quantum hall effect [18,19,20,21,22,23,24] and in $AdS_3\times \mathbf{S}^3\times T^4$ one is naturally led to wonder about the extension of the above remarks to a collection of abelian gauge fields A^α, $\alpha=1,\dots,d$. We take the action

$$\int -\frac{1}{2e^2}\lambda^{-1}_{\alpha\beta}dA^\alpha * dA^\beta + 2\pi K_{\alpha\beta}A^\alpha dA^\beta \tag{4.1}$$

and the gauge fields are normalized so that F^α has integral periods. The gauge group is $U(1)^d$. The Euclidean version is e^{-S_E} with

$$\int \frac{1}{2e^2}\lambda^{-1}_{\alpha\beta}dA^\alpha * dA^\beta - 2\pi i K_{\alpha\beta}A^\alpha dA^\beta \tag{4.2}$$

The coupling e^2 has dimensions of mass, while $\lambda^{-1}_{\alpha\beta}$ is a dimensionless positive definite symmetric matrix. Without loss of generality we may assume it has fixed determinant, say determinant one.

We will assume that $K_{\alpha\beta}$ is nondegenerate. As we have seen above, by adding total derivatives, we can assume that $K_{\alpha\beta}$ is symmetric, and these total derivatives do not affect the quantization of the theory. In order that the action makes sense on arbitrary manifolds we must have

$$\int_{M_4} K_{\alpha\beta}c_1^\alpha c_1^\beta \in \mathbf{Z} \tag{4.3}$$

where c_1^α is a vector of integer cohomology classes on the four-manifold M_4. Clearly $K_{\alpha\alpha}\in\mathbf{Z}$. Using the example of $\mathbf{S}^2\times\mathbf{S}^2$ we see that $K_{\alpha\beta}+K_{\beta\alpha}\in\mathbf{Z}$ for $\alpha\neq\beta$, and this is sufficient for well-definedness in general. [6] Thus, we conclude that $2K_{\alpha\beta}$ is a nondegenerate, even, integral, symmetric matrix. It can have any signature. This matrix defines an integral lattice which we denote $\bar{\Lambda}$. We will denote the integral lattice generated by $-2K_{\alpha\beta}$ by Λ.

The matrix of Chern-Simons couplings $\lambda^{-1}_{\alpha\beta}$ has inverse $\lambda^{\alpha\beta}$. The topological limit is obtained by taking $e^2 \to \infty$. Thus we expect both $\lambda^{\alpha\beta}$ and $K_{\alpha\beta}$ to show up in constructing the wavefunctions for the topological Hilbert space.

4.1. Quantization of the purely topological theory

The quantization of the pure Chern-Simons theory is completely straightforward and was in fact already analyzed to some extent in [9]. We have

$$[A^\alpha_j, A^\beta_k] = \frac{\epsilon_{jk} K^{\alpha\beta}}{2\pi i} \tag{4.4}$$

Choosing a real polarization on the torus we have wavefunctions $\Psi(A^\alpha_1)$. Implementing the Gauss law for transformations in the σ^2 direction we find the wavefunctions are supported on gauge potentials such that $K_{\alpha\beta} A^\beta_1 \omega^\alpha_2 \in \mathbf{Z}$, that is, on points in Λ^*. The Gauss law for transformations in the σ^1 direction shows that the wavefunction descends to Λ^*/Λ. This leads to a standard representation of a finite Heisenberg group, and is associated to a representation of $SL(2,\mathbf{Z})$ in a natural way.

4.2. Hamiltonian analysis and Gauss law

The conjugate momentum is

$$\Pi^i_\alpha = \tilde{\Pi}^i_\alpha + 2\pi K_{\beta\alpha} \epsilon^{ij} A^\beta_j \tag{4.5}$$

where $\tilde{\Pi}^i_a = \lambda^{-1}_{\alpha\beta} g^{ij} \sqrt{g} (\partial_0 A^\beta_j - \partial_j A^\beta_0)$ is the electric field. The Hamiltonian is

$$H = \int_X \frac{g_{ij}}{2\sqrt{g}} \lambda^{\alpha\beta} \tilde{\Pi}^i_\alpha \tilde{\Pi}^j_\beta + \tfrac{1}{2} \lambda^{-1}_{\alpha\beta} F^\alpha *_2 F^\beta \tag{4.6}$$

The classical Gauss law is

$$\partial_i \Pi^i_\alpha + 2\pi K_{\alpha\beta} \partial_i A^\beta_j \epsilon^{ij} = 0 \tag{4.7}$$

[6] By choosing a spin structure and only considering bounding manifolds compatible with the spin structure we can allow theories with more general Chern-Simons couplings [35]. This involves several new issues, and we will not investigate that case here.

Implementing the quantum Gauss law one encounters a subtlety. Let ω^α be a 1-form with integral periods. Define the operator

$$\mathcal{G}_\alpha(\omega^\alpha) := i\int \omega^\alpha_i \Pi^i_\alpha + 2\pi\sum_\beta \omega^\alpha K_{\alpha\beta} A^\beta \tag{4.8}$$

where *there is no sum on* α. One easily computes that

$$e^{\mathcal{G}_\alpha(\omega^\alpha)} e^{\mathcal{G}_\alpha(\tilde\omega^\alpha)} = e^{\mathcal{G}_\alpha(\omega^\alpha+\tilde\omega^\alpha)+2\pi i\int K_{\alpha\alpha}\omega^\alpha\tilde\omega^\alpha} = e^{\mathcal{G}_\alpha(\omega^a+\tilde\omega^a)} \tag{4.9}$$

since $K_{\alpha\alpha}$ is integral. Similarly, if $\alpha\neq\beta$ then

$$e^{\mathcal{G}_\alpha(\omega^\alpha)} e^{\mathcal{G}_\beta(\omega^\beta)} = e^{\mathcal{G}_\alpha(\omega^\alpha)+\mathcal{G}_\beta(\omega^\beta)-2\pi i\int K_{\alpha\beta}\omega^\alpha\omega^\beta} = e^{\mathcal{G}_\beta(\omega^\beta)} e^{\mathcal{G}_\alpha(\omega^\alpha)} \tag{4.10}$$

Since $K_{\alpha\beta}\in\frac{1}{2}\mathbf{Z}$, the operators $e^{\mathcal{G}_\alpha}$ are simultaneously commuting and can all be imposed as constraints. However, one *cannot* enforce the Gauss laws

$$e^{\mathcal{G}_\alpha(\omega^\alpha)+\mathcal{G}_\beta(\omega^\beta)}\Psi = \Psi \tag{4.11}$$

because they have a nontrivial cocycle in the group law. This is the origin of the B-field (4.22) in the holographically dual theory.

Enforcing all the Gauss laws $e^{\mathcal{G}_\alpha}\Psi=\Psi$ for $\alpha=1,\dots,d$ is equivalent to the quantum Gauss law:

$$\Psi(A^1+\omega^1,\dots,A^d+\omega^d) = e^{-2\pi i\int\sum_{\alpha<\beta}K_{\alpha\beta}\omega^\alpha\omega^\beta} e^{-2\pi i\int\sum_{\alpha,\beta}K_{\alpha\beta}\omega^\alpha A^\beta}\Psi(A^1,\dots,A^d) \tag{4.12}$$

4.3. Landau levels

On the flat torus we have Hamiltonian

$$H = -\int \frac{1}{2\mathrm{Im}\tau}\lambda^{\alpha\beta}\left(\frac{\partial}{\partial A^\alpha_z} - 4\pi\mathrm{Im}\tau\ K_{\gamma\alpha}A^\gamma_{\bar z}\right)\left(\frac{\partial}{\partial A^\beta_{\bar z}} + 4\pi\mathrm{Im}\tau\ K_{\gamma\beta}A^\gamma_z\right) \tag{4.13}$$

where we have chosen a normal ordering. On the plane a complete set of functions for the lowest Landau level is generated by the wavefunctions

$$\Psi_{v,\bar v} = \exp\left[-4\pi\mathrm{Im}\tau\ \mu_{\alpha\beta}A^\alpha_z A^\beta_{\bar z} + \bar v_\alpha A^\alpha_z + v_\alpha A^\alpha_{\bar z}\right] \tag{4.14}$$

where $\bar v_\alpha, v_\alpha$ are independent complex vectors.

One finds that (4.14) is an eigenfunction of (4.13) if and only if

$$\begin{aligned} [\lambda K, \lambda\mu] &= 0 \\ (\lambda\mu)^2 &= (\lambda K)^2 \\ (\mu + K)\lambda v &= 0 \\ (\mu - K)\lambda \bar{v} &= 0 \end{aligned} \tag{4.15}$$

where for simplicity we have assumed that $\mu_{\alpha\beta}$ is symmetric.

Now, for normalizable wavefunctions we want $\mu_{\alpha\beta}$ to be positive hermitian. In this case, the last two equations in (4.15) involve projection operators. Now, note that $\lambda^{1/2} K \lambda^{1/2}$ is a symmetric form and therefore can be diagonalized by a real orthogonal matrix $\mathcal{O}$:

$$K = \lambda^{-1/2}\mathcal{O}\begin{pmatrix} \Delta^+ & 0 \\ 0 & \Delta^- \end{pmatrix}\mathcal{O}^{tr}\lambda^{-1/2} \tag{4.16}$$

where $\Delta^\pm$ are diagonal matrices with $\Delta^+_{ii} > 0$ and $\Delta^-_{ii} < 0$. We therefore can solve our equations by letting

$$\mu = \lambda^{-1/2}\mathcal{O}\begin{pmatrix} \Delta^+ & 0 \\ 0 & -\Delta^- \end{pmatrix}\mathcal{O}^{tr}\lambda^{-1/2} \tag{4.17}$$

Thus, μ is positive definite. The energy eigenvalue with our normal ordering is $-8\pi\sum_i \Delta^-_{ii}$.

It is useful to introduce the vector space $V \cong \mathbf{R}^d$ where A^α is valued. We can regard $\mu, K \in V^* \otimes V^*$ while $\lambda, \mu^{-1} \in V \otimes V$. Note that $v = v_\alpha, \bar{v} = \bar{v}_\alpha$ are valued in V_c^*. The subscript c means that we have complexified. Note that $\Gamma^\alpha{}_\beta = \mu^{\alpha\gamma}K_{\gamma\beta}$ is an operator $\Gamma : V \to V$ and satisfies $\Gamma^2 = 1$. Here $\mu^{\alpha\gamma}\mu_{\gamma\beta} = \delta^\alpha{}_\beta$. We define projection matrices

$$P_\pm := \tfrac{1}{2}(1 \pm \mu^{-1}K) \tag{4.18}$$

and accordingly we have subspaces $V_\pm := P_\pm V$. With this choice $\lambda v \in V_-$, and $\lambda\bar{v} \in V_+$. The following identities are useful. Since μ is symmetric, $P_\pm^{tr}$ are also projection matrices, and $\mu P_\pm = P_\pm^{tr}\mu$. Moreover, $(\mu^{-1}K)^{tr} = \lambda^{-1}(\mu^{-1}K)\lambda$, so $P_\pm^{tr} = \lambda^{-1}P_\pm\lambda$, and so we can also say that

$$\begin{aligned} v^{tr}P_+ &= 0 \\ \bar{v}^{tr}P_- &= 0. \end{aligned} \tag{4.19}$$

4.4. Averaged wavefunction

Now we can proceed as before with the average

$$\overline{\Psi_{v,\bar{v}}} = \sum_{\omega^\alpha} \Psi_{v,\bar{v}}(A+\omega) e^{2\pi i \sum_{\alpha<\beta} \int \omega^\alpha K_{\alpha\beta}\omega^\beta + 2\pi i \int \omega^\alpha K_{\alpha\beta} A^\beta} \tag{4.20}$$

Expanding out we find the soliton sum of a theory of bosons $\phi \in V$, with periodicity $\phi^\alpha \sim \phi^\alpha + 1$. The action is

$$S = 2\pi \int d\phi^\alpha \mu_{\alpha\beta} * d\phi^\beta - i\pi \int B_{\alpha\beta}\omega^\alpha \wedge \omega^\beta \tag{4.21}$$

where $B_{\alpha\beta} = -B_{\beta\alpha}$ is a B-field defined by

$$B_{\alpha\beta} = K_{\alpha\beta} \qquad\qquad \alpha < \beta \tag{4.22}$$

The chiral coupling to the gauge fields is

$$-4\pi i \int \partial\phi^\alpha \mu_{\alpha\beta}(P_- A^{0,1})^\beta + 4\pi i \int \bar{\partial}\phi^\alpha \mu_{\alpha\beta}(P_+ A^{1,0})^\beta \tag{4.23}$$

Thus, only holomorphic currents valued in V_- couple to $A_{\bar{z}}$, while only antiholomorphic currents valued in V_+ couple to A_z. Similarly, the coupling to $v, \bar{v}$ in (4.20) is just:

$$\exp[\bar{v}^{tr} P_+ \omega_z + v^{tr} P_- \omega_{\bar{z}}] \tag{4.24}$$

We stressed above in the AB theory that ϕ^+, ϕ^- were not scalars with definite periodicity. The generalization of this statement is that the gauge group (or periodicity lattice for ϕ) defines a lattice $\mathbf{Z}^d \subset V$. The subspaces $V_\pm$ in general do not contain any lattice vectors. Thus, the chiral scalars $P_-\phi_L$ and $P_+\phi_R$ in general do not form a single well-defined scalar. Indeed, the lattice $\bar{\Lambda}$ in general has signature (r_+, r_-) with $r_+ \neq r_-$.

4.5. Vector space of states on T^2

One can quantize the theory of chiral bosons as before. The averaged wavefunction may be expressed in terms of a sum over an even unimodular Narain lattice of signature (d,d). We endow the real vector space $V \oplus V$ with the quadratic form:

$$(p_L; p_R) \cdot (q_L; q_R) := p_L^\alpha \mu_{\alpha\beta} q_L^\beta - p_R^\alpha \mu_{\alpha\beta} q_R^\beta \tag{4.25}$$

Note that there are now two totally independent projections in the game. We have $P_\pm$ projecting onto subspaces of V determined by the Chern-Simons coupings λ, K. In addition we have the left- and right-moving projections of Narain theory, related to the chirality of the bosons. The latter projections are denoted by L, R. The embedding of $II^{d,d} \otimes \mathbf{R} \subset V \oplus V$ is accomplished by the basis vectors:

$$
\begin{aligned}
e_\alpha &= \frac{1}{\sqrt{2}}(\delta^\gamma{}_\alpha - \mu^{\gamma\zeta} B_{\zeta\alpha}; \delta^\gamma{}_\alpha + \mu^{\gamma\zeta} B_{\zeta\alpha}) \qquad \alpha = 1, \ldots, d \\
f^\alpha &= \frac{1}{\sqrt{2}}(\mu^{\gamma\alpha}; -\mu^{\gamma\alpha}) \qquad \alpha = 1, \ldots, d
\end{aligned}
\tag{4.26}
$$

In the above formulae we denote the components of the L, R projection by the superscript γ. One easily checks that

$$
\begin{aligned}
e_\alpha \cdot e_\beta &= 0 \\
f^\alpha \cdot f^\beta &= 0 \\
e_a \cdot f^\beta &= \delta^\alpha_\beta
\end{aligned}
\tag{4.27}
$$

and hence integral combinations of these vectors define an embedding of the even unimodular lattice $II^{d,d}$ into $V \oplus V$.

Now, by examining (4.20) or by quantizing (4.21)(4.23) one finds that only the projection of $A_{\bar z}$ into V_- couples to p_L while only the projection of A_z into V_+ couples to p_R. Similarly, in the averaged wavefunction, the projection of $\bar v$ into V_+ couples to p_L while the projection of v into V_- couples to p_R. Thus we define two collections of d vectors:

$$
\begin{aligned}
\nu_\alpha &= \sqrt{2}\Big((P_-)^\gamma{}_\alpha; (P_+)^\gamma{}_\alpha \Big) \qquad \alpha = 1, \ldots, d \\
\bar\nu_\alpha &= \sqrt{2}\Big((P_+)^\gamma{}_\alpha; (P_-)^\gamma{}_\alpha \Big) \qquad \alpha = 1, \ldots, d
\end{aligned}
\tag{4.28}
$$

The real span of the ν_α is a subspace of $V_L \oplus V_R$ which we can denote $V_{-,L} \oplus V_{+,R}$ while the real span of the $\bar\nu_\alpha$ is $V_{+,L} \oplus V_{-,R}$.

Moreover, one easily computes that

$$
\begin{aligned}
\nu_\alpha \cdot \nu_\beta &= -2K_{\alpha\beta} \\
\bar\nu_\alpha \cdot \bar\nu_\beta &= +2K_{\alpha\beta} \\
\nu_\alpha \cdot \bar\nu_\beta &= 0
\end{aligned}
\tag{4.29}
$$

and hence integral combinations of ν_α generate a lattice Λ, while integral combinations of $\bar{\nu}_\alpha$ generate a lattice $\bar{\Lambda}$. Furthermore,

$$\begin{aligned} f^\beta \cdot \nu_\alpha &= \delta^\beta_{\ \alpha} \\ f^\beta \cdot \bar{\nu}_\alpha &= \delta^\beta_{\ \alpha} \\ e_\alpha \cdot \nu_\beta &= -K_{\alpha\beta} + B_{\alpha\beta} \\ e_\alpha \cdot \bar{\nu}_\beta &= K_{\alpha\beta} + B_{\alpha\beta} \end{aligned} \tag{4.30}$$

Since $II^{d,d}$ is unimodular, Λ and $\bar{\Lambda}$ are sublattices of the Narain lattice generated by e_α, f^α. The lattice $\Lambda \oplus \bar{\Lambda}$ is of finite index in $II^{d,d}$. We can now uniquely decompose any Narain vector in terms of its projection into $\Lambda \otimes \mathbf{R} \oplus \bar{\Lambda} \otimes \mathbf{R}$. These projections consist of a vector in Λ plus a glue vector in Λ^*/Λ. To be specific, there exist a finite set of vectors $\beta \in \Lambda^*$, $\bar{\beta} \in \bar{\Lambda}^*$ such that $\beta + \bar{\beta} \in II^{d,d}$ and such that we can write:

$$p = n^\alpha e_\alpha + m_\alpha f^\alpha = p_\Lambda + p_{\bar{\Lambda}} \tag{4.31}$$

where

$$\begin{aligned} p_\Lambda &= (\ell^\alpha - \tfrac{1}{2}K^{\alpha\beta}\delta_\beta)\nu_\alpha = \ell^\alpha \nu_\alpha + \beta \\ p_{\bar{\Lambda}} &= (\bar{\ell}^\alpha + \tfrac{1}{2}K^{\alpha\beta}\delta_\beta)\bar{\nu}_\alpha = \bar{\ell}^\alpha \bar{\nu}_\alpha + \bar{\beta} \end{aligned} \tag{4.32}$$

Here $\ell^\alpha, \bar{\ell}^\alpha$ are independent vectors of integers. Moreover, δ_α runs over a finite set of integral vectors. Put differently, we can make a 1-1 transform on the integers n^α, m_α in (4.31) in such a way that and we use a finite set of vectors δ_α representing Λ^*/Λ. To be specific, every vector of integers m_α can be uniquely written in terms of a vector of integers ℓ^α and the vectors δ_α as

$$m_\alpha = 2K_{\alpha\beta}\ell^\beta + \delta_\alpha \tag{4.33}$$

We may take $\beta = -\frac{1}{2}K^{\alpha\beta}\delta_\beta\nu_\alpha$ and $\bar{\beta} = +\frac{1}{2}K^{\alpha\beta}\delta_\beta\bar{\nu}_\alpha$. The mapping $\beta \to \bar{\beta}$ should be viewed as an isomorphism of dual quotient groups $\Lambda^*/\Lambda \to \bar{\Lambda}^*/\bar{\Lambda}$. Indeed, the Nikulin embedding theorem [26] describes the embedding of an even integral lattice, such as Λ, into any even unimodular lattice, such as $II^{d,d}$, in terms of an isomorphism of dual quotient groups between Λ and its complementary lattice $\bar{\Lambda}$. Here we have made that isomorphism explicit.

Now, it turns out that the left- and right- projections to p_L, p_R are compatible with the projections $P_\pm$ onto the subspaces $V_\pm$. Thus, for example, we have

$$\nu_{\alpha,L} \cdot \bar{\nu}_{\beta,L} = 0 \qquad \nu_{\alpha,R} \cdot \bar{\nu}_{\beta,R} = 0 \tag{4.34}$$

Thus, we can split the sum over the Narain lattice $II^{d,d}$ into a finite sum over Λ^*/Λ of factorized wavefunctions coupling only to A and $v, \bar{v}$, respectively. The averaged wavefunction can be written in terms of higher-level Siegel-Narain theta functions as:

$$\overline{\Psi_{v,\bar{v}}} = e^{-2\pi \int \mu_{\alpha\beta} A^\alpha * A^\beta} \frac{\mathrm{Im}\tau^{\ d/2}}{\sqrt{\det\mu}} \sum_{\beta \in \Lambda^*/\Lambda} \Theta_\Lambda(\tau, 0, \beta; P; \xi(A)) \Theta_{\bar{\Lambda}}(\tau, 0, \bar{\beta}; P; \xi(v)) \tag{4.35}$$

where

$$\xi(A) = -\sqrt{8}\Big(P_-(i\mathrm{Im}\tau\ A_{\bar{z}}); P_+(i\mathrm{Im}\tau\ A_z) \Big) \tag{4.36}$$

$$\xi(v) = \frac{\sqrt{2}}{2\pi i}\Big(P_+(\mu^{-1}\bar{v}); -P_-(\mu^{-1}v) \Big) \tag{4.37}$$

As in the previous case, (4.35) only gives the wavefunctional of the gauge fields up to a normalization constant. As before, a basis of wavefunctions for the topological theory can be given in the form

$$\Psi_\beta = e^{-2\pi \int \mu_{\alpha\beta} A^\alpha * A^\beta} \frac{\Theta_\Lambda(\tau, 0, \beta; P; \xi(A))}{\eta^{r_+} \bar{\eta}^{r_-}} \tag{4.38}$$

where (r_+, r_-) is the signature of Λ. The representation of the modular group is precisely analogous to what we had before:

$$T_{\beta,\beta'} = e^{-2\pi i (r_+ - r_-)/24} e^{i\pi(\beta,\beta)} \delta_{\beta,\beta'} \tag{4.39}$$

$$S_{\beta,\beta'} = \frac{1}{\sqrt{|\Lambda^*/\Lambda|}} e^{-2\pi i (\beta,\beta')} \tag{4.40}$$

where Λ^*/Λ inherits a quadratic form defined by

$$q(\beta \bmod \Lambda) := (\beta, \beta) \bmod 2, \tag{4.41}$$

where β is any lift of β mod Λ to a vector in Λ^*.

Remarks:

1. We can say precisely in what sense this is a generalization of the chiral splitting of RCFT. The latter case corresponds to the case where Λ and $\bar{\Lambda}$ are lattices of definite signature, hence Λ is purely left-moving and $\bar{\Lambda}$ is purely right-moving. We would like to stress that, despite the notation, $\Theta_\Lambda(\tau, \cdots)$ is *not* holomorphic in τ if Λ is not of definite signature.
2. In [24] there are some related computations. However, these authors assume that the edge state bosons have well-defined periodicity, and hence are not describing the dual to the most general abelian Chern-Simons theory.

4.6. Generalization to higher genus surfaces

The above computations generalize to higher genus surfaces X. Our wavefunctions are functions on the vector space of real harmonic one-forms on X. We define coordinates by choosing a basis $\omega^a = \omega^a_z dz$ of holomorphic 1-forms, while $\bar{\omega}^{\bar{a}}$ is a basis of anti-holomorphic 1-forms, with $a, \bar{a} = 1, \ldots h$, h is the genus of X.

Recall that the momentum is a vector-valued density, so

$$\Pi^i_\alpha \frac{\partial}{\partial x^i} \otimes d^2x, \qquad \alpha = 1, \ldots, d \tag{4.42}$$

is coordinate invariant. The Hamiltonian is

$$H = \int_X \frac{g_{ij}}{2\sqrt{g}} \lambda^{\alpha\beta} \tilde{\Pi}^i_\alpha \tilde{\Pi}^j_\beta d^2x + \tfrac{1}{2} \lambda^{-1}_{\alpha\beta} F^\alpha *_2 F^\beta \tag{4.43}$$

Our phase space is the cotangent space $T^*\Gamma(\Omega^1(X))$. Our strategy is to restrict to the sub-phase space of the cotangent bundle to the space of real harmonic forms. We refer to this as the "small phase space" for brevity. Just as on the torus, we can introduce complex coordinates so that

$$g_{ij} d\sigma^i \otimes d\sigma^j = g_{z\bar{z}} (dz \otimes d\bar{z} + d\bar{z} \otimes dz) \tag{4.44}$$

Now, in restricting to the small phase space we take

$$A^\alpha = \sum_{a=1}^{h} \left(A^\alpha_a \omega^a_z dz + A^\alpha_{\bar{a}} \omega^{\bar{a}}_{\bar{z}} d\bar{z} \right) \tag{4.45}$$

Note that $A^\alpha_{\bar{a}} = (A^\alpha_a)^*$ are complex coordinates on phase space and are $z, \bar{z}$-independent. The symplectic form is

$$\Omega = \int_X \delta\Pi_\alpha \wedge \delta A^\alpha \tag{4.46}$$

Restricting to the subspace (4.45) we define the conjugate coordinates on phase space by

$$\begin{aligned} \delta\Pi^z_\alpha &= \delta\Pi^b_\alpha (\tau^{-1})_{\bar{c}b} \omega^{\bar{c}}_{\bar{z}} \\ \delta\Pi^{\bar{z}}_\alpha &= \delta\Pi^{\bar{b}}_\alpha (\tau^{-1})_{\bar{b}c} \omega^c_z \end{aligned} \tag{4.47}$$

Here we have introduced the period matrix

$$\tau^{a\bar{c}} := \int_X \omega^a \wedge \omega^{\bar{c}} \tag{4.48}$$

The symplectic form on the small phase space is

$$\Omega = \delta \Pi^a_\alpha \wedge \delta A^\alpha_a + cplx.conj. \tag{4.49}$$

and this fixes the quatization:

$$\Pi^a_\alpha = -i \frac{\partial}{\partial A^a_\alpha} \qquad \Pi^{\bar{a}}_\alpha = -i \frac{\partial}{\partial A^a_{\bar{a}}} \tag{4.50}$$

There is no misprint here. We have a minus sign on the RHS of both expressions.

Now we find

$$\begin{aligned}
\tilde{\Pi}^z_\alpha &= = -i\omega^{\bar{a}}_{\bar{z}} (\tau^{-1})_{\bar{a}b} \left[\frac{\partial}{\partial A^\alpha_b} - 2\pi K_{\beta\alpha} i \tau^{b\bar{c}} A^\beta_{\bar{c}} \right] \\
\tilde{\Pi}^{\bar{z}}_\alpha &= - i\omega^c_z (\tau^{-1})_{\bar{b}c} \left[\frac{\partial}{\partial A^\alpha_{\bar{b}}} + 2\pi K_{\beta\alpha} i \tau^{d\bar{b}} A^\beta_d \right]
\end{aligned} \tag{4.51}$$

Finally we substitute into (4.43) with $F = 0$. We get a $(1,1)$ form, and the integral over X is

$$H = -\tfrac{1}{2} \lambda^{\alpha\beta} \tau^{-1}_{\bar{d}b} \left\{ \left(\frac{\partial}{\partial A^\alpha_b} - 2\pi i K_{\gamma\alpha} \tau^{b\bar{e}} A^\gamma_{\bar{e}} \right), \left(\frac{\partial}{\partial A^\beta_{\bar{d}}} + 2\pi i K_{\delta\beta} \tau^{f\bar{d}} A^\delta_f \right) \right\} \tag{4.52}$$

Thus, we see that the above discussion easily generalizes to arbitrary genus. Roughly speaking τ becomes the period matrix, and we replace $\lambda \to \lambda \otimes \tau^{-1}$ while $K \to K \otimes \tau$.

5. Open problems and further questions

The present paper will appear somewhat trivial to many readers. While the computations are elementary — after all we are discussing free field theory — we think it is important to have a clear idea of the wavefunctions which naturally come up in the study of holography of massive Chern-Simons theory. To conclude, we discuss briefly some natural continuations of the above results.

First, much of the structure of the rational Gaussian model can be understood in terms of the extended chiral algebra, where one extends the $u(1)$ chiral algebra generated by $i\partial\phi(z)$ by the operators $e^{\pm i\sqrt{2k}\phi(z)}$. This defines the "level k $U(1)$ chiral algebra" in the sense of RCFT. The conformal blocks of the RCFT are the holomorphic theta functions which are characters of this chiral algebra. Is there an analogous nonholomorphic algebra in the present case? A related question is to understand in detail how Wilson lines piercing the cylinder/torus correspond to vertex operator insertions in the boundary conformal field theory.

Second, there might be some interesting connections with the idea of integrable structures in the AdS/CFT correspondence. In the above discussion we have always assumed that μ is irrational. However, when μ is rational the dual *is* an RCFT. By the correspondence there is an infinite set of "extra" holomorphic conserved charges in the string dual on $AdS_3 \times K_7$. It would be worth seeing if this enhanced symmetry gives useful information on the holographic correspondence and how, in detail, it leads to greater solvability of the string theory.

A natural question one can ask is what the nonabelian generalization of the AB-type theory might be. In fact, Kaluza-Klein reduction of six-dimensional supergravity on $AdS_3 \times \mathbf{S}^3$ yields a very interesting and subtle generalization of $SU(2)$ massive Chern-Simons theory, which deserves to be understood better than it is at present [36,37,38,39,40,41].

The simple free field theory we are discussing might offer a useful laboratory to explore some issues of holography. In the massive Chern-Simons theory, which is not holographic, there is a many-to-one map from "interior data" such as the choice of metric within the solid torus, or the presence of local operators, to the coefficients ζ^{β} of the wavefunction appearing in (1.10). Some aspects of this map (such as the metric dependence) could in principle be made quite explicit. When embedded in string theory the analogous $\zeta^{\beta}_{\rm string}$ in (1.14) is supposed to be a "1-1 map" between the internal data and the data of the boundary conditions of all the string fields. Understanding this better, in the present context might be useful in addressing the puzzles raised in the recent paper [42]. Let $X_3 = \mathbf{H}^3/\Gamma$ be the quotient of hyperbolic 3-space by a quasi-Fuchsian group. Then there are Riemann surfaces X, X' at the two ends. The partition function of the massive abelian Chern-Simons theory on this manifold has the "entangled" form:

$$\sum_{\beta,\beta'} \zeta^{\beta\beta'} \Psi_{\beta}(A) \Psi_{\beta'}(A') \tag{5.1}$$

where $\zeta^{\beta\beta'}$ depends on the details of what operators have been inserted in the interior of the 3-manifold. According to our general conjecture, $\zeta^{\beta\beta'}_{\rm string}$ should only depend on (arbitrary) boundary conditions on the two end surfaces X, X'. AdS/CFT leads us to expect that it is an outer product of two vectors. We see no *a priori* reason why this cannot be true, and we believe this is the resolution of the puzzles described in [42].

It is quite natural to try to extend the discussion here to two higher dimensional analog systems. The first natural generalization is to the (B_{NS}, B_{RR}) system on spacetimes which are asymptotically hyperbolic and have boundary $X_4 \times \mathbf{S}^5$, where X_4 is a 4-manifold. The analysis of the associated topological field theory was undertaken in [28]. In [28] the kinetic terms were neglected, as is appropriate for the study of the representation of $SL(2,\mathbf{Z})$. However, we have seen that for finer questions involving natural bases of wavefunctions one should retain the kinetic terms. A computation analogous to that above indeed produces the partition function of a boundary theory of a $U(1)$ gauge field coupling to a "chiral" combination of (B_{NS}, B_{RR}). In this case, the "new" parameters, analogous to μ above, include the complex dilaton τ of the type IIB string and the conformal class of the metric on X_4. We expect that the full string theory partition function gives an analog of the decomposition (1.14), where $\zeta^{\beta}_{\rm string}$ is the partition function of $SU(N)/\mathbf{Z}_N$ SYM theory in different 't Hooft flux sectors, and $Z_{\rm string}$ is the partition function of $U(N)$ SYM theory.

Finally, we hope that the method of this paper will help in understanding better the pairing between the 5-brane partition function and the supergravity path integral for the C-field and that there will be a nice combination of the results of [32] with the techniques of this paper.

Acknowledgements:

We would like to thank J. Maldacena and E. Witten for discussions and C. Schweigert and N. Read for correspondence. G.M. would like to that the KITP for hospitality during the course of some of this work. This work was conducted during the period S.G. served as a Clay Mathematics Institute Long-Term Prize Fellow. The work of E.M. is supported in part by DOE grant DE-FG02-90ER40560, that of G.M. by DOE grant DE-FG02-96ER40949 and that of A.S. by DE-FG02-91ER40654.

Appendix A. Siegel-Narain Theta functions

Let Λ be a lattice of signature (b_+, b_-). Let P be a decomposition of $\Lambda \otimes \mathbf{R}$ as a sum of orthogonal subspaces of definite signature:

$$P : \Lambda \otimes \mathbf{R} \cong \mathbf{R}^{b_+,0} \perp \mathbf{R}^{0,b_-} \tag{A.1}$$

Let $P_{\pm}(\lambda) = \lambda_{\pm}$ denote the projections onto the two factors. We also write $\lambda = \lambda_+ + \lambda_-$. With our conventions $P_-(\lambda)^2 \leq 0$.

Let $\Lambda+\gamma$ denote a translate of the lattice Λ. We define the Siegel-Narain theta function

$$
\begin{aligned}
\Theta_{\Lambda+\gamma}(\tau,\alpha,\beta;P,\xi) \equiv & \exp[\frac{\pi}{2y}(\xi_+^2-\xi_-^2)] \\
\sum_{\lambda\in\Lambda+\gamma} \exp\Big\{ & i\pi\tau(\lambda+\beta)_+^2+i\pi\bar\tau(\lambda+\beta)_-^2+2\pi i(\lambda+\beta,\xi)-2\pi i(\lambda+\tfrac{1}{2}\beta,\alpha)\Big\} \\
= & e^{i\pi(\beta,\alpha)}\exp[\frac{\pi}{2y}(\xi_+^2-\xi_-^2)] \\
\sum_{\lambda\in\Lambda+\gamma} \exp\Big\{ & i\pi\tau(\lambda+\beta)_+^2+i\pi\bar\tau(\lambda+\beta)_-^2+2\pi i(\lambda+\beta,\xi)-2\pi i(\lambda+\beta,\alpha)\Big\}
\end{aligned}
\tag{A.2}
$$

where $y=\mathrm{Im}\tau$.

The main transformation law is:

$$
\Theta_\Lambda(-1/\tau,\alpha,\beta;P,\frac{\xi_+}{\tau}+\frac{\xi_-}{\bar\tau})=\sqrt{\frac{|\Lambda|}{|\Lambda'|}}(-i\tau)^{b_+/2}(i\bar\tau)^{b_-/2}\Theta_{\Lambda'}(\tau,\beta,-\alpha;P,\xi) \tag{A.3}
$$

where Λ' is the dual lattice. If there is a characteristic vector, call it w_2, such that

$$
(\lambda,\lambda)=(\lambda,w_2) \bmod 2 \tag{A.4}
$$

for all λ then we have in addition:

$$
\Theta_\Lambda(\tau+1,\alpha,\beta;P,\xi)=e^{-i\pi(\beta,w_2)/2}\Theta_\Lambda(\tau,\alpha-\beta-\tfrac{1}{2}w_2,\beta;P,\xi) \tag{A.5}
$$

References

[1] E. Witten, "Quantum Field Theory And The Jones Polynomial," Commun. Math. Phys. **121**, 351 (1989).

[2] G. W. Moore and N. Seiberg, "Lectures On Rcft," RU-89-32 *Presented at Trieste Spring School 1989*

[3] S. Axelrod, S. Della Pietra and E. Witten, "Geometric Quantization Of Chern-Simons Gauge Theory," J. Diff. Geom. **33**, 787 (1991).

[4] M. Manoliu, "Abelian Chern-Simons theory," J. Math. Phys. **39** (1998) 170; "Quantization of symplectic tori in a real polarization," dg-ga/9609012.

[5] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity," Phys. Rept. **323**, 183 (2000) [arXiv:hep-th/9905111].

[6] S. Deser, R. Jackiw and S. Templeton, "Topologically Massive Gauge Theories," Annals Phys. **140**, 372 (1982) [Erratum-ibid. **185**, 406.1988 APNYA,281,409 (1988 APNYA,281,409-449.2000)].

[7] S. Deser, R. Jackiw and S. Templeton, "Three-Dimensional Massive Gauge Theories," Phys. Rev. Lett. **48**, 975 (1982).

[8] M. Bos and V. P. Nair, "U(1) Chern-Simons Theory And C = 1 Conformal Blocks," Phys. Lett. B **223** (1989) 61; "Coherent State Quantization Of Chern-Simons Theory," Int. J. Mod. Phys. A **5** (1990) 959.

[9] S. Elitzur, G. W. Moore, A. Schwimmer and N. Seiberg, Nucl. Phys. B **326**, 108 (1989).

[10] E. Witten, "SL(2,Z) action on three-dimensional conformal field theories with Abelian symmetry," arXiv:hep-th/0307041.

[11] G. W. Moore and N. Seiberg, "Classical And Quantum Conformal Field Theory," Commun. Math. Phys. **123**, 177 (1989).

[12] G. W. Moore and N. Seiberg, "Naturality In Conformal Field Theory," Nucl. Phys. B **313**, 16 (1989).

[13] R. Dijkgraaf and E. Verlinde, "Modular Invariance And The Fusion Algebra," Nucl. Phys. Proc. Suppl. **5B**, 87 (1988).

[14] J. Fuchs, A. N. Schellekens and C. Schweigert, "Galois modular invariants of WZW models," Nucl. Phys. B **437**, 667 (1995) [arXiv:hep-th/9410010].

[15] T. Gannon, "Boundary conformal field theory and fusion ring representations," Nucl. Phys. B **627**, 506 (2002) [arXiv:hep-th/0106105].

[16] J. Fuchs, I. Runkel and C. Schweigert, "TFT construction of RCFT correlators. I: Partition functions," Nucl. Phys. B **646**, 353 (2002) [arXiv:hep-th/0204148].

[17] S. Carlip and I. I. Kogan, "Three-Dimensional Topological Field Theories And Strings," Mod. Phys. Lett. A **6**, 171 (1991).

[18] N. Read, "Excitation structure of the hierarchy scheme in the fractional quantum Hall effect," Phys. Rev. Lett. **65**(1990)1502.

[19] B. Blok and X. G. Wen, "Effective Theories Of Fractional Quantum Hall Effect: The Hierarchy Construction," Phys. Rev. B **42**, 8145 (1990).

[20] J. Frohlich, U. M. Studer and E. Thiran, "A Classification of quantum Hall fluids," KUL-TF-94-35

[21] J. Frohlich, T. Kerler, U. M. Studer and E. Thiran, "Structure the set of imcompressible quantum hall fluids," Nucl. Phys. B **453**, 670 (1995).

[22] J. Frohlich *et al.*, "The Fractional Quantum Hall Effect, Chern-Simons Theory, And Integral Lattices," ETH-TH-94-18

[23] A. Zee, "Quantum Hall Fluids," cond-mat/9501022

[24] Joel E. Moore, Xiao-Gang Wen, "Classification of Disordered Phases of Quantum Hall Edge States," cond-mat/9710208 ; Phys. Rev. B 57, 10138-10156 (1998)

[25] M. Freedman et. al. "A class of P,T-invariant topological phases of interacting electrons," cond-mat/0307511

[26] V.V. Nikulin, "Integral Symmetric Forms and Some of Their Applications," Math. USSR Izvestiya **Vol. 14**(1980) 103.

[27] S. Gukov, E. Martinec, G. Moore and A. Strominger, "The search for a holographic dual to AdS(3) x S(3) x S(3) x S(1)," arXiv:hep-th/0403090.

[28] E. Witten, "AdS/CFT correspondence and topological field theory," JHEP **9812**, 012 (1998) [arXiv:hep-th/9812012].

[29] J. M. Maldacena, G. W. Moore and N. Seiberg, "D-brane charges in five-brane backgrounds," JHEP **0110**, 005 (2001) [arXiv:hep-th/0108152].

[30] A. S. Schwarz, Commun. Math. Phys. **67**, 1 (1979).

[31] G. W. Moore and N. Seiberg, "Taming The Conformal Zoo," Phys. Lett. B **220**, 422 (1989).

[32] E. Diaconescu, G. Moore and D. S. Freed, "The M-theory 3-form and E(8) gauge theory," arXiv:hep-th/0312069.

[33] D. Kutasov and N. Seiberg, "More comments on string theory on AdS(3)," Published in JHEP 9904:008,1999 e-Print Archive: hep-th/9903219

[34] F. Larsen and E. J. Martinec, "U(1) charges and moduli in the D1-D5 system," JHEP **9906**, 019 (1999) [arXiv:hep-th/9905064].

[35] R. Dijkgraaf and E. Witten, "Topological Gauge Theories And Group Cohomology," Commun. Math. Phys. **129**, 393 (1990).

[36] S. Deger, A. Kaya, E. Sezgin and P. Sundell, "Spectrum of D = 6, N = 4b supergravity on AdS(3) x S(3)," Nucl. Phys. B **536**, 110 (1998), hep-th/9804166.

[37] H. Lu, C. N. Pope and E. Sezgin, "SU(2) reduction of six-dimensional (1,0) supergravity," hep-th/0212323.

[38] H. Lu, C. N. Pope and E. Sezgin, "Yang-Mills-Chern-Simons supergravity," hep-th/0305242.

[39] G. Arutyunov, A. Pankiewicz and S. Theisen, "Cubic couplings in D = 6 N = 4b supergravity on AdS(3) x S(3)," Phys. Rev. D **63**, 044024 (2001), hep-th/0007061.

[40] S. D. Mathur, "Gravity on AdS(3) and flat connections in the boundary CFT," hep-th/0101118.

[41] H. Nicolai and H. Samtleben, "Kaluza-Klein supergravity on AdS(3) x S(3)," hep-th/0306202.

[42] J. Maldacena and L. Maoz, "Wormholes in AdS," arXiv:hep-th/0401024.