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Calculation of the electron magnetic moment in Fried-Yennie-gauge QED

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The two-loop contribution to the electron magnetic moment is calculated in the Fried-Yennie gauge. This is the first treatment of the magnetic moment beyond one-loop order in a gauge other than the Feynman gauge. The Fried-Yennie gauge is infrared safe, and the calculation is done without introducing an infrared cutoff or photon mass. The Fried-Yennie-gauge result agrees with the Feynman-gauge result, as expected.

I. INTRODUCTION

The electron magnetic moment is one of the oldest and most precise tests of QED. The "anomalous moment" or "$(g-2)$" correction was first calculated at the one-loop level by Schwinger,[1] with the famous result (in units of the Bohr magneton)

$$\mu^{(1)}=\frac{\alpha}{2\pi} . \quad (1)$$

The two-loop calculation was done by Karplus and Kroll,[2] and was corrected by Sommerfield[3] and Petermann.[4] The two-loop result is

$$\mu^{(2)}=[\tfrac{3}{4}\zeta(3)-3\zeta(2)\ln 2+\tfrac{1}{2}\zeta(2)+\tfrac{197}{144}]\left[\frac{\alpha}{\pi}\right]^2 , \quad (2)$$

where $\zeta(n)$ is the Riemann zeta function with

$$\zeta(2)=\frac{\pi^2}{6},\quad \zeta(3)=\sum_{m=1}^{\infty}\frac{1}{m^3}=1.202\,056\,903 .$$

Three- and four-loop results are summarized by Kinoshita.[5]

In this report, I describe a calculation of the two-loop magnetic moment in the Fried-Yennie gauge. This gauge is infrared safe: it allows on-shell renormalization without the generation of infrared divergences.[6] All terms in the Fried-Yennie-gauge calculation are infrared finite, and there is no need to introduce an infrared-regulating parameter as was done in the previous evaluations of $\mu^{(2)}$ (Refs. 2–4 and 7–9).

The Fried-Yennie-gauge photon propagator is given by

$$D^{\mu\nu}_{\rm FY}(q)=\frac{-1}{q^2}\left[g^{\mu\nu}+\beta\frac{q^\mu q^\nu}{q^2}\right] , \quad (3)$$

where $\beta=2$. In this work dimensional regularization is used to regulate ultraviolet divergences, and it is convenient to use $\beta=2/(1-2\epsilon)$, where $n=4-2\epsilon$ is the number of spacetime dimensions. This choice of β results in simple forms for the finite parts of the electron self-energy and vertex functions discussed in Sec. II.

II. CALCULATIONAL TECHNIQUES

A. Extraction of the magnetic moment

The electron magnetic moment can be obtained from the renormalized vertex function $\Gamma^\lambda_R(p',p)$ shown in Fig. 1. This function contains a description of the interaction of an electron with an external magnetic field. The general form of the vertex function is[10]

$$\Gamma^\lambda_R(p',p)=(\gamma p'-m)M^\lambda+M'^\lambda(\gamma p-m) + A\gamma^\lambda+B\Sigma^\lambda+Ck^\lambda , \quad (4)$$

where M^λ, M'^λ, A, B, and C are functions of p' and p, and

$$\Sigma^\lambda=\frac{i\sigma^{\lambda\kappa}k_\kappa}{2m} . \quad (5)$$

When the incoming and outgoing electron lines are physical one can write

$$\bar{u}'(p')\Gamma^\lambda_R(p',p)u(p)=\bar{u}'(p')[F_1(k^2)\gamma^\lambda+F_2(k^2)\Sigma^\lambda]u(p) , \quad (6)$$

since C vanishes by current conservation under these conditions. Here, F_1 and F_2 are the electric and magnetic form factors, with

$$F_1(0)=1 , \quad (7a)$$

$$F_2(0)=(\mu-1)=\frac{\alpha}{2\pi}+O(\alpha^2) . \quad (7b)$$

The electron magnetic moment is extracted from Γ^λ_R by

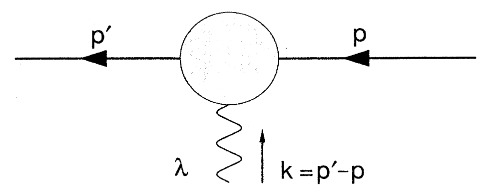

FIG. 1. The vertex function.

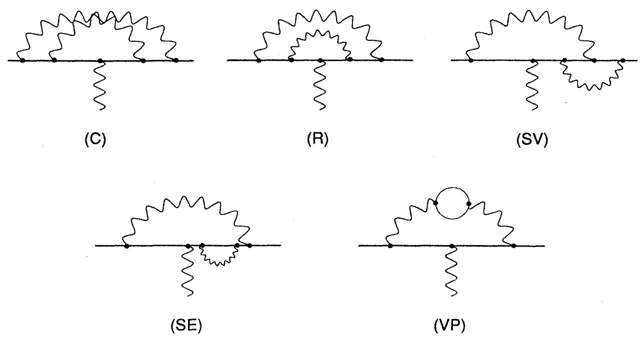

FIG. 2. The five Feynman diagrams contributing to the vertex function at order α^2. They are the crossed (C), rainbow (R), side vertex (SV), self-energy (SE), and vacuum-polarization (VP) graphs. In Karplus and Kroll (Ref. 2) they are labeled I, IIa, IIc, IId, and IIe.

identifying the term proportional to Σ^λ when the electron lines are physical in the limit $k^2 \to 0$.

B. Renormalization

Although the magnetic moment μ is contained in the renormalized vertex function Γ_R^λ, it is convenient to calculate with unrenormalized quantities, so the connection between the two must be considered. The renormalized vertex function is related to the bare vertex function by

$$\Gamma_R^\lambda(p',p) = Z_1 \Gamma^\lambda(p',p) , \tag{8}$$

where $Z_1 = (1-B)^{-1}$ is the vertex renormalization constant. Expanding all quantities in powers of the fine-structure constant α one has

$$\Gamma_R^\lambda = [1-(B^{(1)}+B^{(2)}+\cdots)]^{-1} \times(\gamma^\lambda+\Gamma^{\lambda(1)}+\Gamma^{\lambda(2)}+\cdots) . \tag{9}$$

The order-α^2 term in Γ_R^λ is

$$\Gamma_R^{\lambda(2)} = \Gamma^{\lambda(2)} + B^{(1)}\Gamma^{\lambda(1)} + (B^{(2)}+B^{(1)2})\gamma^\lambda . \tag{10}$$

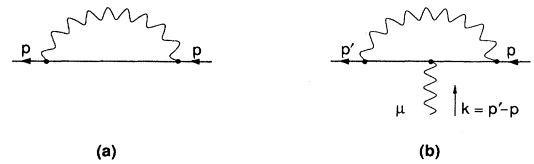

FIG. 3. The one-loop self-energy and vertex functions.

The γ^λ term in $\Gamma_R^{\lambda(2)}$ does not contribute to the magnetic moment, so we will not need to know the order-α^2 part of the renormalization constant $B^{(2)}$. The order-α contribution to μ is the Schwinger result (1), so the second term in Eq. (10) contributes $B^{(1)}(\alpha/2\pi)$ to $\mu^{(2)}$. There are five contributions to $\Gamma^{\lambda(2)}$ coming from the five Feynman diagrams in Fig. 2. Writing μ_x for the corresponding contribution to the magnetic moment, we have

$$\mu^{(2)} = \mu_C + \mu_R + \mu_{\rm SV} + \mu_{\rm SE} + \mu_{\rm VP} + B^{(1)}\left[\frac{\alpha}{2\pi}\right] \tag{11}$$

as the implication of Eq. (10) for the magnetic moment.

C. Lower-order functions

The rainbow, side vertex, self-energy, and vacuum-polarization graphs are all reducible, containing lower-order vertex, self-energy, or vacuum-polarization parts. The vacuum-polarization function is gauge independent. It can be found in the standard textbooks and will not be discussed here. The one-loop self-energy function of Fig. 3(a) is given in the dimensionally regularized Fried-Yennie gauge by[11]

$$\Sigma^{(1)}(p) = B^{(1)}(\gamma p - m) + C^{(1)}(p)(\gamma p - m)^2 , \tag{12}$$

where the one-loop contribution to the renormalization constant is

$$B^{(1)} = -\frac{\alpha}{4\pi}(4\pi)^\epsilon \Gamma(\epsilon)\left[\frac{3-2\epsilon}{1-2\epsilon}\right] \tag{13}$$

and the finite part is (in the limit $\epsilon \to 0$)

$$C^{(1)}(p) = -\frac{\alpha}{4\pi}(6\gamma p)\int_0^1 dx \int_0^1 dz \frac{x(1-x)}{xm^2+(1-x)z(m^2-p^2)} . \tag{14}$$

Note that there are no infrared divergences induced in $B^{(1)}$ and $C^{(1)}(p)$ as there are in other gauges. The choice $\beta = 2/(1-2\epsilon)$ instead of simply $\beta=2$ results in a form for $C^{(1)}(p)$ with just a γp term and no term proportional to m. The one-loop vertex function of Fig. 3(b) is more complicated. It has the form

$$\Gamma^{\mu(1)}(p',p) = -B^{(1)}\gamma^\mu + \Gamma_s^{\mu(1)}(p',p) , \tag{15}$$

where the subtracted vertex function (in the $\epsilon \to 0$ limit) is

$$\Gamma_s^{\mu(1)}(p',p) = -\frac{\alpha}{4\pi}\int_0^1 dx \int_0^1 du \int_0^1 dt \left[-4x(1-x)S^\mu \frac{1}{H^2} + 2\bar{R}^\mu \frac{1}{H} + 2\gamma^\mu x^2(H-xm^2)\frac{m^2}{\bar{H}^2} + 6\gamma^\mu x(H-xm^2)\frac{1}{\bar{H}}\right] \tag{16}$$

with

$$H = xm^2 - xu(1-u)k^2 + (1-x)u(m^2-p'^2) + (1-x)(1-u)(m^2-p^2) , \tag{17a}$$

$$\bar{H} = xm^2 + t(H-xm^2) . \tag{17b}$$

The gamma matrix factors are

$$S^{\mu}=q^2(\gamma p'-m)\gamma^{\mu}(\gamma p-m)-mq^{\mu}(\gamma p'-m)(\gamma p-m)$$
$$+(\gamma p'-m)\{[m\gamma^{\mu}-2(1-u)p^{\mu}]p\cdot q-(1-u)q^{\mu}(m^2-p^2)\}+[(m\gamma^{\mu}-2up'^{\mu})p'\cdot q-uq^{\mu}(m^2-p'^2)](\gamma p-m)$$
$$+\gamma^{\mu}[2(p'\cdot qp\cdot q-m^4)+(1-u)(p'\cdot q+2m^2)(m^2-p^2)+u(p\cdot q+2m^2)(m^2-p'^2)+2u(1-u)m^2k^2]\ , \tag{18a}$$
$$\bar{R}^{\mu}=(3-5x)(\gamma p'-m)\gamma^{\mu}(\gamma p-m)$$
$$+(\gamma p'-m)\{(2-2x-x^2)m\gamma^{\mu}+(-2+3x-x^2u)P^{\mu}+[2-3x+2xu+x^2u(1-2u)]k^{\mu}\}$$
$$+\{(2-2x-x^2)m\gamma^{\mu}+[-2+3x-x^2(1-u)]P^{\mu}+[-2+3x-2x(1-u)+x^2(1-u)(1-2u)]k^{\mu}\}(\gamma p-m)$$
$$+\gamma^{\mu}\{[-2+3x-x^2u(1-u)]k^2+[-2+3x+x(1-x)u](m^2-p'^2)+[-2+3x+x(1-x)(1-u)](m^2-p^2)\}$$
$$+x(1+x)(1-2u)mk^{\mu}-x(1-x)mi\sigma^{\mu\nu}k_{\nu}\ , \tag{18b}$$

where

$$q=up'+(1-u)p\ , \tag{19a}$$
$$P=p'+p\ , \tag{19b}$$
$$k=p'-p\ . \tag{19c}$$

The one-loop Fried-Yennie-gauge self-energy and vertex functions are similar in complexity to the corresponding Feymman-gauge functions.

III. RESULTS

In this section, I will briefly discuss the calculation of the various diagrams and give the results.

The crossed diagram C is completely infrared and ultraviolet safe, so one can set $\epsilon=0$ immediately. There are two photon propagators in C, so there are terms with two, one, and zero factors of the gauge parameter β. The terms with two and one factors of β are easy to evaluate. Their contributions to μ_C are $(-\frac{1}{4})(\alpha/\pi)^2$ and $[\frac{1}{3}\zeta(2)+\frac{2}{3}](\alpha/\pi)^2$. The term with zero factors of β is just the Feynman-gauge result for the crossed graph, which is given by Petermann.[4] The total contribution of the crossed graph is

$$\mu_C=[\tfrac{5}{4}\zeta(3)-5\zeta(2)\ln 2+\tfrac{5}{2}\zeta(2)+\tfrac{7}{12}]\left[\frac{\alpha}{\pi}\right]^2\ . \tag{20}$$

The evaluation of the rainbow diagram R involves a slight subtlety. The $i\sigma^{\mu\nu}k_{\nu}$ term in $\Gamma_s^{\mu(1)}$ contains no factor that is small in the infrared region. (All the rest of the terms in $\Gamma_s^{\mu(1)}$ that contribute to μ_R do contain such factors.) The fact that the spanning photon is in the Fried-Yennie gauge becomes crucial in order to avoid an infrared problem, and one must keep $\beta=2/(1-2\epsilon)$ for the spanning photon and perform the momentum integration in $n=4-2\epsilon$ dimensions. The result for this diagram is

$$\mu_R=-B^{(1)}\left[\frac{\alpha}{2\pi}\right]+\left[-\frac{3}{16}\right]\left[\frac{\alpha}{\pi}\right]^2\ . \tag{21}$$

The remaining diagrams are straightforward (although evaluation of the side vertex is lengthy), and one finds

$$\mu_{SV}=-2B^{(1)}\left[\frac{\alpha}{2\pi}\right]+[-\tfrac{1}{2}\zeta(3)+2\zeta(2)\ln 2-\tfrac{29}{24}]\left[\frac{\alpha}{\pi}\right]^2\ , \tag{22}$$

$$\mu_{SE}=2B^{(1)}\left[\frac{\alpha}{2\pi}\right]+\left[-\frac{9}{8}\right]\left[\frac{\alpha}{\pi}\right]^2\ , \tag{23}$$

$$\mu_{VP}=[-2\zeta(2)+\tfrac{119}{36}]\left[\frac{\alpha}{\pi}\right]^2\ . \tag{24}$$

The final result for the magnetic moment at order α^2 is

$$\mu^{(2)}=[\tfrac{3}{4}\zeta(3)-3\zeta(2)\ln 2+\tfrac{1}{2}\zeta(2)+\tfrac{197}{144}]\left[\frac{\alpha}{\pi}\right]^2 =(-0.328\,479)\left[\frac{\alpha}{\pi}\right]^2 \tag{25}$$

in agreement with the earlier calculations.

IV. CONCLUSION

The gauge independence of physical results guarantees that the Fried-Yennie gauge result for $\mu^{(2)}$ will agree with the known (and well verified[7–9]) Feynman-gauge result. The value of the present work lies in two areas. First, it is conceptually simpler than the Feynman-gauge approach since no infrared cutoff or photon mass is needed. Second, it provides a check of the Fried-Yennie gauge formalism. This is important since the Fried-Yennie gauge promises to be of great use in bound-state QED, where control of the infrared without the use of a nonzero photon mass is imperative.

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[10]The gamma matrix conventions and natural units [$\hbar=c=1$, $\alpha=e^2/4\pi\simeq(137)^{-1}$] of J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), are used throughout. The symbol m represents the electron mass $m\simeq0.511$ MeV.
[11]I have included the effect of a mass counterterm along with the self-energy shown in Fig. 3(a) so that no term of order zero in $(\gamma p-m)$ appears in $\Sigma^{(1)}(p)$.

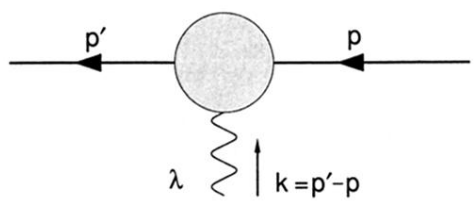

FIG. 1. The vertex function.