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Solar system tests *do* rule out $1/R$ gravity

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Shortly after the addition of a $1/R$ term to the Einstein-Hilbert action was proposed as a solution to the cosmic-acceleration puzzle, Chiba showed that such a theory violates Solar System tests of gravity. A flurry of recent papers have called Chiba's result into question. They argue that the spherically-symmetric vacuum spacetime in this theory is the Schwarzschild-de Sitter solution, making this theory consistent with Solar System tests. We point out that although the Schwarzschild-de Sitter solution exists in this theory, it is not the *unique* spherically-symmetric vacuum solution, and it is *not* the solution that describes the spacetime in the Solar System. The solution that correctly matches onto the stellar-interior solution differs from Schwarzschild-de Sitter in a way consistent with Chiba's claims. Thus, $1/R$ gravity is ruled out by Solar System tests.

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The discovery of an accelerated cosmic expansion [1,2] has led to a flurry of theoretical activity. One class of solutions to the cosmic-acceleration puzzle consists of modifications to the general-relativistic theory of gravity. One particular proposal is the addition of a $1/R$ term to the Einstein-Hilbert action [3,4]. Such a term gives rise to a vacuum solution with constant curvature, the de Sitter spacetime, rather than the Minkowski vacuum of the usual Einstein-Hilbert action.

Shortly after this proposal, Chiba [5] argued that this theory is inconsistent with Solar System tests of gravity. In particular, he showed that the theory is equivalent to a scalar-tensor theory that is known to make Solar System predictions that conflict with measurements.

Since then, however, there have been a number of papers arguing or implying that Chiba's analysis is flawed [6–10]. The crux of the counter-argument is that $1/R$ theories admit as a static spherically-symmetric solution the usual vacuum Schwarzschild-de Sitter spacetime. Apart from a cosmological constant that is too small by many orders of magnitude to affect anything observable in the Solar System, these solutions are just the usual Schwarzschild solution. Consequently, they argue, there is no effective difference between the Solar System spacetime in these models and that in ordinary general relativity.

Here we point out that these arguments are incorrect, and that Chiba was right. The crucial point is that although the Schwarzschild-de Sitter spacetime is indeed a spherically-symmetric vacuum solution to the $1/R$ equations of motion, it is not the *unique* spherically-symmetric vacuum solution in this theory. The correct solution is determined by matching onto the solution in the interior of the star. When this is done correctly, it is found that the Schwarzschild-de Sitter spacetime does *not* describe the spacetime around the Sun, and that Chiba's result stands. This misunderstanding has now propagated through a number of papers. There are moreover a number of other papers that cite these incorrect papers in a way that suggests that they may be onto something. We thought it worthwhile to correct the error before it propagates any further.

Before describing the correct spherically-symmetric spacetime for $1/R$ gravity, we consider a very simple and analogous problem that illustrates what is going on. Suppose we wanted to know the electric field around a spherically-symmetric charge distribution $\rho(r)$ confined to radii $r < R$. For radii $r > R$, the Poisson equation $\nabla^2\phi = 4\pi\rho$ relating the electric potential ϕ to the charge-density distribution ρ reduces to $\nabla^2\phi = 0$. A spherically-symmetric solution to this equation, one might argue, is $\phi = 0$, implying no electric field. This is clearly incorrect.

What went wrong? Although $\phi = 0$ is indeed a spherically-symmetric solution to $\nabla^2\phi = 0$, it is not the *unique* solution. Another solution is $\phi = c/r$, for $r > R$. The constant c in this equation is furthermore fixed in this case to be $c = Q$, where $Q = \int \rho d^3x$ is the total charge, by integrating the right- and left-hand sides of the Poisson equation $\nabla^2\phi = 4\pi\rho$ over the entire volume.

In brief, something similar happens in $1/R$ gravity. The differential equations for the metric components $g_{tt}(r)$ and $g_{rr}(r)$ are supplemented by a differential equation for the curvature R, as we will see below. The three differential equations have the Schwarzschild-de Sitter spacetime as a solution, but these vacuum solutions do not match onto the solutions in the presence of a source (i.e., the Sun). There is an additional vacuum solution that correctly matches onto the solution in the presence of the source.

Now the details: the gravitational action of $1/R$ gravity,

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(R - \frac{\mu^4}{R}\right) + \int d^4x\sqrt{-g}\mathcal{L}_M, \quad (1)$$

may be varied with respect to the metric $g_{\mu\nu}$ to obtain the field equation [3]

$$8\pi G T_{\mu\nu} = \left(1 + \frac{\mu^4}{R^2}\right)R_{\mu\nu} - \frac{1}{2}\left(1 - \frac{\mu^4}{R^2}\right)R g_{\mu\nu} + \mu^4(g_{\mu\nu}\nabla_\alpha\nabla^\alpha - \nabla_\mu\nabla_\nu)R^{-2}. \quad (2)$$

We begin by using the trace of the field equation to determine the Ricci scalar R. Contracting Eq. (2) with the inverse metric yields

$$\square \frac{\mu^4}{R^2} - \frac{R}{3} + \frac{\mu^4}{R} = \frac{8\pi G T}{3}, \tag{3}$$

where $T \equiv g^{\mu\nu}T_{\mu\nu}$.

The constant-curvature vacuum solution is obtained by setting $T = 0$ and $\nabla_\mu R = 0$. It is $R^2 = 3\mu^4$, corresponding to the de Sitter spacetime with Hubble parameter $H^2 = \mu^2/(4\sqrt{3})$, equivalent to the general-relativistic vacuum solution with a cosmological constant $\Lambda = 3H^2 = \sqrt{3}\mu^2/4$. The metric for this spacetime can be written as a static spherically-symmetric spacetime:

$$ds^2 = -(1 - H^2r^2)dt^2 + (1 - H^2r^2)^{-1}dr^2 + r^2 d\Omega^2. \tag{4}$$

To match the observed acceleration of the universe, the effective cosmological constant must be set to $\Lambda \sim \mu^2 \sim H^2 \sim 10^{-56}$ cm^{-2}.

We now consider the spacetime in the Solar System in this theory. First of all, the distances ($\sim 10^{13}$ cm) in the Solar System are tiny compared with the distance $\mu^{-1} \sim 10^{28}$ cm, so $\mu r \ll 1$ everywhere in the Solar System. Moreover, the densities and velocities in the Solar System are sufficiently small that we can treat the spacetime as a small perturbation to the de Sitter spacetime. The spacetime should also be spherically symmetric and static. The most general static spherically-symmetric perturbation to the vacuum de Sitter spacetime given by Eq. (4) can be written

$$ds^2 = -[1 + a(r) - H^2r^2]dt^2 + [1 + b(r) - H^2r^2]^{-1}dr^2 + r^2 d\Omega^2, \tag{5}$$

where the metric-perturbation variables $a(r)$, $b(r) \ll 1$. In the following, we work to linear order in a and b, and also recall that $\mu r \ll 1$. However, a, b are *not* necessarily small compared with μr.

We now return to the trace of the field equation, given by Eq. (3), and solve it for the Ricci scalar $R(r)$ in the presence of the Sun. We write the trace equation in terms of a new function,

$$c(r) \equiv -\frac{1}{3} + \frac{\mu^4}{R^2(r)}, \tag{6}$$

and demand that $c(r) \to 0$ as $r \to \infty$ so that R approaches its background value of $\sqrt{3}\mu^2$ far from the source of the perturbation. Therefore, $c(r)$ parameterizes the departure of R from the vacuum solution, and we anticipate that $c(r)$ will be the same order in the perturbation amplitude as the metric perturbations $a(r)$ and $b(r)$. In terms of $c(r)$, Eq. (3) becomes an *exact* equation,

$$\square c(r) + \frac{\mu^2 c}{\sqrt{c + \frac{1}{3}}} = \frac{8\pi G}{3}T. \tag{7}$$

In the Newtonian limit appropriate for the Solar System, the pressure p is negligible compared to the energy density ρ, and so $T = -\rho$. Neglecting terms that are higher order in $a(r)$, $b(r)$, and $\mu^2 r^2$, we are able to rewrite Eq. (7) as

$$\nabla^2 c + \sqrt{3}\mu^2 c = -\frac{8\pi G}{3}\rho, \tag{8}$$

where ∇^2 is the flat-space Laplacian operator. Note that in writing this equation, which is linear in $c(r)$, we have also neglected higher-order terms in $c(r)$. Below, we will check that the solutions we obtain have $c(r) \ll 1$ everywhere, consistent with our assumptions. The Green's function for Eq. (8) is $-\cos(3^{1/4}\mu r)/(4\pi r)$. Convolving this with the density gives us the solution to Eq. (8). However, we are restricting our attention to the region where $\mu r \ll 1$, so the Green's function reduces to that for the Laplacian operator. Therefore the equation we need to solve is $\nabla^2 c = -(8\pi G\rho)/3$. Integrating the right-hand side over a spherical volume of radius r gives us $-8\pi G m(r)/3$, where $m(r)$ is the mass enclosed by a radius r. Using Gauss's law to integrate the left-hand side gives us $4\pi r^2 c'(r)$, where the prime denotes differentiation with respect to r. Thus, the equation for $c(r)$ becomes

$$\frac{\mathrm{d}c}{\mathrm{d}r} = -\frac{2Gm(r)}{3r^2}[1 + \mathcal{O}(\mu r)]. \tag{9}$$

Integrating Eq. (9) and using the boundary condition that $c \to 0$ as $r \to \infty$ gives us the solution $c(r) = (2/3) \times (GM/r)[1 + \mathcal{O}(\mu r)]$ for $r > R_\odot$. Note also that integration of the equation for $c'(r)$ to radii $r < R_\odot$ inside the star implies that the scalar curvature R remains of order μ^2, even inside the star. We thus see that $c \ll 1$, so we were justified in using the linearized equation for $c(r)$.

This solution for $c(r)$ implies that

$$R = \sqrt{3}\mu^2\left(1 - \frac{GM}{r}\right), \qquad r > R_\odot. \tag{10}$$

We have thus shown that R is not constant outside the star and have already arrived at a result at odds with the constant-scalar-curvature Schwarzschild-de Sitter solution. Notice that had we (*incorrectly*) used $\rho = 0$ in Eq. (8), then the equations would have admitted the solution $c(r) = 0$; i.e. the constant-scalar-curvature solution. However, *this would be incorrect, because even though* $\rho = 0$ *at* $r > R_\odot$, *the solution to the differential equation at* $r > R_\odot$ *depends on the mass distribution* $\rho(r)$ *at* $r < R_\odot$. In other words, although the Schwarzschild-de Sitter solution is a static spherically-symmetric solution to the vacuum Einstein equations, *it is not the solution that correctly matches onto the solution inside the star.* Note further that the solution for R both inside and outside the star is (to linear order in c),

$$R = \sqrt{3}\mu^2\left[1 - \frac{3}{2}c(r)\right]. \quad (11)$$

Clearly, $1/R$ gravity produces a spacetime inside the star that is *very* different from general relativity. This result shows that in this theory one should not assume that $R = 8\pi G\rho$; this has lead to some confusion [11–13].

To proceed to the solutions for $a(r)$ and $b(r)$, we rearrange the field equation for $1/R$ gravity [Eq. (2)] to obtain equations,

$$R_{\mu\nu} = \left(1 + \frac{\mu^4}{R^2}\right)^{-1}\left[8\pi G T_{\mu\nu} + \frac{1}{2}\left(1 - \frac{\mu^4}{R^2}\right)R g_{\mu\nu} - \mu^4(g_{\mu\nu}\nabla_\alpha\nabla^\alpha - \nabla_\mu\nabla_\nu)R^{-2}\right], \quad (12)$$

for the Ricci tensor in terms of the Ricci scalar. When the expression for R obtained from the trace equation is inserted into the right-hand side, we obtain equations for the nonzero components of the Ricci tensor,

$$R^t_t = 3H^2 - 6\pi G\rho - \frac{3}{4}\nabla^2 c, \quad (13)$$

$$R^r_r = 3H^2 - \frac{3c'(r)}{2r}, \quad (14)$$

$$R^\theta_\theta = R^\phi_\phi = 3H^2 - \frac{3}{4}\left(\frac{c'(r)}{r} + c''(r)\right), \quad (15)$$

where we have neglected terms of order $\mu^2 c$, $G\rho c$ and c^2 in all three expressions.

For the perturbed metric given by Eq. (5), the tt component of the Ricci tensor is (to linear order in small quantities) $R^t_t = 3H^2 - (1/2)\nabla^2 a(r)$. Applying $\nabla^2 c = -(8\pi G\rho)/3$ to Eq. (13) leaves us with an equation for $a(r)$,

$$\frac{1}{2}\nabla^2 a = 4\pi G\rho, \quad (16)$$

plus terms that are higher order in GM/r and μr. The solution to this equation parallels that for $c(r)$; it is

$$\frac{\mathrm{d}a}{\mathrm{d}r} = 2G\frac{m(r)}{r^2} \quad (17)$$

both inside and outside the star. Outside the star, this expression may be integrated, subject to the boundary condition $a(r) \to 0$ as $r \to \infty$, to obtain the metric perturbation,

$$a(r) = -\frac{2GM}{r}, \qquad r > R_\odot, \quad (18)$$

exterior to the star. Note that this recovers the Newtonian limit for the motion of nonrelativistic bodies in the Solar System, as it should.

The rr component of the Ricci tensor is (to linear order in small quantities) $R^r_r = 3H^2 - (b'/r) - (a''/2)$. Given our solution for $a'(r)$ and $c'(r) = -(2/3)Gm(r)/r^2$, Eq. (14) becomes a simple differential equation for $b(r)$,

$$\frac{\mathrm{d}b}{\mathrm{d}r} = \frac{Gm(r)}{r^2} - \frac{Gm'(r)}{r} = \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{-Gm(r)}{r}\right]. \quad (19)$$

Integrating this equation subject to the boundary condition $b(r) \to 0$ as $r \to \infty$ gives an expression for $b(r)$ that is applicable both inside and outside the star:

$$b(r) = -\frac{Gm(r)}{r}. \quad (20)$$

This expression for $b(r)$ and Eq. (17) for $a'(r)$ also satisfy Eq. (15) for the angular components of the Ricci tensor. The Ricci scalar [Eq. (10)] is recovered from the Ricci tensor components if terms higher order in $\mathcal{O}(\mu r^2 GM/r)$ are included in our expressions for $a(r)$ and $b(r)$.

The linearized metric outside the star thus becomes

$$ds^2 = -\left(1 - \frac{2GM}{r} - H^2r^2\right)dt^2 + \left(1 + \frac{GM}{r} + H^2r^2\right)dr^2 + r^2d\Omega^2. \quad (21)$$

Noting that in the Solar System, $Hr \ll 1$ and that the PPN parameter γ is defined by the metric,

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 + \frac{2\gamma GM}{r}\right)dr^2 + r^2d\Omega^2, \quad (22)$$

we find that $\gamma = 1/2$ for $1/R$ gravity, in agreement with Chiba's claims [5,14], and prior calculations; e.g., Refs. [15,16]. We note that recent measurements give $\gamma = 1 + (2.1 \pm 2.3) \times 10^{-5}$ [17,18].

Other authors have noted that Birkhoff's theorem—that the unique static spherically-symmetric vacuum spacetime in general relativity is the Schwarzschild spacetime—is lost in $1/R$ gravity, and that there may be several spherically-symmetric vacuum spacetimes. Although this is true, what we have shown here is that the Solar System spacetime is determined uniquely by matching the exterior vacuum solution to the interior solution. When this is done correctly, it is found that the theory predicts a PPN parameter $\gamma = 1/2$ in gross violation of the measurements, which require γ to be extremely close to unity.

A few final comments: It is important to note that the structure of $1/R$ gravity (for example, the way matter sources the metric) is completely different than the structure of general relativity, even in the limit $\mu \to 0$. In particular, the theory does not reduce to general relativity in the $\mu \to 0$ limit, and this can lead to confusion. This is due to the fact that the introduction of additional terms in the Einstein-Hilbert action brings to life a scalar degree of freedom that lies dormant in general relativity. We also note that Chiba's mapping of $f(R)$ theories to scalar-tensor theories is perfectly valid; it amounts to no more than a variable change, from R to $\phi \equiv 1 + \mu^4/R^2$. The trace equation, Eq. (3), is then equivalent to the scalar-field equation of motion in the scalar-tensor theory. Also, the fact that general relativity is not recovered in the $\mu \to 0$ limit becomes particularly apparent in the scalar-tensor theory, as we will discuss elsewhere. Although we have restricted our analysis, for clarity, to $1/R$ theory, similar results can also be derived for other $f(R)$ theories. For example, the correct matching of the exterior and interior solutions can be used to distinguish between the spherically-symmetric vacuum spacetimes for $R^{1+\delta}$ gravity discussed in Ref. [19]. We plan to present more details in a forthcoming publication [20].

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