

Matrix Factorizations and Kauffman Homology

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Abstract

The topological string interpretation of homological knot invariants has led to several insights into the structure of the theory in the case of $sl(N)$. We study possible extensions of the matrix factorization approach to knot homology for other Lie groups and representations. In particular, we introduce a new triply graded theory categorifying the Kauffman polynomial, test it, and predict the Kauffman homology for several simple knots.

1. Introduction

In this paper, we study from the physical perspective knot invariants of the homological type. The exciting progress in this relatively young mathematical field opens many new directions for the interaction between different branches of mathematics and physics, such as gauge theory, topological string theory, symplectic geometry, representation theory, and low-dimensional topology. The present status of the field suggests several lines of connection, which include the appearance of matrix factorizations in the definition of Khovanov-Rozansky theory and the evidence for a triply-graded theory unifying various knot homologies. The purpose of the present work is to develop both of these connections and to present further evidence for the existence of a much richer underlying structure.

Recall that Chern-Simons gauge theory, in which polynomial knot invariants are obtained as expectation values of Wilson lines [1], is connected to topological strings on Calabi-Yau manifolds. Building on earlier work of Witten [2], Gopakumar and Vafa [3,4] proposed a relation between Chern-Simons theory on the three-sphere and the closed topological string on a particular Calabi-Yau three-fold. In this setup, knots can be incorporated by introducing D-branes in the closed string geometry, so that the corresponding polynomial invariants are related to topological string amplitudes in the D-brane background [5]. Alternatively, they can be viewed as generating functions that count dimensions of Hilbert spaces of BPS states in the physical string theory with plus-minus signs [5,6].

As pointed out in [7], this interpretation is conceptually similar to a “categorification”, which is a lift of a polynomial invariant to a homology theory whose graded Euler characteristic is the polynomial at hand. Over the past years, several such homological knot invariants have been discovered, including Khovanov homology [8] whose Euler characteristic is the Jones polynomial, knot Floer homology [9,10] whose associated classical invariant is the Alexander polynomial, Khovanov-Rozansky homology [11,12] which categorifies the quantum $sl(N)$ invariant, and several others. The connection between these homological invariants and the topological string has led to many new predictions and insights into the structure of the theory. For example, it was found [7,13], that the invariants associated with knots decorated with the fundamental representation of $sl(N)$ should unify at large N into a single, triply-graded structure, with well-controlled deviations at small N . This triply graded theory is a categorification of one of the two-variable polynomial invariants, the HOMFLY polynomial [14]. (Another, conceivably related, categorification of the HOMFLY polynomial was proposed in [15].)

The physical picture suggests the existence of homological knot invariants associated with a large class of Lie algebras and representations. It is likely that many of them can be constructed using matrix factorizations, as in [12,15]. Using the connection to the physics of two-dimensional Landau-Ginzburg models, we will argue in this paper that this is indeed the case. In particular, we will present a list of Landau-Ginzburg potentials which can be used for the construction of homological invariants associated with a large class of Lie algebras and representations. Some of these potentials are new. The connection to Landau-Ginzburg models will also be useful to us for understanding relations between different knot homologies.

In the topological string picture, we can introduce orientifolds to naturally define invariants associated with the fundamental representation of the other classical Lie algebras with a large N expansion, $so(N)$ and $sp(N)$. This modification was used to study the Chern-Simons partition function in [16], and the polynomial knot invariants in [17,18]. At this level, the $so(N)/sp(N)$ invariants are known to unify into the Kauffman polynomial [19], which is the second two-variable polynomial knot invariant. We will argue that this can be lifted to the homological level as well. Moreover, we will present evidence based on the Landau-Ginzburg picture that the resulting triply-graded homology theory can not only be reduced to the homological invariants associated with the fundamentals of $so(N)/sp(N)$, but in fact also contains the triply graded HOMFLY homology. In this sense, the Kauffman homology we are proposing in this paper might be the most fundamental homological invariant to date.

The paper is organized as follows. We start in section 2 with a summary of notations and definitions. In section 3, we explain the role of Landau-Ginzburg potentials in knot homology and use the intuition from physics to write new potentials associated with various representations of classical Lie algebras. We further use the relation with Landau-Ginzburg theories in section 4 to study various properties of knot homologies. In particular, we find families of differentials associated with deformations of Landau-Ginzburg potentials. In section 5, we discuss physical interpretation of $so(N)/sp(N)$ knot homologies in the context of topological strings with an orientifold. As in the $sl(N)$ case, this interpretation leads us to a new triply-graded theory that unifies $so(N)/sp(N)$ knot homologies for all N . The study of this theory is the subject of section 6.

2. Preliminaries

Our notations are summarized in the following table:

K	a knot (a link)
\mathfrak{g}	semisimple Lie algebra
R	a representation of \mathfrak{g}

Polynomial invariants

two-variables polynomials

$P(\lambda, q)$	normalized HOMFLY polynomial
$F(\lambda, q)$	normalized Kauffman polynomial

one-variable polynomials (“quantum” invariants)

$\overline{P}^{\mathfrak{g}, R}(q)$	quantum invariant associated with Lie algebra \mathfrak{g} and representation R
$P_N(q), \overline{P}_N(q)$	normalized/unnormalized quantum $sl(N)$ invariant ($\mathfrak{g} = sl(N), R = \square$)
$J_n(q)$	colored Jones polynomial ($\mathfrak{g} = sl(2), R = n$ -dimensional representation)
$F_N(q), \overline{F}_N(q)$	quantum $so(N)$ invariant
$C_N(q), \overline{C}_N(q)$	quantum $sp(N)$ invariant

Homological invariants and their Poincaré polynomials

triply-graded

$\mathcal{H}^{\text{HOMFLY}}$	$\mathcal{P}(\lambda, q, t), \overline{\mathcal{P}}(\lambda, q, t)$	triply-graded theory and reduced/unreduced superpolynomial from [13]
$\mathcal{H}^{\text{Kauffman}}$	$\mathcal{F}(\lambda, q, t), \overline{\mathcal{F}}(\lambda, q, t)$	triply-graded Kauffman theory and its reduced/unreduced superpolynomial

doubly-graded

$\mathcal{H}^{\mathfrak{g}, R}$	$\mathcal{P}^{\mathfrak{g}, R}$	homological invariant associated with \mathfrak{g} and R
$\mathcal{H}^{sl(N), \square}$	$KhR_N(q, t)$	Khovanov-Rozansky $sl(N)$ homology
$\mathcal{H}^{sl(2), \square}$	$Kh(q, t)$	Khovanov $sl(2)$ homology
$\mathcal{H}^{so(N), \square}$	$HSON(q, t)$	$so(N)$ homology
$\mathcal{H}^{sp(N), \square}$	$HSp_N(q, t)$	$sp(N)$ homology

Here, we have assumed the underlying knot to be fixed, which if need arises, we include as an additional variable, as in $P(K; \lambda, q)$. As a general rule, this notation refers to the *normalized* version of the invariants (wherever this notion applies). When appropriate, the *unnormalized* version will be denoted by over-lining it.

Let us explain our conventions in more detail.

Two-variable polynomials: The *unnormalized* HOMFLY polynomial, $\overline{P}(K; \lambda, q)$ is the polynomial invariant of unoriented knots in S^3 defined by the skein relations of oriented planar diagrams

$$\lambda \overline{P}(\text{crossing}) - \lambda^{-1} \overline{P}(\text{crossing}) = (q - q^{-1}) \overline{P}(\text{two crossings}) \quad (2.1)$$

with the normalization

$$\overline{P}(\text{unknot}) = \frac{\lambda - \lambda^{-1}}{q - q^{-1}} \quad (2.2)$$

The *unnormalized* Kauffman polynomial $\overline{F}(K; \lambda, q)$ is another invariant of unoriented knots which is defined by a similar set of combinatorial rules. We first define an invariant of planar diagrams $\tilde{F}(L; \lambda, q)$ via the skein relations

$$\begin{aligned} \tilde{F}(\text{crossing}) &= \lambda \tilde{F}(\text{crossing}) & \tilde{F}(\text{crossing}) &= \lambda^{-1} \tilde{F}(\text{crossing}) \\ \tilde{F}(\text{crossing}) - \tilde{F}(\text{crossing}) &= (q - q^{-1})(\tilde{F}(\text{two crossings}) - \tilde{F}(\text{two crossings})) \end{aligned} \quad (2.3)$$

and the normalization

$$\tilde{F}(\text{unknot}) = \frac{\lambda - \lambda^{-1}}{q - q^{-1}} + 1 \quad (2.4)$$

Then, if $w(K) = (\text{number of “+” crossings}) - (\text{number of “-” crossings})$ is the writhe of K , the Kauffman polynomial is given by

$$\overline{F}(K; \lambda, q) = \lambda^{-w(K)} \tilde{F}(K; \lambda, q) \quad (2.5)$$

The normalized versions of the HOMFLY and Kauffman polynomials, $P(K)$ and $F(K)$, can be defined by the same combinatorial rules, with the normalization where the unknot evaluates to 1. In particular, we have

$$\begin{aligned} \overline{P}(K) &= \overline{P}(\text{unknot}) P(K) \\ \overline{F}(K) &= \overline{F}(\text{unknot}) F(K) \end{aligned} \quad (2.6)$$

Remark: In the knot theory literature (and, *e.g.*, the *Mathematica* Package *KnotTheory*), the HOMFLY and Kauffman polynomials are often expressed in terms of variables a and z , in conventions which are related to ours via

$$\begin{aligned} P_{\text{ours}}(\lambda, q) &= P_{\text{KnotTheory}}(a = \lambda, z = q - q^{-1}) \\ F_{\text{ours}}(\lambda, q) &= F_{\text{KnotTheory}}(a = i\lambda, z = -i(q - q^{-1})) \end{aligned} \quad (2.7)$$

Symmetries: The HOMFLY and Kauffman polynomial have certain symmetries. Let us denote by \overline{K} the mirror image of the knot K , which is obtained by exchanging positive with negative crossings. We have

$$\begin{aligned} P(\overline{K}; \lambda^{-1}, q^{-1}) &= P(K; \lambda, q) = P(K; -\lambda, q^{-1}) = P(K; -\lambda, q) \\ F(\overline{K}; \lambda^{-1}, q^{-1}) &= F(K; \lambda, q) = F(K; -\lambda, q^{-1}) \end{aligned} \quad (2.8)$$

Quantum Invariants: The (unnormalized) quantum invariants $P^{\mathfrak{g}, R}$ are best defined as expectation values of Wilson loop operators in representation R in Chern-Simons theory based on Lie algebra \mathfrak{g} ,

$$\overline{P}^{\mathfrak{g}, R}(q) = \langle W_R(K) \rangle_{\mathfrak{g}, k} \quad (2.9)$$

where, the level k of the Chern-Simons theory is related to q via

$$q = e^{\frac{\pi i}{k+h}} \quad (2.10)$$

where h is the dual Coxeter number of \mathfrak{g} .

Specializations: As is well-known [20], the quantum invariants for the fundamental representations of the classical Lie algebras can be obtained as specializations of the 2-variable polynomials, F and P . With the above conventions, we have

$$\begin{aligned} sl(N) : \quad P_N(q) &= P(\lambda = q^N, q) = P_{\text{KnotTheory}}(a = q^N, z = q - q^{-1}) \\ so(N) : \quad F_N(q) &= F(\lambda = q^{N-1}, q) = F_{\text{KnotTheory}}(iq^{N-1}, -i(q - q^{-1})) \\ sp(N) : \quad C_N(q) &= F(\lambda = -q^{N+1}, q) = F_{\text{KnotTheory}}(iq^{N+1}, i(q - q^{-1})) \end{aligned} \quad (2.11)$$

where in the sp case N is even and the rank of it is $N/2$. In [20], another specialization is mentioned which is related to the quantum invariant associated with twisted Kac-Moody algebra $A_{N-1}^{(2)}$; it is given by

$$F(q^{N-1}, q) = F_{\text{KnotTheory}}(iq^{N-1}, -i(q - q^{-1})) \quad (2.12)$$

Special cases: The relation to the classical Jones polynomial (in the *KnotTheory* conventions) is

$$P_2(q) = J(q^{-2})$$

The isomorphism $so(4) \cong sl(2) \times sl(2)$ yields the relation

$$F_4(q) = P_2(q)^2 \quad (2.13)$$

The isomorphism $sp(2) \cong sl(2)$ yields

$$C_2(q) = P_2(q^2) \tag{2.14}$$

which is equivalent to the classical relation between the Jones and the Kauffman polynomial,

$$J(q^{-4}) = J((iq)^{-4}) = P_2(q^2) = C_2(q) = F(iq^3, i(q - q^{-1})) = F(-q^3, q + q^{-1})$$

Finally, we comment that neither $so(6) \cong su(4)$ nor $sp(4) \cong so(5)$ yield any particular kind of relationship because different representations are involved.

Let us also note that in virtue of the symmetry relations (2.8), the specializations (2.11) imply the relationship

$$\begin{aligned} C_N(\overline{K}; q) &= F(K; \lambda = q^{-N-1}, q) \\ F_N(\overline{K}; q) &= F(K; \lambda = -q^{-N+1}, q) \end{aligned} \tag{2.15}$$

In other words, the continuation of $so(N)/sp(N)$ invariants to negative N can be viewed as the $sp(N)/so(N)$ invariant *for the mirror knot*. We will interpret this in the physical setup in section 5.

3. ABDE of Matrix Factorizations

As we already mentioned in the introduction, we believe that the categorification of polynomial knot invariants can be extended to a much larger class of invariants associated with different Lie algebras and representations than what has been considered so far. The motivation for this comes, on one hand, from the realization of knot homologies in topological string theory which will be the subject of section 5, and, on the other hand, from the connection with Landau-Ginzburg models which will be discussed here. For some further aspects of the relation between matrix factorizations and the physics of Landau-Ginzburg models, see refs. [21,22,23,24,25,26,27,28].

We denote the homology theory associated with a simple Lie algebra \mathfrak{g} and a representation R by $\mathcal{H}_{i,j}^{\mathfrak{g},R}(K)$, or simply by $\mathcal{H}_{i,j}^{\mathfrak{g}}(K)$ when a particular representation is clear from the context. The Poincare polynomial of this theory is denoted $\overline{\mathcal{P}}^{\mathfrak{g},R}$, and the graded Euler characteristic is equal to the quantum group invariant

$$\overline{\mathcal{P}}^{\mathfrak{g},R}(q) = \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim \mathcal{H}_{i,j}^{\mathfrak{g},R}(K) \tag{3.1}$$

When R is a N -dimensional vector representation of $sl(N)$ (resp. $so(N)$ or $sp(N)$) we refer to $\mathcal{H}_{i,j}^{\mathfrak{g},R}(K)$ as the $sl(N)$ (resp. $so(N)$ or $sp(N)$) knot homology. We often denote the vector representation as V and the spinor representation (of $so(N)$) as S . We hope that some of the knot homologies $\mathcal{H}_{i,j}^{\mathfrak{g},R}(K)$ can be constructed using matrix factorizations, as in [12,15].

Let us briefly recall the construction of the Khovanov-Rozansky homology for the fundamental representation of $sl(N)$ [12]. Given a planar diagram of a knot (or link, or tangle), in this construction one defines a certain complex of matrix factorizations of (mostly degenerate) potentials in variables associated with the components of the planar diagram. The cohomology of this $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$ graded complex defines $\mathcal{H}_{i,j}^{sl(N)} \equiv \mathcal{H}_{i,j}^{sl(N),\square}$. For example, to a single crossing-less line starting at a point labeled x and ending at a point labeled by y one associates the potential $x^{N+1} - y^{N+1}$ and the factorization $(x - y)\pi_{x,y} = x^{N+1} - y^{N+1}$ for the appropriate choice of $\pi_{x,y}$. To compute the homology of the simplest knot, the unknot, we identify the two ends of this line. This leads us to the factorization $0 \cdot x^N = 0$ of the trivial potential. The cohomology of the two-periodic complex

$$C(\text{unknot}) = (\cdots \longrightarrow \mathbb{C}[x] \xrightarrow{0} \mathbb{C}[x] \xrightarrow{x^N} \mathbb{C}[x] \longrightarrow \cdots) \quad (3.2)$$

is just the Jacobi ring of the potential $W_{sl(N),\square} = x^{N+1}$,

$$\begin{aligned} \mathcal{H}_{i,j}^{sl(N)}(\text{unknot}) &= H(C(\text{unknot})) \\ &= \mathcal{J}(x^{N+1}) \\ &= \mathbb{C}[x]/x^N \\ &= \{1, x, \dots, x^{N-1}\} \end{aligned} \quad (3.3)$$

The polynomial grading of these homology groups is shifted down by $N - 1$ units, so that the Poincare polynomial of the unknot is

$$\overline{KhR}_N(\text{unknot}) = q^{-N+1} + q^{-N+3} + \cdots + q^{N+1} = \frac{q^N - q^{-N}}{q - q^{-1}}$$

The knot homology of the unknot in this case can also be interpreted as the cohomology ring of complex projective space \mathbf{CP}^{N-1} . It has been suggested in [12] that the extension of this result to the k -th antisymmetric representation $\Lambda^k V$ is given by the cohomology ring of the Grassmannian¹ of k -planes in \mathbb{C}^N :

$$\mathcal{H}_*^{sl(N), \left. \begin{matrix} \square \\ \square \end{matrix} \right\}^k}(\text{unknot}) = H^*(Gr(k, N)) \quad (3.4)$$

¹ Sigma-model with target space $Gr(k, N)$ appears as a theory on the intersection of k compact D-branes with N non-compact D-branes in the string theory realization.

and that the generalization of the matrix factorization construction to these representations should be based on the multi-variable potential

$$W_{sl(N),\Lambda^k}(z_1, \dots, z_k) = x_1^{N+1} + \dots + x_k^{N+1} \quad (3.5)$$

The right-hand side of this expression should be viewed as a function of the variables z_i , which are the elementary symmetric polynomials in the x_j ,

$$z_i = \sum_{j_1 < j_2 < \dots < j_i} x_{j_1} x_{j_2} \dots x_{j_i}$$

In this case, the matrix factorization associated with the unknot is the tensor product of factorizations $0 \cdot \partial_{z_i} W_{sl(N),\Lambda^k}$ over $i = 1, 2, \dots, k$. Its cohomology is again the Jacobi ring of the potential $W_{sl(N),\Lambda^k}$, which is well-known to be the cohomology of the Grassmannian $H^*(Gr(k, N))$.

To summarize, one of the key elements in this approach is the problem of constructing a potentials $W_{\mathbf{g},R}(x_i)$ for given \mathbf{g} and R , such that the Jacobi ring of $W_{\mathbf{g},R}$ is isomorphic to the homology of the unknot

$$\mathcal{H}^{\mathbf{g},R}(\text{unknot}) \cong \mathcal{J}(W_{\mathbf{g},R}(x_i)) \quad (3.6)$$

This isomorphism involves a shift of grading (by $N - 1$ units in the case of $sl(N)$), which corresponds to the spectral flow from NS to Ramond sector. At present, $W_{\mathbf{g},R}(x_i)$ is known only in the special cases we mentioned above: the fundamental and the totally antisymmetric representations of $sl(N)$. In what follows, our goal will be to expand this list and to derive potentials for other Lie algebras and representations using insights from conformal field theory and topological strings. In particular, we shall think of $W_{\mathbf{g},R}(x_i)$ as the potential in the topological Landau-Ginzburg model, in which topological D-branes are described by matrix factorizations [21,22]. Of course, not all ring relations can be derived from a potential, and we will give an example of this below.

Another key elements in the construction of [12] is a set of combinatorial rules (skein relations) which allow us to define homological invariants of planar graphs and, eventually, knots and links. Unfortunately, even for polynomial knot invariants such skein relations are not known except in special cases. If R is a spinor or a vector representation of $so(N)$ or $sp(N)$, it is natural to start with the skein relations for the Kauffman polynomial discussed above. Sometimes these skein relations can be simplified further to a set of rules analogous to the Murakami-Ohtsuki-Yamada rules [29] for $\mathbf{g} = sl(N)$. For example, for $so(5)$ such rules were constructed by Kuperberg [30]. Similarly, for $so(6) \cong sl(4)$ the spinor (resp. vector) of $so(6)$ is identified with the vector (resp. antisymmetric) representation of $sl(4)$. Hence, the $so(6)$ calculus is identical to the usual MOY calculus for $su(4)$. A set of rules also exists for the fundamental of $sp(N)$ [31].

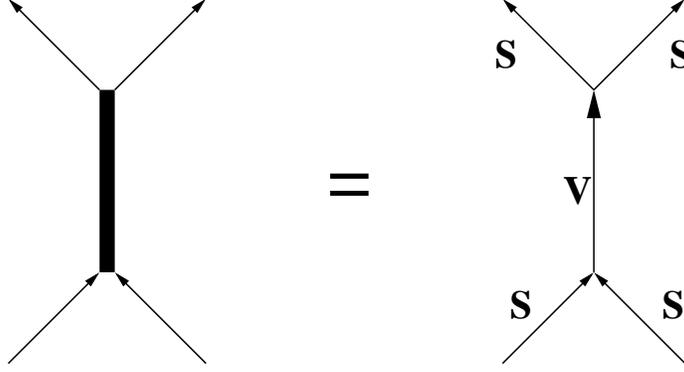


Fig. 1: Planar trivalent graph near a wide edge.

Motivated by these special cases, it is natural to expect that in the $so(N)$ case for general N , a knot diagram can be reduced to (sums of) planar graphs built out of two types of edges, corresponding to the spinor and vector representations, respectively, see fig. 1:

$$\begin{aligned}
 S &: \text{ thin edge} \\
 V &: \text{ wide edge}
 \end{aligned}
 \tag{3.7}$$

By analogy with the $sl(N)$ case studied in [12], to thin/wide edges we associate, respectively, potentials $W_{so(N),S}$ and $W_{so(N),V}$ (see below). More precisely, we have sums of those potentials for all incoming and outgoing edges. It should then be possible to find an appropriate matrix factorization corresponding to such a planar diagram, as well as to concatenate those factorizations into the complex which will compute $\mathcal{H}^{so(N)}$ for an arbitrary knot or link.²

Example. $so(6)$

The potential for the spinor of $so(6)$ (fundamental of $sl(4)$) is $W_{so(6),S} = x^5$. Let us consider a tensor product of two such representations. It corresponds to the potential

$$\begin{aligned}
 W_{so(6),S}(x_1) + W_{so(6),S}(x_2) &= (x_1 + x_2)^5 - 5x_1x_2(x_1 + x_2)^3 + 5(x_1x_2)^2(x_1 + x_2) \\
 &= z^5 - 5yz^3 + 5y^2z
 \end{aligned}$$

where $z = x_1 + x_2$ and $y = x_1x_2$. After a linear change of variables, this is the same as $W_{so(6),V} = z^5 + zy^2$.

² For $N = 5$, one can try to construct the $so(5)$ knot homology using web cobordisms, following the steps of [11]. In this construction, web cobordisms should have different types of edges, faces, and dots.

3.1. Knot Homologies and Hermitian Symmetric Spaces

One of our main examples is the $so(N)$ knot homology, that is the case of $\mathfrak{g} = so(N)$ and $R = V$. This theory should be a categorification of the quantum $so(N)$ invariant introduced in the previous section,

$$\overline{F}_N(q) = \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim \mathcal{H}_{i,j}^{so(N)}(K) \quad (3.8)$$

Using the physical picture, we shall see that the corresponding potential is

$$W_{so(N)} = x^{N-1} + xy^2 \quad (3.9)$$

Indeed, this potential leads to the correct homology of the unknot, which is isomorphic to the Jacobi ring of the D_N singularity,

$$\mathcal{H}_*^{so(N)}(\text{unknot}) = \mathbb{Q}[x, y]/\{x^{N-2} + y^2, xy\} \quad (3.10)$$

It is N -dimensional, in agreement with (2.4), generated by x of (q, t) -degree $(2, 0)$ and y of degree $(N - 2, 0)$. The homology (3.10) is also isomorphic to the cohomology of the homogeneous coset space

$$\frac{SO(N)}{SO(N-2) \times U(1)} \quad (3.11)$$

This relation and the relation between the unknot homology for the anti-symmetric representations of $sl(N)$ and the cohomology of the Grassmannian (3.4) is suggestive of the following generalization. There is a well-known relationship between the chiral rings of certain Landau-Ginzburg potentials, the cohomology of compact hermitian symmetric spaces, and the representations of minimal fundamental weights of simply laced Lie algebras. Namely, among all Kähler coset spaces of the form G/H , those which are hermitian symmetric and for which G is simple and simply laced are distinguished by the fact that the cohomology ring of G/H is integrable. In other words, there is a potential $W_{G/H}$ such that

$$H^*(G/H) \cong \mathcal{J}(W_{G/H}) \quad (3.12)$$

These Landau-Ginzburg models are known from [32,33] as SLOHSS models (the O stands for “level one” and refers to the level of the associated Kazama-Suzuki model which describes the conformal fixed point of the Landau-Ginzburg theory).

The space G/H is compact hermitian symmetric precisely if H is a regular, diagram subalgebra coming from deleting a node of the Dynkin diagram of G with dual Coxeter number equal to one. The weight Ξ of G with Dynkin label 1 on this deleted node and zero elsewhere is a so-called minimal fundamental weight, and as it turns out, the grading on the cohomologies (3.12) is precisely such that the graded dimension of \mathcal{H} gives the polynomial invariant of the unknot associated with the representation R with highest weight Ξ . More precisely [32],

$$\text{gdim}(\mathcal{H}) = q^{(\rho, \Xi)} \chi_{\Xi}(q^{\rho}) \quad (3.13)$$

where ρ is the Weyl vector of G and χ_{Ξ} is the character of the representation, R . Up to the shift in grading by (ρ, Ξ) this is nothing but the unknot invariant associated with R .

Besides the Grassmannians

$$\text{Gr}(k, N) = \frac{SU(N)}{SU(k) \times SU(N-k) \times U(1)} \quad (3.14)$$

associated with the k -th anti-symmetric representation of $SU(N)$, and the cosets (3.11), associated with the vector representation of $SO(N)$, there is another series of SLOHSS models built on the classical Lie groups, associated with the spinor representations of $so(N)$ (for N even),

$$\frac{SO(N)}{U(N/2)} \quad (3.15)$$

The cohomology of the coset (3.15) has generators z_{2i-1} in degree $2i-1$ for $i = 1, 2, \dots, [\frac{N}{4}]$ and relations which are integrable to a potential of total degree $N-1$. Although there is no closed general form for this potential, there is a straightforward algorithm which allows computation of the potential for any given N . The relations are simplest to see by viewing the coset (3.15) as the space of complex structures on \mathbb{R}^N . As for the Grassmannians, the cohomology is generated by the Chern classes of the tautological bundle E in the exact sequence

$$E \rightarrow \mathbb{C}^N \rightarrow E^* \quad (3.16)$$

Expanding

$$c(E) = 1 + \sum_{i=1}^{N/2} t^i z_i \quad c(E^*) = 1 + \sum_{i=1}^{N/2} (-1)^i t^i z_i,$$

the relations amongst the z_i can be obtained from the equation

$$c(E) \cdot c(E^*) = 1$$

The variables z_{2i} for $i = 1, \dots, N/2$ can be immediately eliminated, leaving us with relations in degree $N-2i$ for z_{2i-1} for $i = 1, 2, \dots, [\frac{N}{4}]$.

Example. $so(10)$

In this case, there are two variables z_1 and z_3 in degree 1 and 3, respectively, and the potential looks like

$$W_{so(10),S} = \frac{5}{576}z_1^9 - \frac{1}{8}z_1^6z_3 + \frac{1}{2}z_1^2z_3^2 - \frac{1}{3}z_3^3$$

'Exceptional' Knot Homology

Finally, there are two 'exceptional' cosets,

$$\frac{E_6}{SO(10) \times U(1)} \quad , \quad \frac{E_7}{E_6 \times U(1)} \quad (3.17)$$

corresponding to the 27 and 56-dimensional representations of E_6 and E_7 , respectively. The superpotentials can be computed by the same methods as above, using a tautological exact sequence similar to (3.16). This was done explicitly in [33], and the result is (the indices on the variables indicate their degrees):

$$W_{E_6,27} = z_1^{13} - \frac{25}{169}z_1z_4^3 + z_4z_1^9 \quad (3.18)$$

$$W_{E_7,56} = \frac{2791}{19}z_1^{19} + 37z_1^{14}z_5 - 21z_1^{10}z_9 + z_5^2z_9 + z_1z_9^2 \quad (3.19)$$

3.2. Totally Symmetric Representations

In this subsection, we will derive Landau-Ginzburg potentials corresponding to the totally symmetric representations of $sl(N)$. These representations do not correspond to minimal fundamental weights, and there is (as far as we know) no interpretation in terms of the cohomology of some homogeneous space. To the best of our knowledge, the potentials of this subsection are new.

We recall the generating function of the potentials (3.5) for the cohomology of the Grassmannians $Gr(k, N)$ [34,35]. The tautological sequence

$$E \rightarrow \mathbb{C}^N \rightarrow F$$

says $c(E) \cdot c(F) = 1$. The cohomology ring is generated by the cohomology classes of E , $c(E) = 1 + \sum_{i=1}^k t^i z_i$, and the relations are

$$R_{N+1-i} = 0 \quad \text{for } i = 1, \dots, k, \quad (3.20)$$

where the R_i are defined by

$$c(F) = c(E)^{-1} = 1 + \sum_{i \geq 1} t^i R_i(z_i)$$

It is easy to see that if we define $W_{sl(N), \Lambda^k}$ by the generating function

$$\sum_{N=1}^{\infty} (-1)^N t^{N+1} W_{sl(N), \Lambda^k}(z_i) = \log\left(1 + \sum_i t^i z_i\right) \quad (3.21)$$

then

$$\frac{\partial W_{sl(N), \Lambda^k}}{\partial z_i} = R_{N+1-i}(z_i)$$

Introducing the roots x_i of the Chern polynomial $c(E) = \prod_{i=1}^k (1 + tx_i)$, we obtain the form (3.5) for $W_{sl(N), \Lambda^k}$.

There is an alternative derivation of $W_{sl(N), \Lambda^k}$ which is closer in spirit to Landau-Ginzburg theory and which will be our approach to derive the potentials for the totally symmetric representations. Let us first illustrate this in the simplest case $k = 2$.

Rank 2:

The derivation of the potentials is based on the tensor product decomposition of irreducible representations of $sl(N)$,

$$V \otimes V = \Lambda^2 \oplus S^2 \quad (3.22)$$

As before, V is the vector representation, and Λ^2 and S^2 are the rank 2 anti-symmetric and symmetric representations of $sl(N)$, respectively.

We can understand the decomposition (3.22) at the level of the cohomology by looking at the ground states of the Landau-Ginzburg theory. In the tensor product, with superpotential $W_{N \otimes N} = x_1^{N+1} + x_2^{N+1}$, those ground states can be obtained by acting with the elements of the chiral ring $\mathcal{H}^*(N \otimes N) \cong \mathcal{J}_{N \otimes N} = \mathbb{C}[x_1, x_2] / \langle x_1^N, x_2^N \rangle$ on the unique groundstate with lowest R-charge $|0\rangle = |0\rangle_1 \otimes |0\rangle_2$. This ground state is clearly symmetric with respect to exchange of x_1 and x_2 . Acting with only the symmetric combinations $z = x_1 + x_2$ and $w = x_1 x_2$, we obtain all symmetric ground states. Similarly, we obtain all anti-symmetric ground states by acting with z and w on the lowest one, $|x_1 - x_2\rangle = |x_1\rangle_1 \otimes |0\rangle_2 - |0\rangle_1 \otimes |x_2\rangle_2$.

The relations among z and w are obtained by restricting the tensor product relations $x_1^N \equiv 0$, $x_2^N \equiv 0$ to the symmetric/anti-symmetric sector. In the anti-symmetric sector,

the relations are generated by $x_1^N - x_2^N \equiv 0$, $x_1^{N+1} - x_2^{N+1} \equiv 0$, which in terms of z and w become

$$\begin{aligned}\frac{x_1^N - x_2^N}{x_1 - x_2} &= x_1^{N-1} + x_1^{N-2}x_2 + \cdots + x_2^{N-1} \equiv 0 \\ \frac{x_1^{N+1} - x_2^{N+1}}{x_1 - x_2} &= x_1^N + x_1^{N-1}x_2 + \cdots + x_2^N \equiv 0\end{aligned}\tag{3.23}$$

It is easy to check that these are precisely the relations deduced from the potentials $W_{sl(N),\Lambda^2}$ with generating function

$$\sum_N (-1)^N t^{N+1} W_{sl(N),\Lambda^2}(z, w) = \log(1 + tz + t^2w)\tag{3.24}$$

To get the symmetric cohomology, we concentrate on the symmetric ground states of the tensor product theory. The relations are simply $x_1^N + x_2^N \equiv 0$, $x_1^{N+1} + x_2^{N+1} \equiv 0$, which we can write in terms of z and w as

$$W_{sl(N-1),\Lambda^2}(z, w) \equiv 0 \quad W_{sl(N),\Lambda^2}(z, w) \equiv 0\tag{3.25}$$

These relations can be integrated to a potential $\widetilde{W}_{sl(N),S^2}$ which is encoded in the generating function

$$\sum (-1)^N t^{N+2} \widetilde{W}_{sl(N),S^2}(z, w) = (1 + tz + t^2w) \log(1 + tz + t^2w)\tag{3.26}$$

Indeed, by taking derivatives with respect to z and w on both sides, and using (3.24), we find $\partial_z \widetilde{W}_{sl(N),S^2} = W_{sl(N),\Lambda^2}$ and $\partial_w \widetilde{W}_{sl(N),S^2} = -W_{sl(N-1),\Lambda^2}$.

Higher Symmetric Representations

The foregoing has an immediate generalization to all totally symmetric/anti-symmetric representations, obtained as subsectors of the tensor product theory with potential $W_{N^{\otimes k}} = \sum x_i^{N+1}$. The k -th anti-symmetric representation is obtained by acting with the elementary symmetric functions on the anti-symmetric state

$$|\Delta\rangle = \prod_{i < j} (x_i - x_j) |0\rangle$$

where $\Delta = \det_{r,s} x_r^{k-s}$ stands for the Vandermonde determinant. The relations of the chiral ring $\mathcal{H}^*(N^{\otimes k}) = \mathbf{C}[x_1, \dots, x_k] / \langle x_1^N, \dots, x_k^N \rangle$, when restricted to the anti-symmetric sector,

are generated for $i = 1, 2, \dots, k$ by $\det_{r,s} x_r^{\lambda_s+k-s} \equiv 0$, where $\widetilde{\lambda}_s = N + 1 - i$ for $s = 1$ and 0 else. For the z_i 's, this can be written in terms of the Schur polynomials

$$S_{N+1-i} = \frac{\det_{r,s} x_r^{\lambda_s+k-s}}{\Delta} \equiv 0$$

By a well-known formula (Giambelli's formula), S_i is also equal to the coefficient of t^i in the expansion of $1/\prod(1+tx_j)$ (see, eg, [36]). This is what we have been calling R_i in (3.20), thus completing the derivation of the potential for the anti-symmetric representation.

In the symmetric sector, the relations are generated by

$$\begin{aligned} x_1^N + \dots + x_k^N &= 0 \\ x_1^{N+1} + \dots + x_k^{N+1} &= 0 \\ &\vdots \\ x_1^{N+k-1} + \dots + x_k^{N+k-1} &= 0 \end{aligned} \tag{3.27}$$

When expressed in terms of z_1, \dots, z_k , the relations are $W_{sl(N+k-i-1), \Lambda^k} = 0$ for $i = 1, \dots, k$. Those relations can be integrated, $W_{sl(N+k-i-1), \Lambda^k} = \partial_{z_i} \widetilde{W}_{sl(N), S^k}$ where the \widetilde{W} 's are encoded in the generating function

$$\sum t^{N+k} (-1)^N \widetilde{W}_{sl(N), S^k}(z_1, \dots, z_k) = \left(1 + \sum_{i=1}^k t^i z_i\right) \log\left(1 + \sum_{i=1}^k t^i z_i\right) \tag{3.28}$$

3.3. Representations without Potentials

Finally, let us give an example of a representation for which the relations in the homology ring of the unknot cannot be integrated to a potential. We consider the tensor product decomposition of three fundamentals

$$\square^{\otimes 3} = \square \oplus 2\boxplus \oplus \boxtimes$$

We know already that by acting with the symmetric functions on the ground state, we obtain the symmetric representation \boxtimes , while acting on the totally anti-symmetric state $(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$, we obtain the representation \square . The remaining states must comprise two copies of the representation \boxplus . In degree 1, we have (modulo the symmetric ones) the two states

$$|x_1 - x_2\rangle \quad |x_2 - x_3\rangle$$

We can identify those with the highest weight vectors and build the representation by acting with polynomials of a certain symmetry type. But the resulting relations are not integrable to a potential.

For instance, if we act only with the totally symmetric polynomials, we obtain one states in degree 2 from each of $x_1 - x_2$ and $x_2 - x_3$. Since the total number of states in degree 2 is 6, and two are contained in \square , we would be missing 2 states.

On the other hand, if we act for example on $|x_1 - x_2\rangle$ with polynomials that are only symmetric with respect to $1 \leftrightarrow 2$, namely $x_1 + x_2$, x_3 and x_1x_2 . Then we get just the right number of states in degree 2. But in degree 3, we will also get the totally anti-symmetric state $(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$. This means that we have an additional relation between $x_1 + x_2$, x_3 and x_1x_2 in degree 2, together with some other relations in higher degrees. These relations cannot be integrated to a potential.

4. Deformations

Given an effective Landau-Ginzburg description, it is natural to ask for the behavior of the theory under deformations. In the present context, we could in principle study the dependence of the knot homologies on deformations of the matrix factorizations used to defining them, or on the deformation of the potentials. In general, we will need to deform both.

Although we know the requisite matrix factorizations only in a limited number of cases, we can obtain useful information just from the deformation of the potential associated with a single crossing-less strand with a single marking,

$$W \rightsquigarrow W + \Delta W \tag{4.1}$$

where ΔW is a polynomial in the same variables as W . In those cases in which the matrix factorizations are known, we can also study how they must change under (4.1), and this will give us further valuable information as well.³

³ In general, we cannot exclude the possibility that after the deformation, the factorizations lose some of their important properties for the definition of knot homologies. It could also happen that the factorization cannot be deformed together with the potential, thereby obstructing the deformation altogether. By experience, however, such phenomena are rare, and we will ignore them. See [25,23] for some aspects of the deformation theory of matrix factorizations.

As we have reviewed above, the definition of Khovanov-Rozansky homology involves, in physical terminology, vector spaces of open strings between D-branes which are described by matrix factorizations of Landau-Ginzburg potentials. These vector spaces are endowed with a multiplication structure by associating linear maps with cobordisms of knots (see [12] for details). This defines a ring structure on the homology of the unknot, and the homology of a general knot becomes a module over $\mathcal{H}(\text{unknot})$. On general grounds, we are expecting that under a marginal deformation (formally, a deformation respecting the grading, $\deg \Delta W = \deg W$), the vector spaces themselves will not change, and only the ring/module structure associated with cobordisms of knots will be deformed in a certain way. On the other hand, under relevant deformations (namely, those with $\deg \Delta W < \deg W$), we are expecting that the dimension of the vector spaces will change (more precisely, it should decrease).

The purpose of this section is to describe some of these deformations, and their expected relation to structural elements of the knot homology theories.

4.1. Deformations $sl(N) \rightsquigarrow sl(M)$

Deformations of homological knot invariants were first studied by Lee [37] in the case of Khovanov homology, and further considered by Bar-Natan [38] and Turner [39]. See also [40] for further explanations. In [41], Gornik considers a deformation of Khovanov-Rozansky's $sl(N)$ homology, in which the potential x^{N+1} associated with a thin edge is deformed to $x^{N+1} + \beta^N x$ (where β is a scalar parameter), but all the other essentials of the definitions of [12] are left unchanged. Because the deformed potential is not homogeneous, one thereby obtains instead of a bigraded complex a filtered chain complex $\mathcal{C}^{\text{def}}(K)$ for any given knot K . Gornik concentrates on the unreduced version of the theory and proves two statements about $\mathcal{C}^{\text{def}}(K)$ (similar statements hold for Lee's deformation of Khovanov homology). The second term in the spectral sequence associated with $\mathcal{C}^{\text{def}}(K)$ is isomorphic to the undeformed $sl(N)$ homology of Khovanov-Rozansky. Moreover, the cohomology of $\mathcal{C}^{\text{def}}(K)$ is, for any knot K , and for any N , isomorphic to the $sl(N)$ -homology of the unknot. In the reduced theory, the cohomology of the deformed complex is expected to be one-dimensional.

In ref. [13], it was proposed that the deformed theories of Lee and Gornik could be usefully mounted into the triply-graded homology theory $\mathcal{H}^{\text{HOMFLY}}$ categorifying the (normalized) HOMFLY polynomial. Recall that this theory is expected to come equipped with a family of anti-commuting differentials $\{d_N\}$, with $N \in \mathbb{Z}$. The cohomology of

$\mathcal{H}^{\text{HOMFLY}}$ with respect to d_N with $N > 1$ is isomorphic to the (reduced) $sl(N)$ homology $\mathcal{H}^{sl(N)}$ of Khovanov-Rozansky. Differentials $d_{\pm 1}$ are “canceling”, in the sense that their cohomology is one-dimensional in a particular degree. By restricting $d_{\pm 1}$ to $\mathcal{H}^{sl(N)}$, one induces canceling differentials on Khovanov-Rozansky homology. The existence of precisely such a differential follows from the work of Lee and Turner on $sl(2)$ and can be derived from Gornik’s deformation in the general case [13].

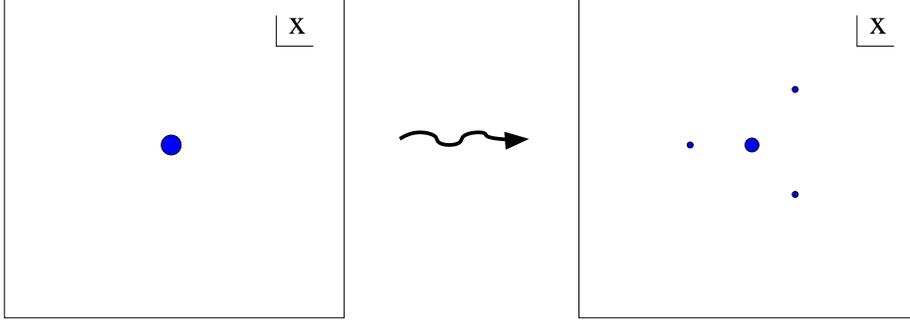


Fig. 2: Behavior of critical points under the deformation (4.2) ($N - M = 3$).

More generally, we can consider deformations of the $sl(N)$ potential $W_{sl(N)} = x^{N+1}$ by any monomial of lower degree,

$$W = x^{N+1} \rightsquigarrow x^{N+1} + \beta^{N-M} x^{M+1} \quad (4.2)$$

Under this deformation, the critical point of order N at the origin $x = 0$ is resolved into $N - M$ non-degenerate critical points at $x^{N-M} = -(M + 1)/(N + 1)$ and a degenerate critical point of order M at the origin, which is equivalent to the potential $W_{sl(M)} = x^{M+1}$ (see fig. 2). One therefore expects that the deformed theory will be related to the $sl(M)$ theory, and at the reduced level, give rise to a differential $d_{N \rightarrow M}$, identified with the restriction of d_M to $\mathcal{H}^{sl(N)}$.

Let us make this deformation of Khovanov-Rozansky theory more explicit. As in [41], we keep the planar graph calculus intact. In a given planar graph, we assign the factorization

$$x^{N+1} + \beta^{N-M} x^{M+1} - y^{N+1} - \beta^{N-M} y^{M+1} = (x - y)\pi(x, y) = (x - y)(\pi_N(x, y) + \pi_M(x, y)) \quad (4.3)$$

to a single arc (oriented edge) with endpoints labeled x and y . Here,

$$\pi_N(x, y) = x^N + x^{N-1}y + \cdots + y^N \quad (4.4)$$

Now recall that to a wide edge with four thin edges attached carrying labels x_1, x_2, x_3, x_4 , Khovanov-Rozansky associate the factorization

$$x_1^{N+1} + x_2^{N+1} - x_3^{N+1} - x_4^{N+1} = (x_1 + x_2 - x_3 - x_4)u_{1,N} + (x_1x_2 - x_3x_4)u_{2,N} \quad (4.5)$$

where

$$\begin{aligned} u_{1,N} &= \frac{x_1^{N+1} + x_2^{N+1} - g(x_3 + x_4, x_1x_2)}{x_1 + x_2 - x_3 - x_4} \\ u_{2,N} &= \frac{g(x_3 + x_4, x_1x_2) - x_3^{N+1} - x_4^{N+1}}{x_1x_2 - x_3x_4} \end{aligned} \quad (4.6)$$

In the deformed case, we simply write

$$\sum_{i=1}^4 \pm (x_i^{N+1} + \beta^{N-M} x_i^{M+1}) = (x_1 + x_2 - x_3 - x_4)u_1 + (x_1x_2 - x_3x_4)u_2 \quad (4.7)$$

where $u_i = u_{i,N} + \beta^{N-M} u_{i,M}$. So, to the two resolutions of a crossing in a planar diagram of a knot, we have associated matrix factorization

$$\begin{aligned} Q_1 &= \left[\begin{pmatrix} \pi_{14} & x_2 - x_3 \\ \pi_{23} & x_4 - x_1 \end{pmatrix}, \begin{pmatrix} x_1 - x_4 & x_2 - x_3 \\ \pi_{23} & -\pi_{14} \end{pmatrix} \right], \\ Q_2 &= \left[\begin{pmatrix} u_1 & x_1x_2 - x_3x_4 \\ u_2 & x_3 + x_4 - x_1 - x_2 \end{pmatrix}, \begin{pmatrix} x_1 + x_2 - x_3 - x_4 & x_1x_2 - x_3x_4 \\ u_2 & -u_1 \end{pmatrix} \right] \end{aligned} \quad (4.8)$$

The maps χ_0 and χ_1 used by Khovanov-Rozansky have the form (this corresponds to setting $\mu = 0$ and $\lambda = 1$ in the definitions of [12].)

$$\begin{aligned} \chi_0 &= \left[\begin{pmatrix} x_4 - x_2 & 0 \\ a_1 & 1 \end{pmatrix}, \begin{pmatrix} x_4 & -x_2 \\ -1 & 1 \end{pmatrix} \right], \\ \chi_1 &= \left[\begin{pmatrix} 1 & 0 \\ -a_1 & x_4 - x_2 \end{pmatrix}, \begin{pmatrix} 1 & x_2 \\ 1 & x_4 \end{pmatrix} \right] \end{aligned} \quad (4.9)$$

χ_0/χ_1 is in the cohomology $\text{Hom}(Q_1, Q_2)/\text{Hom}(Q_2, Q_1)$ if

$$\begin{aligned} \begin{pmatrix} u_1 & x_1x_2 - x_3x_4 \\ u_2 & x_3 + x_4 - x_1 - x_2 \end{pmatrix} \begin{pmatrix} x_4 - x_2 & 0 \\ a_1 & 1 \end{pmatrix} &= \begin{pmatrix} x_4 & -x_2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \pi_{14} & x_2 - x_3 \\ \pi_{23} & x_4 - x_1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ -a_1 & x_4 - x_2 \end{pmatrix} \begin{pmatrix} x_1 + x_2 - x_3 - x_4 & x_1x_2 - x_3x_4 \\ u_2 & -u_1 \end{pmatrix} &= \begin{pmatrix} x_1 - x_4 & x_2 - x_3 \\ \pi_{23} & -\pi_{14} \end{pmatrix} \begin{pmatrix} 1 & x_2 \\ 1 & x_4 \end{pmatrix} \end{aligned} \quad (4.10)$$

All these equations are solved by

$$a_1 = -u_2 + \frac{u_1 + x_1 u_2 - \pi_{23}}{x_1 - x_4} \quad (4.11)$$

and it is clear that this works in the deformed case as well. Note that the property $\chi_0 \chi_1 = (x_4 - x_2)$ is also preserved.

So we can follow all the steps in ref. [12] to define a deformed $sl(N)$ homology of links. Let us denote it by $\mathcal{H}_{N \rightarrow M}$. What are its properties?

First of all, as in Gornik's case, the deformed differential is a sum of two terms of different degree of homogeneity, so that we obtain a filtered chain complex instead of a bigraded one. The proof of "Theorem 1" of [41] can then be extended without much difficulty to show the analogous statement for the present deformation. Namely, the second term in the spectral sequence associated with the filtered chain complex is isomorphic to the undeformed $sl(N)$ homology.

Gornik's second result, concerning the cohomology of the deformed complex, is slightly more complicated to generalize. In the unreduced case, we are expecting that the $sl(M)$ homology will be contained as a summand in the deformed cohomology, with remaining pieces being $N - M$ dimensional, independent of the knot.

$$\mathcal{H}_{N \rightarrow M}(K) \cong \mathcal{H}_M(K) \oplus \mathbb{C}^{N-M} \quad (4.12)$$

Intuitively, the extra terms are associated with the non-degenerate critical points of the deformed potential (see fig. 2). Consequently, going to reduced theory will remove these extra terms.

Thus, we see that there is a correspondence between the differentials d_N of the triply graded HOMFLY theory, and the relevant deformations of the Landau-Ginzburg potentials $W_{sl(N)} = x^{N+1}$. The degree of the differentials can also be (partially) understood from this interpretation: When restricted to $\mathcal{H}^{sl(N)}$, d_M has q -degree $2(M - N)$ (see [13]), which matches the "relevance" of the deformation,

$$\deg x^{M+1} - \deg x^{N+1} = 2(M - N) \quad (4.13)$$

(recall that $\deg x = 2$).

It is natural to generalize these considerations and to look for differentials on other homology theories which can be induced from relevant deformations of the Landau-Ginzburg potentials of section 3.

4.2. Deformations $so(N) \rightsquigarrow so(M)$ and $so(N) \rightsquigarrow sl(N - 2)$

For example, let us consider the $so(N)$ potential $W_{so(N)} = x^{N-1} + xy^2$. Among the relevant deformations, those with a definite grading are the monomials in x as well as the monomials y , xy , and y^2 . We are expecting that each of these deformations will give rise to a differential on $\mathcal{H}^{so(N)}$, and we can predict the cohomology of these differentials by looking at the type of singularity of the deformed potential. For instance, adding x^{M-1} deforms $W_{so(N)}$ into $W_{so(M)}$ so we expect that the cohomology of the corresponding differential, again denoted $d_{N \rightarrow M}$, will be isomorphic to the $so(M)$ knot homology.

$$(\mathcal{H}^{so(N)}, d_{N \rightarrow M}) \cong \mathcal{H}^{so(M)} \quad (4.14)$$

In particular, for $M = 2$, we obtain a canceling differential.

The deformation which we find most interesting for our present purposes is the one relating the so series of potentials with the sl series. Consider the deformed potential

$$x^{N-1} + xy^2 + y^2 \quad (4.15)$$

obtained from adding to $W_{so(N)}$ a quadratic term for y . Under this deformation, the isolated critical point of order N at $x = y = 0$ is resolved into two non-degenerate critical points at $x = -1$, $y^2 = (N - 1)(-1)^{N-1}$, and a degenerate critical point of order $N - 2$ at the origin. Around this degenerate critical point, the deformed potential is equivalent to the $sl(N - 2)$ potential $W_{sl(N-2)} = x^{N-1}$.

We are conjecturing that this deformation can be extended to the entire $so(N)$ homology theory. The deformation will lead to a differential d_{y^2} on the $so(N)$ homology with cohomology equivalent to $sl(N - 2)$ homology,⁴

$$(\mathcal{H}^{so(N)}, d_{y^2}) \cong \mathcal{H}^{sl(N-2)} \quad (4.16)$$

The expected degree of this differential can be determined by noting that the undeformed potential is homogeneous of degree $2N - 2$ if we assign degree 2 and $N - 1$ to x and y ,

⁴ We should mention an important caveat here. It is known from the context of “Knörrer periodicity” that the categories of matrix factorizations associated with $W(x)$ and $W(x) + y^2$ are *not* strictly equivalent, unless the latter is equivariantized with respect to $y \mapsto -y$ [22]. It remains to be seen how this will affect the conjectured relation between $so(N)$ and $sl(N - 2)$ homologies.

respectively. Thus, d_{y^2} will have degree -2 , independent of N . We will find evidence for the existence of such a differential in the triply-graded theory considered in section 6.

We summarize the relevant deformations of $W_{so(N)}$ and associated differentials on $\mathcal{H}^{so(N)}$ in the following table:

<u>deformed potential</u>	<u>differential</u>
$x^{N-1} + xy^2 + x^{M-1}$	$d_{N \rightarrow M} \cong d_M _{\mathcal{H}^{so(N)}}$
$x^{N-1} + xy^2 + x$	canceling
$x^{N-1} + xy^2 + y$	canceling
$x^{N-1} + xy^2 + xy$	canceling
$x^{N-1} + xy^2 + y^2$	d_{y^2}

4.3. Marginal Deformations

Until now, we have considered only relevant deformations which change the knot homologies as vector spaces, and relate categorifications of different type. As we have noted above, marginal deformations are expected to change only the algebraic structure on the knot homologies, leaving the vector spaces untouched.

For example, recall from section 3 that the potentials $W_{sl(N), S^k}$ associated with the k -th symmetric representation of $sl(N)$ are homogeneous functions of total degree $N+k$ in k variables of degree $1, 2, \dots, k$. These are precisely the same degrees as for the potential $W_{sl(N+k-1), \Lambda^k}$ associated with the k -th anti-symmetric representation of $sl(N+k-1)$. Since the Poincare polynomial of the Landau-Ginzburg model depends only on the degrees of the variables and the total degree of the potential, the corresponding homologies are equal as vector spaces,

$$\mathcal{H}^{sl(N), S^k}(\text{unknot}) \cong \mathcal{H}^{sl(N+k-1), \Lambda^k}(\text{unknot}) \quad (4.17)$$

However, since the potentials are not equivalent by a change of variables, the ring structure on the two sides of (4.17) will not be the same.

Based on the existence of such a deformation, it is tempting to speculate that it can be extended beyond the unknot. It is admittedly difficult to make this precise at the moment, given that we do not know the combinatorial definition of, e.g., the $(sl(N), S^k)$ knot homologies. In fact, the combinatorics will most likely not be equivalent to the MOY calculus relevant for the anti-symmetric representations. Hence the corresponding categorifications are not expected to be equal at the level of the knot invariants, and can at most be related at a more subtle level. We find it plausible that such a relation could exist.

5. Topological Strings and $so(N)$ Knot Homology

Now let us consider a string theory realization of the polynomial and homological knot invariants associated with gauge groups $SO(N)$ and $Sp(N)$. As in the $SU(N)$ case, this will lead us to important structure theorems for these knot homologies and, eventually, to a reformulation based on a new triply-graded theory, that will be discussed in more details in section 6.

5.1. Embedding in Topological String

The physical setup for SO/Sp gauge groups can be obtained from the one for $SU(N)$ by introducing a suitable orientifold projection. Namely, recall [2], that Chern-Simons gauge theory with a unitary gauge group can be realized in the topological A-model by considering D-branes wrapped around \mathbf{S}^3 in the deformed conifold geometry $T^*\mathbf{S}^3$. This space can be described as a hypersurface in \mathbb{C}^4 ,

$$z_1 z_4 - z_2 z_3 = \mu \tag{5.1}$$

where μ is a complex deformation parameter. A theory with $SO(N)$ (resp. $Sp(N)$) gauge group can be obtained by starting with N D-branes wrapped around \mathbf{S}^3 and introducing an orientifold, which acts on space-time by an involution

$$\tau : (z_1, z_2, z_3, z_4) \rightarrow (\bar{z}_4, -\bar{z}_3, -\bar{z}_2, \bar{z}_1) \tag{5.2}$$

More precisely, in order for (5.2) to be an anti-holomorphic symmetry of the deformed conifold (5.1), we have to restrict μ to be real. The fixed point set of the involution (the location of the \mathcal{O} -plane) is the locus $|z_1|^2 + |z_2|^2 = \mu$ in \mathbb{C}^2 , which for $\mu > 0$ is precisely the three-sphere, while for $\mu < 0$ it is empty. The minimal supersymmetric three-cycle in the latter case is $\mathbf{S}^3/\mathbb{Z}_2 \cong \mathbb{RIP}^3$. Wrapping D-branes on the supersymmetric 3-cycle of this orientifolded conifold leads to a gauge theory on the brane world-volume. For $\mu > 0$ the result is the Chern-Simons gauge theory on \mathbf{S}^3 with gauge group $SO(N)$ or $Sp(N)$, depending on the orientifold action on the Chan-Paton factors, while for $\mu < 0$ we obtain $SU(N)$ Chern-Simons theory on \mathbb{RIP}^3 .

The topological A-model that we are considering here does not depend on complex structure deformations such as μ . We are therefore led to the conclusion that there is a duality between knot invariants obtained in $SO(N)/Sp(N)$ gauge theory on \mathbf{S}^3 and $SU(N)$ gauge theory on \mathbb{RIP}^3 . In the latter case, the distinction between the two different

orientifold projections corresponds in the gauge theory to the choice of a discrete Wilson line associated with $H^1(\mathbb{R}P^3) = \mathbb{Z}_2$.

It is instructive to contrast this expected behavior with the situation in the physical string, which was recently studied in detail in [42]. First of all, in that case, one is mostly interested in studying the gauge theory on the four-dimensional world-volume transverse to the Calabi-Yau. Because of the reduction to the zero modes, this gauge theory is of SO/Sp type also for $\mu < 0$. Secondly, the charge of the \mathcal{O} -plane on \mathbf{S}^3 is non-zero and is equal to minus twice the charge of a brane wrapped on the 3-cycle. Therefore, fixing the flux at infinity leads to a jump in the rank of the gauge group by 2 when going from $\mu > 0$ to $\mu < 0$. Finally, and most importantly, the transition through $\mu = 0$ is expected to be possible dynamically only for some special cases of flux and orientifold projection. Here, we are interested in the topological string on the deformed conifold with an orientifold and branes, and the story is slightly different.

As in the situation without the orientifold [5], one can incorporate Wilson loop observables associated with knots and links by introducing additional Lagrangian D-branes. For example, given a knot $K \subset \mathbf{S}^3$, the corresponding Lagrangian submanifold $\mathcal{L}_K \subset T^*\mathbf{S}^3$ is defined as a conormal bundle to the knot. In $SO(N)$ or $Sp(N)$ Chern-Simons theory, we have to further divide by the involution τ , which reverses the orientation in the fiber directions of $T^*\mathbf{S}^3$ and acts trivially on the \mathbf{S}^3 . Generically, this leads to a singular Lagrangian submanifold \mathcal{L}_K , which has a conical singularity along the knot.⁵

In order to see the relation with homological knot invariants, we need to take N to be large. In this limit, the theory is dual to topological string theory on the *resolved* conifold X , the total space of the $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ bundle over \mathbf{CP}^1 . The space X can be described as a toric variety (gauged linear sigma model), $X = \mathbb{C}^4/\mathbb{C}^*$, where \mathbb{C}^4 is parametrized by X_i , $i = 1, \dots, 4$, with charges $(1, 1, -1, -1)$ with respect to the \mathbb{C}^* action. In these variables, the space X is

$$X = \{|X_1|^2 + |X_2|^2 - |X_3|^2 - |X_4|^2 = r\}/U(1) \tag{5.3}$$

where $t = r - i\theta$ is the FI parameter (Kähler structure modulus). The value of t determines the volume of the \mathbf{CP}^1 cycle inside X , and is given by the coupling constant and the rank

⁵ In order to avoid the singularity, one can consider a small deformation of the Lagrangian submanifold \mathcal{L}_K by moving it away from the \mathbf{S}^3 . On the covering space, this corresponds to considering two copies of the knot (mapped to each other by the involution τ).

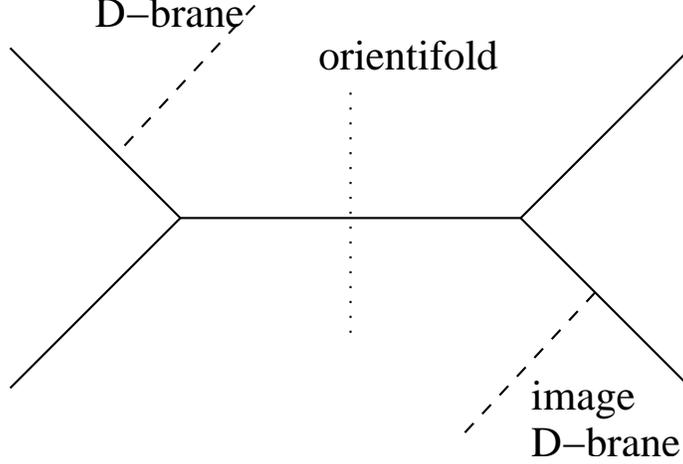


Fig. 3: Toric diagram representing a D-brane and its image on the covering space, the total space of the $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ bundle over \mathbf{CP}^1 .

of the dual Chern-Simons theory, $t = g_s(N + c)$. Instead of g_s and t we shall use the following variables

$$\begin{aligned}
 q &= e^{g_s} = \exp\left(\frac{\pi i}{k+h}\right) \\
 \lambda &= e^t = q^{N+c}
 \end{aligned}
 \tag{5.4}$$

where h is the dual Coxeter number of the gauge group and c is the charge of the orientifold.

Gauge group	h	c
$U(N)$	N	0
$SO(N)$	$N - 2$	-1
$Sp(N)$	$\frac{N}{2} + 1$	1

Table 1: The dual Coxeter number h and the orientifold charge c .

For $SO(N)$ or $Sp(N)$ theories we also need to divide by the involution τ , which acts on the space X as

$$\tau : (X_1, X_2, X_3, X_4) \rightarrow (\bar{X}_2, -\bar{X}_1, \bar{X}_4, -\bar{X}_3)
 \tag{5.5}$$

In particular, it acts freely on X , so that the quotient space X/τ contains a 2-cycle \mathbf{RP}^2 instead of \mathbf{CP}^1 .

After the large N transition, the branes on \mathbf{S}^3 disappear and we are left only with Lagrangian branes on $\mathcal{L}_K \subset X$. This leads to a reformulation of Chern-Simons invariants in terms of open topological string amplitudes on X . The open string amplitudes, in turn, can be expressed via integer BPS invariants that we discuss next.

5.2. BPS States and Homological Knot Invariants

The open topological string amplitudes on X have certain integrality properties, which will be important to us below in understanding homological knot invariants. These properties can be seen by realizing the setup discussed above in superstring theory. Namely, following [5,6,17], we consider type IIA string theory on $\mathbf{R}^4 \times X$ together with D4-branes on $\mathbf{R}^2 \times \mathcal{L}_K$, where $\mathbf{R}^2 \subset \mathbf{R}^4$ and \mathcal{L}_K is the Lagrangian submanifold in X . For SO/Sp theories, we also need to introduce an orientifold plane.

In the case of topological strings, we had only two types of orientifolds, \mathcal{O}^\pm , which have opposite charge and lead to SO and Sp gauge groups, respectively. On the other hand, in the superstring we have orientifolds \mathcal{O}^\pm as well as anti-orientifolds $\overline{\mathcal{O}}^\pm$, which lead to same gauge groups but have opposite Ramond-Ramond charge. In order to preserve supersymmetry, we must choose D-branes together with orientifold planes when the total amount of flux, N , is positive, and anti-D-branes together with anti-orientifold planes $\overline{\mathcal{O}}^\pm$ when $N < 0$. Therefore, in total we have four different choices summarized in the table below. Note that the chirality of the knot is also correlated with the sign of the flux.

A-model	N	Type IIA theory	Gauge group	Knot invariant
\mathcal{O}^-	$N > 0$	\mathcal{O}^-	SO	$F(K; \lambda = q^N, q)$
	$N < 0$	$\overline{\mathcal{O}}^+$	Sp	$F(\overline{K}; \lambda = q^N, q)$
\mathcal{O}^+	$N > 0$	\mathcal{O}^+	Sp	$F(K; \lambda = -q^N, q)$
	$N < 0$	$\overline{\mathcal{O}}^-$	SO	$F(\overline{K}; \lambda = -q^N, q)$

Table 2: *Different types of orientifolds in the A-model and type IIA string theory.*

These four choices correspond to different specializations of the Kauffman polynomial at $\lambda = \pm q^N$, where the choice of sign is correlated with the charge of the orientifold plane.

This interpretation is consistent with eqs. (2.11) and (2.15),

$$\begin{aligned}\overline{F}_{N+1}(q) &= \overline{F}(\lambda = q^N, q) = \overline{F}(\overline{K}; \lambda = -q^{-N}, q) \\ \overline{C}_{N-1}(q) &= \overline{F}(\lambda = -q^N, q) = \overline{F}(\overline{K}; \lambda = q^{-N}, q)\end{aligned}\tag{5.6}$$

Embedding the topological string setup in type IIA string theory allows to express open string amplitudes and, therefore, polynomial knot invariants in terms of integer numbers that count degeneracies of BPS states [5,6]. For example, for the HOMFLY polynomial we have

$$\begin{aligned}\overline{P}(\lambda, q) &= \sum_{g, Q} \widehat{N}_{g, Q} \lambda^Q (q^{-1} - q)^{2g-1} \\ &= \frac{1}{q - q^{-1}} \sum_{Q, s} N_{Q, s} \lambda^Q q^s\end{aligned}\tag{5.7}$$

where $\widehat{N}_{g, Q}$ (resp. $N_{Q, s}$) denote the degeneracies of BPS states, counted with \pm signs. Roughly speaking, these BPS states are represented by genus g holomorphic Riemann surfaces in X with boundary on the Lagrangian submanifold \mathcal{L}_K . More precisely, the space of BPS states on the Lagrangian D4-brane is $\mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ -graded; it is graded by three integer quantum numbers and the fermion number F ,

$$\mathcal{H}_{BPS} = \mathcal{H}_{BPS}^{F, Q, s, r}\tag{5.8}$$

Therefore, one can introduce the following “index”

$$N_{Q, s} = \sum_{r, F} (-1)^{r+F} \dim \mathcal{H}_{BPS}^{F, Q, s, r}\tag{5.9}$$

which appears in (5.7), and its refinement

$$D_{Q, s, r} = \sum_F (-1)^F \dim \mathcal{H}_{BPS}^{F, Q, s, r}\tag{5.10}$$

which is related to the categorification of the HOMFLY polynomial [7].

Similarly, in the presence of the orientifold, there are additional BPS states represented by unoriented Riemann surfaces. This naturally leads to the well-known relation between the HOMFLY and the Kauffman polynomial [17]:

$$\overline{F}(\lambda, q) - \overline{P}(\lambda, q) = \sum_{g, Q} N_{g, Q}^{c=1} (q - q^{-1})^{2g} \lambda^Q + \sum_{g, Q} N_{g, Q}^{c=2} (q - q^{-1})^{2g+1} \lambda^Q\tag{5.11}$$

where the integer coefficients $N_{g,Q}^{c=1}$ and $N_{g,Q}^{c=2}$ are interpreted as BPS degeneracies associated with the contribution of one and two crosscaps, respectively. This relation simplifies further for torus knots since [43],

$$N_{\square,g,Q}^{c=2} = 0 \quad (5.12)$$

Extending (5.9) – (5.10) to the orientifold case, we can also write the Kauffman polynomial $\overline{F}(\lambda, q)$ for any knot K in terms of the refined BPS invariants $D_{Q,s,r}^{\text{Kauffman}}$:

$$\overline{F}(\lambda, q) = \frac{1}{q - q^{-1}} \sum_{Q,s,r} (-1)^r \lambda^Q q^s D_{Q,s,r}^{\text{Kauffman}} \quad (5.13)$$

Notice, that the specialization of this expression to $\lambda = q^{N-1}$ is very similar to the categorification of the quantum $so(N)$ invariants, *cf.* (3.8). In order to make this relation more precise, let us introduce the graded Poincare polynomials for the so/sp knot homologies

$$\begin{aligned} \overline{HSO}_N(q, t) &= \sum_{i,j \in \mathbb{Z}} t^i q^j \dim \mathcal{H}_{i,j}^{so(N)}(K) \\ \overline{HSp}_N(q, t) &= \sum_{i,j \in \mathbb{Z}} t^i q^j \dim \mathcal{H}_{i,j}^{sp(N)}(K) \end{aligned} \quad (5.14)$$

and similar polynomials, $HSO_N(q, t)$, $HSp_N(q, t)$ for the reduced homologies. Then, comparing (3.8) and (5.13) we naturally arrive to the following conjecture, which parallels the conjecture of [7] as formulated in [13]:

Conjecture 1: *For a knot K , there exists a finite polynomial $\overline{\mathcal{F}}(K) \in \mathbb{Z}[\lambda^{\pm 1}, q^{\pm 1}, t^{\pm 1}]$ such that*

$$\begin{aligned} \overline{HSO}_N(K; q, t) &= \frac{1}{q - q^{-1}} \overline{\mathcal{F}}(K; \lambda = q^{N-1}, q, t) \\ \overline{HSp}_N(\overline{K}; q, t) &= \frac{1}{q - q^{-1}} \overline{\mathcal{F}}(K; \lambda = q^{-N-1}, q, t) \end{aligned} \quad (5.15)$$

for all sufficiently large N .

Notice, in order to be consistent with the polynomial specializations (2.11), the three-variable polynomial $\overline{\mathcal{F}}(K; \lambda, q, t)$ must be a refinement of the Kauffman polynomial, in a sense that

$$\overline{F}(K; \lambda, q) = \frac{1}{q - q^{-1}} \overline{\mathcal{F}}(K; \lambda, q, t = -1) \quad (5.16)$$

In particular, from (5.13) and (5.16) it follows that the polynomial $\overline{\mathcal{F}}(K; \lambda, q, t)$ — which we shall call the Kauffman superpolynomial below — is simply the generating function of the refined BPS invariants,

$$\overline{\mathcal{F}}(\lambda, q, t) = \sum_{Q,s,r} \lambda^Q q^s t^r D_{Q,s,r}^{\text{Kauffman}} \quad (5.17)$$

According to Conjecture 1, there should exist many regularities and relations between $so(N)/sp(N)$ knot homologies. In particular, in the rest of this section, we demonstrate how this conjecture can be used to predict the $so(N)/sp(N)$ homological invariants for all values of N . As a starting point, we shall use the homological $so(N)/sp(N)$ invariants for small values of the rank N , which can be deduced using certain isomorphisms between so , sp , and sl Lie algebras of small rank. For example, using the isomorphism $so(4) \cong sl(2) \times sl(2)$ and the fact that a vector representation of $so(4)$ corresponds to the representation $(\mathbf{2}, \mathbf{2})$ of $sl(2) \times sl(2)$, we conclude that $so(4)$ homology is isomorphic to the “square” of the Khovanov homology:

$$\mathcal{H}^{so(4)} \cong \mathcal{H}^{sl(2)} \otimes \mathcal{H}^{sl(2)} \quad (5.18)$$

Note, that this isomorphism holds in both reduced and unreduced theories. In particular,

$$\overline{HSO}_4(q, t) = \overline{Kh}(q, t)^2 \quad (5.19)$$

Similarly, the isomorphism $sp(2) \cong sl(2)$ leads to the relation between the $sp(2)$ and $sl(2)$ homological invariants:

$$\overline{HSp}_2(q, t) = \overline{Kh}(q^2, t) \quad (5.20)$$

The relations (5.19) and (5.20) generalize, respectively, the relations (2.13) and (2.14) between the corresponding polynomial invariants.

Example. *The Trefoil Knot*

The unnormalized Kauffman polynomial for the trefoil knot looks like

$$\begin{aligned} \overline{F}(3_1) = \frac{1}{q - q^{-1}} & \left[-\lambda(q^2 + q^{-2}) + \lambda^2(q^3 - q^{-3}) + \lambda^3(q^2 + 1 + q^{-2}) \right. \\ & \left. - \lambda^4(q^3 - q^{-3}) - \lambda^5 + \lambda^6(q - q^{-1}) \right] \end{aligned} \quad (5.21)$$

On the other hand, its $so(4)$ homological invariant can be obtained from the relation (5.19) with Khovanov homology:

$$\overline{HSO}_4(3_1) = q^2 + 2q^4 + q^6 + 2q^6t^2 + 2q^8t^2 + 2q^{10}t^3 + 2q^{12}t^3 + q^{10}t^4 + 2q^{14}t^5 + q^{18}t^6 \quad (5.22)$$

The simplest form of the Kauffman superpolynomial $\overline{\mathcal{F}}(\lambda, q, t)$ consistent with (5.21) and (5.22) is given by the following expression:

$$\begin{aligned} \overline{\mathcal{F}}(3_1) = & -\lambda(q^{-2} + q^2t^2) + \lambda^2(-q^{-3} + q^{-1} - q^{-1}t^2 - q^3t^3) + \lambda^3(q^{-2} + t^2 - t^3 - t^4 + q^2t^4) \\ & + \lambda^4(q^{-3}t^2 + qt^3 - qt^5 + q^3t^5) + \lambda^5(q^{-2}t^3 + t^5 - q^{-2}t^5) + \lambda^6(-q^{-1}t^6 + qt^6) \end{aligned} \quad (5.23)$$

Indeed, it is easy to verify that specializations of this expression to $\lambda = q^3$ and $t = -1$ yield, respectively, the homological $so(4)$ invariant and the Kauffman polynomial, in agreement with (5.15) and (5.16).

This example can easily be generalized to all torus knots of type $(2, 2k + 1)$:

$$\begin{aligned}
\overline{\mathcal{F}}(T_{2k+1,2}) = & (\lambda - \lambda^{-1} + q - q^{-1})(\lambda/q)^{2k} \\
& + (1 + \lambda qt)\lambda^{2k+1} \left[(1 - q^2\lambda^{-2})(1 + \lambda q^{-3}) \sum_{i=1}^k t^{2i} q^{4i-2k-2} \right. \\
& \left. + (1 - q^{-2})(1 + \lambda^2 q^{-2}t) \sum_{j=0}^{k-1} \sum_{i=0}^{2k-2j-2} t^{2i+2j+4} \lambda^i q^{i+4j-2k+4} \right]
\end{aligned} \tag{5.24}$$

Several comments are in order regarding the structure of this expression. First, as prescribed by (5.15) and (5.16), it reduces to the square of the Khovanov homology at $\lambda = q^3$, and to the Kauffman polynomial at $t = -1$. The first term in (5.24) corresponds to the homology of the unknot, while both the second and the third term correspond to parts in the homology that can be “killed” by the differential of the appropriate degree. Moreover, the structure of the second term is very similar to the result eq. (89) in [13] for the unreduced superpolynomial $\overline{\mathcal{P}}$ of $T_{2k+1,2}$. Also, the last term in (5.24) is multiplied by a factor $(q - q^{-1})$. This structure suggests that, in the topological string interpretation, the second term comes from the oriented worldsheets and the third term from the unoriented ones, *cf.* (5.11).

Let us consider a more complicated example:

Example. *The Figure-eight Knot*

The unnormalized Kauffman polynomial for the figure-eight knot is

$$\begin{aligned}
\overline{F}(4_1) = & \frac{1}{q - q^{-1}} \left[\lambda^{-3}(1 - q^2 - q^{-2}) + \lambda^{-1}(q^4 + q^{-4}) + q - q^{-1} - \lambda(q^4 + q^{-4}) \right. \\
& \left. + \lambda^3(-1 + q^2 + q^{-2}) \right]
\end{aligned} \tag{5.25}$$

Using constraints from various specializations we are led to the following prediction for unreduced Kauffman superpolynomial:

$$\begin{aligned}
\overline{\mathcal{F}}(4_1) = & -\lambda^{-3}q^{-2}t^{-4} + \lambda^{-3}t^{-4} - \lambda^{-3}q^2t^{-2} - \lambda^{-2}q^{-1}(t^{-2} + t^{-3}) - \lambda^{-1}q^{-4}t^{-3} \\
& + \lambda^{-2}q(t^{-3} - t^{-1}) + \lambda^{-1}q^{-2}(t^{-3} - t^{-1}) + \lambda^{-1}(-1 + t^{-2}) + \lambda^{-2}q^5(-1 - t) \\
& + \lambda^{-1}q^2(-1 - t) - q^{-1} - \lambda q^{-4} + \lambda^2q^{-7}(-1 + t^{-2}) + \lambda^{-2}q^7(1 - t^2) \\
& + \lambda^{-1}q^4 + q + \lambda q^{-2}(1 + t^{-1}) + \lambda^2q^{-5}(1 + t^{-1}) + \lambda(1 - t^2) + \lambda q^2(t - t^3) \\
& + \lambda^2q^{-1}(t - t^3) + \lambda q^4t^3 + \lambda^2q(t^2 + t^3) + \lambda^3q^{-2}t^2 - \lambda^3t^4 + \lambda^3q^2t^4
\end{aligned} \tag{5.26}$$

It reduces to the Kauffman polynomial at $t = -1$ and to the Poincare polynomial of the $so(4)$ knot homology, $\overline{HSO}_4(4_1) = \overline{Kh}(4_1)^2$, at $\lambda = q^3$.

5.3. Comparison with Colored Khovanov Homology

Another useful relation follows from the isomorphism $so(3) \cong sl(2)$. Namely, since the vector representation of $so(3)$ corresponds to a 3-dimensional (spin-1) representation of $sl(2)$, we have an isomorphism

$$\mathcal{H}^{so(3)}(K) \cong \mathcal{H}^{sl(2);V_3}(K) \quad (5.27)$$

where we use the notation $\mathcal{H}^{sl(2);V_n}(K)$ for the $sl(2)$ homology of the knot K colored by the n -dimensional representation V_n . This theory should provide a categorification of the colored Jones polynomial,

$$\bar{J}_n(q) = \sum_{i,j} (-1)^i q^j \dim \mathcal{H}_{i,j}^{sl(2);V_n}(K) \quad (5.28)$$

Recall, that the colored Jones polynomial $\bar{J}_n(K)$ can be expressed in terms of the ordinary Jones polynomial of the cables of the knot K :

$$\bar{J}_n(K) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n-k-1}{k} \bar{J}(K^{n-2k-1}) \quad (5.29)$$

Recently, Khovanov [44] proposed a similar definition of the homology $\mathcal{H}^{sl(2);V_n}(K)$ in terms of the ordinary $sl(2)$ homology of the cables of K . For example, the homology $\mathcal{H}^{sl(2);V_3}(K)$ associated with the three-dimensional representation of $sl(2)$ fits into the following long exact sequence [44]:

$$\dots \longrightarrow \mathcal{H}^{sl(2);V_3}(K) \longrightarrow \mathcal{H}^{sl(2)}(K^2) \xrightarrow{u} \mathbb{Z} \longrightarrow \dots \quad (5.30)$$

where $\mathcal{H}^{sl(2)}(K^2)$ is the ordinary Khovanov homology of the 2-cable K^2 of K . In particular, for the 0-framed unknot we have

$$\mathcal{H}_{i,j}^{sl(2);V_3}(\text{unknot}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \text{ and } j \in \{-2, 0, 2\} \\ 0 & \text{otherwise} \end{cases} \quad (5.31)$$

This indeed agrees with the $so(3)$ homology of the unknot, if we identify the q -grading in the $sl(2)$ theory with twice the q -grading in the $so(3)$ theory. Therefore, combining this with the isomorphism (5.27), we expect the following formula for the Poincare polynomial of the $so(3)$ knot homology:

$$\bar{\mathcal{P}}^{sl(2);V_3}(K; q, t) = \overline{Kh}(K^2; q^{1/2}, t) - 1 \quad (5.32)$$

We can compare this with our prediction for the $so(3)$ knot homology based on Conjecture 1. For example, for the trefoil knot, the specialization of the Kauffman superpolynomial (5.23) to $\lambda = q^2$ gives

$$\begin{aligned} \frac{\overline{\mathcal{F}}(\mathfrak{3}_1; \lambda = q^2, q, t)}{(q - q^{-1})} &= q + q^2 + q^3 + q^4 t^2 + q^5 t^2 \\ &+ q^7(t^3 + t^4) + q^8 t^3 + q^9 t^5 + q^{10} t^5 + q^{12} t^6 \end{aligned} \quad (5.33)$$

while (5.32) gives

$$\begin{aligned} \overline{\mathcal{P}}^{sl(2); V_3}(\mathfrak{3}_1; q, t) &= q(2 + t) + q^2 + q^3(2t^2 + t^3) + q^4 t^4 + q^5(t^3 + 2t^4) + q^6(t^5 + t^6) \\ &+ q^7(t^5 + t^8) + q^8 t^7 + q^9 t^9 + q^{10} t^{11} + q^{12} t^{12} \end{aligned} \quad (5.34)$$

It is easy to see that the structure of (5.33) and (5.34) is very similar. However, the expression (5.34) based on the Khovanov homology of the 2-cable of the trefoil contains extra terms of the form $(1 + t)Q^+(q, t)$. It is natural to expect that in the formulation of the colored Khovanov homology as a cohomology of the complex $\mathcal{C}^{sl(2); V_3}$ these extra terms will be killed by the differential. The remaining terms in (5.34) agree with (5.33) after a change of t -grading in some of the terms (roughly speaking, in going from (5.34) to (5.33), the t -grading is reduced by a factor of 2, in such a way that it does not affect specialization to $t = -1$). The analysis of other simple knots leads to similar conclusions.

5.4. Comparison with Khovanov Homology

Finally, let us test the relation (5.20), which is the categorification of (2.14). Recall from section 2 that the $sp(2)$ quantum invariant can be obtained in two different specializations of the Kauffman polynomial:

$$C_2(K; q) = \overline{F}(K; \lambda = -q^3, q) = \overline{F}(\overline{K}; \lambda = q^{-3}, q) \quad (5.35)$$

Correspondingly, there are in principle two ways in which $sp(2)$ knot homology could be obtained from the Kauffman homology. We have included one of them in Conjecture 1. Taking the trefoil as an example, one can see that the naive specialization to $\lambda = q^{-3}$ does not reproduce the $sp(2)$ homology predicted from the isomorphism (5.20). This is a first indication that $N = 2$ is not “large enough” as far as $sp(N)$ homology is concerned, and there are corrections to the naive specialization. We will see this again in section 6 when we consider the reduced homology, for which the small N corrections are under better control.

We cannot resist, however, to offer instead an observation concerning the categorification of the other possible specialization in (5.35), to $\lambda = -q^3$. For the trefoil,

$$\frac{\overline{\mathcal{F}}(3_1; \lambda = -q^3, q, t)}{q - q^{-1}} = -q^2 - q^6 - q^{10}t^4 + q^{18}t^6 \quad (5.36)$$

which, remarkably, is related to the ordinary Khovanov homology in a simple way,

$$\frac{\overline{\mathcal{F}}(3_1; \lambda = -q^3, q, t)}{q - q^{-1}} = -\overline{Kh}(3_1; q^2, -t^2). \quad (5.37)$$

Because of the sign changes, a homological interpretation of this relation is not immediately obvious. But it is hard to believe that it is a pure coincidence. The relation holds for all knots for which we have been able to determine the unreduced Kauffman homology.

6. Kauffman Homology

There is also a reduced version of Conjecture 1:

Conjecture 1': *For a knot K , there exists a finite polynomial $\mathcal{F}(K) \in \mathbb{Z}[\lambda^{\pm 1}, q^{\pm 1}, t^{\pm 1}]$ such that*

$$\begin{aligned} HSO_N(K; q, t) &= \mathcal{F}(K; \lambda = q^{N-1}, q, t) \\ HSp_N(\overline{K}; q, t) &= t^s \mathcal{F}(K; \lambda = q^{-N-1}, q, t) \end{aligned} \quad (6.1)$$

for all sufficiently large N .⁶

In order for this latter version of the conjecture to be true, all the coefficients of the reduced superpolynomial $\mathcal{F}(\lambda, q, t)$ need to be non-negative. As in the $sl(N)$ case [13], this suggests that $\mathcal{F}(K)$ is itself a Poincare polynomial of a triply-graded theory, whose Euler characteristic is the normalized Kauffman polynomial. Combining this with the additional structure of differentials inferred from the analysis of the Landau-Ginzburg theory in section 4, we come to the following:

Conjecture 2: *There exists a triply-graded homology theory, $\mathcal{H}_*^{\text{Kauffman}} = \mathcal{H}_{i,j,k}^{\text{Kauffman}}(K)$, categorifying the Kauffman polynomial. It comes with a family of differentials $\{d_N\}$, two further, “universal” differentials, d_{\rightarrow} and d_{\leftarrow} , and has the following properties:*

- categorification:

$$\chi(\mathcal{H}_*^{\text{Kauffman}}(K)) = F(K) \quad (6.2)$$

⁶ In the formula for the $sp(N)$ homology, s denotes an invariant of the knot similar to Rasmussen’s invariant [45]. See comment below.

- anticommutativity:

$$d_N d_M = -d_M d_N \quad (6.3)$$

- finite support:

$$\dim(\mathcal{H}_*^{\text{Kauffman}}(K)) < \infty \quad (6.4)$$

- specializations

$$(\mathcal{H}_*^{\text{Kauffman}}(K), d_N) \cong \begin{cases} \mathcal{H}_*^{so(N)}(K) & N > 1 \\ \mathcal{H}_*^{sp(-N)}(K) & N < 0 \end{cases} \quad (6.5)$$

- “universal” and “canceling” differentials: The properties we expect of d_0 , d_1 and d_2 , as well as the two additional differentials d_{\rightarrow} and d_{\leftarrow} , are explained in detail below. Roughly speaking, the cohomology of $\mathcal{H}_*^{\text{Kauffman}}$ with respect to d_{\rightarrow} and d_{\leftarrow} should be isomorphic (with a simple regrading) to the HOMFLY homology $\mathcal{H}_*^{\text{HOMFLY}}$,

$$(\mathcal{H}_*^{\text{Kauffman}}(K), d_{\rightleftharpoons}) \cong \mathcal{H}_*^{\text{HOMFLY}} \quad (6.6)$$

while the cohomology with respect to d_i , for $i = 0, 1, 2$ should be “trivial” and depend in a particular simple way on the knot K .

Notice, that in order to be consistent with the specialization to $\lambda = q^{N-1}$, the q -degree of the differential d_N must be proportional to $(N - 1)$. In particular, this implies that the differentials d_N are all trivial for sufficiently large values of N , since $\mathcal{H}_*^{\text{Kauffman}}(K)$ has finite support. Therefore, Conjecture 2 implies Conjecture 1', where the superpolynomial $\mathcal{F}(K; \lambda, q, t)$ is simply the Poincare polynomial of $\mathcal{H}_*^{\text{Kauffman}}$,

$$\mathcal{F}(K; \lambda, q, t) = \sum_{i,j,k} \lambda^i q^j t^k \dim \mathcal{H}_{i,j,k}^{\text{Kauffman}}(K) \quad (6.7)$$

In terms of $\mathcal{F}(K; \lambda, q, t)$, the condition (6.2) can be expressed as

$$\mathcal{F}(K; \lambda, q, -1) = \mathcal{F}(K; \lambda, q) \quad (6.8)$$

We believe there should exist a combinatorial definition of the triply-graded Kauffman homology with all the properties listed here. In practice, while such a definition is not available, one can use any combination of the above axioms as the definition, and the others as consistency checks. This will be our approach below. Specifically, in the rest of this section we explain in more detail various aspects of this conjecture, use it to make predictions, and present some non-trivial checks. We start with the discussion of the differentials.

6.1. Differentials

so/sp differentials:

These are the differentials which justify the idea that $\mathcal{H}_*^{\text{Kauffman}}$ is a unified theory for *so/sp* knot homologies. Namely, for every $N > 1$ we expect the differentials d_N , such that the cohomology of $\mathcal{H}_*^{\text{Kauffman}}$ with respect to d_N yields $so(N)$ homology $\mathcal{H}_*^{so(N)}$. These differentials are expected to have degrees

$$so(N) \quad (N \geq 2) : \quad \deg(d_N) = (-1, N - 1, -1) \quad (6.9)$$

consistent with the specialization $\lambda = q^{N-1}$. Indeed, acting on the bigraded chain complex

$$\bigoplus_{i(N-1)+j=p} \mathcal{H}_{i,j,k}^{\text{Kauffman}} \quad (6.10)$$

the differential d_N has q -degree zero and t -degree -1 . The evidence for these differentials comes from the analysis of the Landau-Ginzburg potentials in section 4, where we found that the $so(N)$ knot homology is equipped with a family of differentials $\{d_{N \rightarrow M}\}$, $M < N$, which correspond to the deformations of the Landau-Ginzburg potential,

$$W_{so(N)} \rightarrow W_{so(N)} + x^{M-1}$$

As we have explained in section 4, the cohomology with respect to $d_{N \rightarrow M}$ is expected to be isomorphic to the $so(M)$ knot homology. These are precisely the properties of the differentials induced on $(\mathcal{H}_*^{\text{Kauffman}}, d_N)$ from the differentials $\{d_M\}$ in the triply-graded theory.

Similarly, for even values of $N \leq -2$ we expect the differentials, such that the cohomology of the bigraded complex (6.10) with respect to d_N yields $sp(-N)$ homology $\mathcal{H}_*^{sp(-N)}$. As in the $sl(N)$ case [13], we expect the λ - and q -degree of these differentials to be given by the same formula as for $N > 1$. Specifically,

$$sp(-N) \quad (N \leq -2) : \quad \deg(d_N) = (-1, N - 1, -1 + N) \quad (6.11)$$

Notice, that when the differential d_N is trivial — *e.g.* when its degree is too large — the corresponding $so(N)/sp(N)$ homology is given by (6.10), up to a simple re-grading. In this case, the Poincare polynomial of the corresponding knot homology is simply a specialization of the Kauffman superpolynomial, *cf.* (6.1).

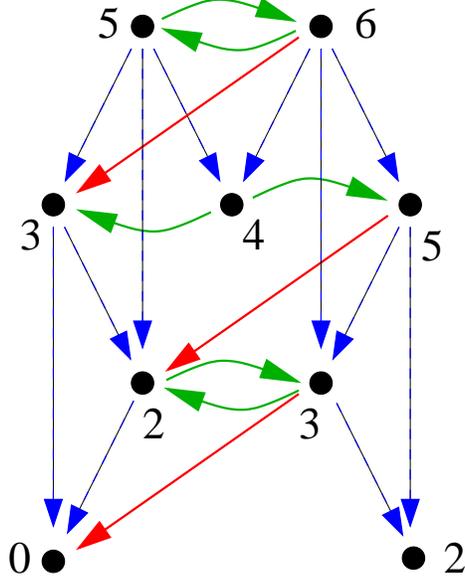


Fig. 4: Dot diagram for the trefoil knot. Each dot represents a term in the Kauffman superpolynomial; its horizontal (resp. vertical) position encodes the power of q (resp. the power of λ). The bottom row has λ -grading 2. The differential d_{-2} is represented by a solid red arrow, while the differentials d_0 , d_1 , and d_2 are shown by dashed blue arrows. The universal differentials d_{\rightleftharpoons} are depicted by curved green arrows.

Example. *The Trefoil Knot*

The normalized Kauffman polynomial for the trefoil knot has 9 terms:

$$F(3_1) = \lambda^2(q^2 + q^{-2}) - \lambda^3(q - q^{-1}) + \lambda^4(1 - q^2 - q^{-2}) + \lambda^5(q - q^{-1}) \quad (6.12)$$

Similarly, using the isomorphism (5.18), we find that the reduced $so(4)$ knot homology also has rank 9, and the Poincare polynomial

$$HSO_4(3_1) = q^4 + 2t^2q^8 + 2t^3q^{10} + t^4q^{12} + 2t^5q^{14} + t^6q^{16} \quad (6.13)$$

Moreover, there is a unique way to identify each term in (6.12) with the corresponding term in (6.13), so that their specializations to $\lambda = q^3$ and $t = -1$ agree. Assuming that there are no “hidden” terms, we obtain the following prediction for the reduced Kauffman superpolynomial of the trefoil knot:

$$\mathcal{F}(3_1) = \lambda^2(q^{-2} + q^2t^2) + \lambda^3(q^{-1}t^2 + qt^3) + \lambda^4(q^{-2}t^3 + t^4 + q^2t^5) + \lambda^5(q^{-1}t^5 + qt^6) \quad (6.14)$$

This result, based on specializations to the Kauffman polynomial and to the $so(4)$ homological invariant, can now be used to make predictions for other so/sp homologies

of the trefoil, as well as to test the consistency of our axioms. For example, it is interesting to note that even for such a simple knot as the trefoil, the differential d_{-2} acts non-trivially on $\mathcal{H}_*^{\text{Kauffman}}$, so that the resulting homology has rank three,

$$HSp_2(\overline{3}_1) = q^4 + q^{12}t^2 + q^{16}t^3 \quad (6.15)$$

in agreement with (5.20).

Since $so(2)$ is abelian, the specialization to $N = 2$ is expected to give a very simple theory. In other words, it means that most of the terms in the Kauffman homology are killed by the differential d_2 . A differential with this property, namely such that its cohomology is one-dimensional, is called *canceling*. In the triply-graded theory categorifying the HOMFLY polynomial [13], an example of such differential is d_1 , which leads to the deformed theory of Lee [37]. It turns out that, in the theory we are considering here, there are several candidates for such differentials, which we discuss now.

canceling differentials: In a multiply-graded theory such as the knot superhomologies, the existence of a canceling differential of a certain degree is a rather non-trivial feature with interesting origin and consequences. For all simple knots for which we have been able to obtain a prediction for the triply-graded theory $\mathcal{H}_*^{\text{Kauffman}}$ (see below), there is room for three canceling differentials, which can be naturally identified as d_2 , d_1 and d_0 , and which have degrees,

$$\begin{aligned} d_2 : & \quad (-1, 1, -1) \\ d_1 : & \quad (-2, 0, -3) \\ d_0 : & \quad (-1, -1, -2) \end{aligned} \quad (6.16)$$

Moreover, the one-dimensional piece surviving the differentials sits in a particular degree, which depends in a simple way on the knot K . Namely, we find that, for all the knots that we considered, the reduced superpolynomial for the Kauffman homology can be written in three different ways

$$\begin{aligned} \mathcal{F}(\lambda, q, t) &= (\lambda/q)^{-s} + (1 + \lambda q^{-1}t)Q_2^+(\lambda, q, t) \\ \mathcal{F}(\lambda, q, t) &= (\lambda t)^{-2s} + (1 + \lambda^2 t^3)Q_1^+(\lambda, q, t) \\ \mathcal{F}(\lambda, q, t) &= (\lambda q t)^{-s} + (1 + \lambda q t^2)Q_0^+(\lambda, q, t) \end{aligned} \quad (6.17)$$

where the polynomials Q_i^+ ($i = 0, 1, 2$) have integer non-negative coefficients. As an example, we show how the canceling differentials act on the Kauffman homology for the trefoil knot in fig. 4.

It is interesting to note that the triple-grading of the term surviving any given canceling differential is related in a simple way to the degree of that differential. Namely,

$$\deg[(\mathcal{H}^{\text{Kauffman}}, d_i)] = -s[\deg(d_i) - (0, 0, 1)] \quad \text{for } i = 0, 1, 2 \quad (6.18)$$

A similar relation holds for the canceling differentials of the HOMFLY theory [13].

Based on our examples and on the experience with the $sl(N)$ case [13], it is natural to conjecture that the Kauffman homology of any knot should admit three canceling differentials. A more conservative form of the conjecture would allow the cohomology with respect to d_i 's, $i = 0, 1, 2$, to be not strictly one-dimensional, or only so after addition of a non-homogeneous correction.

Universal differentials: The existence of the universal differentials is perhaps the most novel and intriguing aspect of Conjecture 2. Indeed, equipped with these differentials, the Kauffman homology should not only be a unified framework for so/sp homologies, but in fact should also contain the triply graded HOMFLY theory!

The evidence for at least one universal differential comes from the “universal” deformation of the Landau-Ginzburg potential discussed in section 4:

$$W_{so(N)} \rightarrow W_{so(N)} + y^2$$

This deformation suggests the existence of a differential d_{y^2} that takes $so(N)$ homology to $sl(N - 2)$ homology. Moreover, since this differential should exist for all values of N , it is natural to expect that it is induced from a differential in the triply graded theory that relates $\mathcal{H}_*^{\text{Kauffman}}$ and $\mathcal{H}_*^{\text{HOMFLY}}$, so that the differentials relating $so(N)$ and $sl(N - 2)$ homology are just different specializations of this universal differential.

Surprisingly, the Kauffman homology appears to have two universal differentials. Namely, for all knots that we considered, the Kauffman superpolynomial can be related to the HOMFLY superpolynomial in two different ways:

$$\begin{aligned} \mathcal{F}(\lambda, q, t) &= q^{-s}\mathcal{P}(\lambda/q, q, t) + (1 + q^2t)Q_{\leftarrow}^+(\lambda, q, t) \\ \mathcal{F}(\lambda, q, t) &= (qt)^s\mathcal{P}(\lambda qt, q, t) + (1 + q^{-2}t^{-1})Q_{\rightarrow}^+(\lambda, q, t) \end{aligned} \quad (6.19)$$

where Q_{\leftarrow}^+ and Q_{\rightarrow}^+ are polynomials with integer non-negative coefficients. These two relations suggest two universal differentials, that we denote d_{\leftarrow} and d_{\rightarrow} , respectively, with gradings

$$\deg(d_{\leftarrow}) = (0, -2, -1) \quad \deg(d_{\rightarrow}) = (0, 2, 1) \quad (6.20)$$

It is curious to note that by specializing (6.19) to $t = -1$, we obtain a non-trivial relation between the Kauffman and HOMFLY polynomial, which to the best of our knowledge is new.

Finally, we should point out that the numbers s in eqs. (6.1), (6.17), and (6.19) might be different knots invariants. However, as in the $sl(N)$ case [13], we conjecture that they are all equal and use the same notation $s(K)$. Furthermore, in all the examples that we considered, this invariant is actually equal to the knot signature, $s(K) = \sigma(K)$, which is also familiar from the structure of the $sl(N)$ homological invariants.

6.2. Thin Knots and δ -grading

Taking some of the axioms (6.2) - (6.6) as a definition of the triply-graded homology $\mathcal{H}_*^{\text{Kauffman}}$, we can predict what it should be for a number of simple knots, as we did for the trefoil in (6.14). For example, in the following table below we write the Kauffman superpolynomial for all knots with less than seven crossings. These results can be used to test other properties of the Kauffman homology, in particular, the structure implied by the canceling, universal, and $sp(2)$ differentials.

These predictions reveal another property of the Kauffman homology, which it shares with other theories. Namely, all the existing knot homologies have an interesting property (which, in a sense, is a hint for unification) that the structure of a theory often becomes simpler if instead of the ordinary homological grading (t -grading) one introduces a new grading — usually called δ -grading — which is a linear combination of the original gradings. Roughly speaking, the δ -grading tells us about the homological complexity of a knot. For example, different homological invariants of a knot with small number of crossings are all localized in one value of the corresponding δ -grading. Such knots are called *homologically thin*, or *thin* for short. The first example of a knot which is not thin⁷ is the 8-crossing knot 8_{19} .

The δ -grading in the Kauffman homology is a linear combination of λ , q and t gradings,

$$\delta = \frac{3}{2}\lambda + \frac{1}{2}q - t \tag{6.21}$$

In particular, it is easy to verify that all the knots listed in Table 3 are thin in Kauffman homology, and their δ -grading coincides with minus the signature of the knot. The first example of a thick knot in Kauffman homology is again the knot 8_{19} .

⁷ Sometimes, such knots are called *thick*.

Knot	s	$\mathcal{F}(\lambda, q, t)$
3_1	-2	$\lambda^2(q^{-2} + q^2t^2) + \lambda^3(q^{-1}t^2 + qt^3) + \lambda^4(q^{-2}t^3 + t^4 + q^2t^5) + \lambda^5(q^{-1}t^5 + qt^6)$
4_1	0	$\lambda^{-2}(q^{-2}t^{-4} + t^{-3} + q^2t^{-2}) + \lambda^{-1}(q^3 + q^{-3}t^{-3} + 2q^{-1}t^{-2} + 2qt^{-1})$ $+2q^{-2}t^{-1} + 3 + 2q^2t + \lambda(q^{-3} + 2q^{-1}t + 2qt^2 + q^3t^3) + \lambda^2(q^{-2}t^2 + t^3 + q^2t^4)$
5_1	-4	$\lambda^4(q^{-4} + t^2 + q^4t^4) + \lambda^5(q^{-3}t^2 + q^{-1}t^3 + qt^4 + q^3t^5)$ $+ \lambda^6(q^{-4}t^3 + q^{-2}t^4 + 2t^5 + q^2t^6 + q^4t^7) + \lambda^7(q^{-3}t^5 + 2q^{-1}t^6 + 2qt^7 + q^3t^8)$ $+ \lambda^8(q^{-2}t^7 + 2t^8 + q^2t^9) + \lambda^9(q^{-1}t^9 + qt^{10})$
5_2	-2	$\lambda^2(q^{-2} + t + q^2t^2) + \lambda^3(q^{-3}t + 3q^{-1}t^2 + 3qt^3 + q^3t^4)$ $+ \lambda^4(q^{-4}t^2 + 3q^{-2}t^3 + 5t^4 + 3q^2t^5 + q^4t^6) + \lambda^5(2q^{-3}t^4 + 4q^{-1}t^5 + 4qt^6 + 2q^3t^7)$ $+ \lambda^6(q^{-4}t^5 + 2q^{-2}t^6 + 3t^7 + 2q^2t^8 + q^4t^9) + \lambda^7(q^{-3}t^7 + q^{-1}t^8 + qt^9 + q^3t^{10})$
6_1	0	$\lambda^{-2}(q^{-2}t^{-4} + t^{-3} + q^2t^{-2}) + \lambda^{-1}(q^3 + q^{-3}t^{-3} + 3q^{-1}t^{-2} + 3qt^{-1}) + q^{-4}t^{-2}$ $+4q^{-2}t^{-1} + 6 + 4q^2t + q^4t^2 + \lambda(3q^{-3} + q^{-5}t^{-1} + 6q^{-1}t + 6qt^2 + 3q^3t^3 + q^5t^4)$ $+ \lambda^2(2q^{-4}t + 4q^{-2}t + 5t^3 + 4q^2t^4 + 2q^4t^5)$ $+ \lambda^3(q^{-5}t^2 + 2q^{-3}t^3 + 3q^{-1}t^4 + 3qt^5 + 2q^3t^6 + q^5t^7)$ $+ \lambda^4(q^{-4}t^4 + q^{-2}t^5 + t^6 + q^2t^7 + q^4t^8)$
6_2	-2	$q^4 + q^{-4}t^{-4} + q^{-2}t^{-3} + 2t^{-2} + q^2t^{-1}$ $+ \lambda(4q + q^{-5}t^{-3} + 3q^{-3}t^{-2} + 4q^{-1}t^{-1} + 3q^3t + q^5t^2)$ $+ \lambda^2(6q^{-2} + 3q^{-4}t^{-1} + 8t + 6q^2t^2 + 3q^4t^3)$ $+ \lambda^3(q^{-5} + 5q^{-3}t + 9q^{-1}t^2 + 9qt^3 + 5q^3t^4 + q^5t^5)$ $+ \lambda^4(2q^{-4}t^2 + 6q^{-2}t^3 + 9t^4 + 6q^2t^5 + 2q^4t^6)$ $+ \lambda^5(2q^{-3}t^4 + 5q^{-1}t^5 + 5qt^6 + 2q^3t^7) + \lambda^6(q^{-2}t^6 + 2t^7 + q^2t^8)$
6_3	0	$\lambda^{-3}(q^{-3}t^{-6} + 2q^{-1}t^{-5} + 2qt^{-4} + q^3t^{-3})$ $+ \lambda^{-2}(2q^{-4}t^{-5} + 5q^{-2}t^{-4} + 7t^{-3} + 5q^2t^{-2} + 2q^4t^{-1})$ $+ \lambda^{-1}(6q^3 + q^{-5}t^{-4} + 6q^{-3}t^{-3} + 11q^{-1}t^{-2} + 11qt^{-1} + q^5t)$ $+ 4q^{-4}t^{-2} + 10q^{-2}t^{-1} + 15 + 10q^2t + 4q^4t^2$ $+ \lambda(6q^{-3} + q^{-5}t^{-1} + 11q^{-1}t + 11qt^2 + 6q^3t^3 + q^5t^4)$ $+ \lambda^2(2q^{-4}t + 5q^{-2}t^2 + 7t^3 + 5q^2t^4 + 2q^4t^5)$ $+ \lambda^3(q^{-3}t^3 + 2q^{-1}t^4 + 2qt^5 + q^3t^6)$

Table 3: *Kauffman superpolynomial for some simple knots.*

Further examples of thin knots are torus knots of type $(2, 2k + 1)$. Their reduced Kauffman superpolynomial is given by the following general formula:

$$\mathcal{F}(T_{2k+1,2}) = (\lambda t)^{4k} + (\lambda^2 t^3 + 1) \left(\frac{\lambda}{q} \right)^{2k} \left[\sum_{i=0}^k q^{4i} t^{2i} + \sum_{j=1}^{2k-1} \sum_{i=1}^{2k-j+1} \lambda^j q^{j+2i-2} t^{2j+i-1} \right] \quad (6.22)$$

It is easy to verify that the structure of this result is consistent with all the specializations and the differentials that we proposed. Also, it is curious to note that, at least for these torus knots, there is a relation between the reduced and unreduced Kauffman superpolynomials, which is similar to the relation between $\mathcal{P}(K)$ and $\overline{\mathcal{P}}(K)$ implied by the existence of the canceling differential d_1 [13]. Specifically, we can write (6.22) in the following form:

$$\mathcal{F}(K) = (\lambda/q)^{2k} + (1 + \lambda q^{-1}t) \left[(1 + \lambda q^{-3})Q_a + (1 + q^2t)Q_b \right] \quad (6.23)$$

As in the HOMFLY case, the structure of this expression also reflects the existence of the canceling differential d_2 in the Kauffman homology, but it is strictly stronger than (6.17) and is written in terms of *two* polynomials:

$$\begin{aligned} Q_a &= \lambda^{2k} \sum_{i=1}^k t^{2i} q^{4i-2k} \\ Q_b &= \lambda^{2k+2} \sum_{j=0}^{k-1} \sum_{i=0}^{2k-2j-2} t^{2i+2j+4} \lambda^i q^{i+4j-2k+2} \end{aligned} \quad (6.24)$$

Notice, that Q_a also appears in the expression for the HOMFLY superpolynomial of the same torus knots [13]. The unreduced Kauffman superpolynomial (5.24) for these torus knots can be also written in terms of Q_a and Q_b :

$$\begin{aligned} \overline{\mathcal{F}}(K) &= (\lambda - \lambda^{-1} + q - q^{-1})(\lambda/q)^{2k} + (1 + \lambda qt) \left[(1 + \lambda q^{-3})(1 - q^2 \lambda^{-2}) \lambda q^{-2} Q_a \right. \\ &\quad \left. + (1 + \lambda^2 q^{-2}t)(1 - q^{-2}) \lambda^{-1} q^2 Q_b \right] \end{aligned} \quad (6.25)$$

Unfortunately, this relation between $\mathcal{F}(K)$ and $\overline{\mathcal{F}}(K)$ does not extend to other thin knots. For example, even though the reduced Kauffman superpolynomial for the figure-eight knot has the form analogous to (6.23), the corresponding expression does not agree with (5.26) (in fact, it does not even reduce to the $so(4)$ knot homology at $\lambda = q^3$). It would be interesting to study the relation between the reduced and unreduced Kauffman superpolynomials further.

As another application of the δ -grading, let us derive the regrading in the relation (6.19) between the Kauffman and HOMFLY superpolynomials. The δ -grading of a thin knot in the HOMFLY theory is $\delta = \lambda + \frac{1}{2}q - t = -s/2$, while the δ -grading of the same knot in the Kauffman theory is $\delta = \frac{3}{2}\lambda + \frac{1}{2}q - t = -s$. Since q and t have δ -grading $\frac{1}{2}$ and -1 in both theories, the factors q^{-s} and $(qt)^s$ shift the δ -grading by $-s/2$, as required for consistency. Similarly, since in the Kauffman theory λ has δ -grading $\frac{3}{2}$, the combinations λ/q and λqt have δ -grading $\frac{1}{2}$, which is precisely what the δ -grading of λ should be in the HOMFLY theory.

Finally, notice that the differentials d_N and d_{-N} , $|N| > 1$, have the same δ -grading. Moreover, note that the differential d_{-2} , the universal differentials d_{\pm} , and the canceling differentials d_0 , d_1 , and d_2 all have zero δ -grading, which means that they can be non-trivial even for thin knots (see *e.g.* fig. 4).

Acknowledgments

We would like to thank M. Khovanov, M. Mariño, C. Vafa, and E. Witten for valuable discussions. We are grateful to the KITP, Santa Barbara for warm hospitality during the program “Mathematical Structures in String Theory”, where part of this work was carried out. This work was conducted during the period S.G. served as a Clay Mathematics Institute Long-Term Prize Fellow. This work was also supported in part by the DOE under grant number DE-FG02-90ER40542, in part by RFBR grant 04-02-16880 and in part by the NSF under Grant No. PHY99-07949.

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