$SL(2,\mathbb{C})$ CHERN-SIMONS THEORY AND THE ASYMPTOTIC BEHAVIOR OF THE COLORED JONES POLYNOMIAL

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Abstract. It has been proposed that the asymptotic behavior of the colored Jones polynomial is equal to the perturbative expansion of the Chern-Simons gauge theory with complex gauge group $SL(2,\mathbb{C})$ on the hyperbolic knot complement. In this note we make the first step toward verifying this relation beyond the semi-classical approximation. This requires a careful understanding of some delicate issues, such as normalization of the colored Jones polynomial and the choice of polarization in Chern-Simons theory. Addressing these issues allows us to go beyond the volume conjecture and to verify some predictions for the behavior of the subleading terms in the asymptotic expansion of the colored Jones polynomial.

1. Introduction

The original volume conjecture [11, 17] is a remarkable relation between the limit of the colored Jones polynomial, $J_N(K;q)$, of a knot K and the volume of the knot complement $S^3 \setminus K$:

Conjecture 1.1 (Volume Conjecture)**.** *For a knot K,*

$$\lim_{N\to\infty} \frac{\log \left| J_N\left(K; \exp\left(2\pi\sqrt{-1}/N\right)\right)\right|}{N} = \frac{1}{2\pi}\mathrm{Vol}(S^3 \setminus K), \tag{1.1}$$

where $\mathrm{Vol}(S^3 \setminus K)$ is the simplicial volume of the knot complement. In particular, if K is hyperbolic, then $\mathrm{Vol}(S^3 \setminus K)$ is the hyperbolic volume.

The physical interpretation of this relation was proposed in [9], where it was conjectured that the asymptotic expansion of the colored Jones polynomial $J_N(K;q)$ in the limit $N \to \infty$, $q \to 1$ should be equal to the partition function of the $SL(2,\mathbb{C})$ Chern-Simons theory on the knot complement $S^3 \setminus K$:

Conjecture 1.2. *As $N \to \infty$ and $k \to \infty$ with $u := 2\pi\sqrt{-1}(\frac{N}{k} - 1)$ kept fixed, the colored Jones polynomial has the asymptotic expansion*

$$\log J_N(K; \exp(2\pi\sqrt{-1}/k)) \underset{N,k\to\infty}{\sim} \frac{k}{\sqrt{-1}} S(u) + \frac{1}{2}\delta_K(u)\log k + \frac{1}{2}\log\left(\frac{T_K(u)}{2\pi^2}\right) + \sum_{n=1}^{\infty}\left(\frac{2\pi}{k}\right)^n S_{n+1}(u) \tag{1.2}$$

where the function $S(u)$ in the first term is the classical action of the Chern-Simons theory; $T_K(u)$ is the Ray–Singer torsion of the knot complement twisted by the flat connection corresponding to the representation $\rho : \pi_1(S^3 \setminus K) \to SL(2;\mathbb{C})$ determined by u (see [21, 22] and §3 below); the number $\delta_K(u) \in \mathbb{Z}$ is determined by the topology of the knot complement and the representation ρ; finally, the function $S_n(u)$ denotes the n-loop contribution [26, 1, 2].

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In particular, this physical interpretation of the volume conjecture opens an avenue for several generalizations. First, it suggests that, for a knot K, there exists a 1-parameter family of relations like (1.1) — sometimes called the "generalized" or "parameterized" volume conjecture — which relate a family of limits of the colored Jones polynomial to the volume function $\mathrm{Vol}(K;u)$ on the character variety of the knot complement, see Conjecture 1.3 below. Moreover, the interpretation via $SL(2,\mathbb{C})$ Chern-Simons theory predicts the structure of the subleading terms in the asymptotic expansion of the colored Jones polynomial, where each term in (1.2) has an *a priori* definition and can be computed independently, before comparing to the colored Jones polynomial. In what follows, first we shall discuss the leading term in the expansion (1.2) and then return to the subleading terms in §3.

Notice, the leading term in the expansion (1.2) gives precisely the generalization of the volume conjecture which can be stated as follows:

Conjecture 1.3. *There exists an open subset $\mathcal{O}_K$ of $\mathbb{C}$ such that for any $u \in \mathcal{O}_K$ the following limit exists:*

$$\lim_{N\to\infty} \frac{\log J_N\left(K;\exp\left(\left(u+2\pi\sqrt{-1}\right)/N\right)\right)}{N} = \frac{2\pi}{\left(u+2\pi\sqrt{-1}\right)} S(u). \tag{1.3}$$

The precise formulation of the Conjectures 1.2 and 1.3 requires fixing normalizations of the colored Jones polynomial and the partition function of the Chern-Simons theory. Moreover, in the geometric quantization approach to Chern-Simons gauge theory, the partition function is determined by quantizing the moduli space of flat connections which involves an extra ambiguity, the choice *polarization*.[1] In principle, all these choices should be fixed by *a priori* arguments, such as a non-perturbative formulation of the $SL(2,\mathbb{C})$ Chern-Simons theory. However, with the lack of such arguments one might use the fact that all these choices do not depend on the knot K and, therefore, can be fixed once and for all by considering, say, the leading term $S(u)$ for a particular knot. Thus, the simplest choice of polarization (that is the choice of the symplectic potential $S(u)$) consistent with the asymptotic behavior of the colored Jones polynomial for the figure-eight knot at $\mathrm{Re}(u)=0$ gives $\mathrm{Im}\, S(u) = \frac{1}{2\pi}\mathrm{Vol}(K;u)$, where $\mathrm{Vol}(K;u)$ is the volume function [19, 5],

$$d\ \mathrm{Vol}(K;u) = -\frac{1}{2}\left(\mathrm{Re}(u) d\,\mathrm{Im}(v) - \mathrm{Re}(v) d\,\mathrm{Im}(u)\right). \tag{1.4}$$

Notice, this determines $\mathrm{Im}\, S(u)$ only up to terms proportional to $\mathrm{Re}(u)$ (which are highly constrained, though, see §2 below). Unfortunately, until recently there was no example of a hyperbolic knot for which the asymptotic behavior of the colored Jones polynomial would be studied for $\mathrm{Re}(u)\neq 0$. One simple possibility is to assume that there are no additional terms proportional to $\mathrm{Re}(u)$, so that $\mathrm{Im}\, S(u) = \frac{1}{2\pi}\mathrm{Vol}(K;u)$. This gives the generalized volume conjecture originally proposed in [9].

Recently, it was realized [14] that the generalized volume conjecture proposed in [9] requires a modification, precisely by the terms proportional to $\mathrm{Re}(u)$. Indeed, the explicit study of the limit (1.3) for the figure-eight knot [18] and torus knots at $\mathrm{Re}(u)\neq 0$ leads to the version of the Conjecture 1.3 with the function $S'(u)$, where [14]:

$$\mathrm{Im}\, S'(u) = \frac{1}{2\pi}\mathrm{Vol}(K;u) + \frac{1}{2}\mathrm{Re}(u) + \frac{1}{4\pi}\mathrm{Re}(u)\,\mathrm{Im}(v). \tag{1.5}$$

We believe it is this generalization of the volume conjecture — called the parameterized volume conjecture in [14] — which has a chance of being true. We note (and explain in more detail in §2) that in the geometric quantization approach to

[1] For a general introduction to geometric quantization see [27].

the $SL(2,\mathbb{C})$ Chern-Simons theory, both $S(u)$ and $S'(u)$ can be interpreted as the semi-classical action of the $SL(2,\mathbb{C})$ Chern-Simons theory, obtained in a different polarization. For this reason, the physical considerations in [9] can not tell us whether it is $S(u)$ or $S'(u)$ which should appear in the right-hand side of (1.2) and (1.3). However, given the evidence in [14, 18], we believe it is the function $S'(u)$ which is the correct semi-classical action of the $SL(2,\mathbb{C})$ Chern-Simons theory, and both Conjectures 1.2 and 1.3 should be considered with $S'(u)$ given by (1.5). We present further evidence for this in §3.

One of the main goals of the present paper is to show that the Conjectures 1.2 and 1.3 with the function $S'(u)$ are consistent with the proposal that the asymptotic expansion of the colored Jones polynomial is equal to the loop expansion of the partition function of the $SL(2,\mathbb{C})$ Chern-Simons theory. In particular, in §2 we show that the function $S'(u)$ can be interpreted as the semi-classical action in the $SL(2,\mathbb{C})$ Chern-Simons theory, and differs from $S(u)$ by a choice of polarization. Note, that $S(u) = S'(u)$ for $\mathrm{Re}(u) = 0$, which for hyperbolic knots is expected to correspond to the case of cone-manifolds. Once we identify the correct choice of polarization, in §3 we present further evidence for the Conjecture 1.2 by studying the subleading terms in the asymptotic expansion (1.2) of the colored Jones polynomial and compare with the expected behavior of the partition function of the $SL(2,\mathbb{C})$ Chern-Simons theory.

Before we proceed, let us describe the representation $\rho\colon \pi_1(S^3 \setminus K) \to SL(2,\mathbb{C})$ determined by u. Following [14], we define

$$v_K(u) := 4\pi \frac{d\,S'(u)}{d\,u} - 2\pi\sqrt{-1}, \tag{1.6}$$

where $S'(u)$ is defined by (1.3). Then, ρ can be defined as a representation from $\pi_1(S^3 \setminus K)$ to $SL(2,\mathbb{C})$ sending the longitude and the meridian to the elements whose eigenvalues are $(l,m) = (-\exp(-v_K(u)/2), \exp(u/2))$. The representation ρ defines a flat $SL(2,\mathbb{C})$ bundle over $S^3 \setminus K$ which we denote E_ρ. This bundle will play an important role in §3.

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2. Choice of polarization

In this section we show that the version of the generalized volume conjecture 1.3 proposed in [14] with the function $S'(u)$ is consistent with the interpretation in terms of $SL(2;\mathbb{C})$ Chern-Simons theory suggested in [9]. In particular, we show that $S'(u)$ can be interpreted as the semi-classical action in the $SL(2,\mathbb{C})$ Chern-Simons theory, and differs from $S(u)$ by a choice of polarization. To explain this in detail, let us start by fixing notations. We use $\tilde{u}$ and $\tilde{v}$ instead of u and v used in [9] respectively. The relation between $(\tilde{u},\tilde{v})$ and (u,v) will be explained later.

Let us recall the argument in [9]. Fix an oriented (not necessarily hyperbolic) knot K in S^3. Let M be its complement $S^3 \setminus \mathrm{int}\, N(K)$, where $N(K)$ is the tubular neighborhood of K and $\mathrm{int}\, N(K)$ is its interior. The boundary of M is denoted by Σ. Note that $\pi_1(\Sigma) \cong \mathbb{Z} \times \mathbb{Z}$ is generated by the meridian μ and the longitude λ, where μ bounds a disk in $N(K)$ and is oriented so that the linking number of K

and μ is -1, and λ is null homologous in M and parallel to K in $N(K)$. (So our orientation of the meridian here is *different* from usual one in knot theory.)

We consider a representation ρ of $\pi_1(M)$ to $SL(2;\mathbb{C})$ and denote by $m = \exp\tilde{v}$ and $l = \exp\tilde{u}$ the eigenvalues of its images of μ and λ respectively. Then the pair (m,l) is a zero of the A-polynomial [5]. The zero locus of the A-polynomial defines a Lagrangian submanifold L of $\mathcal{P}$ with respect to the 2-form ω, where $\mathcal{P} = \mathbb{C}^* \times \mathbb{C}^*$ is the representation space of $\pi_1(\Sigma)$ and ω is defined as follows [9, §3].

$$\omega := -\frac{1}{\pi} d\tilde{u} \wedge d\tilde{v}$$

so that $\omega\big|_L = 0$. (Note that here we put $\sigma = k$ in [9, (3.7)], and rescaled ω by a factor of k which now explicitly appears as the coefficient of the classical action in (1.2).) If we put

$$\theta := \frac{1}{2\pi}\left(\tilde{v}\, d\tilde{u} - \tilde{u}\, d\tilde{v} + d\left(\tilde{u}\overline{\tilde{v}}\right)\right),$$

we have $d\theta = \omega$ [9, (3.26)]. Let S be the classical Chern–Simons action corresponding to ρ. Then S can be obtained by integrating θ over a path on L, that is, we have

$$S = \int \theta.$$

Since the Lagrangian submanifold L is quantizable the integral above is well-difined [9, §3]. Note that S depends on the choice of θ satisfying $d\theta = \omega$. So we can define S only up to a total derivative on $\mathcal{P}$ (the choice of polarization).

One possible choice of θ, consistent with $d\theta = \omega$, gives

$$S = \frac{\sqrt{-1}}{2\pi}\big(\mathrm{Vol}(l,m) + 2\pi^2\sqrt{-1}\,\mathrm{CS}(l,m)\big), \tag{2.1}$$

where $\mathrm{Vol}(l,m)$ and $\mathrm{CS}(l,m)$ are the volume and the Chern–Simons invariant of the representation ρ. According to (1.2), the leading term of the log of the N-colored Jones polynomial evaluated at $e^{2\pi\sqrt{-1}/k}$ should be $-\sqrt{-1}kS$ when $N \to \infty$ and $k \to \infty$. This leads to a generalization of the volume conjecture [9, (5.12)]

$$\lim_{N\to\infty, k\to\infty} \frac{\log J_N\left(K; \exp\left(2\pi\sqrt{-1}/k\right)\right)}{k} = \frac{1}{2\pi}\left(\mathrm{Vol}(l,m) + 2\pi^2\sqrt{-1}\,\mathrm{CS}(l,m)\right). \tag{2.2}$$

Now, as we pointed out earlier, the dependence on the choice of polarization is related to the choice of the 1-form θ such that $d\theta = \omega$. In particular, we can consider the following 1-form:

$$\theta' := \frac{1}{2\pi}\left(-2\tilde{u}\, d\tilde{v} + 2\pi\sqrt{-1}d\tilde{v}\right).$$

Note that $d\theta = d\theta' = \omega$. Let S' be the classical Chern–Simons action obtained from θ' ($S' := \int \theta'$). Then

$$\begin{aligned} dS - dS' &= \theta - \theta' \\ &= \frac{1}{2\pi} d\left(\tilde{u}\tilde{v} + \tilde{u}\overline{\tilde{v}} - 2\pi\sqrt{-1}\tilde{v}\right) \\ &= \frac{1}{\pi} d\left(\tilde{u}\,\mathrm{Re}(\tilde{v}) - \pi\sqrt{-1}\tilde{v}\right). \end{aligned}$$

Therefore from (2.1) we have

$$\begin{aligned} \mathrm{Vol}(l,m) &= 2\pi\,\mathrm{Im}(S) \\ &= 2\pi\,\mathrm{Im}(S') + 2\,\mathrm{Im}(\tilde{u})\,\mathrm{Re}(\tilde{v}) - 2\pi\,\mathrm{Re}(\tilde{v}). \end{aligned} \tag{2.3}$$

Now we consider the pair (u, v) used in the previous sections. As described in [18], u and v are related to $\tilde{u}$ and $\tilde{v}$ as follows:

$$\begin{cases} \tilde{v} &= \dfrac{u}{2}, \\ \tilde{u} &= -\dfrac{v}{2}. \end{cases}$$

So from (2.3) we have

$$\mathrm{Vol}(l, m) = 2\pi \,\mathrm{Im}(S') - \pi \,\mathrm{Re}(u) - \frac{1}{2}\,\mathrm{Re}(u)\,\mathrm{Im}(v).$$

Comparing with (1.5) and (1.2), $-\sqrt{-1}kS'$ gives the leading term of the log of the N-colored Jones polynomial if we use θ' to define the classical action S',

$$\lim_{N\to\infty, k\to\infty} \frac{\log \left| J_N\left(K; \exp\left(2\pi\sqrt{-1}/k\right)\right)\right|}{k} = \frac{1}{2\pi}\,\mathrm{Vol}(K; u) + \frac{1}{2}\,\mathrm{Re}(u) + \frac{1}{4\pi}\,\mathrm{Re}(u)\,\mathrm{Im}(v)$$

This is precisely the version of the generalized volume conjecture proposed in [14].

3. Beyond the Leading Order

Now let us discuss the subleading terms in the asymptotic expansion (1.2) of the colored Jones polynomial.

The simplest knot to consider is the unknot U. Since $J_N(U; q) = [N] := \left(q^{N/2} - q^{-N/2}\right) / \left(q^{1/2} - q^{-1/2}\right)$, we have

$$\begin{aligned} &\log J_N(U; \exp(2\pi\sqrt{-1}/k)) \\ &= \log\sin(N\pi/k) - \log\sin(\pi/k) \\ &\sim \log k - \log\pi + \log\sin\left(u/(2\sqrt{-1})\right) + \text{terms of the order } k^{-1} \text{ or lower} \end{aligned} \tag{3.1}$$

for large N and k with fixed $N/k = u/(2\pi\sqrt{-1}) + 1$ and $u \neq 0$.

Since the colored Jones polynomial vanishes at $u = 0$, in this case one should use the *reduced* colored Jones polynomial $V_N(K; q) := J_N(K; q)/J_N(U; q)$ to study the asymptotic expansion (1.2). We should note, however, that it is $J_N(K; q)$ which naturally appears in Chern-Simons theory.

3.1. The logarithmic term.

In this subsection we study the logarithmic term in the asymptotic expansion (1.2).

For a knot K in S^3, let M_K be the complement of the interior of the regular neighborhood of K. Denote by Σ_K the boundary torus of M_K. Let $H^i\left(M_K; E_\rho\right) \cong H^i\left(S^3 \setminus K; E_\rho\right)$ be the i-th cohomology group of M_K with coefficients in the flat $SL(2,\mathbb{C})$ bundle E_ρ which was described in the end of §1. It is well known that $H^i\left(M_K; E_\rho\right)$ is isomorphic to the cohomology $H^i\left(M_K; sl_2(\mathbb{C})\right)$ with coefficient the Lie algebra $sl_2(\mathbb{C})$ twisted by the adjoint action of ρ. We will mainly use $H^i\left(M_K; sl_2(\mathbb{C})\right)$ for calculation.

Define

$$\delta_K^{\mathrm{rep}}(\rho) := 3 + h^1(M_K; \rho) - h^0(M_K; \rho), \tag{3.2}$$

where $h^0(M_K; \rho) := \dim H^0(M_K; \rho)$ and

$$h^1(M_K; \rho) := \dim\left(\mathrm{Ker}[H^1(M_K; \rho) \to H^1(\Sigma_K; \rho')]\right),$$

where the map is induced by the inclusion $\Sigma_K \to M_K$ and ρ' is the restriction of ρ to $\pi_1(\Sigma_K)$. From [10, Page 42] and [20, Démonstration de Proposition 3.7 (Page

72)], we have

$$
\begin{aligned}
h^1(M_K;\rho) &= \dim H^1(M_K;\rho) - \frac{1}{2}\dim H^1(\Sigma_K;\rho') \\
&= \begin{cases} \dim H^1(M_K;\rho) - 3 & \text{if } \rho' \text{ is trivial,} \\ \dim H^1(M_K;\rho) - 1 & \text{if } \rho' \text{ is non-trivial,} \end{cases}
\end{aligned} \tag{3.3}
$$

where ρ' is trivial if it sends every element of $\pi_1(\Sigma_K)$ to $\pm I$ with I the identity matrix.

Among other things, the identification of the asymptotic expansion (1.2) with the partition function in the Chern-Simons theory implies (see *e.g.* [7, 9, 26])

$$\delta_K(u) = \delta_K^{\mathrm{rep}}(\rho), \tag{3.4}$$

where ρ is determined by u as described in §1. The first term (equal to 3) in (3.2) comes from the normalization by the partition function of S^3,

$$
\begin{aligned}
\log Z(S^3) &= \log \sqrt{2/k}\sin(\pi/k) \\
&\sim -\frac{3}{2}\log k + \cdots .
\end{aligned}
$$

To evaluate (3.2), it is useful to note that

$$h^0(K;\rho) = \dim(H_\rho), \tag{3.5}$$

where H_ρ is the isotropy group of E_ρ, the subgroup of $SL(2,\mathbb{C})$ that commutes with the holonomies of flat connections on E_ρ. In other words,

$$H_\rho = \{g \in SL(2;\mathbb{C}) \mid g\rho(\gamma) = \rho(\gamma)g \text{ for any } \gamma \in \pi_1(M_K)\}.$$

Another useful fact is that $h^1(M_K;\rho)$ counts the infinitesimal deformations of E_ρ for a fixed representation of $\pi_1(\Sigma_K)$ [23, 24, 25].

We can show that $h^1(M_K;\rho) = 0$ if K is a hyperbolic knot or a torus knot for a typical representation ρ.

Definition 3.1. [20, Definition 3.21] Let γ be a simple closed curve on Σ_K for a knot K. Then a representation ρ of $\pi_1(M_K)$ in $SL(2;\mathbb{C})$ is called γ-regular if

(1) the map $H^1(M_K;\rho) \to H^1(\gamma;\rho_0)$ induced by the inclusion $\gamma \to M_K$ is injective, and
(2) if $\mathrm{Tr}(\rho(\pi_1(\Sigma_K))) = \pm 1$, then $\rho(\gamma) \neq \pm I$.

The following representations are known to be γ-regular [20, Page 83].

(1) The holonomy representation corresponding to the complete hyperbolic structure of a hyperbolic knot is μ-regular, where μ is the meridian.
(2) For a hyperbolic knot, if the Dehn surgery along a simple closed curve γ is hyperbolic then the holonomy representation induced by γ is γ-regular.
(3) For a torus knot, any irreducible representation is both μ-regular and λ-regular, where λ is the longitude [6, Example 1].

We can show the following proposition.

Proposition 3.2. *Let ρ be a γ-regular representation of $\pi_1(M_K)$ for a knot K for some simple closed curve γ on Σ_K. Then $\delta_K^{rep}(\rho) = 3 - h^0(K;\rho)$.*

Moreover if ρ is non-Abelian, then $\delta_K^{rep}(\rho) = 3$.

Proof. Let ρ' be the restriction of ρ to $\pi_1(\Sigma_K)$, where Σ_K is the boundary torus of the regular neighborhood of K.

Since ρ is γ-regular, the composition

$$H^1(M_K;\rho) \to H^1(\Sigma_K;\rho') \to H^1(\gamma;\rho_0)$$

is injective and so is the map $H^1(M_K;\rho) \to H^1(\Sigma_K;\rho')$. Therefore we have $h^1(M_K;\rho) = \dim \mathrm{Ker}[H^1(M_K;\rho) \to H^1(\Sigma_K;\rho')] = 0$ and the formula follows.

If ρ is non-Abelian, we have $h^0(K;\rho) = 0$ (see for example [20, Lemme 0.7 (ii)]). □

As a corollary we have

Corollary 3.3. *If K is a hyperbolic knot or a torus knot and ρ is non-Abelian representation of $\pi_1(S^3 \setminus K)$ in $SL(2;\mathbb{C})$, then $\delta_K^{rep} = 3$.*

3.1.1. *Abelian representations.* We will study the case where ρ is Abelian.

Lemma 3.4. *For any knot K, there exists open sets $U_1 \ni I$ and $U_2 \ni -I$ of $SL(2;\mathbb{C})$ such that if ρ is an Abelian representation sending the meridian into U_1 or U_2 then $\delta_K^{rep}(\rho) = 2$.*

Proof. First note that $H_1(M_K;\mathbb{Z}) \cong \mathbb{Z}$ is generated by the meridian. We choose an element $\mu \in \pi_1(M_K)$ that is mapped to the meridian by the Abelianization. If ρ is Abelian, it is determined by the image $A := \rho(\mu) \in SL(2;\mathbb{C})$, since it factors through $H_1(M_K;\mathbb{Z})$.

We can calculate $H^i(M_K;\rho)$ by using the infinite cyclic covering space $\widetilde{M_K}$ of M_K. We have the following chain complex of $\widetilde{M_K}$ as $\mathbb{C}[t,t^{-1}]$-modules by using the Fox differential calculus as described in the proof of [3, Theorem 6.1]:

$$C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0,$$

where C_2 is generated by $\{r_1, r_2, \dots, r_{n-1}\}$, C_1 is generated by $\{s_1, s_2, \dots, s_n\}$, C_0 is generated by $\{p\}$, and d_2 and d_1 are given as follows. We define d_1 by $d_1(s_i) := (t-1)p$ for $1 \le i \le n$. Let $F(t)$ be the $n \times (n-1)$ matrix with entries in $Z[t,t^{-1}]$ given by the Fox free differential calculus. Then $d_2(r_i) = \sum_{j=1}^n F_{ji}(t)s_j$, where $F_{ji}(t)$ is the (j,i) entry of $F(t)$. Note that the sum of each colums of $F(t)$ is zero (and so $d_1 \circ d_2 = 0$), and that the determinant of any $(n-1)\times(n-1)$ matrix obtained from $F(t)$ by deleting any row gives the Alexander polynomial $\Delta(K;t)$ of K. (See for example [12, Chapter 11].)

Then the twisted cohomology $H^i(M_K;sl_2(\mathbb{C}))$ is calculated from the following cochain complex:

$$\{0\} \to \mathrm{Hom}_{\mathbb{C}[t,t^{-1}]}(C_0;sl_2(\mathbb{C})) \xrightarrow{d_1^*} \mathrm{Hom}_{\mathbb{C}[t,t^{-1}]}(C_1;sl_2(\mathbb{C})) \xrightarrow{d_2^*} \mathrm{Hom}_{\mathbb{C}[t,t^{-1}]}(C_2;sl_2(\mathbb{C})). \tag{3.6}$$

Here an element $\varphi \in \mathrm{Hom}_{\mathbb{C}[t,t^{-1}]}(C_0;sl_2(\mathbb{C}))$ is given by $\varphi(p) \in sl_2(\mathbb{C})$ in such a way that

$$\varphi(g(t)p) := g(\mathrm{Ad}(A))\varphi(p),$$

where $g(t) \in \mathbb{C}[t,t^{-1}]$ is a Laurent polynomial and $\mathrm{Ad}(A)$ is the adjoint representation. Since

$$d_1^*(\varphi)(s_i) := \varphi(d_1(s_i)) = \varphi((t-1)p) = (\mathrm{Ad}(A) - I)\varphi(p) = A\varphi(p)A^{-1} - \varphi(p)$$

for any i, we have $\dim \mathrm{Ker}\, d_1^* = 1$ if $A \neq \pm I$. Thus we have $h^0(M_K;\rho) = 1$. (Another way to show this is to use (3.5).)

Similarly, if $\psi \in \mathrm{Ker}\, d_2^*$, then

$$d_2^*(\psi)(r_j) := \psi(d_2(r_j)) = \psi\left(\sum_{k=1}^n F_{kj}(t)s_k\right) = \sum_{k=1}^n F_{kj}\big(\mathrm{Ad}(A)\big)\psi(s_k) = O$$

for $j = 1, 2 \dots n-1$, where O is the 2×2 zero matrix. Since $\sum_{k=1}^{n} F_{kj} = 0$ for any j, we have

$$\sum_{k=1}^{n-1} F_{kj}\big(\mathrm{Ad}(A)\big)\{\psi(s_k) - \psi(s_n)\} = O. \tag{3.7}$$

By conjugation we may assume that A is of the form $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$. Note that for any $X := \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \in sl_2(\mathbb{C})$ the $(2,1)$-entry of $A^m X A^{-m}$ is equal to $a^{-2m} z$. So comparing the $(2,1)$-entries of (3.7) we have

$$\sum_{k=1}^{n-1} F_{kj}\left(a^{-2}\right)\{\psi_{2,1}(s_k) - \psi_{2,1}(s_n)\} = 0$$

for $j = 1, 2, \dots, n-1$, where $\psi_{2,1}(s_k)$ is the $(2,1)$ entry of $\psi(s_k)$. Now we know the determinant of the $(n-1) \times (n-1)$ matrix $(F_{kj}(t))_{1 \le k,j \le n-1}$ is $\Delta(K;t)$. Since $\Delta(K;\pm 1)$ is odd for any knot K (see for example [12, Chapter 6]), there exist open sets $U_1 \ni I$ and $U_2 \ni -I$ in $SL(2;\mathbb{C})$ such that $\Delta(K;a^{-2}) \neq 0$ if a is an eigenvalue of a matrix in $U_1 \cup U_2$. Thus if $A \in U_1 \cup U_2$, the matrix $(F_{kj}(t))_{1 \le k,j \le n-1}$ is non-singular and we have $\psi_{2,1}(s_k) = \psi_{2,1}(s_n)$ for any $1 \le k \le n-1$ from (3.7). Similarly, we have $\psi_{1,1}(s_k) = \psi_{1,1}(s_n)$ and $\psi_{1,2}(s_k) = \psi_{1,2}(s_n)$, which implies $\psi(s_k) = \psi(s_n)$. This means that the 1-cocycle group Z^1 of (3.6) is given by

$$Z^1 = \big\{\psi \in \mathrm{Hom}_{\mathbb{C}[t,t^{-1}]}(C_1; sl_2(\mathbb{C})) \mid \psi(s_1) = \psi(s_2) = \cdots = \psi(s_n)\big\}$$

and so $\dim Z^1 = 3$. Since $\dim \mathrm{Ker}\, d_1^* = 1$, the dimension of 1-coboundary group B^1 is 2. We finally have $h^1(M_K;\rho) = \dim Z^1 - \dim B^1 = 1$.

From (3.3) we have $\delta_K^{\mathrm{rep}}(\rho) = 2$. □

In [8], S. Garoufalidis and T. Le prove that for any knot K the limit

$$\lim_{N\to\infty} V_N\big(K; \exp((u + 2\pi\sqrt{-1})/N)\big)$$

exists if u is sufficiently close to $-2\pi\sqrt{-1}$, which was first proved for the figure-eight knot by the second author in [15]. This means that $\delta_K(u) = 2$ from (3.1). We expect that such u determines an Abelian representation ρ.

3.1.2. *Connected-sums.* We will discuss the behavior of $\delta_K(u)$ and $\delta_K^{\mathrm{rep}}(\rho)$ under connected-sum and satallite. Let us denote by $K_1 \sharp K_2$ the connected-sum of two knots K_1 and K_2. Then we have $J_N(K_1 \sharp K_2; q) = J_N(K_1;q) J_N(K_2;q)/J_N(U;q)$. Therefore we have

$$\delta_{K_1 \sharp K_2}(u) = \delta_{K_1}(u) + \delta_{K_2}(u) - 2 \tag{3.8}$$

if $u \neq 0$. Correspondingly, for $u = 0$ we have $V_N(K_1 \sharp K_2; q) = V_N(K_1;q) V_N(K_2;q)$ which implies

$$\delta_{K_1 \sharp K_2}(0) = \delta_{K_1}(0) + \delta_{K_2}(0) \tag{3.9}$$

Now, let us compare this with $\delta_{K_1 \sharp K_2}^{\mathrm{rep}}(\rho)$. Note that the complement $M_{K_1 \sharp K_2}$ is obtained from M_{K_1} and M_{K_2} by glueing along an annulus. More precisely, $M_{K_1 \sharp K_2}$ is obtained from M_{K_1} and M_{K_2} by identifying annuli $A_1 \in T_{K_1}$ and $A_2 \in T_{K_2}$, where A_1 (resp. A_2) is the regular neighborhood of the meridian μ_1 (resp. μ_2) of K_1 (resp. K_2).

We first calculate $H^0(S^1;\rho)$.

Lemma 3.5. *If a representation $\rho\colon \pi_1(S^1) \to SL(2;\mathbb{C})$ is not $\pm I$ then we have*

$$H^0(S^1;\rho) = \mathbb{C},$$

Otherwise, we have

$$H^0(S^1;\rho) = \mathbb{C}^3,$$

Proof. We use the interpretation of h^0 described in (3.5). If ρ is not $\pm I$ its isotropy group is one-dimensional and so $h^0(S^1;\rho) = 1$. The case where $\rho = \pm I$ the equality follows since the cohomology group is the usual one. □

Next we calculate $H^i(M_{K_1\sharp K_2};\rho)$ for $i = 0$ and 1 under some assumption.

Lemma 3.6. *Let $\rho\colon \pi_1(M_{K_1\sharp K_2}) \to SL(2;\mathbb{C})$ be a representation, and ρ_1, ρ_2, and ρ_0 restrictions of ρ to $\pi_1(M_{K_1})$, $\pi_1(M_{K_2})$, and $\pi_1(A)$ respectively, where $A := M_{K_1}\cap M_{K_2}(\cong A_1 \cong A_2)$. Suppose that ρ_1 and ρ_2 are μ_1- and μ_2-regular respectively.*

Then

$$\delta^{rep}_{K_1\sharp K_2}(\rho) = \delta^{rep}_{K_1}(\rho_1) + \delta^{rep}_{K_2}(\rho_2) - 2 \tag{3.10}$$

if $\rho_0 \neq \pm I$, and

$$\delta^{rep}_{K_1\sharp K_2}(\rho) = \delta^{rep}_{K_1}(\rho_1) + \delta^{rep}_{K_2}(\rho_2) \tag{3.11}$$

if $\rho_0 = \pm I$.

Proof. The Mayer–Vietoris exact sequence for $M_{K_1\sharp K_2} = M_{K_1} \cup M_{K_2}$, M_{K_1}, M_{K_2} and A gives the following exact sequence:

$$\begin{aligned}\{0\} &\to H^0(M_{K_1\sharp K_2};\rho) \to H^0(M_{K_1};\rho_1) \oplus H^0(M_{K_2};\rho_2) \to H^0(A;\rho_0)\\ &\to H^1(M_{K_1\sharp M_2};\rho) \to H^1(M_{K_1};\rho_1) \oplus H^1(M_{K_2};\rho_2) \xrightarrow{j_1^*-j_2^*} H^1(A;\rho_0),\end{aligned}$$

where j_1^* and j_2^* are induced by the inclusions $j_1\colon A \to M_{K_1}$ and $j_2\colon A \to M_{K_2}$ respectively. Therefore we have

$$\begin{aligned}&\dim H^0(A;\rho_0)\\ &= \left\{\dim H^0(M_{K_1};\rho_1) + \dim H^0(M_{K_2};\rho_2) - \dim H^0(M_{K_1\sharp K_2};\rho)\right\}\\ &+ \left\{\dim H^1(M_{K_1\sharp K_2};\rho) - \dim \operatorname{Ker}(j_1^* - j_2^*)\right\}.\end{aligned}$$

If $\rho_0 \neq \pm I$, then from Lemma 3.5 and (3.3) we have

$$\begin{aligned}\delta^{\mathrm{rep}}_{K_1\sharp K_2} &= 3 + \dim \operatorname{Ker}\left[H^1(M_{K_1\sharp K_2};\rho) \to H^1(T_{K_1\sharp K_2};\rho')\right] - \dim H^0(M_{K_1\sharp K_2};\rho)\\ &= 3 + \dim H^1(M_{K_1\sharp K_2};\rho) - 1 - \dim H^0(M_{K_1\sharp K_2};\rho)\\ &= 2 + \dim H^0(A;\rho_0) + \dim \operatorname{Ker}(j_1^* - j_2^*) - h^0(M_{K_1};\rho_1) - h^0(M_{K_2};\rho_2)\\ &= 3 + \dim \operatorname{Ker}(j_1^* - j_2^*)\end{aligned}$$

Now since j_1^* and j_2^* are injective and $\dim H^1(A;\rho_0) = 1$, we conclude that $\dim \operatorname{Ker}(j_1^* - j_2^*) = \dim(\operatorname{Im} j_1^* \cap \operatorname{Im} j_2^*) = 1$. Therefore

$$\delta^{\mathrm{rep}}_{K_1\sharp K_2} = 4 = \delta^{\mathrm{rep}}_{K_1}(\rho_1) + \delta^{\mathrm{rep}}_{K_2}(\rho_2) - 2.$$

If $\rho = \pm I$, then since $\dim H^0(A;\rho_0) = 3$ we have

$$\begin{aligned}\delta^{\mathrm{rep}}_{K_1\sharp K_2} &= \dim H^1(M_{K_1\sharp K_2};\rho) - \dim H^0(M_{K_1\sharp K_2};\rho)\\ &= 3 + \dim \operatorname{Ker}(j_1^* - j_2^*) - h^0(M_{K_1};\rho_1) - h^0(M_{K_2};\rho_2).\end{aligned}$$

But in this case we have

$$\begin{aligned}&\dim \operatorname{Ker}(j_1^* - j_2^*)\\ =&3 + \dim \operatorname{Ker}[H^1(M_{K_1};\rho_1) \to H_1(T_{K_1})] + \dim \operatorname{Ker}[H^1(M_{K_2};\rho_2) \to H_1(T_{K_2})]\end{aligned}$$

and so

$$\begin{aligned}\delta^{\mathrm{rep}}_{K_1\sharp K_2} &= 6 + h^1(M_{K_1},\rho_1) + h^1(M_{K_2};\rho_2) - h^0(M_{K_1},\rho_1) + h^0(M_{K_2};\rho_2)\\ &= \delta^{\mathrm{rep}}_{K_1}(\rho_1) + \delta^{\mathrm{rep}}_{K_2}(\rho_2).\end{aligned}$$

□

Observation 3.7. *Equation* (3.8) *should be compared with its "physics" counterpart, Equation* (3.10)*. On the other hand, Equations* (3.9) *and* (3.11) *which appear to have the same form actually have very different origin.*

Remark 3.8. The assumption of Lemma 3.6 is true for hyperbolic knots and for torus knots. In particular, it is true for hyperbolic knots with representations close to the holonomy representations corresponds to their complete hyperbolic structures.

3.1.3. *Satellite knots.* We consider satellite knots.

Let K be a knot in a solid torus D. If $e\colon D \to S^3$ is an embedding, then the image $e(K)$ forms a knot. We call $e(K)$ a satellite of the knot C with companion K, where C is the image of the core of D. Then the complement $M_{e(K)}$ of the interior of the regular neighborhood of K is obtained from $S^3 \setminus \operatorname{Int} D$ and $D_K := D \setminus \operatorname{Int} N(K)$ by pasting along their boundaries ∂M_C and ∂D, where Int denotes the interior, $N(K)$ is the regular neighborhood of K in D, and $M_C := S^3 \setminus \operatorname{Int} N(C)$. Note that ∂D_K consists of two tori; ∂D and $\partial M_{e(K)}$.

We can compute $\delta^{\mathrm{rep}}_{e(K)}(\rho)$ under some assumptions.

Lemma 3.9. *Let $e(K)$ be a satellite of a knot C with companion K and $\rho\colon \pi_1(S^3 \setminus e(K)) \to SL(2;\mathbb{C})$ a non-trivial representation. We assume the following four conditions:*

(i) $$h^0(M_C;\rho_1) = 0,$$

(ii) $$h^1(M_C;\rho_1) = 0,$$

(iii) $$h^0(D_K;\rho_2) = 0,$$

(iv) $$\operatorname{Ker}\left[H^1(D_K;\rho_2) \to H^1(\partial D_K;{\rho'}_2)\right] = 0,$$

where ρ_1, ρ_2, and ${\rho'}_2$ are the restrictions of ρ to $\pi_1(S^3 \setminus C)$, $\pi_1(D \setminus K)$ and ∂D_K respectively, and the map in (iv) *is induced by the includion. We also assume all the induced representations ρ_0, ρ_1, and ρ_2 are non-trivial.*

Then we have $\delta^{rep}_K(\rho) = 4$.

Proof. From the Mayer–Vietoris exact sequence for $M_{e(K)} = M_C \cup D_K$, M_C, D_K and $T = M_C \cap D_K$, we have

$$\begin{aligned}\{0\} &\to H^0(M_{e(K)};\rho) \to H^0(M_C;\rho_1) \oplus H^0(D_K;\rho_2) \to H^0(T;\rho_0)\\ &\to H^1(M_{e(K)};\rho) \to H^1(M_C;\rho_1) \oplus H^1(D_K;\rho_2) \xrightarrow{j_1^* - j_2^*} H^1(T;\rho_0),\end{aligned}$$

where ρ_0 is the restriction of ρ to $\pi_1(T)$, and j_1 and j_2 are inclusions. From the assumptions (i) and (iii), we have $h^0(M_{e(K)};\rho) = \dim H^0(M_{e(K)};\rho) = 0$.

We note that $h^1(M_C;\rho) = \dim \operatorname{Ker} j_1^* = 0$. We also note that

$$\begin{aligned}\dim \operatorname{Ker} j_2^* &= \dim \operatorname{Ker}\left[H^1(D_K;\rho_2) \to H^1(\partial D_K;{\rho'}_2) \to H^1(T;\rho_0)\right]\\ &= \dim \operatorname{Ker}\left[H^1(\partial D_K;{\rho'}_2) \to H^1(T;\rho_0)\right]\\ &= 1\end{aligned}$$

since $\dim H^1(T;\rho_0) = 1$ from [20, Démonstration de Proposition 3.7 (Page 72)]. Therefore the kernel of the map $j_1^* - j_2^*$ is one-dimensional, and we have $H^1(M_{e(K)};\rho) = H^0(T;\rho_0) \oplus \mathbb{C}$. Since $\pi_1(T)$ is abelian and $\rho_0 \neq \pm I$, we have $H^0(T;\rho_0) = \mathbb{C}$ from (3.5). From (3.3), we finally get $h^1(M_{e(K)};\rho) = 1$.

Therefore $\delta_{e(K)}(u) = 4$ as stated. □

Observation 3.10. *This can be compared with a result of Zheng* [28, Theorem 1.4]*, where he studies Whitehead doubles of non-trivial torus knots and proves that* $\delta_K(0) = 4$ *for such knots.*

In this case D_K *is homeomorphic to the complement of (the interior of the regular neighborhood of) the Whitehead link, which is hyperbolic. So from* [10, Corollary 1.2]*, the assumption* (iv) *is satisfied if* ρ *is a small deformation of the holonomy representation. The other conditions also hold.*

Remark 3.11. The assumption $h^1(S^3 \setminus C; \rho_1) = h^0(S^3 \setminus C; \rho_1) = 0$ of Lemma 3.9 is true if C is a hyperbolic knot and ρ_1 is close to the holonomy representation corresponding to the complete hyperbolic structure. The assumption $h^1(D \setminus K; \rho_2) = h^0(D \setminus K; \rho_2) = 0$ may be true if $D \setminus K$ possesses a complete hyperbolic structure and ρ_2 is close to the holonomy representation.

Remark 3.12. It would be interesting to study the asymptotic behavior of the colored Jones polynomial of the Hopf link H. Since $J_N(H; q) = [N^2]$ (see for example [12, Lemma 14.2]), we have

$$\begin{aligned}\log J_N(H; \exp 2\pi\sqrt{-1}/k) &= \log \sin\left(\pi\left(\frac{N}{k}\right)^2 k\right) - \log\sin(\pi/k)\\ &\sim 2\log k + 2\log(N/k)\end{aligned}$$

if N/k is *very small*. This formula would mean that $\delta_H(u) = 4$, which is consistant with the fact that $3 + h^1(T \times I) - h^0(T \times I) = 4$ since the complement of the Hopf link is $T \times I$.

3.2. **Ray–Singer torsion.** The next interesting term in (1.2) is the term containing the Ray–Singer torsion $T_K(u) := T(S^3 \setminus K, \rho)$. Following the conventions used in the literature on Chern-Simons theory, we define the Ray–Singer torsion of a 3-manifold M with respect to a flat bundle E_ρ corresponding to ρ by

$$T(M, \rho) = \exp\left(-\frac{1}{2}\sum_{n=0}^{3} n(-1)^n \log\det{}'\Delta_n^{E_\rho}\right), \tag{3.12}$$

where $\Delta_n^{E_\rho}$ is the Laplacians on n-forms with coefficients in E_ρ, and $\det{}'\Delta_n^{E_\rho}$ is the regularized determinant of the restriction of the orthocomplement of its kernel. Using Poincaré duality, one finds

$$T(M, \rho) = \frac{\left(\det{}'\Delta_0^{E_\rho}\right)^{3/2}}{\left(\det{}'\Delta_1^{E_\rho}\right)^{1/2}}. \tag{3.13}$$

We remind that the relation between u and the corresponding representation ρ was discussed in the end of §1. Note, that the definition of the Ray–Singer torsion is particularly simple when $h^0(S^3 \setminus K; \rho) = h^1(S^3 \setminus K; \rho) = 0$; in this case the Laplacians $\Delta_n^{E_\rho}$ have empty kernels.

Very much like the leading term in the expansion (1.2), $T_K(u)$ is a non-trivial function on the character variety. In view of the Cheeger-Müller theorem [4, 13], it would be interesting to compare the Ray–Singer torsion as a function on the character variety to the Reidemeister torsion studied by Porti [20]. For example, for the figure-eight knot E and $\mathrm{Re}(u) = 0$ one has

$$T_E(u) = \frac{1}{\sqrt{(3/2 - \cos\alpha)(1/2 + \cos\alpha)}} \tag{3.14}$$

if $u \neq 0$, and

$$T_E(0) = \frac{\pi^2}{\sqrt{(3/2 - \cos\alpha)(1/2 + \cos\alpha)}} \tag{3.15}$$

where $\alpha = |\sqrt{-1}u|$ is the singular (cone) angle of the cone manifold $M = S^3 \setminus K$. Note that in [20, §5.3, Exemple 1] the torsion is given as $\pm 1/T_E(u)$ for $0 < \sqrt{-1}u < 2\pi/3$. Note also that we use the *reduced* colored Jones polynomial V_N when $u = 0$. The difference between (3.14) and (3.15) comes from $-\log\pi$ in (3.1).

Let us consider the following function of N and $r := N/k = 1 + u/(2\pi\sqrt{-1})$.

$$\begin{aligned}&\operatorname{Re}\left\{\log J_N\big(E; \exp(2\pi r\sqrt{-1}/N)\big)\right\}\\ &\qquad - \operatorname{Re}\left\{\frac{N}{r\sqrt{-1}}S(r) + \frac{3}{2}\log\left(\frac{N}{r}\right) + \frac{1}{2}\log\left(\frac{T_E(u)}{2\pi^2}\right)\right\},\end{aligned}$$

for the figure-eight knot E. Here we use the result of [16]. We expect that it vanishes when $N \to \infty$. In Figure 1–6 we use MATHEMATICA to plot the graphs of this function for $N = 100$, 200, 300, 400, 500 and 1000.

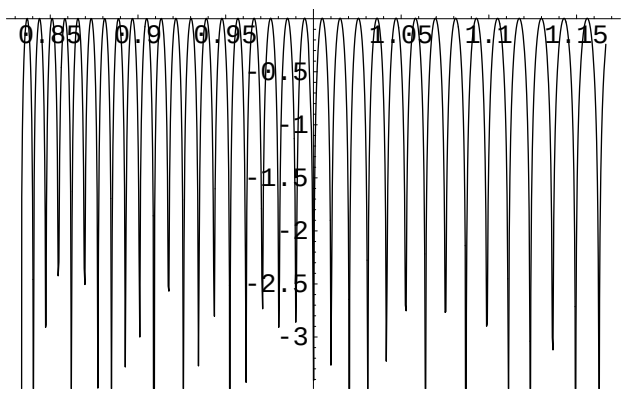

FIGURE 1. $N = 100$

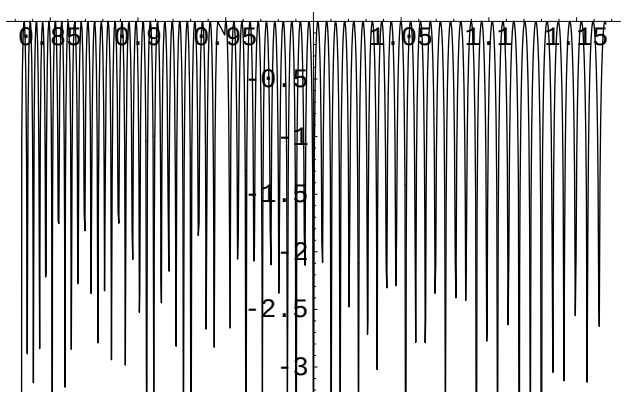

FIGURE 2. $N = 200$

We also draw the graph of

$$\operatorname{Re}\left\{\log J_N\big(E; \exp(2\pi\sqrt{-1}/N)\big)\right\} - \operatorname{Re}\left\{\frac{N}{\sqrt{-1}}S(0) + \frac{3}{2}\log N + \frac{1}{2}\log\frac{T_E(0)}{2\pi^2}\right\}$$

for $N = 100 \times n$ with $1 \leq n \leq 100$.

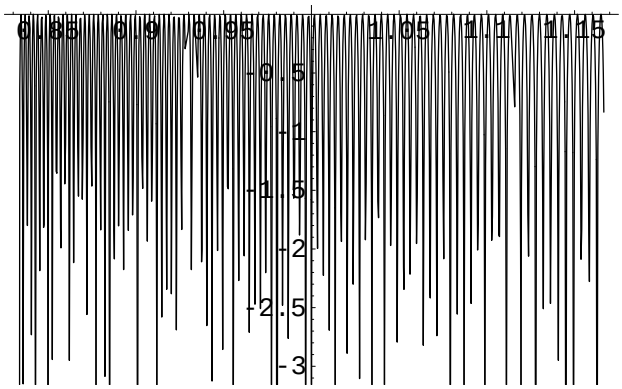

FIGURE 3. $N = 300$

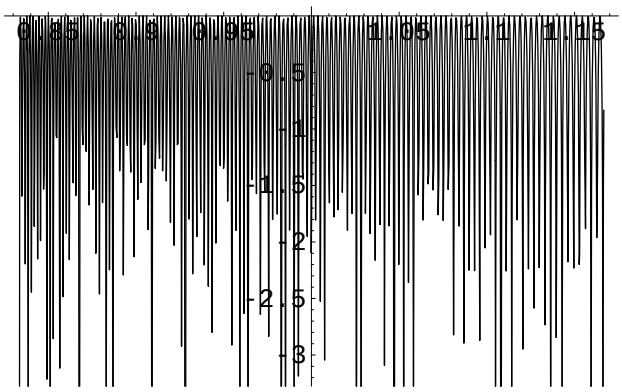

FIGURE 4. $N = 400$

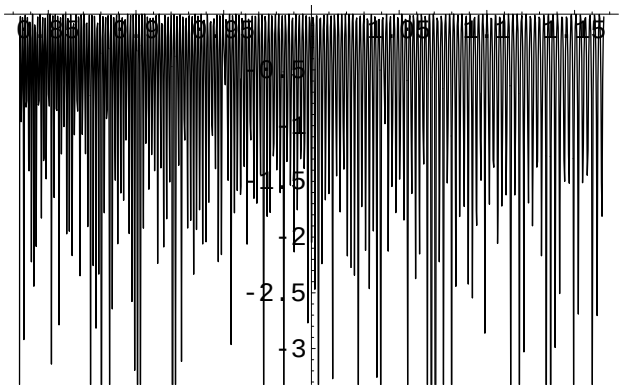

FIGURE 5. $N = 500$

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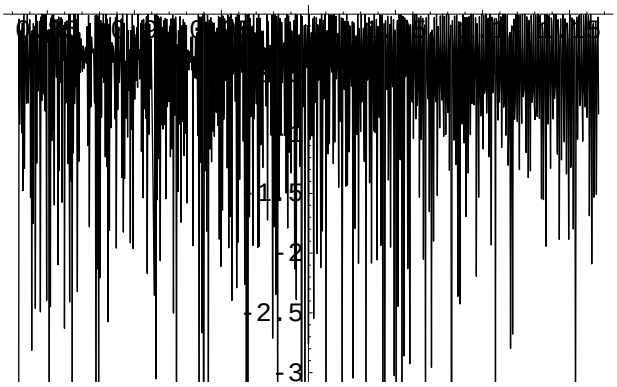

FIGURE 6. $N = 1000$

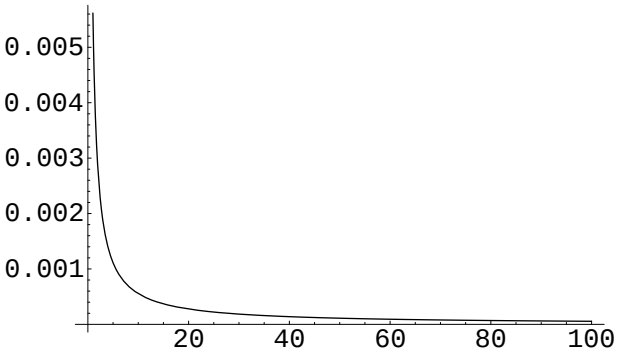

FIGURE 7. The horizontal axis corresponds to $N/100$.

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