

Holomorphic reduction of $\mathcal{N} = 2$ gauge theories, Wilson-'t Hooft operators, and S-duality

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Abstract

We study twisted $\mathcal{N} = 2$ superconformal gauge theory on a product of two Riemann surfaces Σ and C . The twisted theory is topological along C and holomorphic along Σ and does not depend on the gauge coupling or theta-angle. Upon Kaluza-Klein reduction along Σ , it becomes equivalent to a topological B-model on C whose target is the moduli space \mathcal{M}_V of nonabelian vortex equations on Σ . The $\mathcal{N} = 2$ S-duality conjecture implies that the duality group acts by autoequivalences on the derived category of \mathcal{M}_V . This statement can be regarded as an $\mathcal{N} = 2$ counterpart of the geometric Langlands duality. We show that the twisted theory admits Wilson-'t Hooft loop operators labelled by both electric and magnetic weights. Correlators of these loop operators depend holomorphically on coordinates and are independent of the gauge coupling. Thus the twisted theory provides a convenient framework for studying the Operator Product Expansion of general Wilson-'t Hooft loop operators.

1 Introduction

The observation that many supersymmetric field theories can be twisted into topological field theories was first made by E. Witten [1] and has proved very fruitful for understanding properties of such theories as well as their relationship with mathematics (see e.g. [2, 3, 4, 5]). Recently a twisted version of the $\mathcal{N} = 4$ gauge theory in four dimensions has been used to provide a physical derivation of the central statements of the Geometric Langlands Program [6]. The main goal of this paper is to explore to what extent the considerations of [6] can be generalized to gauge theories with $\mathcal{N} = 2$ supersymmetry.

One obvious motivation is finding analogs of the Geometric Langlands Program. It is known that Montonen-Olive duality, which implies the geometric Langlands duality, has close analogs for certain finite $\mathcal{N} = 2$ gauge theories. The simplest example of such a model is the so-called Seiberg-Witten theory [7] which is $\mathcal{N} = 2$ gauge theory with gauge group $SU(2)$ and four hypermultiplets in the fundamental representation. It is natural to inquire whether $\mathcal{N} = 2$ S-duality implies something similar to the geometric Langlands duality.

A somewhat different motivation stems from the observation made in [6] that the twisted $\mathcal{N} = 4$ gauge theory considered there has topological Wilson-'t Hooft loop observables. These observables are the most basic observables in any gauge theory, and the twisted theory, being independent of the gauge coupling, offers an opportunity to compute some of their properties exactly. For example, it was shown in [6] that the OPE algebra of 't Hooft operators in the theory with gauge group G reproduces the fusion rules for irreducible representations of the Langlands-dual group ${}^L G$. This was achieved by identifying the 't Hooft operators with Hecke operators studied by mathematicians and by exploiting existing mathematical results on the algebra of Hecke operators (“the geometric Satake correspondence” [8, 9, 10]). From the physical viewpoint, this is a new nontrivial test of the Montonen-Olive duality conjecture.

In the untwisted theory, Wilson-'t Hooft operators are labelled [11] by pairs $(\mu, \nu) \in \Lambda_{cw} \times \Lambda_w$, where Λ_{cw} and Λ_w are coweight and weight lattices of G ; more precisely, they are labelled by equivalence classes of such pairs under the action of the Weyl group. The usual Wilson loop operators correspond to pairs of the form $(0, \nu)$, while 't Hooft operators have $\nu = 0$ and μ arbitrary. A natural problem is to compute the algebra of general Wilson-'t Hooft operators with both μ and ν nonzero. Unfortunately, it is not possible

to study this question in the framework of the GL-twisted $\mathcal{N} = 4$ theory considered in [6], for the following reason. The GL-twisted theory depends on an extra complex parameter $t \in \mathbb{C} \cup \{\infty\}$, and for each value of t the weight ν must be proportional to the coweight μ , with the coefficient depending on t and the complexified gauge coupling of the theory. (Here we assume that the Cartan subalgebra and its dual have been identified using the Killing metric.) Thus for each particular value of t one can study only a subset of Wilson-'t Hooft operators.

We will see that there exists a nontopological twist which allows arbitrary values for μ and ν in a given theory. The twisted theory is still independent of the gauge coupling, so semiclassical computations of the OPE algebra should give an exact result. We will also see that such a twist makes sense for an arbitrary finite $\mathcal{N} = 2$ gauge theory, and this allows one to formulate an analog of the geometric Langlands duality. More precisely, what can be generalized to the $\mathcal{N} = 2$ case is the ‘‘classical limit’’ of the usual geometric Langlands duality.

The twist we are going to consider works only for manifolds whose holonomy group is reduced to $U(1) \times U(1)$. That is, the four-manifold is a product of two Riemann surfaces C and Σ . The twisted theory depends on a parameter $t \in \mathbb{C} \cup \{\infty\}$ similar to the one in the GL-twisted $\mathcal{N} = 4$ theory. If $t \neq 0, \infty$, the theory is topological on C and holomorphic on Σ . That is, it is invariant under arbitrary diffeomorphisms of C , but depends on the complex structure of Σ . Under the Kaluza-Klein reduction along C it becomes a chiral CFT on Σ . Thus we can attach to any finite $\mathcal{N} = 2$ gauge theory a chiral algebra. In fact, this construction works in greater generality: as first noted in [12] and discussed below, it is sufficient to assume that the theory one starts with has $\mathcal{N} = 1$ supersymmetry and nonanomalous $U(1)_R$ symmetry.

But in order to have Wilson-'t Hooft operators in the twisted theory and be able to formulate an analog of the geometric Langlands duality, it is better to start with an $\mathcal{N} = 2$ theory and reduce along Σ . The resulting effective field theory on C is a 2d TFT, namely the B-model whose target is the moduli space of so-called vortex equations on Σ . Vortex equations generalize Hitchin equations to the case when the Higgs field is in the representation other than the adjoint. Many considerations of [6] have direct analogs for such a B-model; in particular 't Hooft operators are identified with Hecke operators, and they act by functors on the category of branes on the moduli space of vortex equations.

The values $t = 0$ and $t = \infty$ are special in that the twisted theory does

not admit either branes or Wilson-'t Hooft loop observables of any kind. But such a theory is interesting in its own right because it is a holomorphic field theory on $C \times \Sigma$. That is, it depends on the complex structures of both C and Σ and correlators can have holomorphic dependence on coordinates. (It also depends on the complexified gauge coupling τ , but only on the non-perturbative level.) Such a theory can be viewed as a higher-dimensional analog of chiral algebra. It has both local and nonlocal observables; while the OPE of local observables is nonsingular due to Hartogs' theorem, this is not so if one also considers nonlocal observables. From the mathematical viewpoint, the nonlocal observables take values in the sheaf cohomology of certain holomorphic line bundles on $C \times \Sigma$. Thus twisted supersymmetric theories suggest a physically motivated definition of a chiral algebra in two complex dimensions.

Topological reduction of $\mathcal{N} = 2$ gauge theories on a Riemann surface has been considered for the first time by Bershadsky, Johansen, Sadov, and Vafa [13]. It was noted there that the reduced theory is a sigma-model whose target is the moduli space of nonabelian vortex equations. The main difference between the present work and [13] is that Bershadsky et al. consider a topologically twisted theory in four dimensions, while we consider a nontopological (holomorphic) twist. As a consequence, their effective sigma-model is an A-model for the moduli space of vortex equations, while we end up with a B-model with the same target space.

The organization of the paper is as follows. After a very brief review of $\mathcal{N} = 2$ gauge theories, we define in section 2 a twist of a finite $\mathcal{N} = 2$ theory on $C \times \Sigma$ and study its properties. Then in section 3 we perform Kaluza-Klein reduction along both C and Σ and reinterpret our results in two-dimensional terms. We also briefly comment on holomorphic twists of $\mathcal{N} = 1$ gauge theories. In section 4 we discuss observables in the twisted $\mathcal{N} = 2$ theory: local observables and their descendants, as well as Wilson-'t Hooft loop observables. In section 5 we discuss how Wilson and 't Hooft operators act on branes in the twisted theory; the general case of mixed Wilson-'t Hooft operators will be studied elsewhere [34]. Finally, in section 6 we propose an $\mathcal{N} = 2$ analog of the Geometric Langlands Program in the special case of the Seiberg-Witten theory.

We mostly follow the conventions of [6]. One notable difference is that we define the covariant derivative to be $D = d + iA$ instead of $d + A$, so that the connection 1-form A is Hermitian. This is in line with most of the physics literature. Generators T^a of a compact Lie group G are also taken to

be Hermitian, so that $\text{Tr}T^aT^b$ is a positive definite metric on the Lie algebra of G . This accounts for some extra minus signs compared to [6]. Also, the roles of C and Σ are in some sense reversed compared to [6].

2 Partial twists of finite $\mathcal{N} = 2$ gauge theories

2.1 $\mathcal{N} = 2$ gauge theories

In this paper we consider $\mathcal{N} = 2$ gauge theory with gauge group G and hypermultiplets in the representation R of G . The gauge group must be compact and semisimple; the representation R is allowed to be reducible. An irreducible component of R is called a ‘‘flavor’’; for example, if R is a sum of N_f copies of an irreducible representation R_0 , one says that the theory has N_f flavors of hypermultiplets in representation R_0 . If we use $N = 1$ superfield notation, an $\mathcal{N} = 2$ hypermultiplet in representation R is described by a pair of chiral superfields \tilde{Q} and Q in representations R and its dual R^\vee respectively. An $\mathcal{N} = 2$ gauge multiplet is described by an $N = 1$ real superfield V and a chiral superfield Φ , both in the adjoint representation of G . The part of the action describing the gauge fields is

$$I_{gauge} = \frac{1}{2\pi} \text{Im} \left(\tau \int d^4x d^2\theta \text{Tr} W^\alpha W_\alpha \right) + \frac{\text{Im}\tau}{4\pi} \int d^4x d^4\theta \text{Tr} (\Phi^\dagger e^{2V} \Phi).$$

Here $W_\alpha = -\frac{1}{8} \bar{D}^2 e^{-2V} D_\alpha e^{2V}$, as usual, and

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$$

is the complexified gauge coupling which combines the usual gauge coupling e and the theta-angle. The part of the action describing the matter fields is

$$I_{matter} = \int d^4x d^4\theta \left(Q^\dagger e^{2V} Q + \tilde{Q} e^{-2V} \tilde{Q}^\dagger \right) + \sqrt{2} \text{Re} \int d^4x d^2\theta \tilde{Q} \Phi Q.$$

Here and in what follows we set the hypermultiplet mass terms to zero.

When we consider the twisted theory, it will be convenient to rescale all scalar fields (i.e. the lowest components of Φ , Q , and \tilde{Q}) by a factor $\sqrt{2}$. This is also the convention adopted in [6].

The classical theory has $SU(2)_R \times U(1)_N$ group of R-symmetries as well as a global symmetry $U(1)_B$ which multiplies Q and \tilde{Q} by opposite phases. In

the quantum theory, the $U(1)_N$ subgroup, under which Φ has charge 2 and Q and \tilde{Q} have charge zero, is generically anomalous. The anomaly cancellation condition is

$$C(G) - C(R) = 0, \tag{1}$$

where $C(R)$, the index of representation R , is defined by

$$\text{Tr}_R T^a T^b = C(R) \delta^{ab},$$

and $C(G)$ is the index of the adjoint representation. The condition for vanishing of the beta-function is the same as the $U(1)_N$ anomaly cancellation condition. The theories satisfying (1) are superconformal field theories; they are also known as finite $\mathcal{N} = 2$ theories, since the fields and the gauge coupling do not require infinite renormalization.

The simplest way to satisfy (1) is to take R to be the adjoint representation; this gives a theory with $\mathcal{N} = 4$ supersymmetry. For $G = SU(N)$ one can take R to be the sum of $2N$ copies of the fundamental representation; this theory is known as $\mathcal{N} = 2$ super-QCD (with $2N$ flavors) and has been extensively studied beginning with the work of N. Seiberg and E. Witten [7] in the case $N_c = 2$ and P. Argyres and A. Faraggi [14] in general.

In the superconformal case, the gauge coupling e^2 does not run and the theory depends on a single parameter τ taking values in the complex upper half-plane. The upper half-plane is acted upon by the group $SL(2, \mathbb{R})$. In the case of $\mathcal{N} = 4$ super-Yang-Mills, the S-duality conjecture states that the theory is invariant with respect to an action of a certain discrete subgroup of $SL(2, \mathbb{R})$ which is commensurate with $SL(2, \mathbb{Z})$ [15, 16, 17, 18, 19] (if G is simply-laced, this subgroup coincides with $SL(2, \mathbb{Z})$). The $\mathcal{N} = 4$ S-duality conjecture follows from the S-duality of type IIB string theory, as well as from the properties of the $(2, 0)$ superconformal field theory in six dimensions [20].

What about other superconformal $\mathcal{N} = 2$ theories? In the case of $\mathcal{N} = 2$ super-QCD with gauge group $SU(N)$ and $2N$ flavors, there is an S-duality conjecture very similar to the $\mathcal{N} = 4$ S-duality conjecture [7, 21]. In the special case $N = 2$, it can be deduced from the S-duality of Type IIB string theory. The duality group is again $SL(2, \mathbb{Z})$ in this case. For $N > 2$ the conjectured duality group is the subgroup $\Gamma_0(2)$ of $SL(2, \mathbb{Z})$ [21]. It is tempting to conjecture that S-duality exists for all superconformal $\mathcal{N} = 2$ gauge theories.

2.2 Partial twist along C

Consider a superconformal $\mathcal{N} = 2$ field theory on a Euclidean four-manifold of the form $M = C \times \Sigma$, where C and Σ are Riemann surfaces. The curvature of M in general breaks all supersymmetry, and if we wish to preserve some of it, we have to twist the gauge theory so that at least one supercharge becomes a space-time scalar. Twisting amounts to embedding the holonomy group into the R-symmetry group.

Consider first the case when Σ is flat. The holonomy group is $U(1)_C$ in this case, so one needs to consider an embedding of $U(1)_C$ into $SU(2)_R \times U(1)_N$. The two most obvious choices are to identify $U(1)_C$ with the maximal torus of $SU(2)_R$ or with $U(1)_N$. In the first case, the adjoint field ϕ remains a 0-form, while in the second case it becomes a 1-form on C (with values in the adjoint representation of G). Note that the first twist is defined for an arbitrary (not necessarily superconformal) $\mathcal{N} = 2$ theory, while the second one makes sense only in the superconformal case. In what follows we will refer to the first and second possibilities as the α -twist and β -twist, respectively. Note also that a twist along C makes sense whether the metric on Σ has Euclidean or Minkowski signature.

Let us begin with the α -twisted theory. In the untwisted theory, the bosonic fields are the gauge field A_μ , the complex adjoint scalar ϕ (which is the lowest component of the superfield Φ) and a pair of complex scalars q and \tilde{q} in representations R and R^\vee of G (these are the lowest components of Q and \tilde{Q} , respectively). The $U(1)_R$ subgroup of $SU(2)_R$ which we will use for twisting can be chosen so that both q and \tilde{q} have $U(1)_R$ -charge -1 (and ϕ has charge 0). Then in the α -model q and \tilde{q} become sections of $K_C^{-1/2}$, where K_C is the canonical line bundle of C (or rather, its pullback to $\Sigma \times C$).

As for fermionic fields, in the untwisted theory we have gauginos λ_1 and λ_2 (in the adjoint of G) and quark fields ψ and χ (in representations R^\vee and R). All these fields are Weyl fermions; the $U(1)_R$ charges of λ_1 and λ_2 are $+1$ and -1 , respectively, while the $U(1)_R$ charges of ψ and χ are zero. There are also right-handed spinors (quarks and gauginos) with opposite R-charges. In Minkowski signature they are related to the left-handed ones by complex conjugation; in Euclidean signature they are independent fields and have to be considered separately. We will distinguish the right-handed “partners” of left-handed fields with a bar.

In the α -model ψ and χ remain left-handed spinors. Using complex struc-

tures on C and Σ , we can think of left-handed spinors as sections of

$$S_- = K_\Sigma^{-1/2} \otimes K_C^{1/2} + K_\Sigma^{1/2} \otimes K_C^{-1/2}.$$

Similarly, the right-handed spinors $\bar{\psi}, \bar{\chi}$ are sections of

$$S_+ = K_\Sigma^{-1/2} \otimes K_C^{-1/2} + K_\Sigma^{1/2} \otimes K_C^{1/2}.$$

On the other hand, the gauginos λ_1 and λ_2 become sections of the vector bundles

$$K_\Sigma^{-1/2} \otimes K_C + K_\Sigma^{1/2} \otimes \mathcal{O}_C \tag{2}$$

and

$$K_\Sigma^{-1/2} \otimes \mathcal{O}_C + K_\Sigma^{1/2} \otimes K_C^{-1}, \tag{3}$$

tensored with the gauge bundle in the adjoint representation. Their right-handed partners become sections of

$$K_\Sigma^{-1/2} \otimes K_C^{-1} + K_\Sigma^{1/2} \otimes \mathcal{O}_C$$

and

$$K_\Sigma^{-1/2} \otimes \mathcal{O}_C + K_\Sigma^{1/2} \otimes K_C,$$

also tensored with the adjoint gauge bundle.

In flat space-time the theory has eight complex supercharges which can be assembled into two left-handed spinors and two right-handed spinors. In the twisted theory, only those which are scalars on C survive. The transformation properties of the supercharges with respect to R-symmetries are identical to those of the gauginos, so from eqs. (2,3) we conclude that the α -model has four complex supercharges, two of which transform as spinors of one chirality on Σ , and the other two transform as spinors of opposite chirality. If Σ has Minkowski signature, then for each chirality the two supercharges are related by complex conjugation. If in the Kaluza-Klein spirit we regard the twisted theory as a field theory living on Σ , then it has (2, 2) supersymmetry.

Now let us perform the same analysis for the β -twisted theory. The scalar ϕ has $U(1)_N$ charge +2, so after twist it becomes a section of K_C . The scalars q and \tilde{q} have zero $U(1)_N$ charge, so they are unaffected by the β -twist. Both ψ and χ have $U(1)_N$ charge -1 , so they become sections of the vector bundle

$$K_\Sigma^{-1/2} \otimes \mathcal{O}_C + K_\Sigma^{1/2} \otimes K_C^{-1} \tag{4}$$

tensored with the gauge bundle in the representation R^\vee or R . Their right-handed partners become sections of

$$K_\Sigma^{-1/2} \otimes \mathcal{O}_C + K_\Sigma^{1/2} \otimes K_C$$

tensored with the gauge bundle in the representation R or R^\vee . Both gauginos have $U(1)_N$ charge $+1$, so they become sections of the vector bundle

$$K_\Sigma^{-1/2} \otimes K_C + K_\Sigma^{1/2} \otimes \mathcal{O}_C$$

tensored with the adjoint gauge bundle. Their right-handed partners are sections of

$$K_\Sigma^{-1/2} \otimes K_C^{-1} + K_\Sigma^{1/2} \otimes \mathcal{O}_C$$

tensored with the adjoint gauge bundle.

The β -model has four complex supercharges which are spinors of the same chirality on Σ . If Σ has Minkowski signature, they are pairwise related by complex conjugation, so there are four independent real supercharges. This means that the β -model, when regarded as a field theory on Σ , has chiral (left-handed) $(4, 0)$ supersymmetry. We will see later that under Kaluza-Klein reduction it becomes a $(4, 0)$ sigma-model on Σ whose target is the moduli space of Hitchin equations on C , and with right-handed fermions taking values in a certain vector bundle over the Hitchin moduli space.

2.3 Partial twist along Σ

Now let us consider the situation where both Σ and C are curved. Then in order to preserve at least one fermionic symmetry one has to perform a further twist of the holonomy group $U(1)_\Sigma$. Another motivation to twist the theory along Σ is to simplify the dependence of the theory on the metric: we will see below that after Σ -twist the theory becomes independent of the Kähler forms of both C and Σ (although it still depends on their complex structures). Thus the theory twisted both along C and Σ is equivalent to its Kaluza-Klein reduction along C or Σ .

For either α or β models, there are many possibilities for twisting along Σ . In order to remain as close as possible to the GL twist, we will twist the α -model by identifying $U(1)_\Sigma$ with $U(1)_N$. We will call this the α' -model. It is also natural to twist the β -model by identifying $U(1)_\Sigma$ with the maximal torus of $SU(2)_R$. We will call this the β' -model. Obviously, α' and

β' -models are related by the exchange of C and Σ , so from now on we will only consider the β' -model. Let us also mention that if we use the maximal torus of $SU(2)_R$ to twist both along C and Σ , then we get nothing but the Witten-Donaldson topological twist of the $\mathcal{N} = 2$ theory, in the special case when the four-manifold on which the theory lives is taken to be $C \times \Sigma$.

Let us now describe the field content of the β' -model. The bosonic fields are the gauge field A_μ , the adjoint Higgs field ϕ which is a section of K_C , and the squark fields q and \tilde{q} which after the Σ -twist become sections of $K_\Sigma^{-1/2}$ (tensored with the gauge bundle in representation R^\vee or R). That is, the squark fields becomes chiral spinors on Σ . The quark fields ψ and χ are unaffected by the Σ -twist and remain sections of the vector bundle

$$K_\Sigma^{-1/2} \otimes \mathcal{O}_C + K_\Sigma^{1/2} \otimes K_C^{-1}.$$

That is, they remain spinors along Σ . Their right-handed partners are sections of

$$K_\Sigma^{-1/2} \otimes \mathcal{O}_C + K_\Sigma^{1/2} \otimes K_C$$

On the other hand, the gauginos λ_1 and λ_2 become sections of the vector bundles

$$\mathcal{O}_\Sigma \otimes K_C + K_\Sigma \otimes \mathcal{O}_C$$

and

$$K_\Sigma^{-1} \otimes K_C + \mathcal{O}_\Sigma \otimes \mathcal{O}_C.$$

That is, they become differential forms along both Σ and C . Their right-handed partners become sections of

$$K_\Sigma^{-1} \otimes K_C^{-1} + \mathcal{O}_\Sigma \otimes \mathcal{O}_C$$

and

$$\mathcal{O}_\Sigma \otimes K_C^{-1} + K_\Sigma \otimes \mathcal{O}_C.$$

On a general Σ there are two fermionic symmetries (one from left-handed supercharge and one from right-handed supercharge in four dimensions). These two supercharges have the same chirality on Σ and opposite chirality on C .

All the models described so far contain spinor fields on C or Σ and therefore depend on the choice of spin structure. We can avoid this by modifying the twist further. Namely, we add to the generator of $U(1)_C$ or $U(1)_\Sigma$ a multiple of a generator of $U(1)_B$. This has the effect of modifying the spins of the fields in a way which depends on their $U(1)_B$ charge. The fields in the

Field	α		β		β'		β''	
	$U(1)_C$	$U(1)_\Sigma$	$U(1)_C$	$U(1)_\Sigma$	$U(1)_C$	$U(1)_\Sigma$	$U(1)_C$	$U(1)_\Sigma$
ϕ	0	0	2	0	2	0	2	0
λ_{1+}	2	-1	2	-1	2	0	2	0
λ_{1-}	0	1	0	1	0	2	0	2
λ_{2+}	0	-1	2	-1	2	-2	2	-2
λ_{2-}	-2	1	0	1	0	0	0	0
$\bar{\lambda}_{+}^1$	-2	-1	-2	-1	-2	-2	-2	-2
$\bar{\lambda}_{-}^1$	0	1	0	1	0	0	0	0
$\bar{\lambda}_{+}^2$	0	-1	-2	-1	-2	0	-2	0
$\bar{\lambda}_{-}^2$	2	1	0	1	0	2	0	2
q	-1	0	0	0	0	-1	0	-2
\tilde{q}	-1	0	0	0	0	-1	0	0
ψ_{+}	1	-1	0	-1	0	-1	0	-2
ψ_{-}	-1	1	-2	1	-2	1	-2	0
χ_{+}	1	-1	0	-1	0	-1	0	0
χ_{-}	-1	1	-2	1	-2	1	-2	2
$\bar{\psi}_{+}$	-1	-1	0	-1	0	-1	0	0
$\bar{\psi}_{-}$	1	1	2	1	2	1	2	2
$\bar{\chi}_{+}$	-1	-1	0	-1	0	-1	0	-2
$\bar{\chi}_{-}$	1	1	2	1	2	1	2	0

Table 1: The field content of various twisted models. The subscripts \pm on spinor fields refer to upper and lower components.

$\mathcal{N} = 2$ vector multiplet are unaffected, but we can use this new freedom to make the spins of all fields in the hypermultiplet to be integral.

In the case of the β' -model, we modify the generator of $U(1)_\Sigma$ so as to make \tilde{q} a 0-form on both C and Σ . We will refer to the resulting theory as the β'' -model. The field content of α, β, β' , and β'' models is listed in Table 2.3. In what follows, we will focus on the β'' -model, as its properties most closely resemble the properties of the GL-twisted $\mathcal{N} = 4$ theory. Note that even if we take the hypermultiplet to be in the adjoint representation of G , the β'' -model does not reduce to the GL-twisted theory. Instead it reduces to one of the more general twisted theories mentioned in section 5.1 of [6].

2.4 BRST transformations

In the β'' -model there are two independent BRST charges which anticommute and square to zero. If we denote them by Q_ℓ and Q_r , then the most general BRST charge is

$$Q = uQ_\ell + vQ_r.$$

Of course, the resulting theory depends only on the ratio $t = v/u$. This is similar to the GL twist of $\mathcal{N} = 4$ super-Yang-Mills theory. But there is also an important difference: in the case considered here, there is a $U(1)$ symmetry with respect to which Q_ℓ and Q_r transform with opposite weights, so variation of the phase of t can be undone by a global symmetry transformation. In other words, the phase of t is irrelevant. The $U(1)$ symmetry in question is $U(1)_N$. Further, the BRST charge depends holomorphically on t , and we will see below that the same is true about the action of the twisted theory. This implies that t -dependence is trivial (can be undone by a symmetry transformation) provided $t \neq 0, \infty$. We will see below that properties of the β'' -model for $t = 0, \infty$ are very different from those at other values of t .

As explained in [6], under S-duality the left-handed and right-handed supersymmetries are multiplied by opposite phases, which is equivalent to saying that under S-duality the parameter t is multiplied by a phase. But since the dependence on the phase of t is essentially trivial, we conclude that S-duality maps the β'' -model to itself (i.e. t is unchanged). This is unlike the GL-twist, where the phase of t affects the theory in a nontrivial way.

It is straightforward to work out BRST transformations of all the fields. Local complex coordinates on Σ and C will be denoted w and z respectively. Let us begin with the $\mathcal{N} = 2$ vector multiplet. The Higgs field is a section of K_C and will be denoted ϕ_z . The components of λ_1 will be denoted λ_z, λ_w , the components of λ_2 will be $\lambda_{\bar{w}z}, \lambda_{z\bar{z}}$, the components of $\bar{\lambda}_1$ will be $\bar{\lambda}_{\bar{w}\bar{z}}, \bar{\lambda}_{z\bar{z}}$, the components of $\bar{\lambda}_2$ will be $\bar{\lambda}_{\bar{z}}, \bar{\lambda}_w$. The BRST variations of bosons are

$$\begin{aligned} \delta A_z &= -i\bar{\xi}\lambda_z, & \delta A_{\bar{z}} &= -i\xi\bar{\lambda}_{\bar{z}} \\ \delta A_w &= i\xi\bar{\lambda}_w + i\bar{\xi}\lambda_w, & \delta A_{\bar{w}} &= 0 \\ \delta\phi_z &= \xi\lambda_z, & \delta\phi_{\bar{z}} &= \bar{\xi}\bar{\lambda}_{\bar{z}} \end{aligned}$$

The BRST variations of fermions are

$$\begin{aligned}
\delta\lambda_z &= 0, \\
\delta\lambda_w &= 4\xi q_w T \tilde{q}^\dagger, \\
\delta\lambda_{\bar{w}z} &= -4\xi F_{\bar{w}z} - 4i\bar{\xi} D_{\bar{w}}\phi_z, \\
\delta\lambda_{z\bar{z}} &= 2(-i\xi[\phi_z, \phi_{\bar{z}}] + \xi F_{z\bar{z}} - 2i\bar{\xi} D_{\bar{z}}\phi_z) - 2\xi g^{w\bar{w}} g_{z\bar{z}} (F_{w\bar{w}} + i\mu_{w\bar{w}}), \\
\delta\bar{\lambda}_{\bar{z}} &= 0, \\
\delta\bar{\lambda}_w &= -4\bar{\xi} q_w T \tilde{q}^\dagger, \\
\delta\bar{\lambda}_{\bar{w}\bar{z}} &= -4\bar{\xi} F_{\bar{w}\bar{z}} - 4i\xi D_{\bar{w}}\phi_{\bar{z}}, \\
\delta\bar{\lambda}_{z\bar{z}} &= 2(i\bar{\xi}[\phi_z, \phi_{\bar{z}}] - \bar{\xi} F_{z\bar{z}} - 2i\xi D_z\phi_{\bar{z}}) - 2\bar{\xi} g^{w\bar{w}} g_{z\bar{z}} (F_{w\bar{w}} + i\mu_{w\bar{w}}).
\end{aligned}$$

Here we denoted

$$\mu_{w\bar{w}} = q_w T q_{\bar{w}} - 2g_{w\bar{w}} \tilde{q} T \tilde{q}^\dagger.$$

and T denotes the intertwiner between $R \otimes R^\vee$ and the adjoint representation.

From these formulas, we see that the fields $\lambda_z, \bar{\lambda}_{\bar{z}}, \lambda_w, \bar{\lambda}_{\bar{w}}$ have ghost number $+1$, while the fields $\lambda_{z\bar{z}}, \bar{\lambda}_{z\bar{z}}, \lambda_{\bar{w}z}, \bar{\lambda}_{\bar{w}\bar{z}}$ have ghost number -1 . (By definition, BRST transformation increases the ghost number by 1).

Now let us turn to the matter fields. The BRST variations of squarks are

$$\begin{aligned}
\delta q_w &= 0, & \delta q_{\bar{w}} &= \xi \psi_{\bar{w}} + \bar{\xi} \bar{\chi}_{\bar{w}}, \\
\delta \tilde{q} &= \frac{i}{2} g^{w\bar{w}} (\xi \chi_{w\bar{w}} - \bar{\xi} \bar{\psi}_{w\bar{w}}), & \delta \tilde{q}^\dagger &= 0.
\end{aligned}$$

The BRST transformations of quark fields are

$$\begin{aligned}
\delta\psi_{\bar{w}} &= -4\bar{\xi} D_{\bar{w}} \tilde{q}^\dagger, & \delta\psi_{\bar{z}} &= -4(\bar{\xi} D_{\bar{z}} - \xi \phi_{\bar{z}}) \tilde{q}^\dagger, \\
\delta\chi_{w\bar{w}} &= -4i\bar{\xi} D_{\bar{w}} q_w, & \delta\chi_{w\bar{z}} &= -4i(\bar{\xi} D_{\bar{z}} q_w + \xi q_w \phi_{\bar{z}}), \\
\delta\bar{\psi}_{w\bar{w}} &= -4i\xi D_{\bar{w}} q_w, & \delta\bar{\psi}_{wz} &= -4i(\xi D_z q_w + \bar{\xi} q_w \phi_z), \\
\delta\bar{\chi}_{\bar{w}} &= 4\xi D_{\bar{w}} \tilde{q}^\dagger, & \delta\bar{\chi}_z &= 4(\xi D_z - \bar{\xi} \phi_z) \tilde{q}^\dagger.
\end{aligned}$$

The ghost number assignments of matter fields are not unique, because of the freedom to add to the ghost number a multiple of $U(1)_B$ charge. One fairly natural choice is to require $q_{\bar{w}}$ and \tilde{q} to have the same ghost number gh . Then the ghost numbers of all matter fields are uniquely determined: q_w and \tilde{q}^\dagger have $gh = 1$, $q_{\bar{w}}$ and \tilde{q} have $gh = -1$, and all fermionic matter fields have $gh = 0$. As we will see below, this definition of the ghost number is natural

from the viewpoint of the effective field theory on Σ . From the viewpoint of the effective field theory on C , it is more natural to let q_w and $q_{\bar{w}}$ to have $gh = 0$. Then \tilde{q} and \tilde{q}^\dagger have ghost numbers -2 and 2 , respectively. As for fermionic matter fields, all 1-form fermions have $gh = 1$, while all 2-form fermions have $gh = -1$.

The above formulas describe how fields transform under both Q_ℓ and Q_r . The transformation law under the BRST charge Q is obtained by taking $\bar{\xi}$ to be proportional to ξ :

$$\bar{\xi} = -it\xi, \quad t \in \mathbb{C} \cup \{\infty\}$$

We will denote the corresponding BRST variation δ_t .

The BRST transformation δ_t is almost off-shell nilpotent: without using equations of motion, the equation $\delta_t^2\Phi = 0$ holds for all fields except $\lambda_{z\bar{z}}$ and $\bar{\lambda}_{z\bar{z}}$, but if we use the equations of motion for these two fields we get $\delta_t^2\Phi = 0$ for all fields. In fact, one can do slightly better by introducing the linear combinations

$$\Upsilon_{z\bar{z}} = \lambda_{z\bar{z}} - it^{-1}\bar{\lambda}_{z\bar{z}}, \quad \Omega_{z\bar{z}} = \lambda_{z\bar{z}} + it^{-1}\bar{\lambda}_{z\bar{z}},$$

because then we have $\delta_t^2\Upsilon_{z\bar{z}} = 0$, and therefore δ_t fails to be off-shell nilpotent only on $\Omega_{z\bar{z}}$.

To check the nilpotency of δ_t , it is very useful to note that the complex connection \mathcal{A} defined by

$$\mathcal{A}_z = A_z + t\phi_z, \quad \mathcal{A}_{\bar{z}} = A_{\bar{z}} - t^{-1}\phi_{\bar{z}}, \quad \mathcal{A}_w = A_w, \quad \mathcal{A}_{\bar{w}} = A_{\bar{w}}$$

is BRST-invariant for all values of t . If we denote by \mathcal{F} the curvature 2-form of this connection, then the BRST-variations of gauginos take the form

$$\delta_t\lambda_{\bar{w}z} = -4\mathcal{F}_{\bar{w}z}, \tag{5}$$

$$\delta_t\bar{\lambda}_{\bar{w}\bar{z}} = 4it\mathcal{F}_{\bar{w}\bar{z}}, \tag{6}$$

$$\delta_t\Upsilon_{z\bar{z}} = 4\mathcal{F}_{z\bar{z}}, \tag{7}$$

$$\delta_t\Omega_{z\bar{z}} = 4(t^{-1}D_z\phi_{\bar{z}} - tD_{\bar{z}}\phi_z) - 4g^{w\bar{w}}g_{z\bar{z}}(F_{w\bar{w}} + i\mu_{w\bar{w}}). \tag{8}$$

The off-shell nilpotency of δ_t when acting on $\lambda_{\bar{w}z}$, $\lambda_{\bar{w}\bar{z}}$, and $\Upsilon_{z\bar{z}}$ is manifest.

In order to write down BRST-invariant actions, it is very convenient to make the BRST transformation nilpotent off-shell by introducing suitable

auxiliary fields. In the present case we need just one such field which we call $H_{z\bar{z}}$. We will change the action of δ_t on $\Omega_{z\bar{z}}$ and postulate

$$\delta_t \Omega_{z\bar{z}} = H_{z\bar{z}}, \quad \delta_t H_{z\bar{z}} = 0.$$

If we want the new BRST transformations to be equivalent on-shell to the old ones, we have to ensure that on-shell $H_{z\bar{z}}$ is equal to the old BRST variation of $\Omega_{z\bar{z}}$, i.e. we want the equation of motion for $H_{z\bar{z}}$ to read

$$H_{z\bar{z}} = 4 \left(t^{-1} D_z \phi_{\bar{z}} - t D_{\bar{z}} \phi_z \right) - 4g^{w\bar{w}} g_{z\bar{z}} (F_{w\bar{w}} + i\mu_{w\bar{w}}).$$

The modified BRST transformations of the fields in the $\mathcal{N} = 2$ vector multiplet enjoy the nice property that they are independent of the Kähler forms $g_{w\bar{w}}$ and $g_{z\bar{z}}$. This is almost true for the BRST transformations of the hypermultiplet fields: the only offender is \tilde{q} . We can easily repair this by working with the following linear combinations of the quark fields $\chi_{w\bar{w}}$ and $\bar{\psi}_{w\bar{w}}$:

$$\varrho = \frac{1}{2} g^{w\bar{w}} (\chi_{w\bar{w}} + it \bar{\psi}_{w\bar{w}}), \quad \eta_{w\bar{w}} = \chi_{w\bar{w}} - it \bar{\psi}_{w\bar{w}}. \quad (9)$$

Then we have

$$\begin{aligned} \delta_t \tilde{q} &= i\varrho, \\ \delta_t \varrho &= 0, \\ \delta_t \eta_{w\bar{w}} &= -8t D_{\bar{w}} q_w. \end{aligned}$$

2.5 Action and properties of the twisted theory

Recall that the action of a topologically twisted theory typically can be written as a “topological” term plus a BRST-exact term. The “topological” term is independent of the metric and is a locally-constant function in the space of fields. This implies that correlators of BRST-closed operators are independent of the metric.

The β'' -model turns out to behave somewhat differently. We will see that for $t \neq 0, \infty$ its action can be written as a BRST-exact piece plus a fermionic piece which is BRST-closed but not BRST exact. The fermionic piece is not “topological” in the sense that it is not a total derivative, but it is still independent of the Kähler forms of C and Σ . This implies that correlators of BRST-closed operators are independent of the Kähler structure, although

they may still depend on the complex structure. We will also see that the β'' -model is independent of the gauge coupling e^2 and the theta-angle.

For $t = 0, \infty$ in addition to the fermionic piece there is also a more traditional topological term proportional to the instanton number, so the properties of the theory are rather different. In particular for $t = 0$ (resp. $t = \infty$) the observables of the theory may have holomorphic (resp. anti-holomorphic) dependence on the complexified gauge coupling τ .

The action of the twisted $\mathcal{N} = 2$ Yang-Mills theory on \mathbb{C}^2 with a flat product metric $ds^2 = g_{z\bar{z}}dz \otimes d\bar{z} + g_{w\bar{w}}dw \otimes d\bar{w}$ is

$$\begin{aligned}
I_{YM} = & \frac{1}{e^2} \int d^2z d^2w \sqrt{g} \left[\text{Tr} \left(\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + 2g^{z\bar{z}} D_\mu \phi_z D^\mu \phi_{\bar{z}} + (g^{z\bar{z}})^2 [\phi_z, \phi_{\bar{z}}]^2 + (g^{w\bar{w}} \mu_{w\bar{w}})^2 \right. \right. \\
& + 8g^{w\bar{w}} (\tilde{q} T q_{\bar{w}}) (q_w T \tilde{q}^\dagger) + 4D_\mu \tilde{q} D^\mu \tilde{q}^\dagger + 2g^{w\bar{w}} D_\mu q_w D^\mu q_{\bar{w}} + 4g^{z\bar{z}} \tilde{q} \{ \phi_z, \phi_{\bar{z}} \} \tilde{q}^\dagger \\
& + 2g^{z\bar{z}} g^{w\bar{w}} q_w \{ \phi_z, \phi_{\bar{z}} \} q_{\bar{w}} - \text{Tr} (i g^{w\bar{w}} g^{z\bar{z}} \bar{\lambda}_{\bar{w}\bar{z}} (D_w \lambda_z + D_z \lambda_w) + i g^{z\bar{z}} \bar{\lambda}_{z\bar{z}} (g^{z\bar{z}} D_{\bar{z}} \lambda_z - g^{w\bar{w}} D_{\bar{w}} \lambda_w) \\
& + i g^{z\bar{z}} \bar{\lambda}_{\bar{z}} (g^{w\bar{w}} D_w \lambda_{\bar{w}z} + g^{z\bar{z}} D_z \lambda_{z\bar{z}}) + i g^{w\bar{w}} g^{z\bar{z}} \bar{\lambda}_w (D_{\bar{z}} \lambda_{\bar{w}z} - D_{\bar{w}} \lambda_{z\bar{z}}) \\
& + i g^{z\bar{z}} \phi_{\bar{z}} (g^{z\bar{z}} \{ \lambda_z, \lambda_{z\bar{z}} \} - g^{w\bar{w}} \{ \lambda_w, \lambda_{\bar{w}z} \}) + i g^{z\bar{z}} \phi_z (g^{z\bar{z}} \{ \bar{\lambda}_{\bar{z}}, \bar{\lambda}_{z\bar{z}} \} - g^{w\bar{w}} \{ \bar{\lambda}_w, \bar{\lambda}_{\bar{w}\bar{z}} \}) \\
& - i g^{w\bar{w}} \bar{\psi}_{w\bar{w}} (g^{w\bar{w}} D_w \psi_{\bar{w}} + g^{z\bar{z}} D_z \psi_{\bar{z}}) - i g^{w\bar{w}} g^{z\bar{z}} \bar{\psi}_{wz} (D_{\bar{z}} \psi_{\bar{w}} - D_{\bar{w}} \psi_{\bar{z}}) \\
& - i g^{w\bar{w}} \chi_{w\bar{w}} (g^{z\bar{z}} D_{\bar{z}} \bar{\chi}_z + g^{w\bar{w}} D_w \bar{\chi}_{\bar{w}}) + i g^{w\bar{w}} g^{z\bar{z}} \chi_{w\bar{z}} (D_{\bar{w}} \bar{\chi}_z - D_z \bar{\chi}_{\bar{w}}) \\
& - i g^{w\bar{w}} g^{z\bar{z}} q_w (\lambda_{z\bar{z}} \psi_{\bar{w}} - \lambda_{\bar{w}z} \psi_{\bar{z}}) - i g^{w\bar{w}} (g^{z\bar{z}} \bar{\psi}_{wz} \bar{\lambda}_{\bar{z}} - g^{w\bar{w}} \bar{\psi}_{w\bar{w}} \bar{\lambda}_w) q_{\bar{w}} \\
& - g^{z\bar{z}} g^{w\bar{w}} (\chi_{w\bar{z}} \lambda_{\bar{w}z} - \chi_{w\bar{w}} \lambda_{z\bar{z}}) \tilde{q}^\dagger + 2\tilde{q} (g^{w\bar{w}} \bar{\lambda}_w \bar{\chi}_{\bar{w}} - g^{z\bar{z}} \bar{\lambda}_{\bar{z}} \bar{\chi}_z) \\
& - i g^{w\bar{w}} (g^{z\bar{z}} \chi_{w\bar{z}} \lambda_z - g^{w\bar{w}} \chi_{w\bar{w}} \lambda_w) q_{\bar{w}} - i g^{z\bar{z}} g^{w\bar{w}} q_w (\bar{\lambda}_{z\bar{z}} \bar{\chi}_{\bar{w}} - \bar{\lambda}_{\bar{w}\bar{z}} \bar{\chi}_z) \\
& - 2\tilde{q} (g^{w\bar{w}} \lambda_w \psi_{\bar{w}} - g^{z\bar{z}} \lambda_z \psi_{\bar{z}}) + g^{z\bar{z}} g^{w\bar{w}} (\bar{\psi}_{wz} \bar{\lambda}_{\bar{w}\bar{z}} - \bar{\psi}_{w\bar{w}} \bar{\lambda}_{z\bar{z}}) \tilde{q}^\dagger \\
& \left. - i g^{z\bar{z}} g^{w\bar{w}} (\chi_{w\bar{z}} \phi_z \psi_{\bar{w}} - \chi_{w\bar{w}} \phi_z \psi_{\bar{z}} + \bar{\psi}_{wz} \phi_z \bar{\chi}_{\bar{w}} - \bar{\psi}_{w\bar{w}} \phi_z \bar{\chi}_z) \right]
\end{aligned}$$

We let

$$\begin{aligned}
\Psi_1 &= -\frac{1}{2}g^{z\bar{z}}g^{w\bar{w}}\text{Tr}\lambda_{\bar{w}z}(F_{w\bar{z}}+t^{-1}D_w\phi_{\bar{z}}), \\
\Psi_2 &= -\frac{it^{-1}}{2}g^{z\bar{z}}g^{w\bar{w}}\text{Tr}\bar{\lambda}_{\bar{w}\bar{z}}(F_{wz}-tD_w\phi_z), \\
\Psi_3 &= -\frac{1}{4}(g^{z\bar{z}})^2\text{Tr}\Upsilon_{z\bar{z}}(F_{z\bar{z}}-i[\phi_z,\phi_{\bar{z}}]+t^{-1}D_z\phi_{\bar{z}}+tD_{\bar{z}}\phi_z), \\
\Psi_4 &= \frac{1}{16}(g^{z\bar{z}})^2\text{Tr}\Omega_{z\bar{z}}(H_{z\bar{z}}-8(t^{-1}D_z\phi_{\bar{z}}-tD_{\bar{z}}\phi_z)+8g^{w\bar{w}}g_{z\bar{z}}(F_{w\bar{w}}+i\mu_{w\bar{w}})). \\
\Psi_5 &= g^{w\bar{w}}D_w\tilde{q}(\bar{\chi}_{\bar{w}}-it^{-1}\psi_{\bar{w}})-it^{-1}g^{z\bar{z}}(D_z\tilde{q}+it\tilde{q}\phi_z)\psi_{\bar{z}}+g^{z\bar{z}}(D_{\bar{z}}\tilde{q}-it^{-1}\tilde{q}\phi_{\bar{z}})\bar{\chi}_z, \\
\Psi_6 &= -\frac{t^{-1}}{2}(g^{w\bar{w}})^2\eta_{w\bar{w}}D_wq_{\bar{w}}-\frac{t^{-1}}{2}g^{w\bar{w}}g^{z\bar{z}}\chi_{w\bar{z}}(D_zq_{\bar{w}}-it\phi_zq_{\bar{w}})+\frac{i}{2}g^{w\bar{w}}g^{z\bar{z}}\bar{\psi}_{wz}(D_{\bar{z}}q_{\bar{w}}+it^{-1}\phi_{\bar{z}}q_{\bar{w}}), \\
\Psi_7 &= g^{w\bar{w}}\tilde{q}(\lambda_w-it^{-1}\bar{\lambda}_w)q_{\bar{w}}.
\end{aligned}$$

Consider the following ansatz for a BRST-invariant action:

$$I_0 = \frac{1}{e^2} \sum_{i=1}^7 \int d^2z d^2w \sqrt{g} \delta_t \Psi_i$$

Its bosonic part is t -independent and in flat space-time is equal to the bosonic part of I_{YM} . But its fermionic part is t -dependent and does not coincide with the fermionic part of I_{YM} . To write down the difference between the fermionic parts of I_{YM} and I_0 in a compact form, we will make use of the remaining letters of the Greek alphabet, except the really strange ones. We define

$$\begin{aligned}
\Upsilon &= \Upsilon_{z\bar{z}}dz \wedge d\bar{z} & (10) \\
\sigma &= \bar{\chi}_z dz - it^{-1}\psi_{\bar{z}}d\bar{z}, \\
\Psi_w &= -it\bar{\psi}_{wz}dz + \chi_{w\bar{z}}d\bar{z}, \\
\zeta_{\bar{w}} &= \lambda_{\bar{w}z}dz + it^{-1}\bar{\lambda}_{\bar{w}\bar{z}}d\bar{z}, \\
\beta_{\bar{w}} &= \bar{\chi}_{\bar{w}} - it^{-1}\psi_{\bar{w}}, \\
\Lambda_w &= i\bar{\lambda}_w - t\lambda_w
\end{aligned}$$

Thus Υ is a 2-form on C , σ, Ψ_w , and $\zeta_{\bar{w}}$ are 1-forms on C , and $\beta_{\bar{w}}, \Lambda_w$ are 0-forms on C . Let \mathcal{D}_C denote the covariant differential along C ; it maps k -forms on C to $k+1$ -forms. Then the difference between the fermionic parts

of I_{YM} and I_0 is

$$I_1 = \frac{1}{2e^2} \int_{C \times \Sigma} dw \wedge d\bar{w} \left(\text{Tr} (\zeta_{\bar{w}} \mathcal{D}_C \Lambda_w - \Upsilon D_{\bar{w}} \Lambda_w) + 2i\Psi_w D_{\bar{w}} \sigma - i\Psi_w \mathcal{D}_C \beta_{\bar{w}} \right. \\ \left. - i\eta_{w\bar{w}} \mathcal{D}_C \sigma - \eta_{w\bar{w}} \Upsilon \tilde{q}^\dagger - tq_w \Upsilon \beta_{\bar{w}} + 2tq_w \zeta_{\bar{w}} \sigma - 2\Psi_w \zeta_{\bar{w}} \tilde{q}^\dagger \right)$$

One can check that I_1 is BRST-invariant, as it should. Thus if we want the twisted theory to be BRST-invariant and agree with the $\mathcal{N} = 4$ SYM in flat space-time, we can take its action to be

$$I = I_0 + I_1 + I_\theta, \tag{11}$$

where

$$I_\theta = -\frac{i\theta}{8\pi^2} \int \text{Tr} F \wedge F.$$

We immediately observe that the action depends on the Kähler forms of C and Σ only through BRST-exact terms. Therefore the correlators in the β'' -model cannot depend on the Kähler forms.

In fact, a stronger result holds: the theory is independent of the complex structure of C , which makes it topological along C and holomorphic along Σ . Indeed, apart from the BRST-exact term I_0 , the action does not make reference to the decomposition of the forms along C into holomorphic and antiholomorphic components. This is the reason we introduced linear combinations (10) in the first place.

Another important consequence of (11) is that after rescaling all the fermions by a factor e the action depends on e only through BRST-exact terms. Thus correlators of BRST-invariant operators are independent of e . This means that weak-coupling computations in the β'' -model give exact results.

It turns out that for $t \neq 0, \infty$ varying the theta-angle also does not affect the correlators. To see this, recall that the path-integral of the twisted theory localizes on BRST-invariant bosonic field configurations. We will refer to them as BPS configurations. They are solutions of the equations $\delta_t \Phi = 0$, where Φ is any fermionic field. From (5) we see that BPS configurations satisfy

$$\mathcal{F}_{\bar{w}z} = \mathcal{F}_{\bar{w}\bar{z}} = \mathcal{F}_{z\bar{z}} = 0,$$

which implies $\mathcal{F} \wedge \mathcal{F} = 0$. The connections A and \mathcal{A} have the same instanton number, hence BPS configurations are automatically topologically trivial,

and the correlators in the β'' -model are independent of θ .¹

The preceding analysis does not apply to two exceptional cases $t = 0$ and $t = \infty$. We will consider explicitly only the case $t = \infty$, as $t = 0$ is very similar. For $t = \infty$ (which corresponds to setting $\xi = 0$) the BRST transformations close off-shell for all fields except $\bar{\lambda}_{z\bar{z}}$. So we introduce an auxiliary field $G_{z\bar{z}}$ and redefine the BRST transformation on $\bar{\lambda}_{z\bar{z}}$ as

$$\delta_\infty \bar{\lambda}_{z\bar{z}} = G_{z\bar{z}}.$$

We also let

$$\delta_\infty G_{z\bar{z}} = 0$$

and construct the action so that the equation of motion for $G_{z\bar{z}}$ be

$$G_{z\bar{z}} = 2(i[\phi_z, \phi_{\bar{z}}] - F_{z\bar{z}}) - 2g_{z\bar{z}}g^{w\bar{w}}(F_{w\bar{w}} + i\mu_{w\bar{w}}).$$

If we also redefine

$$\varrho = -\frac{i}{2}g^{w\bar{w}}\bar{\psi}_{w\bar{w}}$$

we note that the modified BRST transformations are independent of both Kähler forms.

To write down the action at $t = \infty$ we define

$$\begin{aligned} \Xi_1 &= -g^{z\bar{z}}g^{w\bar{w}}\text{Tr}\bar{\lambda}_{\bar{w}\bar{z}}F_{wz} + ig^{z\bar{z}}g^{w\bar{w}}\text{Tr}\lambda_{\bar{w}\bar{z}}D_w\phi_{\bar{z}}, \\ \Xi_2 &= i(g^{z\bar{z}})^2\lambda_{z\bar{z}}D_z\phi_{\bar{z}} + \frac{1}{4}(g^{z\bar{z}})^2\text{Tr}\bar{\lambda}_{z\bar{z}}(G_{z\bar{z}} - 4(i[\phi_z, \phi_{\bar{z}}] - F_{z\bar{z}}) + 4g_{z\bar{z}}g^{w\bar{w}}(F_{w\bar{w}} + i\mu_{w\bar{w}})), \\ \Xi_3 &= -2g^{z\bar{z}}D_z\tilde{q}\psi_{\bar{z}} - 2g^{w\bar{w}}D_w\tilde{q}\psi_{\bar{w}} + ig^{z\bar{z}}g^{w\bar{w}}\chi_{w\bar{z}}D_zq_{\bar{w}} + i(g^{w\bar{w}})^2\chi_{w\bar{w}}D_wq_{\bar{w}} \\ \Xi_4 &= -2g^{z\bar{z}}\tilde{q}\phi_{\bar{z}}\bar{\chi}_z + ig^{z\bar{z}}g^{w\bar{w}}\bar{\psi}_{wz}\phi_{\bar{z}}q_{\bar{w}} - 2g^{w\bar{w}}\tilde{q}\bar{\lambda}_wq_{\bar{w}}. \end{aligned}$$

Then we consider the following BRST-exact action:

$$I_0^\infty = \frac{1}{e^2} \int d^2z d^2w \sqrt{g} \sum_{i=1}^4 \delta_\infty \Xi_i$$

Its bosonic part is equal to the bosonic part of I_{YM} plus a topological term

$$-\frac{1}{e^2} \int \text{Tr} F \wedge F$$

¹If we allow $C \times \Sigma$ to have boundaries, the difference between $\int \mathcal{F}^2$ and $\int F^2$ need not vanish, and the question of θ -dependence has to be re-examined.

The fermionic part of I_0^∞ is different from that of I_{YM} , but if we define

$$\begin{aligned}
I_1^\infty = & \frac{i}{e^2} \int_{C \times \Sigma} dz \wedge d\bar{z} \wedge dw \wedge d\bar{w} \left[\text{Tr} \left(\lambda_{\bar{w}z} D_{\bar{z}} \bar{\lambda}_w - \lambda_{z\bar{z}} D_{\bar{w}} \bar{\lambda}_w - \phi_z \{ \bar{\lambda}_w, \bar{\lambda}_{\bar{w}z} \} \right) \right. \\
& + \bar{\psi}_{wz} (D_{\bar{z}} \psi_{\bar{w}} - D_{\bar{w}} \psi_{\bar{z}}) + (\chi_{w\bar{w}} D_{\bar{z}} - \chi_{w\bar{z}} D_{\bar{w}}) \bar{\chi}_z - q_w \bar{\lambda}_{\bar{w}z} \bar{\chi}_z + i \bar{\psi}_{wz} \bar{\lambda}_{\bar{w}z} \tilde{q}^\dagger \\
& \left. + q_w \lambda_{z\bar{z}} \psi_{\bar{w}} - q_w \lambda_{\bar{w}z} \psi_{\bar{z}} + i \chi_{w\bar{w}} \lambda_{z\bar{z}} \tilde{q}^\dagger - \chi_{w\bar{w}} \phi_z \psi_{\bar{z}} - i \chi_{w\bar{z}} \lambda_{\bar{w}z} \tilde{q}^\dagger + \chi_{w\bar{z}} \phi_z \psi_{\bar{w}} \right]
\end{aligned}$$

then in flat space-time we have an identity

$$I_{YM} + I_\theta = I_0^\infty + I_1^\infty - \frac{i\bar{\tau}}{4\pi} \int \text{Tr} F \wedge F \quad (12)$$

Therefore we define the action of the β'' -model on a general curved manifold of the form $C \times \Sigma$ to be the right-hand side of (12). By construction, I_1^∞ is BRST-invariant; it is also easy to check this explicitly.

Since I_0^∞ is BRST-exact, and I_1^∞ does not depend on the Kähler forms of C and Σ , we see that for $t = \infty$ the correlators in the twisted theory depend only on the complex structures of C and Σ . In fact, as we will see below, correlators of gauge-invariant observables are holomorphic functions of w and z . (More precisely, they are holomorphic sections of various holomorphic line bundles on $C \times \Sigma$). One can summarize the situation by saying that for $t = \infty$ the β'' -model is a holomorphic field theory on $C \times \Sigma$.

Note that after rescaling all the fermions by a factor e , the action depends on e^2 and θ only through a combination

$$-\frac{i\bar{\tau}}{4\pi} \int \text{Tr} F \wedge F.$$

This implies that correlators are independent of e^2 and θ in perturbation theory, while on the nonperturbative level the correlators are antiholomorphic functions of τ .

For $t = 0$, the situation is similar, except that the correlators turn out to be holomorphic functions of w and τ and antiholomorphic functions of z .

3 Kaluza-Klein reduction

Our next goal is to interpret some features of the β'' -model in two-dimensional terms. To this end, we consider the limit in which either C or Σ has vanishingly small size. In this limit, one expects the twisted gauge theory to reduce

to a field theory on Σ or C , respectively. In the case when C shrinks to zero size, we will show that the effective 2d field theory on Σ is a half-twisted model whose target is the Hitchin moduli space $\mathcal{M}_H(G, C)$, and with right-moving fermions taking values in a certain vector bundle over $\mathcal{M}_H(G, C)$ which depends on the matter content. In the case when Σ shrinks to zero size, we will show that the effective 2d field theory on C is a B -model whose target is the moduli space of the nonabelian vortex equations.

3.1 Reduction along C

Consider first the limit in which the volume of C goes to zero. That is, we rescale

$$g^{z\bar{z}} \rightarrow \epsilon^{-1} g^{z\bar{z}}$$

and take the limit $\epsilon \rightarrow 0$. In this limit, the effective field theory on Σ must be a sigma-model whose target is the moduli space of time-independent zero-energy configurations. We can replace the zero-energy condition by the requirement of BRST-invariance. In the limit $\epsilon \rightarrow 0$ such configurations must be solutions of

$$F_{z\bar{z}} - i[\phi_z, \phi_{\bar{z}}] - tD_{\bar{z}}\phi_z = 0, \quad (13)$$

$$F_{z\bar{z}} - i[\phi_z, \phi_{\bar{z}}] - t^{-1}D_z\phi_{\bar{z}} = 0, \quad (14)$$

$$(D_z + it\phi_z)\tilde{q}^\dagger = 0, \quad (15)$$

$$(D_{\bar{z}} - it^{-1}\phi_{\bar{z}})\tilde{q}^\dagger = 0, \quad (16)$$

$$D_z q_w - itq_w\phi_z = 0, \quad (17)$$

$$D_{\bar{z}} q_w + it^{-1}q_w\phi_{\bar{z}} = 0. \quad (18)$$

The first two equations are equivalent to the Hitchin equations:

$$F_{z\bar{z}} - i[\phi_z, \phi_{\bar{z}}] = 0, \quad D_{\bar{z}}\phi_z = 0.$$

The remaining equations require \tilde{q}^\dagger and q_w to be covariantly constant with respect to the connection 1-form

$$\mathcal{A} = (A_z + t\phi_z) dz + (A_{\bar{z}} - t^{-1}\phi_{\bar{z}}) d\bar{z}$$

Hitchin equations imply that this complex connection is flat. Generically, it is also irreducible, and therefore the only solution of the last four equations is $q_w = \tilde{q}^\dagger = 0$. That is, all squark fields vanish. One also has to identify field

configurations which are related by a gauge transformation. The net result is that the target space of the effective field theory on Σ is the moduli space of Hitchin equations $\mathcal{M}_H(G, C)$.

This moduli space is well-known to be hyperkähler, which means that a left-right symmetric sigma-model with this target has $(4, 4)$ supersymmetry. One can get a topological field theory on Σ by twisting this sigma-model. However, the effective field theory on Σ we get from the β'' -model is not a twisted $(4, 4)$ sigma-model. Rather, it is a twisted version of a $(4, 0)$ sigma-model with the same target. To see this, recall that the β'' -model is a twisted version of the β -model. The latter is a supersymmetric sigma-model with two complex supercharges both of which have the same chirality. That is, it is a $(4, 0)$ sigma-model. The BRST operator of the β'' -model is a linear combination of these two complex supercharges.

To completely specify a twisted $(4, 0)$ theory, one has to specify a holomorphic vector bundle on $\mathcal{M}_H(G, C)$ (because the right-moving fermions take values in this bundle) and a complex structure on $\mathcal{M}_H(G, C)$ (this tells us which linear combination of the two supercharges one takes as the BRST charge).

To determine the complex structure, it is sufficient to examine the equations satisfied by the BRST-invariant configurations. These equations say that the following combinations must be (covariantly) holomorphic functions of the variable w :

$$A_z + t\phi_z, \quad A_{\bar{z}} - t^{-1}\phi_{\bar{z}}.$$

Therefore the relevant complex structure on $\mathcal{M}_H(G, C)$ is the one for which these combinations are holomorphic coordinates. We will call it J_t . By varying t , one can get an arbitrary complex structure on $\mathcal{M}_H(G, C)$.

The right-moving fermions on Σ come from the quark fields $\psi_{\bar{z}}, \bar{\chi}_z$ and $\bar{\psi}_{wz}, \chi_{w\bar{z}}$.² The former are 0-forms on Σ , while the latter are 1-forms on Σ . It is sufficient to consider 0-form fermions, since 1-form fermions automatically take values in the dual vector bundle. The equations of motion for 0-form fermions in the limit $\epsilon \rightarrow 0$ read

$$D_{\bar{z}}\bar{\chi}_z - \phi_z\psi_{\bar{z}} = 0, \quad D_z\psi_{\bar{z}} - \phi_{\bar{z}}\bar{\chi}_z = 0.$$

Solutions of these equations define, roughly speaking, a vector bundle \mathcal{R} over $\mathcal{M}_H(G, C)$. (More precisely, it is a twisted vector bundle, see below). To

²Thus it is natural to define the ghost number so that all these fields have $gh = 0$.

understand its geometric meaning and to verify that it is holomorphic in complex structure J_t , recall that we have previously defined a fermionic field σ with values in $R^\vee(E) \otimes \Omega_C^1$:

$$\sigma = \bar{\chi}_z dz - it^{-1} \psi_z d\bar{z}.$$

The equations of motion for $\bar{\chi}_z$ and ψ_z are equivalent to

$$\mathcal{D}\sigma = 0, \quad \mathcal{D} \star_C \sigma = 0. \tag{19}$$

These equations clearly depend holomorphically on the complex flat connection \mathcal{A} and define, roughly speaking, a holomorphic vector bundle on the moduli space of flat G_C connections on C . (More precisely, it is a twisted vector bundle, see below). The latter moduli space is isomorphic as a complex manifold to $\mathcal{M}_H(G, C)$ in complex structure J_t (for any $t \neq 0, \infty$).

Let \mathcal{R} denote the holomorphic vector “bundle” defined by the equations (19). The equations involve the Hodge operator \star_C on 1-forms, so it is not obvious that \mathcal{R} is independent of the complex structure on C . Nevertheless, this must be so, since we have shown that the β'' -model is independent of it. To see how this comes about (at least, outside of a set of a high codimension on $\mathcal{M}_H(G, C)$), let us consider dropping the second equation in (19) and identifying solutions whose difference is \mathcal{D}_C -exact. This gives a holomorphic vector “bundle” on $\mathcal{M}_H(G, C)$ which we call \mathcal{R}' . \mathcal{R}' is manifestly independent of the complex structure on C . If in each equivalence class there is a unique representative satisfying the equation that we dropped, then \mathcal{R}' is isomorphic to \mathcal{R} , and this would prove that \mathcal{R} is also independent of the complex structure.

As usual, it is much easier to demonstrate uniqueness than existence, and we will only argue the former. Suppose σ satisfies both equations in (19) and has the form $\sigma = \mathcal{D}_C \varrho$ for some section ϱ of the bundle $R^\vee(E)$. Then ϱ satisfies the second order equations

$$D_{\bar{z}} D_z \varrho - \phi_z \phi_{\bar{z}} \varrho = 0, \quad D_z D_{\bar{z}} \varrho - \phi_{\bar{z}} \phi_z \varrho = 0.$$

We would like to show that ϱ vanishes identically. To this end we multiply, say, the second equation by ϱ^\dagger and integrate over C . The resulting identity implies $D_{\bar{z}} \varrho = 0$, $\phi_z \varrho = 0$. If the pair (E, φ) is stable, then these equations have only the trivial solution. This proves the desired result (uniqueness) at least outside the strictly semistable locus on $\mathcal{M}_H(G, C)$.

Let us now explain why \mathcal{R}' is a twisted vector bundle. Since for $t \neq 0, \infty$ $\mathcal{M}_H(G, C)$ in complex structure J_t is isomorphic to the moduli space of flat $G_{\mathbb{C}}$ connections on C , the product $\mathcal{M}_H(G, C) \times C$ carries a universal twisted flat principal $G_{\mathbb{C}}$ -bundle, which we denote \mathcal{E} . As explained in [6], \mathcal{E} is twisted by a pull-back of a certain class $\zeta \in H^2(\mathcal{M}_H, Z(G))$. Given an irreducible representation R of G , we can associate to \mathcal{E} a twisted flat vector bundle $R^\vee(\mathcal{E})$ on $\mathcal{M}_H(G, C) \times C$; it is twisted by a class $R^\vee(\zeta) \in H^2(\mathcal{M}_H, U(1))$. If R is reducible, then $R^\vee(\mathcal{E})$ is a sum of twisted flat vector bundles. The vector bundle $\mathcal{R} \simeq \mathcal{R}'$ is the 1-st fiberwise de Rham cohomology of $R^\vee(\mathcal{E})$ with respect to the projection to $\mathcal{M}_H(G, C)$.

At first sight, the appearance of twisted bundles seems worrisome, because the path integral over fermions only makes sense if the fermions take values in an ordinary bundle over the worldsheet Σ . The potential paradox is resolved by recalling that for a simply-connected gauge group such as $SU(N)$ the path-integral on $S^1 \times \tilde{S}^1 \times C$ involves summation over all electric fluxes on \tilde{S}^1 (and C). As explained in [6], section 7, from the point of view of the effective sigma-model on Σ summing over electric fluxes along \tilde{S}^1 is the same as including only those maps from $\Sigma = S^1 \times \tilde{S}^1$ to $\mathcal{M}_H(G, C)$ for which the pull-back of ζ is trivial. The pull-back of \mathcal{R} by such a map is an ordinary bundle on Σ .

Note that the vector bundle \mathcal{R} has a holomorphic structure with respect to complex structure J_t for arbitrary t . A vector bundle on a hyperkähler manifold can be holomorphic in all complex structures simultaneously only if its first Chern class vanishes. Hence we have $c_1(\mathcal{R}) = 0$, and the right-handed fermion number is conserved. Of course, one can also show that $c_1(\mathcal{R}) = 0$ more directly by using the index theorem for families.

3.2 Reduction along Σ

Now consider the limit in which the volume of Σ goes to zero. More precisely, we rescale

$$g^{w\bar{w}} \rightarrow \epsilon^{-1} g^{w\bar{w}}$$

and take the limit $\epsilon \rightarrow 0$. The resulting effective field theory on C turns out to be rather different from that on Σ : it is a topological sigma-model (B-model) whose target is a certain Kähler manifold $\mathcal{M}_V(G, R, \Sigma)$. To see this, let us temporarily undo the twist along C , or equivalently consider the α -twisted theory with Σ and C exchanged. This theory has two complex supercharges

of opposite chirality on C . That is, it has $(2, 2)$ supersymmetry. The BRST charge of the β'' -model is a linear combination of these two supercharges. If $t \neq 0, \infty$, then the β'' -model reduces to a topologically twisted $(2, 2)$ field theory on C ; otherwise the BRST charge is purely left-moving or right-moving on C , and the β'' -model reduces to a half-twisted $(2, 2)$ field theory on C .

To determine the target space of this $(2, 2)$ sigma-model, we consider BRST-invariant configurations in the limit $\epsilon \rightarrow 0$. The resulting equations for bosonic fields read

$$\begin{aligned} F_{w\bar{w}} + iq_w T q_{\bar{w}} &= 0, \\ D_{\bar{w}} q_w &= 0, \\ D_{\bar{w}} \tilde{q}^\dagger &= 0, \\ D_{\bar{w}} \phi_z &= 0. \end{aligned}$$

The last two of these equations imply that generically $\tilde{q}^\dagger = \phi_z = 0$. The first two equations are called the nonabelian vortex equations. They generalize the Hitchin equations to the case when the Higgs field is in representation R of G other than the adjoint. We will denote the moduli space of solutions $\mathcal{M}_V(G, R, \Sigma)$. Since $q_w, q_{\bar{w}}$ are bosonic coordinates on \mathcal{M}_V , it is natural to define the ghost number so that both of these fields have $gh = 0$. As discussed in section 2.4, this requirement fixes uniquely the ghost numbers of all matter fields.

The moduli space \mathcal{M}_V is well-known to be Kähler, with the holomorphic coordinates being $A_{\bar{w}}$ and q_w , and symplectic form

$$\omega_V = \frac{i}{2\pi} \int_{\Sigma} d^2w (\text{Tr } \delta A_{\bar{w}} \wedge \delta A_w + \delta q_w \wedge \delta q_{\bar{w}}).$$

We denote the corresponding complex structure I_V . The Kähler form of the sigma-model is $\text{Im}\tau \omega_V$.

Another way to describe the complex structure on \mathcal{M}_V is to consider the space of pairs (E, \mathfrak{q}) , where E is a holomorphic G -bundle on Σ and $\mathfrak{q} = q_w dw$ is a holomorphic section of the vector bundle $R(E) \otimes K_{\Sigma}$. We will call such a pair a (G, R) Higgs bundle on Σ . Clearly, any solution of the nonabelian vortex equations defines a (G, R) Higgs bundle.

Similar objects have been studied by mathematicians under the name ‘‘holomorphic pairs’’ [23, 24, 25, 26]. A holomorphic pair on Σ is a holomorphic vector bundle E together with a section of $E \otimes V$, where V is a fixed

vector space. A (G, R) Higgs bundle is a modification of this: instead of a vector bundle we have a principal G -bundle, and instead of $E \otimes V$ we have a vector bundle $R(E) \otimes K_\Sigma$.³

In the theory of holomorphic pairs and related objects, an important role is played by the notion of (semi)stability. One can define the moduli space of semistable holomorphic pairs and show that it is isomorphic to the moduli space of “vortex equations” very similar to our vortex equations [23, 24, 26]. Similarly, one can formulate a notion of (semi)stability for (G, R) Higgs bundles. (For a discussion of stability for holomorphic pairs and related objects, see [27].) For simplicity, we will spell it out only for $G = SU(2)$ and when R is the sum of k copies of the fundamental representation. Such a (G, R) Higgs bundle is called stable (resp. semistable) if any holomorphic line subbundle F of E satisfying $\mathfrak{q} \in H^0(F \otimes K_\Sigma) \otimes \mathbb{C}^k$ has negative (resp. nonpositive) first Chern class. One can easily check that for this special choice of G and R any solution of vortex equations defines a semistable (G, R) Higgs bundle. Presumably, the moduli space of solutions of nonabelian vortex equations is isomorphic to the moduli space of semistable (G, R) Higgs bundles.⁴

We know from the previous section that for $t \neq 0, \infty$ the twisted gauge theory depends on the complex structure on Σ but not on the gauge coupling e^2 . Since the complex structure on $\mathcal{M}_V(G, R, \Sigma)$ depends on the complex structure on Σ , while e^2 sets the overall scale of the Kähler form, it is clear that the topological sigma-model to which the gauge theory reduces must be the B-model with target $\mathcal{M}_V(G, R, \Sigma)$. We can check that the theory on C is indeed the B-model by looking at the equations defining BRST-invariant

³The notion of a “holomorphic pair” as defined in [25] is very general and includes holomorphic pairs of [23] and our (G, R) Higgs bundles as special cases. See also [27] for a survey of various kinds of vortex equations and associated holomorphic data.

⁴A special case, corresponding to the twisted $\mathcal{N} = 4$ gauge theory, is when R is the adjoint representation. In this case a (G, R) Higgs bundle is the same as a Higgs bundle, as defined by Hitchin [22], and the vortex equations become the Hitchin equations. The isomorphism of the moduli space of Hitchin equations and the moduli space of semistable Higgs bundles is known to hold for arbitrary G [22].

bosonic configurations. We find:

$$\begin{aligned} F_{\bar{w}z} &= 0, \\ F_{\bar{w}\bar{z}} &= 0, \\ D_z q_w &= 0, \\ D_{\bar{z}} q_w &= 0. \end{aligned}$$

These equations are overdetermined and require the map from C to \mathcal{M}_V to be both holomorphic and anti-holomorphic. This is precisely what one expects for the B-model on C with target $\mathcal{M}_V(G, R, \Sigma)$.

The two-dimensional interpretation of the theta-angle is the same as in [6]. From the viewpoint of the effective field theory on Σ it produces a flat but topologically nontrivial B-field on $\mathcal{M}_H(G, C)$ of the form

$$-\omega_I \text{Re } \tau ,$$

where $I = J_\infty$. From the viewpoint of the effective field theory on C , it produces a flat B-field on $\mathcal{M}_V(G, R, \Sigma)$ in the same cohomology class as

$$-\omega_V \text{Re } \tau .$$

3.3 Some properties of the reduced theories

Now we can explain some of the results of section 2.5 in two-dimensional terms. We showed there that for $t \neq 0, \infty$ the β'' -model does not depend on the parameter τ . From the viewpoint of the half-twisted model on Σ , this happens because for $t \neq 0, \infty$ the form ω_t is exact, and therefore are no nontrivial worldsheet instantons. From the viewpoint of the B-model on C , there is no dependence on $\text{Im } \tau$ because it affects only the Kähler form of \mathcal{M}_V , and there is no $\text{Re } \tau$ dependence because the B-field is of type $(1, 1)$.

We also showed that the twisted gauge theory is topological along C and holomorphic along Σ . This is also obvious from the two-dimensional viewpoint, since the B-model on C is a 2d TFT, while the half-twisted model on Σ is a chiral CFT.

Let us also comment on the two special cases $t = 0$ and $t = \infty$. From the viewpoint of the half-twisted field theory on Σ , these points are special because the forms $\omega_0 = -\omega_I$ and $\omega_\infty = \omega_I$ are not exact, and therefore worldsheet instanton contributions introduce nontrivial dependence on τ .

More precisely, since instanton contributions depend holomorphically on the combination $B + i\omega$, the correlators at $t = \infty$ (resp. $t = 0$) are antiholomorphic (resp. holomorphic) functions of τ . From the viewpoint of the effective field theory on C , for $t = 0, \infty$ the model degenerates into a half-twisted $(2, 2)$ model with target $\mathcal{M}_V(G, R, \Sigma)$ and complex structure $\pm I_V$. One can see it, for example, from the fact that the BPS equations become elliptic rather than over-determined and require the map to \mathcal{M}_V to be holomorphic. In such a model correlators depend holomorphically on $B + i\omega$, which again translates into holomorphic or anti-holomorphic dependence on τ depending on whether $t = 0$ or $t = \infty$.

3.4 Anomaly cancelation

To define the β'' -model, it was essential that the $U(1)_N$ symmetry is anomaly-free, that is, that the condition (1) is satisfied. We can also explain the significance of this condition from the two-dimensional viewpoint.

From the viewpoint of the B-model on C , this is simply the Calabi-Yau condition on the target $\mathcal{M}_V(G, R, C)$. Indeed, the $U(1)_N$ symmetry is nothing but the R-symmetry of the model, which is nonanomalous precisely when the target manifold is a Calabi-Yau.

If we pretend that $\mathcal{M}_V(G, R, \Sigma)$ is smooth, we can see it more explicitly as follows. After identifying \mathcal{M}_V with the moduli space of stable R -Higgs bundles, we can describe a holomorphic tangent vector at a point $(E, 0) \in \mathcal{M}_V$ ⁵ as the equivalence class of a pair $(\delta A_{\bar{w}}, \delta q_w)$ where $\delta A_{\bar{w}}$ is an arbitrary variation of $A_{\bar{w}}$ and δq_w satisfies

$$D_{\bar{w}}\delta q_w = 0.$$

The equivalence relation is

$$(\delta A_{\bar{w}}, \delta q_w) \sim (\delta A_{\bar{w}} + D_{\bar{w}}a, \delta q_w),$$

where a is a section of $\text{ad}(E)$. Thus the tangent space at a point $(E, 0)$ can be identified with

$$H^1(\text{ad}(E)) \oplus H^0(R(E)).$$

⁵The moduli space \mathcal{M}_V is homotopic to its submanifold $q_w = 0$, therefore it is sufficient to compute the restriction of the first Chern class to this submanifold.

If \mathcal{M}_V were smooth, the first Chern class of this bundle could be equated with the first Chern class of the virtual tangent bundle. The latter can be computed via the family index theorem, giving

$$\begin{aligned} c_1(T\mathcal{M}_V^{virt}) &= - \int_{\Sigma} (\text{ch}_4(\text{ad}(\mathcal{E}_R)) - \text{ch}_4(R(\mathcal{E}_R))) \\ &= - \left(1 - \frac{C(R)}{C(G)}\right) \int_{\Sigma} \text{ch}_4(\text{ad}(\mathcal{E}_R)). \end{aligned} \quad (20)$$

where \mathcal{E}_R is the universal G -bundle on $\mathcal{M}_V(G, R, \Sigma) \times \Sigma$ and ch_4 is the degree-4 part of the Chern character. The anomaly cancelation condition (1) ensures that this vanishes. In fact, $\mathcal{M}_V(G, R, \Sigma)$ is not smooth, and it is reasonable to *define* its first Chern class as the $U(1)_N$ anomaly. The latter is still given by (20).

From the viewpoint of the half-twisted model on Σ , the interpretation of the condition (1) is rather different. In general, for a $(4, 0)$ sigma-model with target X and fermionic bundle \mathcal{R} to be well-defined (even before twisting), the degree-4 parts of the Chern characters of \mathcal{R} and X must be equal:

$$\text{ch}_4(X) = \text{ch}_4(\mathcal{R}). \quad (21)$$

This condition is necessary for the Pfaffian of the right-handed Dirac operator to be well-defined. We claim that in our case this condition is equivalent to (1). Indeed, the bundle \mathcal{R} in our case is equal to the fiberwise cohomology of the (twisted) bundle $R(\mathcal{E})$ with respect to the projection $\mathcal{M}_H(G, C) \times C \rightarrow \mathcal{M}_H(G, C)$. The families index theorem gives the following virtual Chern character for \mathcal{R} in degree four:

$$\text{ch}_4(\mathcal{R}) = (g - 1)\text{ch}_4(R(\mathcal{E}))_{\mathcal{M}_H},$$

where subscript \mathcal{M}_H indicates that we must take the piece of the cohomology class which is pulled back from \mathcal{M}_H . The tangent bundle to \mathcal{M}_H is similarly the fiberwise cohomology of $\text{ad}(\mathcal{E})$, so we get

$$\text{ch}_4(\mathcal{M}_H) = (g - 1)\text{ch}_4(\text{ad}(\mathcal{E}))_{\mathcal{M}_H}.$$

The equality (21) holds precisely if $C(G) = C(R)$.

3.5 Chiral algebras from gauge theories in four dimensions

We have seen that to any $\mathcal{N} = 2$ superconformal gauge theory one can attach a pair of chiral 2d CFTs, one being a half-twisted $(4, 0)$ sigma-model with target \mathcal{M}_H and the other being a half-twisted $(2, 2)$ sigma-model with target \mathcal{M}_V . These chiral algebras encode certain holomorphic sectors of the four-dimensional gauge theory and generalize the familiar chiral ring.

One can generalize this construction in two directions. First, one can start with an $\mathcal{N} = 1$ supersymmetric gauge theory with a nonanomalous $U(1)_R$ symmetry and twist it suitably along C and Σ . This gives a chiral CFT on C or Σ which is a half-twisted $(2, 0)$ sigma-model with target \mathcal{M}_V . Such twisted $\mathcal{N} = 1$ gauge theories have been previously considered by A. Johansen [12].

As an example, let us consider $\mathcal{N} = 1$ super-QCD, i.e. an $\mathcal{N} = 1$ gauge theory with $G = SU(N)$ and $2k$ chiral superfields $Q_i, \tilde{Q}_i, i = 1, \dots, k$, where the superfield Q_i takes values in the fundamental representation of $SU(N)$ and \tilde{Q}_i takes values in its dual. This theory is not superconformal: for $k < 3N$ it is asymptotically free and flows to strong coupling in the infrared, while for $k \geq 3N$ it is free in the infrared and has a Landau pole.

A nonanomalous R-symmetry exists such that the lowest components of both Q_i and \tilde{Q}_i have R-charge $1 - N/k$ [28]. The R-current is unique up to addition of a multiple of the $U(1)_B$ current (Q_i and \tilde{Q}_i have $U(1)_B$ charge 1 and -1 respectively). We consider the theory on $C \times \Sigma$ and shift $U(1)_C$ charge by half the R-charge, so that the left-handed gaugino becomes a section of

$$K_\Sigma^{-1/2} \otimes K_C + K_\Sigma^{1/2} \otimes \mathcal{O}_C,$$

while the right-handed gaugino becomes a section of

$$K_\Sigma^{-1/2} \otimes K_C^{-1} + K_\Sigma^{1/2} \otimes \mathcal{O}_C.$$

We see that the theory twisted along C has two supercharges of the same chirality which are Hermitian-conjugate of each other. That is, the effective field theory on Σ has $(2, 0)$ supersymmetry. We are still free to add a multiple of the $U(1)_B$ current to the R-current; one may choose it so that the lowest component of Q_i becomes a section of K_C (for all i); the lowest component of \tilde{Q}_i will then be a section of $K_C^{1-2N/k}$. Note that \tilde{Q}_i field has a fractional $U(1)_C$ charge, in general; in order to have only integer spins in the twisted

theory one is forced to restrict the values of N and k suitably. The most natural choice is $k = 2N$; then the reduced theory on Σ is similar to what we had in the $\mathcal{N} = 2$ case. Namely, it is a $(2, 0)$ sigma-model whose target is $\mathcal{M}_V(SU(N), R, C)$, where R is the sum of $2N$ copies of the fundamental representation of $SU(N)$.

This theory still depends on the Kähler form of C , so the reduced theory becomes equivalent to the gauge theory on $C \times \Sigma$ only when C shrinks to zero size. We can try to rectify this by twisting it along Σ with a suitable R-current (which is again a linear combination of the canonical R-current described above and the $U(1)_B$ current). In this way one obtains a half-twisted $(2, 0)$ sigma-model with target \mathcal{M}_V . Correlators in this theory depend holomorphically on coordinates on Σ .

Alternatively, one can start with a finite $\mathcal{N} = 2$ gauge theory in four dimensions but allow C to have a boundary, with suitable BRST-invariant boundary conditions, thus halving the number of supercharges. From the viewpoint of the effective theory on C , this corresponds to a B-brane on \mathcal{M}_V . From the viewpoint of the field theory on Σ , we get a half-twisted $(2, 0)$ sigma-model on Σ which depends on the choice of the brane. In particular, to any B-brane on $\mathcal{M}_H(G, \Sigma)$ (in complex structure I in the notation of [6]) one can associate a chiral algebra on Σ .

4 Observables of the twisted theory

4.1 Local observables and their descendants

First, let us assume that $t \neq 0, \infty$. By inspection, the only observables in the gauge theory which are BRST invariant, not BRST-exact, gauge-invariant, and local are gauge-invariant polynomials made of q_w and \tilde{q}^\dagger . Note that covariant derivatives of these operators along \bar{w} are BRST-exact; this implies that all correlators and OPE coefficients of gauge-invariant polynomials are holomorphic functions of w . Similarly, the covariant exterior differential \mathcal{D}_C applied to these operators gives BRST-exact operators; this implies that correlators and OPE coefficients are independent of z . This confirms our claim that the theory is topological along C and holomorphic along Σ .

The OPE of these local observables is nonsingular, because of Hartogs' theorem: a holomorphic function, such as an OPE coefficient function, cannot have a singularity in complex codimension 2. An alternative way to show this

is to note that because of topological invariance along C , when computing the singularities of the OPE coefficient functions we can keep the insertion points on C far from each other. But then the coefficient functions cannot become singular even if the insertion points on Σ collide. Since the OPE coefficients are nonsingular and holomorphic, they must be constant. Therefore local observables form a commutative algebra. This algebra is a subalgebra of the anti-chiral ring of the untwisted theory.

For example, let $G = SU(N)$ and let R be the sum of $2N$ copies of the fundamental representation. The theory has $SU(2N)$ global flavor symmetry which acts only on the matter fields; accordingly, we will denote the fields as \tilde{q}_i^\dagger and q_w^i , where $i = 1, \dots, 2N$ is the $SU(2N)$ flavor index. Then the algebra of gauge-invariant polynomials is generated by the “mesons”

$$M_j^i = q_w^i \tilde{q}_j^\dagger,$$

the “baryons”

$$\tilde{B}_{i_1 \dots i_N} = \tilde{q}_{i_1}^\dagger \dots \tilde{q}_{i_N}^\dagger$$

and the “anti-baryons”

$$B^{i_1 \dots i_N} = q_w^{i_1} \dots q_w^{i_N}.$$

Here we did not show explicitly how the $SU(N)$ color indices are contracted, since there is a unique way to do so.

We can construct nonlocal BRST-invariant and gauge-invariant observables following the descent procedure introduced by E. Witten in the context of topological field theory [1]. In a TFT, this procedure allows one to construct “pre-observables”: differential forms whose BRST-variation is exact. One can obtain BRST-invariant observables by integrating pre-observables over homology cycles.

The β'' -model is topological along C and holomorphic along Σ . The descent procedure along C works in the usual manner, giving observables which are integrals over 1-cycles and the fundamental class of C . On the other hand, descent along Σ gives pre-observables which are $(0, 1)$ forms on Σ whose BRST-variations are $\bar{\partial}_\Sigma$ -exact. We can get true observables by multiplying these by holomorphic 1-forms on Σ and integrating over Σ .

To be concrete, let us again consider the case $G = SU(N)$ and let R be the sum of $2N$ copies of the fundamental representation. The first descendant of the anti-baryon $B = q_w^N$ along C is defined by

$$\delta_t B^{(C,1)} = d_C B.$$

This equation is solved by

$$B^{(C,1)} = -\frac{Nt^{-1}}{4}\Psi_w q_w^{N-1},$$

where the $SU(N)$ indices are completely anti-symmetrized. This pre-observable is a section of $K_\Sigma^N \times \Omega_C^1$ and can be integrated over a 1-cycle $\gamma \in H_1(C)$ to give a true observable

$$B^{(C,1)}(\gamma, p) = \int_{\gamma \times p} B^{(C,1)}, \quad p \in \Sigma$$

Such observables are local on Σ and may have nontrivial OPEs with each other. OPE is necessarily holomorphic in the coordinate w . Another constraint on the OPE comes from the fact that the theory is topological along C : if the homology cycles γ and γ' have vanishing intersection number, one can choose their representatives to be nonintersecting on C , and then there can be no singularity in the OPE on Σ . Similarly, diffeomorphism invariance along C guarantees that the OPE of $B^{(C,1)}$ with any local observable on $C \times \Sigma$ is nonsingular.

Now let us consider descent along Σ . Taking the same observable B , its first (and last) Σ -descendant is defined by

$$\delta_t B_{\bar{w}}^{(\Sigma,1)} = \partial_{\bar{w}} B.$$

We find

$$B_{\bar{w}}^{(\Sigma,1)} = -\frac{Nt^{-1}}{8}\eta_{w\bar{w}} q_w^{N-1},$$

where $\eta_{w\bar{w}}$ was defined in (9). This pre-observable is a $(0, 1)$ form on Σ with values in K_Σ^N . To construct from it a true observable, we must integrate over Σ a product of $B^{(\Sigma,1)}$ and a holomorphic section of K_Σ^{1-N} . In other words, one can view $B^{(\Sigma,1)}$ as an observable taking values in the vector space $H^1(K_\Sigma^N)$ which by Serre duality is dual to $H^0(K_\Sigma^{1-N})$. These observables are local on C ; because of diffeomorphism invariance along C their correlators do not depend on the z -coordinate (i.e. the OPE is nonsingular).

Finally let us discuss the case $t = \infty$ (the case $t = 0$ is very similar). The adjoint Higgs field ϕ_z is now BRST-closed, while λ_z is no longer BRST-exact. We also note that for $t = \infty$ the covariant derivatives $D_{\bar{w}}\phi_z, D_{\bar{z}}\phi_z$, and $D_{\bar{w}}\lambda_z, D_{\bar{z}}\lambda_z$ are BRST-exact. Thus we can build new local observables as gauge-invariant polynomials built out of ϕ_z and λ_z , and their correlators

will be holomorphic functions of both z and w . Note also that ϕ_z and λ_z commute up to BRST-exact terms:

$$[\phi_z, \lambda_z] \sim \delta_\infty D_z \phi_z.$$

Thus we get BRST-invariant and gauge-invariant local observables

$$\text{Tr } \phi_z^m \lambda_z^l.$$

Because of Hartogs' theorem, their OPE coefficients are nonsingular, so these operators generate a supercommutative algebra.

Of course, we still have the observables such as B, \tilde{B} , and M , which are part of the anti-chiral ring.⁶ It may seem surprising that OPE of operators such as $\text{Tr } \phi_z^k$ on one hand and M, \tilde{B}, B on the other hand are nonsingular, since the former belong to the chiral ring of the untwisted theory while the latter belong to the anti-chiral ring. But this becomes quite obvious once we apply an $SU(2)_R$ transformation which in the untwisted theory maps q_w to \tilde{q} and \tilde{q}^\dagger to $-q_{\bar{w}}$ and leaves ϕ_z invariant. After this transformation, all the local operators found above become part of the usual chiral ring, and the regularity of the OPE coefficients follows in the usual manner. This is true even when we take into account the local observables containing λ_z : after the above-mentioned $SU(2)_R$ rotation, λ_z becomes the lowest component of the chiral superfield $W_\alpha = -\frac{1}{8} \bar{D}^2 e^{-2V} D_\alpha e^{2V}$.

For $t = \infty$, the theory is a chiral CFT both along C and Σ . Thus the descent procedure applied to the above operators gives us nonlocal observables valued in $H^1(C \times \Sigma, \mathcal{L})$ and $H^2(C \times \Sigma, \mathcal{L})$ for various holomorphic line bundles \mathcal{L} on $C \times \Sigma$. These observables can have nontrivial holomorphic OPE with local observables and between themselves. By way of an example, consider the observable $\text{Tr } \lambda_z \phi_z$. Its descendant along C is

$$\text{Tr} \left(F_{z\bar{z}} \phi_z + \frac{1}{4} \lambda_z \lambda_{z\bar{z}} \right),$$

while its descendant along Σ is

$$\text{Tr} \left(-F_{\bar{w}z} \phi_z + \frac{1}{4} \lambda_z \lambda_{\bar{w}z} \right).$$

We can combine them into a $(0, 1)$ form on $C \times \Sigma$ with values in K_C^2 ; this defines an observable valued in $H^1(K_C^2)$.

⁶Note that certain other candidate observables which mix matter fields and fields in the gauge multiplet, such as $q_w \phi_z^m \lambda_z^l \tilde{q}^\dagger$, are BRST-exact.

4.2 Wilson-'t Hooft loop observables

For $t \neq 0, \infty$ the twisted theory has another interesting class of nonlocal gauge-invariant observables. The simplest of these are gauge-invariant functions of the holonomy of the BRST-invariant connection

$$\mathcal{A}_C = \mathcal{A}_z dz + \mathcal{A}_{\bar{z}} d\bar{z}.$$

Explicitly, for any closed curve γ on C , any point p on Σ , and any irreducible representation R of the gauge group G we can define an observable

$$W(R, \gamma, p) = \text{Tr}_R P \exp \left(i \int_{\gamma \times p} \mathcal{A} \right)$$

We will call it the Wilson loop observable. It depends only on the homotopy class of γ (unlike superficially similar observables constructed in the previous section, which are labelled by homology classes of C). Unlike the case of the GL-twisted theory, such observables exist for all $t \neq 0, \infty$. Since an irreducible representation is completely determined by its highest weight, sometimes we will label Wilson loops by elements of the weight lattice $\Lambda_w(G)$ modulo the action of the Weyl group.

The Wilson loop observables are local observables in the effective field theory on Σ . From the viewpoint of the twisted $(4, 0)$ model with target \mathcal{M}_H , they generate the algebra of holomorphic functions on \mathcal{M}_H . From the 2d viewpoint, it is clear that the OPE of Wilson loops is nonsingular, irrespective of the choice of γ ; one can also show this directly in the gauge theory, see below.

Analogy with $\mathcal{N} = 4$ gauge theory suggests that there should also be BRST-invariant 't Hooft loop operators, which are magnetic analogs of Wilson loops. In fact, S-duality requires the existence of such operators. As in [11, 6], we can construct them by requiring the fields in the path integral to have a prescribed singularity along $\gamma \times p$. The singularity should be compatible with BRST transformations, in the sense that there should exist bosonic configurations which are BRST-invariant and have the required singular behavior near $\gamma \times p$.

To write down the singularity in the fields corresponding to an 't Hooft operator, we may replace an arbitrary curve $\gamma \subset C$ by a straight line on \mathbb{C} oriented in the direction $x^1 = \text{Re } z$. The gauge field will be required to have a monopole singularity at $x^2 = x^3 = x^4 = 0$:

$$F \sim \frac{\mu}{2} \star_3 d \frac{1}{r}.$$

Here $r^2 = (x^2)^2 + (x^3)^2 + (x^4)^2$, \star_3 is the Hodge star operator in the 234 plane, and $\mu \in \mathfrak{g}$ is the so-called GNO charge of the monopole. It is the value at 1 of a homomorphism from \mathbb{R} to \mathfrak{g} obtained from a homomorphism of Lie groups $U(1) \rightarrow G$. The GNO charge is defined modulo the adjoint action of G , so one can assume that it belongs to a chosen Cartan subalgebra of \mathfrak{g} . Then μ becomes a coweight of G defined modulo the action of the Weyl group \mathcal{W} . We will denote by $T_\mu(\gamma, p)$ the 't Hooft operator associated with a coweight μ .

The singularity in the Higgs field ϕ is uniquely fixed by the requirement of BRST-invariance:

$$\phi_z \sim -\frac{t^{-1}\mu}{4r}, \quad \phi_{\bar{z}} \sim -\frac{t\mu}{4r}.$$

(Note that this is compatible with the reality condition $\phi_z^\dagger = \phi_{\bar{z}}$ only if $|t| = 1$. This does not mean that 't Hooft operators make sense only if $|t| = 1$: when evaluating the path-integral using saddle-point approximation it is quite common to expand about complex solutions to saddle-point equations.) Thus for any $t \neq 0, \infty$ we have 't Hooft operators labelled by equivalence classes of coweights of G , or, equivalently, by the equivalence classes of weights of the Langlands dual group ${}^L G$. This suggests that there could be a dual description of the theory where the gauge group is ${}^L G$. This is believed to be true in the special case when the hypermultiplet is in the adjoint representation of G ; the dual theory is conjectured to be identical to the original one except G is replaced with ${}^L G$. For other theories, such as $G = SU(2)$ and hypermultiplets in the defining representation, the naive duality conjecture appears to be false,⁷ but a somewhat different version of S-duality appears to hold, see section 6 below.

We see that in the β'' -model one can construct Wilson and 't Hooft operators which commute with the same BRST operator. This is contrast to the GL-twisted theory, where for a fixed BRST operator we have either Wilson observables or 't Hooft observables, or neither, but never both. Moreover, in the β'' -model by fusing Wilson and 't Hooft operators we can get more general Wilson-'t Hooft operators labelled [11] by equivalence classes of pairs

$$(\mu, \nu), \quad \mu \in \Lambda_{cw}(G), \quad \nu \in \Lambda_w(G)$$

⁷The putative dual description of the theory with $G = SU(2)$ and four hypermultiplets in the defining representation, would have to have dynamical magnetic sources whose magnetic charge is half the charge of the 't Hooft-Polyakov monopole; this seems impossible in the context of semiclassical gauge theory.

under the action of the Weyl group as well as by elements of $\pi_1(C)$.

All Wilson-'t Hooft operators are local on Σ and in general their correlators are holomorphic functions of the coordinates on Σ . In particular, in general there is a nontrivial OPE between different loop operators, as we will see below. But if we choose a particular element $y \in \pi_1(C)$ and only consider loop operators corresponding to this homotopy class, then the OPE is nonsingular. Indeed, because of diffeomorphism invariance along C , we can choose two loops γ and γ' representing the class y which do not intersect on C . Given two loop operators A and B corresponding to the class y , we may choose A and B to be supported on $\gamma \times p$ and $\gamma' \times p'$, where $p, p' \in \Sigma$. Then the operator product will remain nonsingular as p merges with p' , because the supports of A and B on $C \times \Sigma$ remain disjoint in this limit.

We conclude that loop operators corresponding to a fixed $y \in \pi_1(C)$ form a commutative algebra, and that there is a natural basis in this algebra labelled by $(\Lambda_{cw} \times \Lambda_w) / \mathcal{W}$. It is an interesting problem to determine the structure constants of the algebra in this basis, i.e. determine the ‘‘fusion rules’’ of Wilson-'t Hooft operators. We are planning to discuss this issue elsewhere; here we simply note that the answer will depend not only on the group G but also on the representation R of the hypermultiplet. This is clear from the fact that the algebra structure should be compatible with the action of S-duality, which works differently for different R . In the case when R is the adjoint representation, the algebra seems to be abstractly isomorphic to the \mathcal{W} -invariant part of the group algebra of $\Lambda_{cw} \times \Lambda_w$. The latter has an obvious basis labelled by the correct set: one simply takes an element $(\mu, \nu) \in \Lambda_{cw} \times \Lambda_w$ and averages over the Weyl group. Nevertheless, it is not the basis we want. For example, if we take the naive basis elements of the form $(0, \nu_1)$ and $(0, \nu_2)$, their product will have the form

$$\sum_{x \in \mathcal{W}} (0, \nu_1 + x \cdot \nu_2),$$

where $x \cdot \nu$ denotes the result of applying an element $x \in \mathcal{W}$ to $\nu \in \Lambda_w$. On the other hand, the expected answer (the operator product of Wilson loops labelled by ν_1 and ν_2) is given by the decomposition of the tensor product of representations of G with highest weights ν_1 and ν_2 into irreducibles.

Let us now briefly discuss the OPE of Wilson loop operators which wrap different cycles on C . We claim that for such operators there is no singularity in the OPE. Indeed, if we compute the OPE in perturbation theory using Wick theorem, the singularity in the OPE may come only from the

propagators of A and ϕ_z . Since the propagators are proportional to e^2 , and we know that varying e^2 changes the action only by BRST-exact terms, we conclude that the OPE must be nonsingular.

We conclude this section by displaying an example where the OPE of loop operators is nontrivial. Consider the case $G = U(1)$. The abelian $\mathcal{N} = 4$ gauge theory is free, and the OPE of arbitrary loop observables can be readily computed. As argued above, the OPE of arbitrary Wilson operators is trivial, and by S-duality the same should be true for 't Hooft operators. But as we will now explain, OPE of a Wilson operator and an 't Hooft operator is nontrivial, in general. It is particularly simple to see this from the viewpoint of the effective field theory on Σ . Then we are dealing with a twisted $(4, 0)$ sigma-model whose target is the moduli space of flat \mathbb{C}^* connections on C . As a complex manifold, it is isomorphic to $(\mathbb{C}^*)^{2g} \simeq T^*(S^1)^{2g}$, and therefore the sigma-model has the usual momentum and winding states. The winding charge takes values in the lattice $H_1(C, \mathbb{Z})$; the momentum also takes values in a lattice which is most naturally identified with $H^1(C, \mathbb{Z})$, but for our purposes it is more convenient to make use of the Poincaré isomorphism $H^1(C, \mathbb{Z}) \simeq H_1(C, \mathbb{Z})$ and regard momentum as taking values in $H_1(C, \mathbb{Z})$. There is a natural pairing between winding and momentum lattices which with our conventions can be identified with the intersection pairing; we will denote it $\langle m, n \rangle$, where $m, n \in H_1(C, \mathbb{Z})$ are winding and momentum respectively. As noted in [29], the Wilson loop $W_n(\gamma)$ upon Kaluza-Klein reduction corresponds to a conformal primary with momentum $n[\gamma]$, where $[\gamma]$ is the homology class of γ . Similarly, the 't Hooft operator $T_m(\gamma)$ carries winding $m[\gamma]$. BRST-invariance requires local operators on Σ corresponding to Wilson and 't Hooft loop operators also to carry imaginary momentum in the “cotangent” directions on $T^*(S^1)^{2g}$. To infer the singularity in the OPE, we recall that in the presence of a winding-state operator of charge m an operator with momentum n has a monodromy $2\pi\langle m, n \rangle$. By holomorphy on Σ , this implies that the OPE has a singularity of the form $w^{\langle m, n \rangle}$, where w is the insertion point of the Wilson operator relative to the 't Hooft operator.

5 The action of loop operators on branes

In [6], loop operators in topologically twisted gauge theory have been connected to geometric Langlands program via their action on branes. The fact

that loop operators act by functors on the category of topological D-branes is also very useful for computing the Operator Product Expansion of loop operators. Our goal in this section is to generalize some of these considerations to the β'' -model.

Depending on whether C or Σ is taken to be small, the twisted gauge theory reduces to either a half-twisted $(4, 0)$ sigma-model on Σ , or a B-twisted $(2, 2)$ sigma-model on C . The half-twisted sigma-model on Σ is chiral, and therefore Σ cannot have a nonempty boundary. But it is perfectly possible to allow C to have a boundary. A choice of BRST-invariant boundary conditions on C defines a topological D-brane of type B on the Kähler manifold \mathcal{M}_V .

All this matches very well with the spectrum of BRST-invariant loop operators. We have seen that there exist Wilson-'t Hooft loop operators which are extended on C and point-like on Σ . If one lets such a loop operator run along the boundary of C , one gets a functor from the category of B-branes on \mathcal{M}_V to itself. All such functors necessarily commute. From the point of view of the half-twisted sigma-model on Σ , loop operators are simply local BRST-invariant operators.

The action of Wilson operators on the category of branes is easy to determine. The same arguments as in [6] tell us that a Wilson operator in representation \mathcal{S} of the group G tensors the brane with the holomorphic vector bundle $\mathcal{S}(\mathcal{E}_p)$ where \mathcal{E} is the universal G -bundle on $\mathcal{M}_V \times \Sigma$, $p \in \Sigma$ is the insertion point of the Wilson operator, and \mathcal{E}_p is the restriction of \mathcal{E} to $\mathcal{M}_V \times \{p\}$.

It is more difficult to understand the action of 't Hooft operators. The first step is to consider the BPS equations in the case when $C = \mathbb{R} \times I$, where I is an interval, and the fields are assumed to be time-independent. For definiteness, let $t = i$ (as discussed above, nothing depends on the choice of t , provided $t \neq 0, \infty$). Let x^1 parameterize the ‘‘time’’ direction, and x^2 be the coordinate on I . We will assume that the 't Hooft operator is inserted at $x^2 = 0$. In the gauge $A_2 = 0$ the static BPS equations have the form

$$\begin{aligned}\partial_2 A_{\bar{w}} &= -2D_{\bar{w}}\phi_z, \\ \partial_2 q_w &= -2iq_w\phi_z.\end{aligned}$$

One can regard these equations as describing the evolution of the (G, R) -Higgs bundle (E, \mathfrak{q}) on Σ as one varies x^2 . The equations say that (E, \mathfrak{q}) changes only by a complex gauge transformation until one gets to $x^2 = 0$ where ϕ_z has a singularity. There the G -bundle E undergoes a Hecke transformation, the same one as in the twisted $\mathcal{N} = 4$ theory [6]. We will denote

by E_- and E_+ the bundles for $x^2 < 0$ and $x^2 > 0$. We remind that the Hecke transformation modifies the bundle at one point $p \in \Sigma$, i.e. one is given a holomorphic isomorphism between E_+ and E_- on $\Sigma \setminus p$. But a section of E_- which is holomorphic at p may become meromorphic when regarded as a section of E_+ . Therefore the requirement that the Higgs field q_w is holomorphic both for $x^2 < 0$ and $x^2 > 0$ puts a constraint on the allowed Hecke transformations.

As an illustration, consider $G = SU(2)$, and take R to be the sum of four copies of the fundamental representation. Consider an 't Hooft operator with

$$\mu = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

inserted at the point $w = 0$ on Σ . The Higgs field \mathfrak{q} is a made of four holomorphic sections of the bundle $E_- \times K_\Sigma$. Since we are working locally on Σ , we may assume that E_- and K_Σ are trivial, so we may think of these sections as sections of E_- .

Let V be the subspace of $E_-|_{w=0}$ spanned by the four sections at $w = 0$. If V is zero-dimensional (i.e. if all four sections vanish at $w = 0$), then the 't Hooft operator can effect an arbitrary Hecke transformation on the bundle E_- ; the space of such transformations is isomorphic to $T\mathbb{P}^1$. Let us recall how this comes about [6]. A Hecke transformation is specified once we pick a pair of holomorphic sections of E_- which generate E_- in the neighborhood of $w = 0$. If these sections are s_1 and s_2 , then E_+ is generated by

$$s'_1 = w^{-1}s_1, \quad s'_2 = ws_2.$$

Not every choice of s_1, s_2 gives a different E_+ , since any holomorphic $SL(2)$ gauge transformation of s'_1, s'_2 with holomorphic coefficients gives the same E_+ . Using this freedom, one can show that only the first two terms in the Taylor expansion of s_1 affect E_+ ; moreover, if $s_1(w)$ has the form

$$s_1(w) = \begin{pmatrix} a_1 + b_1w + \dots \\ a_2 + b_2w + \dots \end{pmatrix},$$

then using $SL(2)$ gauge freedom one can shift

$$(b_1, b_2) \mapsto (b_1, b_2) + c(a_1, a_2), \quad c \in \mathbb{C} \tag{22}$$

as well as rescale

$$(a_1, a_2, b_1, b_2) \mapsto \lambda(a_1, a_2, b_1, b_2), \quad \lambda \in \mathbb{C}^* \tag{23}$$

without changing E_+ . The transformation (22) can be gotten rid of by replacing (b_1, b_2) with an invariant combination $y = b_1 a_2 - b_2 a_1$. Then the space of Hecke transformations is parameterized by a triple (a_1, a_2, y) modulo a rescaling

$$(a_1, a_2, y) \mapsto (\lambda a_1, \lambda a_2, \lambda^2 y), \quad \lambda \in \mathbb{C}^*.$$

In addition, we have the condition that $(a_1, a_2) \neq (0, 0)$ (because $s_1(0) \neq 0$). The resulting moduli space is the total space of the bundle $\mathcal{O}(2)$ over the projective line \mathbb{P}^1 with the homogeneous coordinates $(a_1 : a_2)$.

Suppose now that the subspace V is one-dimensional; then we get a restriction on the choices of s_1 and s_2 by requiring the holomorphic section of E_- taking value in V at $w = 0$ to be a linear combination of s'_1 and s'_2 with holomorphic coefficients. This restriction says that at $w = 0$ s_1 should also take value in V . This fixes the parameters (a_1, a_2) up to an overall scaling, and the resulting space of allowed Hecke modifications is isomorphic to the fiber of $T\mathbb{P}^1$ over the point $(a_1 : a_2)$.

Finally, if the subspace V is two-dimensional (which is a generic situation), then the space of allowed Hecke transformations is empty. This means that applying an 't Hooft operator to a generic (G, R) Higgs bundle (regarded as a 0-brane on \mathcal{M}_V) at a generic point $p \in \Sigma$ gives the zero object in the category of branes.

6 S-duality and an $\mathcal{N} = 2$ version of the Geometric Langlands Duality

It is conjectured that some or maybe all finite $\mathcal{N} = 2$ super-Yang-Mills theories enjoy S-duality which exchanges electric and magnetic charges, Wilson and 't Hooft operators, and strong and weak coupling. S-duality is best understood in the case when R is the adjoint representation and G is arbitrary [16], in which case one has $\mathcal{N} = 4$ supersymmetry, or in the case when $G = SU(2)$ and R is a sum of four copies of the fundamental representation. In this section we discuss in turn these two cases.

6.1 Montonen-Olive conjecture and classical geometric Langlands duality

If R is the adjoint of G , the theory has $\mathcal{N} = 4$ supersymmetry. The Montonen-Olive conjecture states that $\mathcal{N} = 4$ theories with gauge groups G and ${}^L G$ are equivalent provided the gauge couplings are related by

$${}^L \tau = -\frac{1}{n_{\mathfrak{g}} \tau},$$

where $n_{\mathfrak{g}}$ is 1, 2, or 3 depending on the maximal multiplicity of edges in the Dynkin diagram of the Lie algebra \mathfrak{g} of G [18, 19]. Langlands duality exchanges weight and coweight lattices of G , and this is compatible with the exchange of Wilson and 't Hooft operators under S-duality.

Let us draw the implications of the Montonen-Olive duality for the β'' -model regarded as an effective sigma-model on C . If R is the adjoint representation, the nonabelian vortex equations on Σ become the usual Hitchin equations, and the target space of the effective sigma-model on C is $\mathcal{M}_H(G, \Sigma)$. This moduli space is hyperkahler, rather than Kähler, because of $\mathcal{N} = 4$ supersymmetry. The β'' -model is equivalent to the B-model with target \mathcal{M}_H taken with a complex structure in which $\delta A_{\bar{w}}$ and δq_w are holomorphic differentials. In this complex structure \mathcal{M}_H is isomorphic to the moduli space $\mathcal{M}_{Higgs}(G, \Sigma)$ of semistable Higgs bundles; in [6] this complex structure was denoted I .

The Montonen-Olive conjecture implies that the B-models for $\mathcal{M}_{Higgs}(G, \Sigma)$ and $\mathcal{M}_{Higgs}({}^L G, \Sigma)$ are isomorphic. This implies that their categories of B-branes (i.e. derived categories of coherent sheaves) are equivalent. More specifically, both moduli spaces are fibered over an affine space (this is known as the Hitchin fibration [22]), and there is an identification of the two affine spaces such that generic fibers for G and ${}^L G$ over the same point are dual abelian varieties. Then if we exclude singular fibers from consideration, the equivalence between the categories of B-branes is given by the fiberwise Fourier-Mukai transform. The duality of Hitchin fibrations for G and ${}^L G$ is known as classical geometric Langlands duality and has been proved for nonsingular fibers in [30, 31, 32, 33].

The classical geometric Langlands duality involves only “commutative objects” (coherent sheaves) on both sides. It is a limit of the usual geometric Langlands duality in the following sense. As explained in [6], the usual geometric Langlands duality is a consequence of mirror symmetry which re-

lates the B-model of $\mathcal{M}_H({}^L G, \Sigma)$ in complex structure J and the A-model of $\mathcal{M}_H(G, \Sigma)$ in complex structure K . One can continuously deform J into I , therefore deforming the B-model in complex structure J into the B-model in complex structure I . On the mirror side, this corresponds to deforming the symplectic structure ω_K into a generalized complex structure, so that in the limit one gets ordinary complex structure I . That is, on the mirror side the A-model in complex structure K is deformed into the B-model in complex structure I .

6.2 Seiberg-Witten duality

Now let us turn to the case $G = SU(2)$, and R being the sum four copies of the fundamental representation. This theory is believed [7] to be self-dual under

$$\tau \mapsto -\frac{1}{4\tau}.$$

Together with a more obvious symmetry $\tau \mapsto \tau + \frac{1}{2}$ this generates $PGL(2, \mathbb{Z})$ duality group.⁸ The action on τ takes a more familiar form if we rescale τ by a factor 2; then τ is acted upon by the group $PGL(2, \mathbb{Z})$ of integral fractional linear transformations in a standard way:

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \quad (24)$$

We will call this conjectural duality the Seiberg-Witten duality. Although it is superficially similar to the Montonen-Olive duality, there are also important differences. First of all, the SW duality does not exchange G and ${}^L G$. Second, on the Coulomb branch it maps the 't Hooft-Polyakov monopole to a massive quark state, while the massive W-boson is mapped to a bound state of two monopoles. This suggests that the dual of the Wilson operator in the fundamental representation will be a topologically trivial 't Hooft operator with $SU(2)$ coweight corresponding to the adjoint representation of ${}^L G = SU(2)/\mathbb{Z}_2$. As for topologically nontrivial 't Hooft operators carrying nonvanishing Stiefel-Whitney class w_2 (called the 't Hooft flux in the physics literature), they are not allowed in the Seiberg-Witten theory because there are fields in the fundamental representation.

⁸The theory is invariant under $\theta \mapsto \theta + \pi$ because amplitudes with odd instanton number vanish because of fermionic zero modes.

Third, the SW duality acts nontrivially on the global “flavor” symmetry currents. For massless hypermultiplets, the theory has $Spin(8)$ symmetry under which four copies of q and four copies of \tilde{q} transform as an eight-dimensional vector representation. The subgroup of $Spin(8)$ which does not mix q and \tilde{q} is isomorphic to $SU(4) \times U(1)_B$. The generator

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

of the S-duality group acts on the $Spin(8)$ currents by an outer automorphism known as triality. Triality maps the vector representation of $Spin(8)$ to the spinor representation. This is compatible with the fact that quarks and charge-one monopoles are vectors and spinors of $Spin(8)$, respectively [7].

The β'' -twist made use of the $U(1)_B$ current. Thus in order to extract the implications of the SW duality for the β'' -model, we must only consider the subgroup of the S-duality group which preserves the $U(1)_B$ current. This subgroup consists of elements of $PGL(2, \mathbb{Z})$ congruent to the identity matrix modulo 2. It is denoted $\Gamma(2)$ in the mathematical literature and is generated by

$$T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

and

$$ST^2S = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}$$

Thus the Seiberg-Witten duality conjecture predicts that $\Gamma(2)$ acts by autoequivalences on the derived category of \mathcal{M}_V .

It is easy to understand the autoequivalence corresponding to the element T^2 . The transformation T^2 is simply a shift of theta-angle by 2π . In the β'' -model, it shifts the B-field by the 2-form

$$-\frac{i}{2\pi} \int_{\Sigma} d^2w \operatorname{Tr} \delta A_{\bar{w}} \wedge \delta A_w = \frac{1}{4\pi} \int_{\Sigma} \operatorname{Tr} \delta A \wedge \delta A.$$

This 2-form is minus the pull-back of the curvature of the determinant line bundle Det on $\operatorname{Bun}(SL(2), \Sigma)$ via the obvious projection $\mathcal{M}_V(SU(2), R, \Sigma) \rightarrow \operatorname{Bun}(SL(2), \Sigma)$. Since the 2-form is of type $(1, 1)$, it has no effect on the β'' -model if C has no boundaries. In the presence of boundaries, such a shift can be undone by tensoring the B-brane by the pull-back of Det . Thus T^2 corresponds to tensoring all B-branes on \mathcal{M}_V by the pull-back of Det .

On the other hand, the transformation ST^2S is not visible at weak coupling, and its effect on a general B-brane is harder to describe. An additional constraint on it comes from considering the action of Wilson-'t Hooft operators on branes. Wilson operators have a very simple effect on 0-branes on \mathcal{M}_V : a 0-brane corresponds to a particular (G, R) Higgs bundle (E, \mathfrak{q}) on Σ , and a Wilson operator in representation \mathcal{S} inserted at a point $p \in \Sigma$ has the effect of tensoring the 0-brane with the vector space $\mathcal{S}(E)_p$. Thus any 0-brane (E, \mathfrak{q}) is an eigenobject for a Wilson operator inserted at any point $p \in \Sigma$, and all the ‘‘eigenvalues’’ fit into a holomorphic vector bundle $\mathcal{S}(E)$. The autoequivalence T^2 obviously commutes with all Wilson operators, but ST^2S maps Wilson operators to mixed Wilson-'t Hooft operators. Thus the image of a 0-brane under ST^2S must be an eigenobject for certain Wilson-'t Hooft operators.

More explicitly, for $SU(2)$ we can identify both the lattice of weights and the lattice of coweights with integers, and the Weyl group $\mathcal{W} \in \mathbb{Z}_2$ acts by negation. Thus a general loop operator is described by a pair $(m, n) \in \mathbb{Z}^2$ modulo an overall sign change, where m and n are the coweight and weight, respectively. A Wilson loop with the same electric charge as that of a quark corresponds to the pair $(0, 1)$, while the 't Hooft-Polyakov monopole has the same magnetic charge as the 't Hooft operator $(1, 0)$. More generally, a Wilson loop in $n+1$ -dimensional irreducible representation of $SU(2)$ corresponds to the pair $(0, n)$. Under the transformation (24) the pair (m, n) transforms as follows:

$$(m \ n) \mapsto (m \ n) \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

The pair $(0, n)$ is mapped by ST^2S to $(2n, -n)$. The image of a 0-brane under the autoequivalence ST^2S must be an eigenobject of such Wilson-'t Hooft operators for all n and all $p \in \Sigma$. To understand what this means concretely, one needs to know how general Wilson-'t Hooft operators act on B-branes; we plan to address this important issue elsewhere [34].

In the case of geometric Langlands duality an important role is played by the Hitchin fibration [22]. As mentioned above (see also [6] for a more detailed discussion), the Hitchin fibration is a holomorphic map

$$\mathcal{M}_{Higgs}(G, \Sigma) \rightarrow \mathcal{V},$$

where \mathcal{V} is a complex affine space of dimension $(g-1) \dim G$. This map is a holomorphic completely integrable system, in the sense that its generic fibers

are complex tori, and there is a canonical holomorphic symplectic form on $\mathcal{M}_{Higgs}(G, \Sigma)$ with respect to which the fibers are Lagrangian. S-duality in this case acts by an involution on the base \mathcal{V} and by T-duality on the fibers. For simply-laced groups, the involution is trivial.

Some aspects of this construction do generalize to the case of general R .⁹ The analog of the Hitchin fibration is a map from $\mathcal{M}_{\mathcal{V}}$ to the space of gauge-invariant polynomials in the Higgs field q_w . In the case we are considering, the Higgs field is really a collection of four holomorphic 1-forms $q_w^i dw, i = 1, \dots, 4$, with values in a holomorphic $SL(2)$ bundle E on Σ . The gauge-invariant polynomials

$$B^{ij} = q_w^i q_w^j$$

form a skew-symmetric 4×4 matrix with values in $H^0(K_{\Sigma}^2)$. The entries of the matrix are not independent but satisfy a constraint

$$\epsilon_{ijkl} B^{ij} B^{kl} = 0.$$

Since $h^0(K_{\Sigma}^4) = 7(g-1)$, we get $7(g-1)$ equations which define a cone \mathcal{C} in the $18(g-1)$ -dimensional affine space $H^0(K_{\Sigma}^2) \otimes \mathbb{C}^6$. This cone is the analog of the base of the Hitchin fibration. It has the same dimension as $\mathcal{M}_{\mathcal{V}}$ and the projection from $\mathcal{M}_{\mathcal{V}}$ to \mathcal{C} is generically one-to-one. That is, a generic pair (E, \mathfrak{q}) is determined by its image in \mathcal{C} .

The full SW duality group acts nontrivially on the fields \tilde{B}, B, M whose expectation values parameterize the Higgs branch of the theory. This follows from the fact that it does not commute with flavor $Spin(8)$ symmetry which acts nontrivially on the Higgs branch. But the subgroup $\Gamma(2)$ that we are considering acts trivially on the base of the Hitchin fibration (because both T^2 and $S^2 = 1$ act trivially). This is again similar to the situation in the $\mathcal{N} = 4$ case.

7 Concluding remarks

In this paper we showed that any finite $\mathcal{N} = 2$ gauge theory can be twisted into a field theory on $C \times \Sigma$ which is topological along C and holomorphic along Σ . The observables in the twisted theory are independent of the gauge coupling and theta-angle and are holomorphic functions of coordinates on Σ .

⁹We are grateful to Nigel Hitchin for explaining this to us.

One can regard the twisted theory as providing a new class of “protected” observables which goes beyond the usual chiral ring.

We also showed that the twisted theory admits Wilson-’t Hooft loop observables labelled by elements of the set $(\Lambda_{cw} \times \Lambda_w) / \mathcal{W}$. BRST symmetry forces the loops to be of the form $\gamma \times p$, where γ is a loop in C and $p \in \Sigma$. The OPE of two parallel loop operators is nonsingular, therefore one gets a commutative algebra of Wilson-’t Hooft operators. For purely electric (Wilson) loop operators, this algebra is identical to the fusion algebra of irreducible representations of G . For purely magnetic (’t Hooft) operators, the algebra depends on the matter content of the theory. In the case when the matter fields transform in the adjoint representation of G , it follows from the results of [6] that the algebra of ’t Hooft operators is isomorphic to the fusion algebra of irreducible representations of ${}^L G$. It remains an interesting problem to describe the algebra of general Wilson-’t Hooft operators. In the case when the matter is in the adjoint of G , it should unify the representation theory of G and ${}^L G$. Perhaps it has something to do with the double-affine Hecke algebra introduced by I. Cherednik [35].

We also proposed an analog of geometric Langlands duality for finite $\mathcal{N} = 2$ gauge theories. It states that the S-duality group (or rather, the subgroup of the S-duality group which commutes with the twist) acts by autoequivalences on the derived category of the moduli space of nonabelian vortex equations. This action should be compatible with the action of Wilson-’t Hooft operators on the derived category. In many respects, the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ dualities are similar, but there are also important differences. Perhaps the most important one is that unlike the Hitchin moduli space, the moduli space of nonabelian vortex equations is not holomorphic symplectic (although it is a Calabi-Yau), and consequently the connection with algebraic integrable systems is lost. An analog of the Hitchin map does exist, but its generic fiber is not a complex torus; for example, for $G = SU(2)$ and hypermultiplets in the fundamental representation the map is generically one-to-one.

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