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K-Theory, Reality, and Orientifolds

Sergei Gukov[1]
Joseph Henry Laboratories, Princeton University
Princeton, New Jersey 08544
gukov@pupgg.princeton.edu

Abstract

We use equivariant K-theory to classify charges of new (possibly non-supersymmetric) states localized on various orientifolds in Type II string theory. We also comment on the stringy construction of new D-branes and demonstrate the discrete electric-magnetic duality in Type I brane systems with $p+q=7$, as proposed by Witten.

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[1] On leave from the Institute of Theoretical and Experimental Physics and the L. D. Landau Institute for Theoretical Physics

1. Introduction

During the past few years, D-branes have been playing a significant role in the study of non-perturbative dynamics of supersymmetric string and field theories. Dirichlet p-branes are themselves 'solitonic' BPS states charged under Ramond-Ramond fields [1]. In turn, SUSY gauge theories appear as low-energy description of parallel D-branes [2]. In numerous applications (extended) supersymmetry was an indispensable ingredient to guarantee stability of the vacuum and to retain control in the strong coupling regime – for a review see [3].

The study of non-supersymmetric string vacua is especially important for making a contact with reality. Some progress in this direction has been achieved by Sen [4,5] who found new states in non-BPS brane systems with tachyon condensation [6]. At least perturbatively, these states are stable because of charge conservation. For example, Type I D-particle, dual to the $SO(32)$ heterotic spinor, is the lightest state with $SO(32)$ spinor quantum numbers [7,8]. In fact, there are topological obstructions preventing a decay of such states.

In the systematic approach via K-theory [9], Witten proposed a new way of looking at D-brane charges [2]. The basic idea that D-branes are equipped with gauge bundles naturally leads to the identification of lower-dimensional branes with topological defects (vortices) in the gauge bundle. Because this argument is purely topological, it does not rely on supersymmetry at all. For this reason, it not only reproduces conventional spectrum of BPS D-branes, but it also yields charges of new non-supersymmetric states. For example, a novel non-BPS eight-brane, a seven-brane and a gauge instanton with $\mathbb{Z}_2$ charges were found in Type I string theory [9].

In the present paper we classify charges of new (possibly non-supersymmetric) states in Type II orientifolds by means of equivariant K-theory. The reason to consider orientifolds rather than orbifolds is that in many cases K-theory of an orbifold does not provide more information than the ordinary cohomology theory of its smooth resolution (we present some arguments and examples in section 6). Thus, in reduction to lower dimension, D-brane charges follow by wrapping over all possible cycles. The statement obviously does not hold if the singularity is 'frozen', *i.e.* if it can not be blown up. Such orbifolds correspond to the non-zero flux of the Neveu-Schwarz anti-symmetric tensor field [11], which we always assume to vanish.

[2] Possible interpretation of BPS charges in terms of K-theory was first considered in [10].

The paper is organized in such a way that the balance between physics and mathematics shifts gradually from one section to another. The next section is a warm-up where we briefly review the results of [9], and prepare to study K-theory of orientifolds. Then, we study in details three types of orientifolds, as in [9]. Section 3 is devoted to Ω orientifolds (type (i)). Depending on the choice of projection, D-brane charges localized on such orientifolds take values in the real K-theory $KR(X)$, or its symplectic analog which we call $KH(X)$. Calculating these groups we find a number of new D-branes with $\mathbb{Z}_2$ charges, *e.g.* a non-BPS 3-brane localized on an $\mathcal{O}^+5$ plane. Of particular interest is Type I string theory which is an example of such orientifolds where the involution acts trivially in the space-time. It was proposed in [9], that there is a (-1) monodromy experienced by a gauge instanton crossing an 8-brane, or by a 0-brane winding around a 7-brane. In section 4 we justify this conjecture in two different ways. First, we observe Berry's phase analyzing degeneracy of the 0-7 fermion spectrum. Second, a gauge-theoretic approach leads to the spectral flow of the Dirac operator. In section 5 we return to the main theme of the paper and classify D-brane charges localized on $(-1)^{F_L}$ orientifolds. The spectrum turns out to be the same for any dimension of an orientifold. Hence, the analogy with Type IIA theory can be used to deduce physical properties of the new states. Even though in this paper we will not try to present a complete analysis of $(-1)^{F_L} \cdot \Omega$ orientifolds, this case will be mentioned in section 6, where some orbifold models will be discussed as well. Finally, we will present our conclusions in section 7.

Close to the completion of this work we received preprints [12,13] which complement and slightly overlap discussion of $(-1)^{F_L}$ orientifolds in section 5, in particular Type IIA string theory.

2. General Aspects

2.1. K-theory and D-brane charges

Before we proceed to the K-theory of orientifolds it is necessary to set notation and formulate the problem. Consider Type IIB superstring[3] propagating in the space-time:

$$\mathbf{R}^{d+1} \times X$$

[3] Generalization to Type IIA theory is straightforward, and we comment on that in the end of each section. In later sections we also clarify the relation between D-branes in IIA and IIB theories, regarding the former as $(-1)^{F_L}$ orientifold of the latter.

with n 9-branes and m $\bar{9}$-branes, the simplest setup to define K-theory of D-brane charges [9]. For a moment we forget about tadpole cancellation condition, and impose it later. The nine-branes are supplemented with gauge bundles E and F respectively. In order to describe a d-brane, we want the configuration (E, F) to be translationally invariant in $(d+1)$ directions. In other words, (E, F) labels a pair of bundles over X.

Of course, brane – anti-brane system described above is unstable which is marked by the presence of a tachyon T in the spectrum of open $9-\bar{9}$ strings. The tachyon is a map:

$$T \colon F \to E \tag{2.1}$$

or put differently, a section of $E \otimes F^*$. Therefore, such system tends to annihilate itself unless there is some topological obstruction. The latter is measured by the K-theory group $K(X)$ which we are about to define.

Assuming that an arbitrary number of brane – anti-brane pairs can be created (or annihilated) from vacuum with isomorphic bundles H and H', we come to the equivalence relation:

$$(E, F) \sim (E \oplus H, F \oplus H') \tag{2.2}$$

which makes a semigroup of pairs (E, F) an abelian group $K(X)$ called Grothendieck group [14,15]. The additive structure of $K(X)$ is induced by the direct sum of bundles.

To keep the discussion less abstract, it is instructive to work out a simple example that will prove useful below. Let us calculate the Grothendieck group of a point $K(\mathrm{pt})$. Any bundle over a point is isomorphic to the trivial bundle of certain dimension n. In this case, the equivalence (2.2) takes the form: $(n, m) \sim (n+k, m+k)$ where n, m and k are non-negative integers representing the dimensions of bundles. Therefore, the elements of $K(X)$ are $(n, m) = n - m$ which constitute a group of integer numbers $\mathbb{Z}$.

Now, using the result $K(\mathrm{pt}) = \mathbb{Z}$, we make a few refinements of the construction. First of all, we notice that a map of X to a point induces the homomorphism $\rho \colon \mathbb{Z} \to K(X)$. Since in physical applications the difference $(n-m)$ is fixed by the tadpole cancellation condition we should be actually interested in the cokernel of ρ, the so-called reduced K-theory group $\tilde{K}(X) \equiv \mathrm{coker}\ \rho$. We also expect a d-brane to have finite tension. This condition translates to the statement that the charges of the physical D-branes take values in the K-theory with compact support [9]. In other words, it tells that (2.1) is an isomorphism outside a set $U \subset X$ such that the closure $\overline{U}$ is compact. Physically, U represents the region in the transverse space where the d-brane is localized. Since this condition automatically implies reduced K-theory, in the rest of the paper (except in section 5) we will omit tilde and use the notation $K(X)$ for the *reduced* K-theory with compact support [4]. If the space-time is flat, $X = \mathbf{R}^{9-d}$, then $K(\mathbf{R}^{9-d})$ with compact support is isomorphic to $K(\mathbf{S}^{9-d})$ by adding a point 'at infinity'. This group is equal to $\mathbb{Z}$ for d-odd, and is trivial otherwise (see *e.g.* [14,15]). Thus, we obtain the standard spectrum of D-brane charges in Type IIB string theory. When d is odd, we take $S_\pm$ to be positive (negative) spinor representation of $SO(9-d)$, the group of rotations in the transverse directions. Then, the explicit form of the tachyon field corresponding to the unit d-brane charge placed at the origin of $\mathbf{R}^{9-d}$ can be written in terms of Gamma matrices $\vec{\Gamma}\colon S_- \to S_+$ [9]:

$$T(\vec{x}) = \vec{\Gamma} \cdot \vec{x} \tag{2.3}$$

where we omit a suitable normalization factor.

Generalization of this construction to other string theories is also possible [9]. Here we state without proof that in Type IIA string theory D-brane charges take values in $K(X \times \mathbf{S}^1)$, while the charges of Type I D-branes are measured by $KO(X)$. For details we refer the reader to the original work [9]. On the other hand, the necessary mathematical background on K-theory can be found in [14,15,17].

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[4] There is a nice definition of such K-theory given by G. B. Segal in terms of *complexes* [16]. A complex is given by a sequence:

$$\mathcal{E}^k\colon 0 \to E^0 \to E^1 \to \ldots \to E^{k-1} \to 0$$

of vector bundles $\{E^i\}$ over X which fails to be exact over the compact support $U \subset X$. The complex $\mathcal{E}$ is called *acyclic* if $U = \emptyset$. Then, $K(X)$ is defined as the set of isomorphism classes of complexes $\mathcal{E}$ on X modulo acyclic complexes. Even though it may sound too abstract, this definition has a clear physical interpretation. For example, an acyclic complex of length 2 represents a pair of isomorphic bundles $E \cong F$. Equivalence modulo such complexes is nothing but the equivalence relation (2.2) which allows the creation/annihilation of brane–anti-brane pairs with isomorphic bundles. Therefore, at length 2, we just recover the standard definition of $K(X)$ given in the text. It might seem that equivalence modulo acyclic complexes of arbitrary length is stronger than the relation (2.2). However it is not the case [16,17], and the two definitions are equivalent. As a next step, acyclic complex $\mathcal{E}^3$ is given by the exact sequence: $0 \to E \to G \to F \to 0$. This is nothing but the charge conservation condition for scattering of (anti-)BPS states $[E] + [F] \to [G]$ found in terms of the ordinary cohomology [18].

2.2. Equivariant K-theory and Orientifolds

In what follows we consider space-time of the form:

$$\mathbf{R}^{p+1} \times (\mathcal{M}^{9-p}/G)$$

where $\mathcal{M}^{9-p}$ is a smooth manifold, and the discrete symmetry group G acts continuously on $\mathcal{M}^{9-p}$. Being interested in the d-brane charges, we also consider vector bundles E over $X = \mathbf{R}^{p-d} \times \mathcal{M}^{9-p}$, such that the projection $E \to X$ commutes with the action of G. The above conditions define the category of G-equivariant bundles over G-space X [16,17]. The corresponding K-theory is called G-equivariant K-theory $K_G(X)$. In many ways, $K_G(X)$ is similar to the ordinary K-theory. For example, such properties of $K(X)$ like Thom isomorphism and Bott periodicity continue to hold in the equivariant case [16,17,19]. Another basic theorem of equivariant K-theory tells that if G acts freely on X, then:

$$K_G(X) \cong K(X/G) \tag{2.4}$$

This isomorphism will prove to be useful in calculations.

So far we have described K-theory of orbifolds [9]. However, it turns out that, compared to the usual cohomology theory, for 'regular' orbifolds it does not provide new states (*cf.* section 6). For this reason we consider G accompanied by a world-sheet symmetry action. We refer to its fixed point set (a number of $\mathbf{R}^{p+1}$ planes) as orientifold p-planes, or $\mathcal{O}p$-planes for short.

Following the approach of [9], we address the following question: What are the charges of stable (possibly non-BPS) states localized at a singularity of $\mathcal{M}^{9-p}/G$ (*i.e.* located on the $\mathcal{O}p$-plane)? To answer this question, we have to recast it explicitly in terms of vector bundles — the language used throughout the paper. Stability of a d-dimensional object just means that it is a topological defect in the gauge bundle of 9-brane – antibrane system, *i.e.* its charge takes values in the G-equivariant K-theory of $X = \mathbf{R}^{p-d} \times \mathcal{M}^{9-p}$ [9]. Provided that $d < p$, a d-brane can be constructed from p-branes placed at the fixed point of $\mathcal{M}/G$. The d-brane is stable if it is the lightest state charged under p-brane gauge group [4]. The condition for such a state to be localized at a singularity translates to the assertion that K-theory a has compact support which includes the singular point. Therefore, it has to be G-equivariant K-theory. Indeed, if in the vicinity of the singularity the tachyon is homotopic to the vacuum[5], and this region is path-connected to the infinity, then one

[5] In other words, $T\colon F \to E$ is an isomorphism.

can deform the compact support (the core of a gauge 'vortex') arbitrarily far from the singularity. Therefore, the state is *not* localized at the singularity and is represented by an element of $K_G(X) \cong K(X_G)$ where G acts freely on X [16,19]. Since for the most of our applications this group is isomorphic to the usual K-theory $K(X)$, we consider only the states localized on the singularity.

Suppose $\mathcal{M}$ is a vector space, and G acts on $\mathcal{M}$ with at most one isolated singularity at the origin. If we define $\mathbf{S}$ to be a unit sphere in $\mathcal{M}$, then the smooth manifold $H = \mathbf{S}/G$ (= unit sphere in $X = \mathcal{M}/G$) is automatically Einstein. In analogy with the AdS/CFT correspondence [20], it is natural to call it a 'horizon', *cf.* [21]. According to [20], gauge theory on p-branes placed at the singularity is dual to the supergravity compactification on H. The counterpart of this relation in the equivariant K-theory with compact support is given by the exact triangle for the pair $(\mathcal{M}, \mathbf{S})$:

$$\begin{array}{ccc} K^*(H) & \xrightarrow{\delta^*} & K^*_G(\mathcal{M}, \mathbf{S}) \\ & \nwarrow \qquad \swarrow & \\ & K^*_G(\mathcal{M}) & \end{array} \tag{2.5}$$

where $\delta: \mathcal{M} \to \mathbf{S}$ is boundary homomorphism. To write the equivariant group $K_G(\mathbf{S})$ we used the fact that G acts freely on $\mathbf{S}$ and the theorem (2.4). Because the relative K-theory $K^*_G(\mathcal{M}, \mathbf{S})$ is canonically isomorphic to the K-theory with compact support, the exact sequence (2.5) will prove to be useful in computations of the groups $K_G(X)$. In mathematical terminology, X is a cone on H, and $\Sigma' H = X/H$ is called unreduced suspension of H [14,15].

3. Orientifolds of type (i): $\mathbf{R}^{p+1} \times (\mathcal{M}^{9-p}/\Omega \cdot \mathcal{I}_{9-p})$

3.1. $\tau^2 = -1$: The Real K-theory

Now we are ready to consider the first example: orientifolds

$$\mathbf{R}^{p+1} \times (\mathcal{M}^{9-p}/\Omega \cdot \mathcal{I}_{9-p}) \tag{3.1}$$

of type (i), as in [9]. In this case the generator of $G = \mathbb{Z}_2$ is a combination of the involution $\mathcal{I}_{9-p}$ on $\mathcal{M}^{9-p}$ and the world-sheet parity transformation Ω. String orientation reversal induces an anti-linear map $\tau: E_x \to E_{\tau(x)}$ on the gauge bundle. There are two consistent orientifold projections in Type IIB string theory [22], corresponding to $\tau^2 = 1$ and $\tau^2 = -1$ respectively. In the first case we obtain KR-theory [9], while in the second case D-brane charge takes values in the group which we denote[6] by $KH(X)$ and study in the next part of this section. There are two types of orientifolds, called $\mathcal{O}^{\pm}$ according to their tadpole contribution. They carry $\mp 2^{p-4}$ units of p-brane charge and produce SO or Sp gauge groups respectively. In what follows, we will see that the choice of projection is determined by τ (whether its square is equal to plus or minus identity), so that the states on the orientifolds are classified by $KR(X)$ or $KH(X)$.

Let us first consider the case $\tau^2 = 1$ corresponding to the quantization of 9-branes with SO Chan-Paton factors. Our major example in this paper will be the simplest case $\mathcal{M}^{9-p} = \mathbf{R}^{9-p}$ where new D-brane charges can be found. Then, orientifolds (3.1) take the following form:

$$\mathbf{R}^{p+1} \times (\mathbf{R}^{9-p}/\Omega \cdot \mathcal{I}_{9-p}) \tag{3.2}$$

It is convenient to introduce the notation $\mathbf{R}^{p,q}$ for the space-time $X = \mathbf{R}^q \times \mathbf{R}^p$ with the involution $\mathcal{I}_p$ acting on the second factor. The convention is chosen to agree with the notation of the corresponding linear space in [23]. We also denote:

$$\begin{array}{c} \mathbf{B}^{p,q} \equiv \text{unit ball in } \mathbf{R}^{p,q} \\ \mathbf{S}^{p,q} \equiv \text{unit sphere in } \mathbf{R}^{p,q} \end{array} \tag{3.3}$$

Note, $\mathbf{S}^{p,q}$ has dimension $p+q-1$, *e.g.* $\mathbf{S}^{o,n} = \mathbf{S}^{n-1}$.

In mathematical terms, the above properties define the real category of vector bundles over X with compact support. Therefore the d-brane charge localized on the orientifold p-plane takes values in the real K-theory [9], which we denote as:

$$KR^{9-p,p-d}(\mathrm{pt}) \equiv KR(\mathbf{B}^{9-p,p-d}, \mathbf{S}^{9-p,p-d}) \tag{3.4}$$

These are the so-called (p,q) suspension groups of a point [23]; compare with the ordinary definition $KR^{-n}(X,Y) \equiv KR(X \times \mathbf{B}^{0,n}, X \times \mathbf{S}^{0,n} \cup Y \times \mathbf{B}^{0,n}) \cong KR(\Sigma^n(X/Y))$ [14,15]. Because the involution acts trivially on a single point, we find helpful the following general relation:

$$KR(X_R) \cong KO(X_R) \tag{3.5}$$

where X_R is the set of fixed points under the involution τ [23].

[6] This is in analogy with symplectic bundles, where τ is multiplication by j over the field of quaternions $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$.

To calculate (3.4), we also need the following periodicity isomorphisms established by Atiyah:

$$KR(X) \cong KR^{-8}(X) \tag{3.6}$$

$$KR^{p,q}(X) \cong KR^{p+1,q+1}(X) \cong KR^{p-q}(X)$$

The first property follows from multiplication by the generator of $KR^{-8}(\mathrm{pt})$, while multiplication by the generator of $KR^{1,1}(\mathrm{pt})$ induces the second isomorphism in (3.6). In the special case (of our interest) when $X = \mathrm{pt}$, one can independently prove the formulas (3.6) via the periodicity of the corresponding Clifford algebras, *cf.* section 6.

To compute $KR(\mathbf{R}^{9-p,p-d})$, we use the periodicity theorem (3.6) which leads to the group $KR(\mathbf{R}^{0,2p-d-1})$ of the real space $\mathbf{R}^{0,2p-d-1}$ with a compact support where the involution acts trivially, $\tau(x) = x$. Hence, by the formula (3.5), we obtain for the d-brane charges:

$$KR(\mathbf{R}^{9-p,p-d}) \cong KO(\mathbf{S}^{2p-d-1}) \tag{3.7}$$

Modulo the Bott periodicity, we list all the KO-groups of spheres in the table below [24]:

n	0	1	2	3	4	5	6	7
$KO(\mathbf{S}^n)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0

Now we turn to the classification of D-brane charges that follow from (3.7) for various values of p. The $p = 9$ orientifold is nothing but Type I unoriented string theory. Apart from the familiar D-strings, 5-branes and 32 nine-branes, the spectrum contains $SO(32)$ D-particle discovered by Sen [4,6]. The other non-BPS states with $\mathbb{Z}_2$-valued charges – a gauge instanton, a seven-brane and an eight-brane – were found in [9] by means of the systematic approach via K-theory. Clearly, all these results are in accordance with the formula (3.7).

The formula (3.7) allowes us to classify stable D-brane charges localized on the $\mathcal{O}^-5$-plane. Due to the Bott periodicity, the spectrum looks very much like in Type I string theory:

$$\mathbb{Z}, \qquad \mathrm{D-string};$$

$$\mathbb{Z}_2, \qquad \text{gauge soliton}; \tag{3.8}$$

$$\mathbb{Z}_2, \qquad \text{gauge instanton.}$$

Among Type IIA orientifolds, a 4-plane has the form (3.2). It was proposed in [9], that Type IIA D-brane charges take values in $K(X \times \mathbf{S}^1) \cong K^{\pm 1}(X)$. Because of the mod 2 periodicity, the uncertainty in the degree of suspension does not affect the answer in the complex K-theory. However, one has to be more accurate in the real category. We claim (and argue in the following sections) that the correct shift is given by one extra suspension, *i.e.* in the real case Type IIA D-brane charges are measured by the group:

$$KR(\mathbf{R}^{9-p,p-d} \times \mathbf{S}^1) \cong KO(\mathbf{S}^{2p-d}) \tag{3.9}$$

Thus, under the T-duality transformation ($p \to p-1$) the dimensions of all the d-branes are reduced by one, compared to Type IIB orientifolds. It means that the only stable objects localized on a 4-plane are D-particles and D-instantons with charges $\mathbb{Z}$ and $\mathbb{Z}_2$ respectively.

3.2. $\tau^2 = -1$*: Symplectic Bundles and Periodicity*

So far we considered 9-branes quantized with SO Chan-Paton factors according to the choice $\tau^2 = 1$ of orientifold projection, $\Omega^2 = 1$ in the notations of [22]. Gimon and Polchinski explained that in Type I sting theory Ω^2 acts as (-1) on the $5-9$ strings. Hence $\Omega^2|5\rangle = -|5\rangle$, and 5-branes must be quantized with Sp Chan-Paton factors. On the other hand, T-dualizing four directions one would get an orientifold 5-plane with 5-branes and 9-branes interchanged because T-duality along the x^i direction maps Ω to $\Omega \cdot \mathcal{I}_{x^i}$, and vice versa. This implies the existence of two kinds of orientifolds $\mathcal{O}^{\pm}$ with the same geometry (3.2), but different gauge groups. Explanation of all these phenomena in terms of K-theory will be the goal of the present section. As a byproduct, we find new non-BPS 3-branes and 4-branes localized on an $\mathcal{O}^{+}$5-plane.

As we have already announced, the two choices of projection $\tau^2 = \pm 1$ give rise to KR and KH groups respectively. While the first choice was the subject of the previous subsection, now we focus on the properties of $KH(X)$. First of all, if the involution acts trivially on X, i.e. $X = X_R$, then $KH(X_R) \cong KSp(X_R)$. This is a symplectic analog of the relation (3.5) in the real case. It follows that the KH-theory inherits many properties of the KSp-theory. Namely, multiplication by the generator of $KH^{-4}(\mathrm{pt}) \cong KSp^{-4}(\mathrm{pt}) = \mathbb{Z}$ induces periodicity isomorphisms:

$$KH^{-4}(X) \cong KR(X), \quad KR^{-4}(X) \cong KH(X) \tag{3.10}$$

Using these formulas, one can always reduce calculation of KH-groups to the real K-theory.

Now we return to the orientifolds (3.2) with $\tau^2 = -1$, and study the spectrum of d-brane charges measured by $KH(\mathbf{R}^{9-p,p-d})$ with a compact support. Using the periodicity (3.10), it is convenient to rewrite (3.5) and (3.6) for the symplectic case at hand:

$$KH(X_R) \cong KSp(X_R)$$
$$KH(X) \cong KH^{-8}(X) \tag{3.11}$$
$$KH^{p,q}(X) \cong KH^{p+1,q+1}(X) \cong KH^{p-q}(X)$$

If X = pt, the case relevant to orientifold applications, these isomorphisms might be derived independently repeating arguments in [14] for $\tau^2 = -1$ or via the relation to Clifford algebras [25,26].

Calculation of the groups $KH(\mathbf{R}^{9-p,p-d})$ is similar to the corresponding computation in the real K-theory. The periodicity isomorphism (the last line in (3.11)) yields $KH(\mathbf{R}^{0,2p-d-1})$ which is isomorphic to $KSp(\mathbf{S}^{2p-d-1})$ in the theory with compact support. Finally, using the standard periodicity theorem $KSp(\mathbf{S}^n) = KO(\mathbf{S}^{n+4})$, we obtain:

$$KH(\mathbf{R}^{9-p,p-d}) \cong KO(\mathbf{S}^{2p-d+3}) \tag{3.12}$$

Of course, this result was expected from the consecutive application of (3.10) and (3.7).

Now we shall discuss the interpretation of the d-brane charges given by (3.12). For instance, if $p = 5$, we get the following d-branes localized on an $\mathcal{O}^+5$-plane with charges:

$$\begin{aligned} &\mathbb{Z}, && 5-\text{brane;} \\ &\mathbb{Z}_2, && 4-\text{brane;} \\ &\mathbb{Z}_2, && 3-\text{brane.} \end{aligned} \tag{3.13}$$

It is instructive to see how the states (3.13) with $d < 5$ can arise from the gauge bundles on the five-branes placed at the singularity. Choosing $\tau^2 = -1$, we start with $KH(\mathbf{R}^{5-d} \times \mathbf{R}^{4,0})$ in a ten-dimensional space-time. Because of eqs. (3.11) and (3.10), this group is isomorphic to $KO(\mathbf{R}^{5-d})$ which implies orthogonal gauge bundles on 5-branes. Indeed, $KO(\mathbf{R}^{5-d})$ with compact support is equivalent to the stable homotopy group $\pi_{4-d}(O(N))$ for sufficiently large N. To exhibit this, one needs to compactify $\mathbf{R}^{5-d}$ by a point 'at infinity' and to regard $\mathbf{S}^{5-d}$ as a union of two hemispheres intersecting over the 'equator'

$\mathbf{S}^{4-d}$. A transition function on $\mathbf{S}^{4-d}$ describes $O(N)$ vector bundles over $\mathbf{S}^{5-d}$, hence the isomorphism $KO(\mathbf{R}^{5-d}) \cong \pi_{4-d}(O(N))$. Because $\pi_0(O(N)) = \pi_1(O(N)) = \mathbb{Z}_2$ we again come to the 3-brane and 4-brane with $\mathbb{Z}_2$ charges (3.13). Similar argument can be used to demonstrate that five-branes at the $\tau^2 = +1$ orientifold discussed earlier carry symplectic gauge bundles, in agreement with Gimon and Polchinski [22]. In that case, non-trivial homotopy groups $\pi_4(Sp) = \pi_5(Sp) = \mathbb{Z}_2$ account for the Sp gauge soliton and instanton (3.8).

It is important to stress here that the orientifold symmetry group $\{1, \Omega\mathcal{I}\}$ consists just of two elements. If we rather considered a larger symmetry group, the charges of D-branes would be classified by another equivariant K-theory. For example, dividing by the group of four elements $\{1, \mathcal{I}, \Omega, \mathcal{I}\Omega\}$, one obtains a theory equivalent to K3 compactification of Type I theory [22]. D-brane charges in the latter theory take values in the group $KO_{\mathbb{Z}_2}(X)$ rather than $KR(X)$.

3.3. Stringy Construction

To conclude this section, we comment on the stringy construction of new non-BPS objects. Non-supersymmetric states (3.8) and (3.13) localized on orientifold 5-planes $\mathcal{O}^{\mp}$ will be our main examples.

Following [9], it is natural to propose that a d-brane for d odd is a bound state of a Type IIB d-brane and an anti-brane exchanged by the Ω action, *i.e.* d could be either -1, 3 or 7. If nine-branes are quantized with orthogonal Chan-Paton factors, it turns out that the tachyon is removed by Ω projection only for $d = -1$, 7 [9]. On the other hand, in the case $\tau^2 = -1$, only $3 — \bar{3}$ system is stable. This is indeed what we found in (3.8) and (3.13).

When interpreting a d-brane with d even, one encounters the same problem as in [9]. Namely, Neveu-Schwarz and Ramond sectors of open $d - p$ string produce odd numbers of fermion zero-modes. Consistent quantization of the corresponding Clifford algebras is obstructed by the absence of the operator $(-1)^F$ that would anti-commute with fermionic modes. To resolve the difficulty, Witten proposed to introduce one extra fermion zero mode η, anti-commuting with the other fermions w_i. Then the operator $(-1)^F$ can be defined as the product $\eta \prod_i w_i$. The appearance of the zero mode η has several effects on string dynamics. Firstly, in effect there is no GSO projection on the string ground state because we have enlarged the original Fock space [8,9]. Secondly, the world-sheet path integral has an extra factor $\sqrt{2}$ from the η mode in the NS sector, so that the masses of all such d-branes are $\sqrt{2}$ times greater than the masses of the corresponding Type IIA D-branes. Furthermore, after adding η field and making the GSO projection, we obtain chiral spinors of $SO(1, d+1)$ in the Ramond sector of $d-p$ string. These fermions must be real or pseudoreal to agree with the orientifold projection. It is easy to see that this is indeed the case [27]. For example, $\mathcal{C}l_{1,5} = \mathbb{H}(4)$ confirms the existence of D-particle on the $\mathcal{O}^{-}5$-plane, in accordance with $KSp(\mathbf{S}^5) = \mathbb{Z}_2$. In turn, an orientifold 5-plane supplemented with an orthogonal gauge group has a $\mathbb{Z}_2$ charge of non-BPS 4-brane (3.13). This is in perfect agreement with the corresponding Clifford algebra $\mathcal{C}l_{1,1} = \mathbb{R}(2)$ which is real.

Relation between fermion zero modes in the Ramond sector and Clifford algebras seems to be more profound, and begs for further investigation.

4. Dynamics of Type I D-branes

Unlike the usual D-branes, new non-supersymmetric branes with $\mathbb{Z}_2$ charges found above do not couple to massless Ramond-Ramond fields. Of particular interest is the question about the interaction of such states in Type I string theory. The interaction amplitudes of Type I D-particle can be found using the set of rules in [8]. Another (topological) sort of interaction could be the discrete electric-magnetic duality in $p-q$ brane systems with $p+q=7$, as proposed by Witten [9].

To justify the conjecture of [9], in this section we demonstrate the (-1) monodromy in two Aharonov-Bohm experiments:

(a) when we parallel transport a D-particle around a 7-brane;

(b) when we parallel transport a gauge instanton across an 8-brane.

We expect the interaction to be mediated by $p-q$ strings and to be topological in the sense that it should not depend on small perturbations, but must feel the relative orientation of the brane system. The last effect can be felt only by fermions that become massless when the branes come close to each other. In the Neveu-Schwarz sector, the $p-q$ string zero point energy equals $-\frac{1}{2} + (\mathrm{DN} + \mathrm{ND})/8 > 0$ [28]. Therefore, we have to focus on the fermions in the Ramond sector where the ground state energy is always zero.

Below we study the fermions in the Ramond sector of $p-q$ string by two different methods. First, we present 'stringy' approach where the monodromy appears as a Berry's phase, and $0-7$ system is the most convenient example to use. On the other hand, case *(b)* is the main example of the second approach via gauge bundles.

4.1. 0 – 7 Strings and Berry's Phase

In order to observe the Berry's phase in the $0-7$ system, we establish the degeneracy of fermion energies in the Ramond sector when the branes coincide. Then we show that the degeneracy is lifted once the D-particle moves away from the 7-brane. We place the 7-brane at $x^8 = x^9 = 0$ and choose the position of the D-particle to be $x^\mu = (0, \ldots 0, \vec{a})$, $\mu > 0$, where $\vec{a}$ is the position vector in the $8-9$ plane. For the time being we put $\vec{a} = 0$.

Type I seven-brane is a bound state of a Type IIB 7-brane and an anti-7-brane where the tachyon is projected out by Ω [9]. Therefore, Type I $0-7$ string spectrum contains two copies of modes, corresponding to a $0-7$ string and a $0-\bar{7}$ string in Type IIB theory. Because these are oriented strings, the fermions are complex. In what follows we will count real fermions, *i.e.* we will distinguish between 0-7 strings and 7-0 strings, the fermions of the last two being real. In total we obtain $0-7$, $7-0$, $0-\bar{7}$ and $\bar{7}-0$ strings. The world-sheet orientation reversal Ω maps Type IIB 7-branes to $\bar{7}$-branes, and vice versa. Therefore, only two sets of the modes listed above are independent: Ω identifies $0-7$ with $\bar{7}-0$ strings, and $0-\bar{7}$ with $7-0$. Let us consider $0-7$ and $0-\bar{7}$ independent string sectors.

Taking into account the extra fermion field η, there are four fermion zero modes in the Ramond sector of the $0-7$ sting[7]: w^0, w^8, w^9 and η. Fixing the light-cone gauge in the $8-9$ directions, we end up with two real fermions [29]. It is convenient to combine them into the creation and annihilation operators $d^\pm = \frac{1}{2}(w^0 \pm \eta)$ which generate two Ramond ground states [28]:

$$|+\frac{1}{2}\rangle \quad \text{and} \quad |-\frac{1}{2}\rangle \tag{4.1}$$

These eigenstates represent two irreducible representations of the two-dimensional rotation symmetry group $SO(2)$ with eigenvalues $s = \pm\frac{1}{2}$ respectively. The GSO projection keeps only one of them, the one with even fermion number. Assuming $d^-|-\frac{1}{2}\rangle = 0$, we end up with the only fermion zero mode $|-\frac{1}{2}\rangle$ in the Ramond sector of the $0-7$ string. The discussion of the $0-\bar{7}$ sector is very similar, and we still get two fermion zero modes (4.1). But this time, since $7-7$ and $7-\bar{7}$ vertex operators undergo the opposite GSO projections, consistent OPE of $0-7-\bar{7}$ string triangle requires the GSO projection in the $0-\bar{7}$ sector to be opposite to that in the $0-7$ sector [30]. Hence now we end up with the zero mode of opposite chirality, $|+\frac{1}{2}\rangle$. To summarize our results, in the system

[7] Discussion of the $0-\bar{7}$ sector requires only minor modifications which we will make later.

of coinciding 0-brane and 7-brane we have found two fermion zero modes with quantum numbers as in (4.1).

Now we argue that the two-fold degeneracy found above is lifted if we perturb the system by small displacement of the D-particle, $\vec{a} \neq 0$. Because prior to the gauge fixing fermion zero modes w^0, w^8, w^9 and η were in the same representation of the four-dimensional Clifford algebra $\mathcal{C}l_{1,3}$, we can choose the $SO(2)$ symmetry group in the previous paragraph to be the rotation symmetry in the $8-9$ plane. Furthermore, physical states (4.1) must satisfy the super-Virasoro constraint:

$$G_0|\psi\rangle = 0 \tag{4.2}$$

which, on the ground states, reduces to the two-dimensional Dirac equation $p_\mu w^\mu|\psi\rangle \simeq \not{D}\psi = 0$. Because the states (4.1) have opposite $SO(2)$ chirality, they have different eigenvalues. It means that degeneracy is lifted as long as $\vec{a} \neq 0$, *i.e.* when $0-7$ string has finite length.

After all, we have the two-level system with parameter space $\{\vec{a}\}$, such that levels cross [8] at the single point $\vec{a} = 0$. This is sufficient information to deduce the Berry's phase acquired by the ground state during adiabatic transport of $\vec{a}$ around the origin [31]. To the first order in perturbation, the general Hamiltonian describing the two levels (4.1) in the real representation of $SO(2)$ can be expressed in terms of real Pauli matrices:

$$H(\vec{a}) = \frac{1}{2}\begin{pmatrix} a^8 & a^9 \\ a^9 & -a^8 \end{pmatrix} = \frac{1}{2}\vec{\sigma}\cdot\vec{a} \tag{4.3}$$

Note, the same Hamiltonian describes 3d spin with $S = \frac{1}{2}$ in the external magnetic field $(a^8, 0, a^9)$, and the so-called dynamical Jahn-Teller effect. It is important to stress here that because of the reality condition "the Berry's phase" is actually a discrete number (0 or π) rather than a continuous phase. And the eigenfunction of the pure state $|s\rangle$ can change the sign via mixing with the orthogonal state during the adiabatic transport, *e.g.*:

$$|+\frac{1}{2}\ (\theta)\rangle = \cos(\frac{\theta}{2})|+\frac{1}{2}\rangle + \sin(\frac{\theta}{2})|-\frac{1}{2}\rangle$$

This is an eigenfunction of the Hamiltonian (4.3) where $\vec{a} = (a\cos\theta, a\sin\theta)$. Analogous pattern takes place in the dynamical Jahn-Teller effect. The topological phase is given

[8] We assume that perturbation of energy levels is first order in $\vec{a}$. Direct calculation in the end of this subsection will confirm this assumption.

by half the 'solid angle' that the adiabatic path subtends at the degeneracy point, *i.e.* $\varphi = \frac{1}{2}(2\pi) = \pi$. This leads to the expected monodromy $\exp(i\varphi) = -1$.

In order to see how the Hamiltonian (4.3) follows from string dynamics, it is convenient to consider string coordinates ($i = \{8, 9\}$):

$$X^\mu(z, \bar{z}) = X^\mu(z) + X^\mu(\bar{z}) = -i\frac{a^\mu}{2\pi}\ln(\frac{z}{\bar{z}}) + \text{oscillators}$$

in the T-dual picture [28]:

$$\tilde{X}^\mu(z, \bar{z}) = X^\mu(z) - X^\mu(\bar{z}) = -i\alpha' p^\mu \ln(z\bar{z}) + \text{oscillators}$$

where $p^\mu = a^\mu/(2\pi\alpha')$. Therefore, small perturbation of the 'Dirac equation' (4.2) leads to the effective Hamiltonian (4.3) in the representation $w^8 = \sigma_x$, $w^9 = \sigma_z$. It follows that energy gap between two states (4.1) is proportional to a which confirms our assumption about conical crossing of energy levels at the origin.

4.2. Approach via Gauge Theory

Now we turn to another face of the $p - q$ strings where the branes are represented by topological defects in the gauge bundles on 9-branes. This approach is reminiscent of the K-theory construction (2.3). Since tadpole cancellation requires 32 nine-branes to present in Type I string theory from the very beginning [22], we don't need to invoke extra anti-branes to construct the $p - q$ system. Following this reasoning, we study $\mathcal{N} = 1$ effective $SO(32)$ gauge theory on the world-volume of parallel 9-branes:

$$\text{Tr}\int(\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + i\bar{\Psi}[\not{D}, \Psi]) \tag{4.4}$$

where Ψ is the Weyl fermion, and $F_{\mu\nu}$ is the field strength of the gauge field. In general, the background of p- and q-branes system ($q = 7 - p$) is given by:

$$A_\mu = \begin{pmatrix} A_\mu^{(p)} & 0 \\ 0 & A_\mu^{(q)} \end{pmatrix} \tag{4.5}$$

and vanishing fermion field. The gauge connection $A_\mu^{(p)}$ describing the p-brane depends on $(9 - p)$ coordinates x^i transverse to the p-brane. This is in accordance with the fact that the corresponding bundle $E_{(p)}$ (together with the trivial bundle of rank 0) represents the non-trivial element of $KO(\mathbf{R}^{9-p})$.

In this language, the fermions in the Ramond sector of $p-q$ strings are represented by the off-diagonal blocks ψ and $\psi^\dagger$ of the fermion field [32]:

$$\begin{pmatrix} 0 & \psi \\ \psi^\dagger & 0 \end{pmatrix} \tag{4.6}$$

The Weyl fermion ψ is a section of $E_{(p)} \otimes E^*_{(q)}$.

Now it is convenient to focus on the $p = 8$ ($q = -1$) system. We are interested in the zero modes of ψ when the gauge instanton and the 8-brane are placed at the same point $x^9 = 0$. An advantage of 8-branes is that rank of the bundle $E_{(8)}$ is equal to 1, *i.e.* we don't have to worry about the corresponding indexes. Hence, according to the index theorem [33,34], in the sector with non-trivial instanton numbers, the Dirac operator $\not{D}(A_{(-1)}) \oplus \not{D}(A_{(8)})$ has one zero mode of definite chirality with respect to the operators $(\prod_{\mu=0}^{9} \Gamma^\mu)$ and Γ^9. Here it is important that we deal with orthogonal gauge group. Consider perturbation of this system by small displacement of the 8-brane: $x^9 \to x^9 - a$. Effective action for the zero mode ψ_0 follows from (4.4):

$$\int \psi_0^\dagger (\Gamma^9 a) \psi_0$$

Because ψ_0 satisfies $\Gamma^9 \psi_0 = +\psi_0$, the eigenvalue of the Dirac operator

$$\not{D}_a = \not{D}(A_{(-1)}) + \not{D}(A_{(8)}) + \Gamma^9 a$$

is equal to $+a$, and changes its sign as the instanton crosses the 8-brane. Hence, fermion contribution to the amplitude $(\mathrm{Det}\ i\not{D}_a)^{\frac{1}{2}}$, defined as the product of the half of the eigenvalues, also changes the sign. The other choice of the disconnected component of the orthogonal group, corresponding to the opposite sign in $\Gamma^9 \psi_1 = -\psi_1$, would result in the fermion mode ψ_1 which always remains massive in the neighborhood of $a \simeq 0$. Therefore, it would not affect the path integral, as well as other massive modes.

Like in the approach via Berry's phase, the (-1) monodromy is produced by the fermions which become massless when the branes coincide. Actually the two methods are equivalent and are based on the spectral flow of the Dirac operator.

In general, using the Thom isomorphism, it is convenient to reduce the problem to two dimensions. Then, a 7-brane and a (-1)-brane become a gauge instanton, while a 0-brane and an 8-brane transform into a two-dimensional soliton. The world-line of the gauge soliton is one-dimensional curve, say $x^1 = a$. We want to demonstrate that the sign of the instanton amplitude is reversed in crossing the curve $x^1 = a$. Even though this system is very similar to the $(-1)-8$ case discussed above, we use a different argument to show that odd number of eigenvalues of the Dirac operator $\not{D}_a$ change sign. As usual, to find the spectral flow uder deformation from $\not{D}_{-\infty}$ to $\not{D}_{+\infty}$, one has to promote a to the independent coordinate, $\mathcal{D} = \not{D}_a + \Gamma^a \partial_a$. Then, the spectral flow of $\not{D}_a$ is equal to the index of $\mathcal{D}$ [35]. Now, to complete the proof, we show that $\mathrm{ind}(\mathcal{D})$ represents a non-trivial element in K-theory[9]. Since $A_{(0)}$ depends only on $(x^1 - a)$, the contribution from the a 'direction' is the same (up to relative sign) as the contribution from the gauge soliton. Therefore, we end up with $\mathrm{ind}(\not{D}_{(-1)})$ corresponding to the gauge instanton class in KO-theory.

5. Orientifolds of type (ii): $\mathbf{R}^{p+1} \times (\mathbf{R}^{9-p}/(-1)^{F_L} \cdot \mathcal{I}_{9-p})$

Now we consider Type IIB orientifolds where involution is combined with the perturbative symmetry group $(-1)^{F_L}$. Acting on 9-branes, it maps a pair of bundles (E, F) to its 'negative' (F, E), in the sense $(E, F) = E - F$. According to [9], charges of d-branes localized at the singularity take values in the corresponding K-theory group $K_\pm(\mathbf{R}^{9-p,p-d})$ that will be the main subject of this section. Because calculation of $K_\pm(\mathbf{R}^{p,q})$ involves both unreduced and reduced K-theories, notations in this section slightly differ from the rest of the paper. Namely, we restore the conventional notation $\tilde{K}(X)$ for reduced cohomology of X with the base point, while the symbol $K(X)$ will denote unreduced K-theory.

It has been shown by M. J. Hopkins that calculation of $K_\pm$-groups can be carried out in terms of the usual $\mathbb{Z}_2$-equivariant K-theory by means of the formula [9]:

$$\tilde{K}_\pm(X) \cong K^{-1}_{\mathbb{Z}_2}(X \times \mathbf{R}^{1,0}) \tag{5.1}$$

Note that we always imply cohomology theory with compact support.

Since the right-hand side of (5.1) represents a functor in the complex category, multiplication by the Thom space of $\mathbb{C}$ (or $\mathbb{C}/\mathbb{Z}_2$) induces the periodicity isomorphisms:

$$\tilde{K}_\pm(\mathbf{R}^{p,q}) \cong \tilde{K}_\pm(\mathbf{R}^{p,q+2}), \qquad \tilde{K}_\pm(\mathbf{R}^{p,q}) \cong \tilde{K}_\pm(\mathbf{R}^{p+2,q}) \tag{5.2}$$

Therefore, $\tilde{K}_\pm(\mathbf{R}^{9-p,p-d})$ depends only on parity of p and d. Consider first the case when p is even. Application of the Hopkins' formula (5.1) leads to the equivariant group:

$$K_\pm(\mathbf{R}^{9-p,p-d}) \cong K^{-1}_{\mathbb{Z}_2}(\mathbf{R}^{10-p,p-d})$$

[9] Here we use equivalence of the topological and the analytical indices [33,34].

which, by the periodicity theorem (5.2), gives the answer for d-brane charges (p-even):

$$\tilde{K}_{\pm}(\mathbf{R}^{9-p,p-d}) \cong K^{-1}_{\mathbb{Z}_2}(\mathbf{R}^{10-p,p-d}) \cong K^{d-1}_{\mathbb{Z}_2}(\mathrm{pt}) \tag{5.3}$$

The last group is isomorphic to the representation ring $R[\mathbb{Z}_2]$ if d is odd, and is trivial if d-even [16,17]. However, p-even is not the case relevant to Type IIB orientifolds discussed in [4,5].

To determine $\tilde{K}_{\pm}(\mathbf{R}^{9-p,p-d})$ for p-odd, we employ the exact sequence (2.5) to the pair $(\mathbf{B}^{9-p,p-d}, \mathbf{S}^{9-p,p-d})$:

$$\ldots \to K^n_{\mathbb{Z}_2}(\mathbf{B}^{9-p+1,0}, \mathbf{S}^{9-p+1,0}) \to K^n_{\mathbb{Z}_2}(\mathbf{B}^{9-p+1,0}) \xrightarrow{\lambda} K^n(\mathbf{S}^{9-p+1,0}/\mathbb{Z}_2) \to \ldots \tag{5.4}$$

where we used the suspension isomorphism to substitute d by a $\mathbb{Z}_2$-graded index n. Let us analyze each term in the part of the sequence (5.4). The first term is obviously isomorphic to the K-theory $\tilde{K}_{\pm}(\mathbf{R}^{9-p,p-n})$ with compact support we are interested in. Since $\mathbf{B}^{9-p+1,0}$ is equivariantly contractable, we also get $K^n_{\mathbb{Z}_2}(\mathbf{B}^{9-p+1,0}) \cong K^n_{\mathbb{Z}_2}(\mathrm{pt})$, the second term in (5.4). Therefore, the sequence (5.4) relates groups in question to the cohomology theory of the horizon $H \cong \mathbf{RP}^{9-p}$ [24]:

$$K^n(\mathbf{RP}^{9-p}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_{2^r}, & r = \left[\frac{9-p}{2}\right],\ n \text{ even;} \\ 0, & n \text{ odd.} \end{cases}$$

Careful analysis of the ring structure shows that λ in (5.4) maps the generator of $K_{\mathbb{Z}_2}(\mathbf{B}^{9-p+1,0})$ to the generator of $K^n(\mathbf{RP}^{9-p})$. Finally, it follows that $\tilde{K}_{\mathbb{Z}_2}(\mathbf{R}^{9-p+1,0}) = \mathbb{Z}$ and $K^1_{\mathbb{Z}_2}(\mathbf{B}^{9-p+1,0}) = 0$. It is convenient to list the results in the following table:

$\tilde{K}_{\pm}(\mathbf{R}^{9-p,p-d})$	d even	d odd
p even	0	$R[\mathbb{Z}_2]$
p odd	$\mathbb{Z}$	0

Since only odd values of p are possible in Type IIB sring theory, d-brane charges localized on $\mathbf{R}^{p+1} \times (\mathbf{R}^{9-p}/(-1)^{F_L} \cdot \mathcal{I}_{9-p})$ orientifolds are classified by the second line of the table. Some states on such orientifolds have already been discussed in the literature. For example, if $p = 9$, we obtain the standard spectrum of Type IIA string theory: even-dimensional branes of arbitrary integer charge. Notice, we obtain a direct argument that

D-brane charges in Type IIA string theory are classified by $K(\Sigma X)$, regarding it as $(-1)^{F_L}$ orientifold of Type IIB theory.

Recently, non-BPS D-particle on such an $\mathcal{O}5$-plane has also been discussed by Sen [4,5]. Note, charge of the D-particle on the orientifold $\mathbf{R}^6 \times (\mathbf{R}^4/\Omega \cdot \mathcal{I}_4)$ takes value in $\mathbb{Z}_2$, while charge of the D-particle that lives on the $\mathbf{R}^6 \times (\mathbf{R}^4/(-1)^{F_L} \cdot \mathcal{I}_4)$ orientifold can be arbitrary integer. Actually there is no discrepancy here, because K-theory classifies charges of topologically stable objects only at weak coupling. On the contrary, S-duality which relates the two types of orientifolds inverts string coupling constant, *i.e.* maps type (i) orientifold at weak coupling to type (ii) orientifold at strong coupling. Hence, spectra of states may not be the same. Below we also show that masses of the states differ by a factor of $\sqrt{2}$.

5.1. Stringy Construction

Using analogy with Type IIA string theory, it is not difficult to provide string theory construction of the new states. In the case $p = 5$ this was done by Bergman and Gaberdiel [36]. Following the notation of [5,36], we define Type IIB closed string boundary state in the light-cone gauge:

$$|Bd, \eta\rangle = \exp\{\sum_{n>0} \frac{1}{n}[\alpha^I_{-n}\tilde{\alpha}^I_{-n} - \alpha^i_{-n}\tilde{\alpha}^i_{-n}] + i\eta \sum_{r>0}[\psi^I_{-r}\tilde{\psi}^I_{-r} - \psi^i_{-r}\tilde{\psi}^i_{-r}]\}|Bd, \eta\rangle^{(0)}$$

where $\eta = \pm$ and $n \in \mathbb{Z}$. Index r labels the fermion oscillators and runs over integers or half-integers ($r \in \mathbb{Z} + \frac{1}{2}$) depending on the sector: untwisted or twisted (U/T); NS or R; Neumann ($i = 1, \ldots, d+1$) or Dirichlet ($I = d+2, \ldots, 8$) boundary conditions. As usual, we choose NS – NS sector ground state $|Bd, \eta\rangle^{(0)}$ to be odd under $(-1)^{F_L}$ and $(-1)^{F_R}$. Therefore, NS – NS boundary state for new d-branes must have the same form as for ordinary Type II D-branes. On the other hand, because d is even, there are no R – R boundary states invariant under $(-1)^{F_L}$ in the untwisted sector of Type IIB string. Nevertheless, the closed string spectrum includes a twisted sector where the left-GSO projection is opposite, and we *do* get invariant R – R boundary states for d-even. It means that the even-dimensional branes found above can be interpreted as twisted states localized at the orientifold plane. Combining the contributions of NS – NS and R – R sectors, we obtain:

$$|Bd\rangle = (|Ud, +\rangle_{\mathrm{NS-NS}} - |Ud, -\rangle_{\mathrm{NS-NS}}) + (|Td, +\rangle_{\mathrm{R-R}} + |Td, -\rangle_{\mathrm{R-R}})$$

This boundary state has precisely the same form as the boundary state of the ordinary Type IIA d-brane. Hence, masses of the corresponding d-branes are also equal (there is no extra factor $\sqrt{2}$). The authors of [36] also noticed that masses of D-particles on orientifolds of type (i) and type (ii) are different. Here we observe that not only the masses of all other states do not match, but also their charges are different. Again, this confirms the idea that we can not simply follow from weak to strong coupling.

6. Miscellany

6.1. Orientifolds of type (iii) and Relation to Clifford Algebras

In the previous sections we considered Type IIB orientifolds where we divided either by Ω or by $(-1)^{F_L}$ perturbative symmetry group. Amalgamating the two cases we obtain orientifolds of type (iii):

$$\mathbf{R}^{p+1} \times (\mathbf{R}^{9-p}/\Omega(-1)^{F_L} \cdot \mathcal{I}_{9-p}) \tag{6.1}$$

Even though we will not try to develop $KR_{\pm}$-theory of such orientifolds, a few comments are in place here. In order to calculate groups $KR_{\pm}(X)$, we need an analog of Hopkins' formula (5.1) in the real category, something like:

$$KR_{\pm}(X) \cong KR_{\mathbb{Z}_2}(X \times \mathbf{R}^{1,1}) \tag{6.2}$$

Validity of such formula would strongly depend on the definition of the appropriate K-theory. For example, (6.2) would be true if we defined $KR_{\pm}(X)$ as a cohomology theory of X that fits into the following exact sequence (the way similar to how M. J. Hopkins defined $K_{\pm}(X)$ group):

$$\ldots \to KR_{\mathbb{Z}_2}(X) \to KR(X) \to KR_{\pm}(X) \to \ldots \tag{6.3}$$

Using the five lemma for (6.3) and the exact sequence in $KR_{\mathbb{Z}_2}$-theory for the pair $(X \times \mathbf{R}^{1,0}, X \times (\mathbf{R}^{1,0} - \mathrm{pt}))$ we come to (6.2). However, (6.3) might not be the suitable definition of $KR_{\pm}$ for orientifold applications.

There is another evidence to (6.2) based on the relation between K-theory of n-dimensional vector space X^n and the corresponding Clifford algebra $\mathcal{C}l_n$ [26]. In fact, in the present paper we are mainly interested in flat space-time orientifolds where $X = \mathbf{R}^{p,q}$. For this reason, in the rest of this subsection we make a short digression on the Clifford algebras of such spaces.

If we define A_n to be the Grothendieck group of graded $\mathcal{C}l_n$-modules modulo those extendable to $\mathcal{C}l_{n+1}$-modules[10], then there exists an isomorphism [25,26,37]:

$$A_n \cong K(X^n)$$

We can use this isomorphism twice, first to convert the problem to the algebraic one, and then to read off the answer for $K(X^n)$. In general, analysis of Clifford algebras is very simple, and many results in the previous sections become manifest once translated to the algebraic language. For example, let us prove the periodicity isomorphism (3.6), namely $\mathcal{C}l_{p,q} \cong \mathcal{C}l_{p-4,q+4}$, $p > 4$. Take an orthonormal basis of $\mathbb{R}^{p,q}$ generated by matrices γ_μ, such that[11]:

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu} \tag{6.4}$$

Now, we define:

$$\begin{cases} \gamma'_\mu = \gamma_\mu (\prod_{\nu=1}^4 \gamma_\nu), & \text{if } \mu = 1 \ldots 4; \\ \gamma'_\mu = \gamma_\mu & \text{otherwise.} \end{cases}$$

Then, according to (6.4), the subset $\{\gamma'_\mu\}$ of $\mathcal{C}l_{p,q}$ generates $\mathcal{C}l_{p-4,q+4}$. QED.

Involutions on X^n induce (anti-)automorphisms of the corresponding Clifford algebra $\mathcal{C}l_n$, and the latter are classified [27]. In the orientifolds (6.1) of type (*iii*) the involution maps a pair of gauge bundles (E, F) to $(\overline{F}, \overline{E})$. Since the tachyon (2.3) defines a scalar product on the spin bundle $S_+ \oplus S_-$, it suggests that the involution induces reversion automorphism of the Clifford algebra $\mathcal{C}l_{p,q}$. Calculation of the corresponding automorphism groups gives an independent argument to (6.2). To be specific we mention an intriguing example of a non-BPS state: a 3-brane with $\mathbb{Z}_2$ charge is localized on the 7-plane. However we will not pursue the analysis any further.

6.2. AdS Orbifolds

In the second section we briefly mentioned the AdS/CFT correspondence [20], which relates the conformal gauge theory on branes placed at the orbifold singularity and supergravity on the horizon manifold H. It would be interesting to investigate further implications of this duality in terms of K-theoretic relation (2.5) between X and H, *cf.* [21].

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[10] The inclusion map $\mathcal{C}l_n \to \mathcal{C}l_{n+1}$ is induced by $X^n \to X^n \oplus \mathbb{R}$.

[11] Note, here we use the equivalence between the Clifford algebra of the real space $\mathbf{R}^{p,q}$ with involution τ, $\tau^2 = +1$, and the Clifford algebra of the linear space $\mathbb{R}^{p,q}$ with signature (p, q) [23].

Let us consider an example of $\mathbb{Z}_3$ AdS orbifold which is dual to $\mathcal{N}=1$ superconformal field theory. Namely, we study Type IIB compactification on $\mathrm{AdS}_5 \times (\mathbf{S}^5/\mathbb{Z}_3)$ where the Lens space $H = L^2(3) = \mathbf{S}^5/\mathbb{Z}_3$ is a genuine horizon in the sense of [20,21]. It is dual to $SU(N)^3$ gauge theory on the boundary (= the gauge theory on N parallel 3-branes placed at the orbifold singularity) with nine chiral multiplets in the bifundamental representation of the gauge factors [38,39]. This SCFT has discrete global symmetry group [40]:

$$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3 \tag{6.5}$$

where $\mathbb{Z}_3$ factors are generated by A, C and B such that:

$$A^{-1}B^{-1}AB = C$$

Extended objects in the boundary theory which are charged under the discrete symmetry group (6.5) can be understood as Type IIB branes wrapped on various cycles in $H = S^5/\mathbb{Z}_3$. Because the horizon H has non-trivial homology groups $H_1(H) = H_3(H) = \mathbb{Z}_3$, we end up with even-dimensional objects propagating in AdS_5 with charges given by (6.5). Let us focus, say, on membranes which look like gauge strings on the boundary. There are three types of membranes corresponding to different $\mathbb{Z}_3$ factors in (6.5) — one can make a membrane by wrapping a 3-brane on a 1-cycle in H, and by wrapping a D5-brane or a NS5-brane on 3-cycles respectively. The charge of the NS5-brane corresponds to the last factor in (6.5), and accurate analysis shows that it does not commute with the other D-brane charges. Since in the present paper we deal with ordinary topological K-theory which does not take into account the Neveu-Schwarz B-field [12], we don't expect to see the last $\mathbb{Z}_3$ charge factor in (6.5). Indeed, calculation of the K-group of the Lens space H gives [41]:

$$K(H) = (\mathbb{Z}_3)^2 \cong H^{\mathrm{even}}(H, \mathbb{Z})$$

Complete agreement with the ordinary cohomology theory tells us that K-theory does not supply new objects for this orbifold example.

[12] Note, in our discussion $K^*(X)$ is always a commutative ring.

6.3. Toric Varieties

In fact, the result of the previous subsection is not very surprising. A number of space-time manifolds X (including the models of [21]) are birationally equivalent to smooth toric varieties. Vector bundles over such X have simple combinatorial description on the dual lattice (see *e.g.* [42]), and $K(X)$ can be examined in the same way [43]. By Lemma 1 of [43], $K(X)$ is free of torsion, that is the Chern character map:

$$ch\colon K(X) \to H^{\mathrm{even}}(X, \mathbb{Z}) \tag{6.6}$$

is an isomorphism [44]. Restriction of bundles to hypersurfaces and complete intersections in toric varieties enlarges the range of possible applications. More generally, (6.6) holds for CW complexes of low dimension [15].

7. Summary

As we have seen, K-theory is a powerful tool which helped us to study charges of non-BPS D-branes localized on the following types of orientifolds:

(*i*) For the orientifolds of the form (3.1), two choices of the projection ($\tau^2 = \pm 1$) lead to different K-theories: $KR(X)$ and $KH(X)$ respectively. In the case of flat space-time orientifolds (3.2), we calculated these groups with the result (3.7), (3.12). For example, we found new D-brane charges (3.8) and (3.13) localized on orientifold 5-planes $\mathcal{O}^-$ and $\mathcal{O}^+$. String theory construction of the new states with $\mathbb{Z}_2$ charges was also discussed. In general, odd-dimensional d-branes are represented by $d - \bar{d}$ configurations in Type IIB theory, while the description of d-branes with d-even involves extra fermion zero mode η. It would be interesting to further investigate the dynamics of such states either by topological methods of section 4, where we proved the discrete electric-magnetic duality in Type I theory [9], or via direct computation of string amplitudes [8].

(*ii*) In this case, calculation of the groups $K_\pm(\mathbf{R}^{9-p,p-d})$, p-odd, resulted in the spectrum of even-dimensional d-branes with arbitrary integer charges, like in Type IIA theory. These states are simply twisted states localized on $(-1)^{F_L}$ orientifolds.

(*iii*) Our discussion of $\Omega \cdot (-1)^{F_L}$ orientifolds is by no means complete. In order to calculate $KR_\pm(X)$, we conjectured the isomorphism (6.2) and made some arguments in favor of it. For the seven-plane example, it predicts the existence of 3-branes with $\mathbb{Z}_2$-valued charge.

Finally, we argued that K-theory of smooth (toric) compactifications and their orbifold limits does not supply new objects.

One can generalize the present analysis to other $\mathcal{M}$, say tori. Another aspect, which is not quite clear yet, is the relation to Clifford algebras mentioned in sections 3.3 and 6.1.

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