

Three Dimensional $N = 2$ Gauge Theories and Degenerations of Calabi-Yau Four-Folds

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Three dimensional $N = 2$ gauge theories with arbitrary gauge group and fundamental flavors are engineered from degenerations of Calabi-Yau four-folds. We show how Coulomb and Higgs branches emerge in the geometric picture. The analysis of instanton generated superpotentials unravels interesting aspects of the five-brane effective action in M theory.

key words: four-folds, gauge theories, degenerations

PACS numbers: 11.25.Mj, 11.30.Pb, 11.15.Tk

April 1998

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1. Introduction

Recently, string theory has provided remarkable insights in non-perturbative field theory dynamics. It has turned out that sophisticated field theory phenomena have a clear interpretation in terms of brane configurations [1,2,3] or local degenerations of Calabi-Yau spaces [4,5,6,7]. In certain situations, it is possible to construct a direct correspondence between the two types of constructions [8].

We consider $N = 2$ supersymmetric gauge theories with fundamental matter in three dimensions. As their $N = 1$ four dimensional counterparts, these theories exhibit very interesting phenomena such as confinement, Seiberg duality, emergence of various phases and instanton effects which require a deeper string theory understanding. While brane and field theoretic aspects have been already discussed in [9,10,11], we focus on geometric constructions.

It is clear that three dimensional QCD with the desired supersymmetry can be geometrically engineered from M theory compactification on Calabi-Yau four-folds. Similar F theory compactifications on specific compact Calabi-Yau spaces leading to four-dimensional theories were investigated in [12,13,14]. However, geometric engineering methods involve specific local (non-compact) degenerations which are effectively described by field theories. Although well understood for three-folds, the analysis is harder in four-fold context. An important simplification would be achieved if the relevant constructions could be carried out in the framework of toric geometry similarly to [15]. This has been realized for two dimensional gauge theories with four supercharges in [16]. The corresponding toric model describes a resolved ADE singularity fibered over P^2 . In order to obtain gauge theories with fundamental quark/anti-quark pairs, the toric diagram must be deformed by adding extra vertices as in [15]. These deformations would correspond to collisions of singularities which are known to yield matter hypermultiplets. It turns out that in toric setup, the matter multiplets are fibered over rational curves, therefore they do not actually give rise to quark multiplets in the low energy effective action [17]. This shows that the construction must be performed beyond toric framework so that the matter hypermultiplets can be fibered over curves of genus one. The relevant four-fold degenerations can be regarded as generalizations of the three-fold models considered in [18,19].

A very important dynamical aspect of gauge theories with four supercharges is the generation of a non-perturbative superpotential which can be exactly determined [20]. In M theory, this effect is dynamically generated by five-brane instantons wrapping compact

divisors of unit arithmetic genus [21]. The arithmetic genus constraint is closely related to the anomalous $U(1)_R$ superselection rules governing the superpotential in gauge theories. In the case studied in [5], this leads to the expected field theory results.

In theories with fundamental matter the superpotential is affected by quark zero modes [10],[11]. This leads to a splitting phenomenon dividing the Coulomb branch into different subwedges characterized by different instanton factors. At geometric level, these can be understood as subcones of the extended relative Kähler cone as in [18,19]. The derivation of the superpotential leads to an apparent paradox since the arithmetic genera of the compact divisors are stable under the flavor deformations. It turns out that the resolution involves certain subtleties related to the dynamics of the anti-symmetric tensor field in the five-brane world-volume. A careful analysis shows that the anomaly controlling the superpotential can receive perturbative contributions from the chiral two-form in the $(2, 0)$ tensor multiplet. This phenomenon has been anticipated in [22].

The organization of the paper is as follows: section 2 contains an outline of the geometric construction avoiding technical details. In section 3, we consider the quantum moduli spaces and the superpotential from geometric point of view including a discussion of the five-brane effective action. Section 4 provides a more rigorous treatment of four-fold degenerations with an accent on Kähler phases and flops interconnecting them. We conclude in section 5 with directions for future work.

2. Outline of the basic construction

We start with pure $SU(N)$ gauge theory on $R^3 \times S^1_R$ engineered in [5]. The three dimensional case is recovered in the limit $R \rightarrow 0$. These theories can be found in F-theory compactifications on elliptic Calabi-Yau four-folds plus a circle of radius R . This can be equivalently seen as a M theory compactification on an elliptic four-fold with the size of the elliptic fiber $\sim \frac{1}{R}$. The relevant degeneration consists of an A_{N-1} elliptic curve fibered over a two dimensional component B of the discriminant locus. The latter must be rational and can be taken either P^2 or a Hirzebruch surface F_n . Resolution of the singularity produces a chain of N P^1 components fibered over B which intersect according to the affine A_{N-1} Dynkin diagram. As noted in [5], each P^1 fibered over B defines a rational threefold divisor D_i , $i = 0 \dots N$. The divisors intersect pairwise along common sections isomorphic to B . We denote the divisor corresponding to the affine node by D_0 . Note that D_0 grows to

infinite size in the limit $R \rightarrow 0$. Therefore, in three dimensions we are left only with an ordinary Dynkin diagram as expected. The bare gauge coupling constant is given by

$$\frac{1}{g^2} = RV_B \quad (2.1)$$

where V_B is the base volume. The real Kähler moduli of the P^1 components combine with the periods of the magnetic six-form of the eleven-dimensional supergravity²

$$\int_{D_i} C^{(6)} \quad (2.2)$$

to yield $N - 1$ complex chiral multiplets Φ . These parameterize the complex one dimensional Coulomb branch of the theory. We can introduce alternative coordinates on the Coulomb branch

$$A_i = \Phi_i - \Phi_{i-1} \quad (2.3)$$

where $\Phi_0 = \Phi_N = 0$ by convention. Then the Coulomb branch is identified with the standard affine Weyl chamber

$$a_1 > a_2 > \dots > a_N \quad (2.4)$$

where a_i denotes the real part of the complex scalar field in the chiral multiplet A_i .

There is a non-perturbative superpotential generated by Euclidean five-brane instantons wrapping the six-cycles D_i [5]. The relevant instanton factors are given by

$$Y_i = \exp\left(\frac{1}{g^2}\phi_i\right), \quad Y = \exp\left(\frac{1}{g^2}\sum_{i=1}^{N-1}\phi_i\right). \quad (2.5)$$

Note that only Y is well defined throughout the entire Coulomb branch due to the splitting phenomenon discussed in [11]. The superpotential reads

$$W = \sum_{i=1}^{N-1} Y_i^{-1} + \gamma Y \quad (2.6)$$

where

$$\gamma = \exp\left(-\frac{1}{Rg^2}\right) \quad (2.7)$$

is related to the four dimensional QCD scale.

² From gauge theoretic point of view, these are compact scalars dual to the low energy photons along the Coulomb branch.

This construction can be generalized to arbitrary gauge groups [5,23]. The resolution of the singularity yields the exceptional locus

$$\sum_{i=0}^r n_i D_i \quad (2.8)$$

where r is the rank of the gauge group and n_i are the Dynkin numbers of the affine diagram. The global instanton factor is then defined by

$$Y = \exp \left(\frac{1}{g^2} \sum_{i=1}^{N-1} n_i \phi_i \right). \quad (2.9)$$

The main question addressed in the present paper is the generalization to gauge theories with matter multiplets. Matter fields are well understood in four and five dimensional field theories engineered from Calabi-Yau threefolds [15,18,19]. As a basic principle, quark multiplets generally arise from collision of singularities [24,25,26,27,28,29]. As opposed to three-folds, the four-fold case relevant here is less studied. We outline the idea of the construction, postponing the technical details for latter sections.

Consider two A_{N-1} and A_{N_f-1} singularities fibered over two components of the discriminant locus B , B_f intersecting transversely along an elliptic curve³ Σ . We take $B \simeq B_f \simeq P^2$ for concreteness. The collision results in a non-abelian conifold singularity [30] fibered over Σ , as detailed in appendix C. Resolution of singularities away from the collision locus gives an A_{N-1} tree fibered over B and an A_{N_f-1} tree fibered over B_f . Over Σ , we obtain a chain of rational components of length $N + N_f - 1$ (see [24] and appendix C). Note that this includes an extra P^1 component arising from the resolution of a conifold singularity. This configuration is actually expected to yield $SU(N) \times SU(N_f)$ gauge theory with bifundamental matter. In order to obtain $SU(N)$ gauge theory with N_f fundamental quarks, the dynamics of $SU(N_f)$ must be weakened by sending the size of B_f to infinity. The three dimensional quark multiplets can be thought as five dimensional hypermultiplets reduced on Σ .

The resulting picture consists of a chain of $N - 1$ divisors $D_1 \dots D_{N-1}$ as in the pure gauge theory case. As shown in section 4 and appendix C, there are different possible resolutions of the enhanced singularity over the collision locus. These lead to different Kähler phases with well defined gauge theory interpretation. For simplicity, we consider

³ We thank Cumrun Vafa for explaining this construction to us.

the case $N_f = 1$. The singularity can be resolved in such a way that the l -th divisor D_l develops a reducible fiber with two rational components intersecting transversely over Σ . In fact, it turns out that the divisor D_l undergoes an *embedded blow-up* along Σ . It can be shown that the components of the reducible fiber define the weights of the fundamental matter. Note that the resulting divisor is nonsingular and birationally equivalent to the rationally ruled threefold D_l .

As explained in [10,11], a $d = 3$ $N = 2$ quark multiplet can have both complex and real mass parameters. The real mass parameters are geometrically realized as Kähler moduli associated to the exceptional components defining the matter representations. In the particular case considered above, we have a single exceptional component whose size defines the quark real mass m . As explained in [18,19], the value of m specifies the particular Kähler phase (subcone) of the resolution. The phase in which the l -th divisor acquires a double fiber corresponds to

$$a_1 > \dots > a_l > m > a_{l+1} > \dots a_N. \quad (2.10)$$

Note that these regions can be identified with the subwedges of the Coulomb branch in gauge theory. The variation of m leads to geometrical phase transitions. The exceptional component is flopped from one divisor in the chain to an adjoint one. It is very suggestive to compare the geometrical flops with the brane picture as in Fig.1.

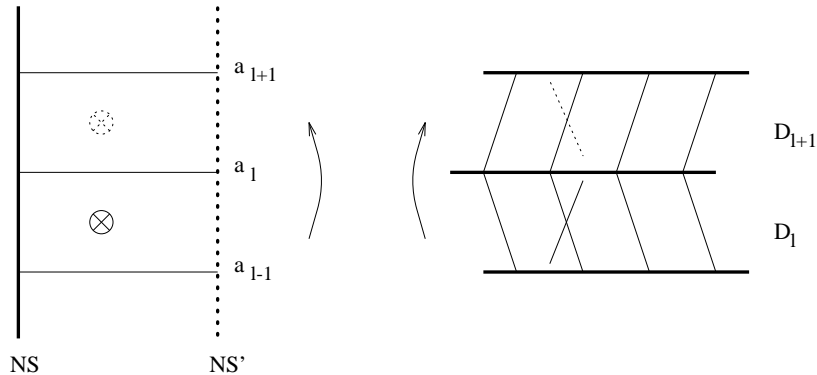


Fig. 1: The flop of the exceptional fiber component between two adjacent divisors is the geometrical counterpart of moving a flavor D5-brane in the brane construction.

3. Quantum Moduli Spaces and Superpotentials

The main dynamical test of the geometric construction is to check the non-perturbative superpotential against the gauge theory expectations. According to [10,11], in the subwedge (2.10), the l -th instanton should be lifted as an effect of the quark fermion zero

modes. In fact, if we have N_f flavors whose bare real masses satisfy (2.10), the number of zero modes (weighted by $U(1)_R$ charge) of the five-brane instanton should be $1 - N_f$. As we will see shortly, it turns out that understanding this effect in geometry involves certain subtleties.

Recall [21] that generation of a superpotential is controlled by the perturbative anomaly of a certain $U(1)$ symmetry of the five-brane effective action. The symmetry is generated by rotations $z \rightarrow e^{i\theta}z$ in the normal bundle $N_{D/X}$ of the six-cycle within the Calabi-Yau space X . Note that the Calabi-Yau condition implies that the normal bundle is isomorphic to the canonical bundle of the divisor $N_{D/X} \simeq K_D$. For pure gauge theories, it can be identified with the $U(1)_R$ symmetry of the gauge theory [31,32]. In general, the five-branes fermions must be twisted by $N_{D/X}^{1/2}$. Therefore, they are sections of

$$\Omega_{D_{-1/2}}^{0,0} \oplus \Omega_{D_{-1/2}}^{0,2} \oplus K_{D_{1/2}} \oplus \Omega_D^{0,2} \otimes K_{D_{1/2}} \quad (3.1)$$

where the lower index represents W charge. The total charge violation is given by the arithmetic genus of the divisor

$$\Delta W = \chi(\mathcal{O}_D) = \sum_{i=0}^3 (-1)^i h^{0,i}(D). \quad (3.2)$$

A superpotential is generated if

$$\chi(\mathcal{O}_D) = 1, \quad h^{0,i}(D) = 0, \quad i = 1, 2, 3. \quad (3.3)$$

In the present situation, this leads to a puzzle since the conditions (3.3) are stable under blow-up. The relevant holomorphic Hodge numbers do not change when the divisor acquires a double fiber over the collision locus. Therefore, it seems that the flavor divisor D_l still contributes to the superpotential, contradicting the gauge theory results. The resolution of this puzzle follows from a careful analysis of the effective action of the five-brane wrapped on the blown-up divisor. There is a problem here⁴ related to the quantization of the field strength G of the supergravity three-form. As shown in [33], the correct quantization of G in a curved background is

$$\int_{\Gamma} \left(G - \frac{\lambda}{2} \right) = 0 \quad (3.4)$$

⁴ We thank E. Witten for explaining this to us.

where $\lambda = p_1/2$ is half the Pontrjagin class of the compactification manifold and Γ is an arbitrary four-cycle. If λ is odd, the quantization condition (3.4) requires a half-integral flux on the four-cycles of the manifold. Since G is related to the five-brane field strength H by $G = dH$, this would prevent the wrapping of the five-brane. At the same time, a non-trivial G -flux would induce three dimensional Chern-Simons terms which would lift the gauge theory Coulomb branch [11]. In the present situation, it can be checked that the restriction of λ to the blown-up divisor is even (appendix D) avoiding these effects.

We consider first the case when the rationally ruled divisor D is blown-up once along the elliptic curve Σ in the base. Let \tilde{D} denote the resulting threefold. To fix notation, let ϵ denote the class of the P^1 fiber and γ^0, γ^1 denote the components of the reducible fiber over the curve Σ satisfying

$$\gamma^0 + \gamma^1 = \epsilon. \quad (3.5)$$

Note that γ^1 is the class of the exceptional P^1 while γ^0 is the proper transform of the original fiber. It will be shown in the next section that this geometry encodes the fundamental weights of the quarks. For now, we focus on the one-loop effective action of a five-brane wrapped on \tilde{D} .

As pointed out in [21,34], the non-perturbative superpotential is of the form

$$\int d^2\theta e^{-(V_D + i\phi_D)} f(m_a). \quad (3.6)$$

Here $V_D + i\phi_D$ is the classical instanton action and $f(m_a)$ is a holomorphic function of the moduli present in the problem obtained by integrating out the five-brane degrees of freedom at one loop. Since it is a determinant of massive modes, the function $f(m_a)$ does not vanish anywhere on the moduli space in generic situations. However, it has been shown in [22] that cancelations can occur⁵ if the six-cycle has $h_3 \neq 0$.

In the present case, the blown-up divisor \tilde{D} has non-trivial homology classes in degree three arising from the exceptional P^1 curve ruled above non-trivial one-cycles of Σ . If a, b denote a canonical set of generators of $H_1(\Sigma, Z)$ then $a \times \gamma^1, b \times \gamma^1$ define a set of generators of $H_3(\tilde{D}, Z)$. In fact the blown-up divisor \tilde{D} has Hodge numbers

$$h^{0,1}(\tilde{D}) = h^{0,3}(\tilde{D}) = 0, \quad h^{1,2}(\tilde{D}) = h^{2,1}(\tilde{D}) = 1. \quad (3.7)$$

Therefore \tilde{D} has a non-trivial intermediate Jacobian $J(\tilde{D}) = H^{1,2}(\tilde{D})/H^3(\tilde{D}, Z)$ which is a principally polarized abelian variety [35]. Moreover, it has been proved in [35] that the

⁵ We thank E. Witten for explanations and very helpful discussions on these points.

intermediate Jacobian of \tilde{D} is isomorphic (as a polarized abelian variety) to the Jacobian of the elliptic curve $J(\Sigma) = H^{0,1}(\Sigma)/H^1(\Sigma, Z)$. Note that a point $C \in J(\tilde{D})$ parameterizes the values of the supergravity three-form on three-cycles $a, b \times \gamma^1$ in $H_3(X, Z)$. The two real periods combine in a complex field which is the lowest component of a chiral multiplet [21]. Under the isomorphism to $J(\Sigma)$, C can be equivalently seen as a background $U(1)$ flat connection on the torus

The analysis of [22] shows that in this case the partition function of the chiral two-form in the $(2, 0)$ tensor multiplet is determined by a complex holomorphic line bundle \mathcal{L} over the intermediate Jacobian $J(\tilde{D})$. The line bundle \mathcal{L} is determined up to translation on $J(\tilde{D})$ by the Chern class, which is equal to the polarization $c_1(\mathcal{L}) = \omega$. The holomorphic structure can be described [22] in terms of a $U(1)$ connection B with curvature $F = 2\pi\omega$. In order to completely determine the line bundle one has to specify a collection of phases $H(C)$ corresponding to the holonomy of B around closed loops in the Jacobian associated to lattice vectors $C \in H^3(\tilde{D}, Z)$. In M-theory context, the phases $H(C)$ are determined by the eleven-dimensional Chern-Simons interaction (including the gravitational corrections) [22]. A precise computation is very hard to perform since the divisor \tilde{D} is not spin and the normal bundle is not trivial. Alternatively, \mathcal{L} can be described as follows. The Jacobian $J(\Sigma)$ has a distinguished holomorphic line bundle \mathcal{L}_0 associated to the Θ divisor. The line bundle \mathcal{L} can be described as the pull back of \mathcal{L}_0 by the map $T \rightarrow T \otimes S$ where S is a fixed flat line bundle on Σ . In most physical applications, S is determined by a fixed spin structure on Σ such that \mathcal{L} is the holomorphic determinant bundle of the corresponding Dirac operator. In the present case, S is in principle uniquely determined by M-theory data, but it is very hard to make explicit. However, this degree of accuracy suffices for our purposes. Since $c_1(\mathcal{L}) = \omega$, a simple application of the index theorem [22] shows that \mathcal{L} has a unique holomorphic section up to multiplication by a constant factor. This is the partition function of the chiral two-form $\theta(C)$ which depends holomorphically on the three-form periods C .

Now we can collect all the pieces and resolve our puzzle. Since the line bundle \mathcal{L} is equivalent to \mathcal{L}_0 up to translation, the holomorphic section $\theta(C)$ vanishes at precisely one point C^0 which is the translate of the Θ divisor. Therefore the superpotential cancels for precisely one special value C^0 of the three-form moduli. Clearly, the cancelation does not take place for generic values of C . This property identifies the periods of the three-form as complex mass parameters in the gauge theory [10]. More precisely the complex masses of the quarks are given $m_c = C - C^0$. For $C \neq C^0$, $m_c \neq 0$ and the superpotential does

not vanish in any subwedge of the Coulomb branch. Note that the line bundle \mathcal{L} and the values C^0 can depend a priori on the metric on the Calabi-Yau space [22]. However, this dependence is constrained by holomorphy. Since the line bundle can depend only on the real Kähler parameters by construction, it follows that it is actually independent of the complex Kähler moduli. Therefore the values C^0 are well determined once the complex structure is fixed.

The discussion can be extended to an arbitrary number of flavors. Let \tilde{D} denote the divisor D blown-up successively N_f times along the elliptic curve Σ . Let $\gamma^1 \dots \gamma^{N_f}$ denote the exceptional P^1 components and γ^0 denote the proper transform of the original fiber over Σ . Then, as shown in section 4, the curve classes corresponding to fundamental weights are given by

$$\sigma^\alpha = \gamma^0 + \dots + \gamma^{\alpha-1}, \quad \alpha = 1 \dots N_f. \quad (3.8)$$

The three-cycles $a, b \times \sigma^\alpha$ constitute a set of generators of $H_3(\tilde{D}, Z)$. Therefore $h^{1,2}(\tilde{D}) = N_f$ and the intermediate Jacobian is a N_f (complex) dimensional polarized abelian variety. A local computation based on appendix B and C shows that the three-cycles $a, b \times \sigma^\alpha$ and $a, b \times \sigma^\beta$ do not intersect for $\alpha \neq \beta$. Therefore, the intermediate Jacobian is isomorphic to the N_f -th direct product $J(\Sigma) \times \dots \times J(\Sigma)$. The complex line bundle is therefore of the form $\mathcal{L} \simeq \otimes_{\alpha=1}^{N_f} \mathcal{L}_\alpha$ where \mathcal{L}_α are line bundles as above on each factor. The two-form partition function reduces accordingly to a product of holomorphic sections $\theta_1(C_1) \dots \theta_{N_f}(C_{N_f})$. The three-form periods C_α determine the complex masses of the N_f flavors. The superpotential vanishes whenever $m_c^\alpha = C_\alpha - C_\alpha^0 = 0$ for some α in agreement with gauge theory expectations [10].

It has been earlier noted that the classical W symmetry corresponds to the $U(1)_R$ symmetry in the low energy gauge theory. In the presence of quarks, the five-brane instanton is expected to have $1 - N_f$ fermion zero modes (weighted by $U(1)_R$ charge). Their geometric realization is rather implicit in the above cancelation mechanism. It is known [36] that the Θ divisor on $J(\Sigma)$ corresponds to a distinguished spin structure S_0 on Σ . The line bundle \mathcal{L}_0 can be represented as the determinant line bundle of the family of twisted Dirac operators $D_{S_0}(C)$ where C is a generic point in $J(\Sigma)$. The zero in the Riemann theta function is then associated to a cancelation of the determinant as a result of a fermionic zero mode. Since \mathcal{L} is obtained from \mathcal{L}_0 by translation, we conclude that the zero of the section of \mathcal{L} can also be thought as a cancelation due to fermionic zero mode. In fact \mathcal{L} can be thought as a determinant of a family of twisted Dirac operators $D_S(C)$, but the

origin $D_S(0)$ being twisted by the line bundle S . The cancelation occurs for the special value of C which undoes the effect of the initial twist.

Since these fermion zero modes appear in a rather abstract manner, the question is how to interpret them. As shown in [21], the W charge violation can be interpreted as a diffeomorphism anomaly which must be cancelled by a classical effect. As a result, the compact magnetic scalar ϕ_D is shifted by $\chi(\mathcal{O}_D)$ under rotations in the normal bundle. In the above discussion, the cancelation of the superpotential can be effectively seen as an effect of chiral fermion zero modes localized along Σ . Since the fermions are locally sections of $K_\Sigma^{1/2}$ (ignoring the twist), these can contribute to the W charge anomaly if the latter is localized accordingly, that is if W acts trivially in normal directions to Σ . In local coordinates, if z is a coordinate on Σ and t, w are normal coordinates, W should act as $dz \rightarrow e^{i\theta} dz$, $dt, dw \rightarrow dt, dw$. Equivalently, the diffeomorphisms generated by W are required to preserve a normal neighborhood of Σ . The net effect is a N_f contribution to the W charge violation, but there is a subtle question related to the sign of this contribution. In the present conventions, the chiral two-form in the $(2, 0)$ tensor multiplet has self-dual field strength H which couples to the anti-self dual part of C , therefore the partition function is a holomorphic section over $J(\tilde{D}) \simeq J(\Sigma)^{N_f}$ rather than anti-holomorphic [22]. This means that the twisted Dirac operator couples to holomorphic flat bundles on Σ , therefore the fermionic zero modes are locally sections of $K_\Sigma^{1/2}$ rather than $\overline{K_\Sigma}^{1/2}$. Therefore the fermion zero modes have $W = 1/2$. Comparing to (3.1), we conclude that

$$\Delta W = 1 - N_f \tag{3.9}$$

for the blown-up divisor. This analysis suggests that in this case, the shift in ϕ_D picks up an extra $-N_f$ contribution from the diffeomorphism anomaly of the chiral two-form. A complete derivation of this effect is very subtle and will not be given here.

3.1. Quantum moduli spaces

We have seen above that the Coulomb branch of the three dimensional gauge theories can be identified with extended Kähler cones of Calabi-Yau degenerations. The geometric nature of Higgs branches can be similarly understood since the boundaries between Kähler subcones correspond to points where the quark multiplets become massless.

Consider an $SU(N)$ gauge theory with N_f flavors in the subwedge (2.10). A Higgs branch emerges from the boundary of the subcone when the flavors becomes massless $a_l =$

m . Since the corresponding exceptional P^1 is of zero size, the Calabi-Yau space is singular at the origin. The expectation values of the quark fields parameterize complex structure deformation of the singular locus. Therefore, we obtain an extremal transition between two branches of the Calabi-Yau moduli space generalizing the known conifold transitions. The discussion has been so far classical. Quantum mechanically, the Coulomb branch is complexified by adding the six-form periods (2.2). The emerging complex structure branch is invariant under arbitrary shifts of the latter, therefore, it should connect to the Coulomb branch along a complex codimension one locus. Thus we have recovered the quantum structure of the moduli space described in [10,11], from geometric point of view. The effective superpotential can be written

$$W_{pert} = -N_f (V_+ V_- \det M)^{\frac{1}{N_f}} \quad (3.10)$$

where V_{\pm} are holomorphic coordinates in the two regions of the Coulomb branch and $\det M$ is a gauge invariant coordinate on the Higgs branch. Similar transitions and the associated superpotentials have been discussed from F-theory point of view in [37]

In general, the exact dependence of the superpotential on the Coulomb branch variables is hard to determine due to the splitting phenomenon discussed in [11]. Once the geometric origin of the $U(1)_R$ charge understood, the analysis follows the same lines as in the gauge theory. We restrict to $SU(N)$ theory with $N_f \leq N - 1$ of equal mass m . As explained in appendix A, the extended Kähler cone is divided into $N - 1$ subcones separated by $N - 2$ boundaries defined by $a_l = m, l = 2 \dots N - 2$. Each boundary introduces a splitting region of the Coulomb branch which connects to the Higgs branch. This imposes certain restrictions on the instanton factors which cannot be analytically continued past the splitting regions. The procedure is explicitly carried out in appendix A.

The maximal region of the Coulomb branch unlifted by non-perturbative effects is semiclassically [11]

$$a_1 > a_2 = \dots = a_{N-1} = m > a_N \quad (3.11)$$

It can be parameterized by the global instanton factor Y . The effective superpotential as a function of $(Y, \det M)$ can be determined [11] by integrating out all the other fields in (A.6)

$$W \sim -(N_f - N + 1) (Y \det M)^{\frac{1}{N_f - N + 1}}. \quad (3.12)$$

Note that this result can be directly determined in M theory as an instanton effect associated to a five-brane wrapping the divisor sum $D_1 + \dots + D_{N-1}$. This corresponds

to a monopole of charges $(1, 1 \dots 1)$ in the gauge theory which is expected to have $N - N_f - 1$ zero modes [10]. The number of zero modes cannot be determined by a direct method because the Calabi-Yau space is singular along the sub-locus (3.11). Since it is independent of Kähler moduli, it can be computed for the blown-up Calabi-Yau space and then extrapolated by an adiabatic argument.

The sum $D_1 + \dots + D_{N-1}$ is a singular divisor with normal crossings whose arithmetic genus can be computed using Grothendieck-Riemann-Roch theorem (appendix E)

$$\chi(\mathcal{O}_D) = 1. \tag{3.13}$$

This apparently leads to a disagreement with the gauge theory result. Note however that the $N - 1$ branches of the five-brane can be separated in the transverse noncompact directions by giving expectation values to the scalars in the tensor multiplet. This corresponds to separating the monopole centers of charge in the gauge theory. Since the branches do not touch each other, each of them can be analyzed separately giving precisely one fermion zero mode. Therefore, the net W charge violation is

$$\Delta W = N - N_f - 1. \tag{3.14}$$

When the $N - 1$ branches come together, there are two sources of extra fermion zero modes. As also pointed out in [32], there are extra fermionic zero modes of negative charge signaling the occurrence of domain walls in four dimensions. The computation in appendix E shows that they are associated with corrections to the arithmetic genus coming from singular points of the sum divisor. At the same time there are new hypermultiplet degrees of freedom when two five-brane branches intersect along a section. The origin of these hypermultiplets has been described in [2]. To see that these fermions contribute to the anomaly, note that the zero mode problem can be analyzed by scaling the size of the base to be much larger than the size of the fiber. In this limit the degrees of freedom in the $(2, 0)$ tensor multiplet can be effectively reduced along the P^1 fiber. This results in a twisted $N = 2$ hypermultiplet localized along the base B . The twist is easily derived by reducing the fermions (3.1)

$$\Omega_{B_{-1/2}}^{0,0} \oplus \Omega_{B_{-1/2}}^{0,2} \oplus \Omega_B^{0,1} \otimes K_{B_{1/2}}. \tag{3.15}$$

This shows that the number of zero modes weighted by W charge is given by $\chi(\mathcal{O}_B)$. In fact we have derived a physical realization of the spectral sequence technique used in [5].

The net effect is that each hypermultiplet contributes $\chi(\mathcal{O}_B)$ to the W charge anomaly. Therefore, the final result is

$$\Delta W = 1 + N - 2 - N_f = N - N_f - 1. \quad (3.16)$$

If $N_f = N - 1$, we have $\Delta W = 0$ and the expected quantum effect is a deformation of the classical moduli space of the form

$$Y \det M = 1. \quad (3.17)$$

If $N_f = N - 2$, $\Delta W = 1$ and we expect a non-perturbative superpotential $\sim \frac{1}{Y}$. The dependence on the Higgs branch variables can be obtained following the instanton through the transition discussed above. Gauge invariance and holomorphy would restrict the dependence to an analytic function of $\det M$. However, since the origin is a singular point on the Higgs branch, the function is actually allowed to have a pole there [6]. Moreover, such a pole is expected on general grounds [38]. We conjecture⁶ that the behavior of the instanton generated superpotential is of the form

$$W \sim \frac{1}{Y \det M}. \quad (3.18)$$

Therefore we obtain the expected gauge theory results.

4. The geometry of Calabi-Yau degenerations

Here we describe in detail the Calabi-Yau four-fold degenerations leading to $d = 3 N = 2$ gauge theories with matter and simply laced gauge groups. The geometry corresponding to non-simply laced gauge groups is more complicated and will not be treated here. The construction is a generalization of the threefold models studied in [18,19].

⁶ It would be very interesting to check this conjecture by explicitly evaluating the determinants of the non-zero modes in the five-brane theory. A similar computation for Ray-Singer analytic torsion has appeared in [39].

4.1. $SU(N)$ with N_f quarks

As presented in section 2, pure $SU(N)$ gauge theory corresponds to a chain $D_1 \dots D_{N-1}$ of rationally ruled divisors over a rational base $B \simeq P^2$. Let $\epsilon_1 \dots \epsilon_{N-1}$ denote the curve classes of the rational fibers. Since the whole construction is embedded in a Calabi-Yau space, it is easy to prove (appendix B) that the intersection matrix

$$D_i \cdot \epsilon_k = \begin{cases} -2 & k = i \\ 1 & |i - k| = 1 \\ 0 & k \neq i, i \pm 1 \end{cases} \quad (4.1)$$

is the negative Cartan matrix of $SU(N)$. The weights of the fundamental matter correspond to extra curve classes $\sigma_1^\alpha \dots \sigma_N^\alpha$, $\alpha = 1 \dots N_f$, satisfying

$$D_i \cdot \sigma_k^\alpha = \begin{cases} -1 & i = k \\ 1 & k = i + 1 \\ 0 & \text{otherwise} \end{cases} \quad (4.2)$$

Real bare mass parameters correspond to extra divisors M_α in the ambient Calabi-Yau space [18,15] such that

$$M_\alpha \cdot \sigma^\beta = \delta_\alpha^\beta, \quad M_\alpha \cdot \epsilon_k = 0. \quad (4.3)$$

An arbitrary divisor supported on the exceptional locus can be written

$$D = \sum_{i=1}^{N-1} \phi_i D_i + \sum_{\alpha=1}^{N_f} m_\alpha M_\alpha \quad (4.4)$$

where ϕ_i are order parameters on the Coulomb branch and m_α are bare masses. Following [19], we can introduce new coordinates on the Coulomb branch

$$a_i = \phi_i - \phi_{i-1}, \quad i = 0 \dots N \quad (4.5)$$

with $\phi_0 = \phi_N = 0$. The negative Kähler cone is divided into subcones defined by the relations

$$\begin{aligned} -D \cdot \epsilon_k &> 0, & k = 1 \dots N-1 \\ -D \cdot \sigma_k &> 0, & k \leq l \\ -D \cdot \sigma_k &< 0, & k \geq l+1. \end{aligned} \quad (4.6)$$

It follows that the extended Kähler cone corresponds to the standard Weyl chamber

$$a_1 > a_2 > \dots > a_N. \quad (4.7)$$

The subcones are subwedges of the Weyl chamber defined by

$$a_1 > \dots > a_l > m_\alpha > a_{l+1} > \dots > a_N. \quad (4.8)$$

In geometrical terms, this means that the weight σ_α is represented by an irreducible holomorphic curve such that

$$\sigma_l \cdot D_l = -1. \quad (4.9)$$

This can be realized by blowing-up the l -th divisor along an elliptic curve Σ in the base. The resulting threefold can be viewed as a P^1 fibration over a two dimensional base with a reducible double fiber over Σ . This is the higher dimensional analogue of the exceptional curves of the first kind introduced in [19]. The resulting configuration is represented schematically in Fig.3. Bold vertical lines represent sections i.e. two dimensional surfaces. Dotted horizontal lines represent the P^1 fibers. The reducible fibers corresponding to matter representations is represented by a continuous line. It should be understood that they are fibered over an elliptic curve Σ in the base.

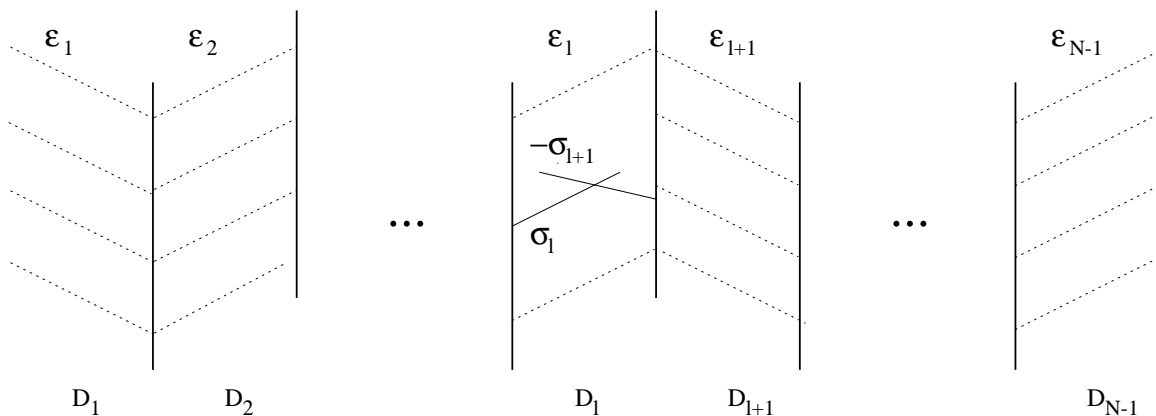


Fig. 2: The four-fold degeneration corresponding to $SU(N)$ theory with one flavor.

The fundamental weights are represented by the connected holomorphic curves

$$\begin{aligned} \sigma_k &= \epsilon_k + \dots + \epsilon_{l-1} + \sigma_l, & k &\leq l \\ -\sigma_k &= -\sigma_{l+1} + \epsilon_{l+1} + \dots + \epsilon_{k-1}, & k &\geq l+1. \end{aligned} \quad (4.10)$$

Note that the irreducible rational curves $\sigma_l, -\sigma_{l+1}$ are the two components of the double fiber

$$\sigma_l + (-\sigma_{l+1}) = \epsilon_l. \quad (4.11)$$

When the number of flavors is $N_f > 1$, the l -th divisor is blown-up N_f times along Σ . This results in a reducible fiber with $N_f + 1$ components $\gamma_l^0, \dots, \gamma_l^{N_f}$ represented in Fig.4. The intersection numbers $\gamma_l^\alpha \cdot D_l$ are $-1, 0 \dots, 0, -1$ (appendix B).

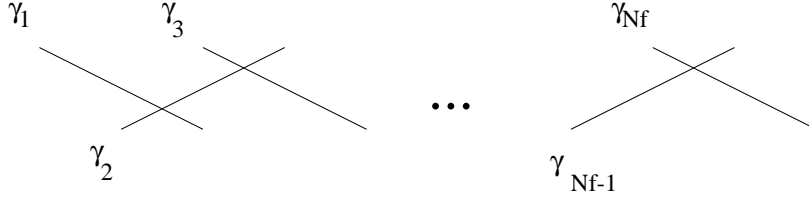


Fig. 3: The reducible fiber over the collision locus for $N_f \geq 2$.

The N_f fundamental weights are determined by the reducible curves

$$\sigma_l^\alpha = \gamma_l^0 + \gamma_l^1 + \dots + \gamma_l^{\alpha-1}. \quad (4.12)$$

Two adjoint subcones are related by a flop of a flavor curve, moving a fiber component from one divisor to the other. Note that this transition passes through a singular point when the flavor component is shrunk to zero size, i.e. when we have

$$m_\alpha = a_l \quad (4.13)$$

for some l and α . At this point it is possible to move on a different branch by a complex deformation of the singularity rather than a Kähler blow-up. In gauge theory, this corresponds classically to a transition to the Higgs branch, as explained in section 3.

Note that the degeneration constructed above can be naturally obtained as resolution of an $A_{N-1} + A_{N_f-1}$ collision [24,25,26,27,28,29]. The two singularities are trivially fibered over two P^2 components of the singular locus intersecting along the elliptic curve Σ . The size of the flavor P^2 must be taken to infinity in order to reduce $SU(N_f)$ to a global symmetry group. The divisors $D_1 \dots D_N$ result from the resolution of the A_{N-1} singularity. The reducible fibers corresponding to fundamental weights arise from the resolution of the A_{N_f-1} over the collision locus, as detailed in appendix C.

4.2. $Spin(2N)$ with fundamental matter

The degeneration corresponding to $Spin(2N)$ consists of N divisors $D_1 \dots D_N$ intersecting along sections as below.

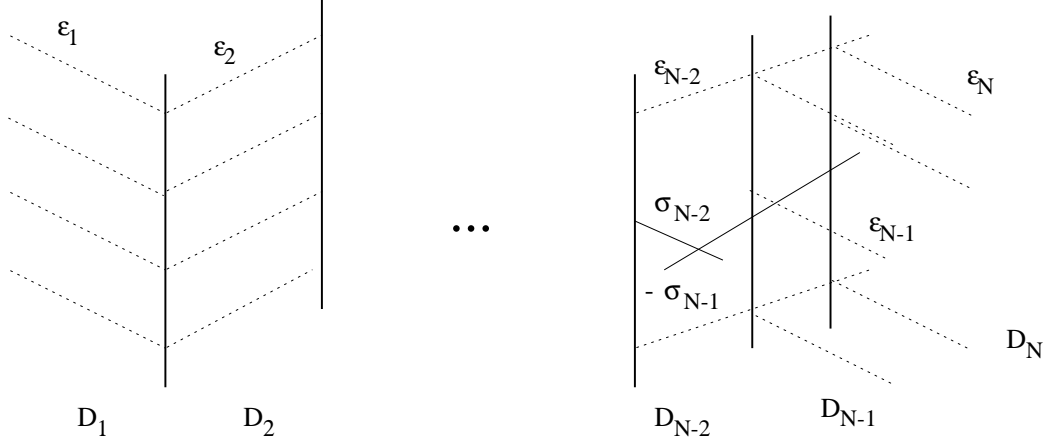


Fig. 4: The Calabi-Yau degeneration corresponding to $Spin(2N)$ with fundamental quarks.

If $\epsilon_1 \dots \epsilon_N$ denote the P^1 fiber classes, the intersection matrix

$$D_i \cdot \epsilon_k = \begin{cases} -2 & k = i \\ 1 & |i - k| = 1, i \neq N, k \neq N \\ 1 & (i, k) = (N - 2, N) \text{ or } (N, N - 2) \\ 0 & \text{otherwise} \end{cases} \quad (4.14)$$

is the negative $SO(2N)$ Cartan matrix. The fundamental weights are represented by connected holomorphic curves $\sigma_1 \dots \sigma_N$ satisfying

$$D_i \cdot \sigma_k = \begin{cases} -1 & k = i \\ 1 & k = i + 1 \\ -1 & (i, k) = (N, N - 1) \\ 0 & \text{otherwise} \end{cases} \quad (4.15)$$

$$\sigma_{N-1} + \sigma_N = \epsilon_N. \quad (4.16)$$

An arbitrary divisor supported on the exceptional locus can be written as

$$D = \sum_{i=1}^{N-1} \phi_i D_i + \frac{1}{2} \phi_N (D_N - D_{N-1}) + mM, \quad (4.17)$$

including the mass divisor M . We introduce alternative Coulomb branch coordinates

$$a_k = \phi_k - \phi_{k-1}. \quad (4.18)$$

Fundamental weights can be realized in the standard manner by blowing-up one of the first $N - 2$ divisor along an elliptic curve in the base. If we blow-up D_l , the relevant curve classes are

$$\begin{aligned} \sigma_k &= \epsilon_k + \dots + \epsilon_{l-1} + \sigma_l, & k \leq l \\ -\sigma_k &= -\sigma_{l+1} + \epsilon_{l+1} + \dots + \epsilon_{k-1}, & l + 1 \leq k \leq N - 1. \end{aligned} \quad (4.19)$$

The corresponding Kähler subcone is defined by

$$\begin{aligned} -D \cdot \sigma_k &> 0, & k \leq l \\ -D \cdot \sigma_k &< 0, & k \geq l + 1 \end{aligned} \tag{4.20}$$

which yield

$$a_1 > \dots > a_l > m > a_{l+1} > \dots > a_N > 0. \tag{4.21}$$

For $l = N - 2$, the components of the double fiber must intersect the divisors D_{N-1}, D_N as in Fig.8. As noted in $SU(N)$ case, the subcones of the Kähler cones are related by flops moving the singular fiber from one divisor to the other. Since the geometry is different, the flop relating the subcones $N - 2, N$ or $N - 2, N - 1$ is expected to have a peculiar behavior. Taking into account the exchange symmetry $D_{N-1} \leftrightarrow D_N$, it suffices to treat only one case, say $N - 2, N$. We first describe the geometry which realizes the fundamental weight σ_N as an irreducible fiber in D_N . The construction is similar to the $Spin(2N)$ threefold degeneration considered in [19].

The weights σ_{N-1}, σ_N must satisfy simultaneously

$$\begin{aligned} \sigma_{N-1} + \sigma_N &= \epsilon_N \\ \sigma_{N-1} &= \epsilon_{N-1} + \sigma_N. \end{aligned} \tag{4.22}$$

Since $\sigma_N \cdot D_N = -1$, it follows that σ_N must be the fiber of an exceptional divisor in D_N introduced by blowing-up along a curve Σ in the base. Moreover, Σ must be an elliptic curve in order to obtain exactly one flavor. We also have $D_{N-1} \cdot \sigma_N = 1$, therefore the divisors D_{N-1}, D_N must have nontrivial intersection. Equations (4.22) imply that

$$\epsilon_N = \epsilon_{N-1} + 2\sigma_N. \tag{4.23}$$

Therefore, the fiber of the ruling of D_N can be written as a sum between a rational component δ and $2\sigma_N$ over Σ . The curve δ is identified with the fiber of D_{N-1} over Σ . We conclude that the divisors D_{N-1}, D_N intersect along a rationally ruled surface over Σ with fiber $\delta \sim \epsilon_{N-1}$. Note that Σ is the collision locus of two sections of D_{N-1}, D_N as in Fig.9.

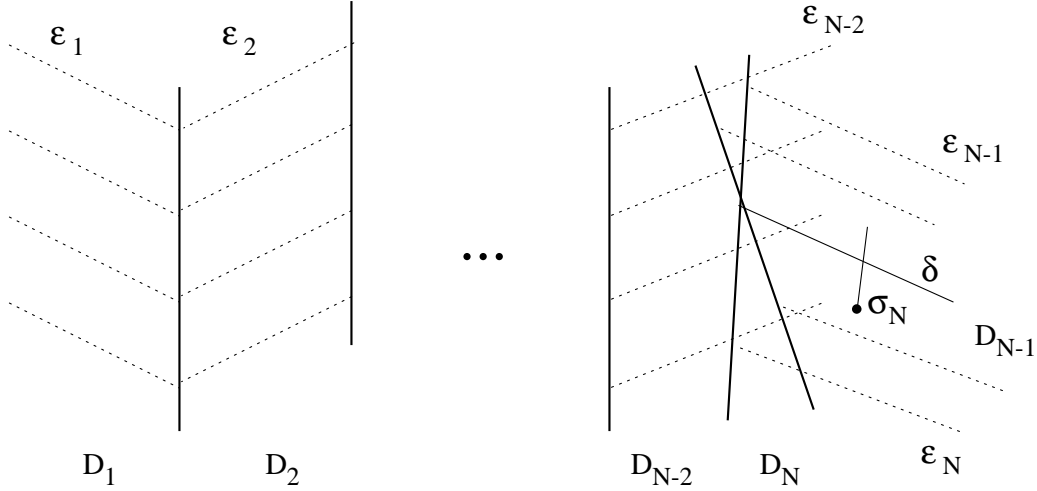


Fig. 5: $Spin(2N)$ with one fundamental flavor in the “special” subcone of the extended Kähler cone.

Finally, the intersection number $\delta \cdot D_N$ must be 0, therefore the normal bundle of δ in D_N must have degree -2 . This is similar to the situation described in [19]. If we restrict to a local analysis in directions normal to the curve Σ , it follows that the curve σ_N must pass through an ordinary double point as in [19]. The difference is that in the present situation, the singularity is fibered over Σ , therefore the divisor D_N has a curve of singularities.

The flop relating the model in figure Fig.8 to that in Fig.9 can be understood as follows. First, we contract the curve σ_{N-1} in Fig.8, introducing a curve Σ of ordinary double points in the ambient Calabi-Yau space. This results in a collision of the divisors D_{N-1} , D_N along a rationally ruled surface over the torus Σ with fiber δ . The common section δ passes through the singular locus of the four-fold. The singularity of the total space can be resolved by performing an embedded blow-up of the divisor D_N . Therefore D_N acquires an exceptional curve of the first kind fibered over the elliptic curve Σ . As shown above, this is configuration cannot be embedded in a smooth Calabi-Yau unless D_N develops at the same time a double point singularity along Σ .

The Kähler subcone corresponding to the present construction is defined by

$$-\epsilon_k \cdot D > 0, \quad -\sigma_N \cdot D_N > 0. \quad (4.24)$$

Hence it is identified with the

$$a_1 > a_2 \dots > a_N > m > 0 \quad (4.25)$$

subwedge of the Coulomb branch.

5. Conclusions

To summarize our results, we have extended the methods of geometric engineering to $N = 2$ gauge theories with fundamental matter. The construction is rather general, including various gauge groups with distinct matter content. Even though we mainly discuss the Coulomb branch of the field theories, we also provide information on the Higgs branch corresponding to complex deformations of the singular locus. For instance, we verify that the dynamically generated superpotentials on both branches match the gauge-theoretic expectations. Along with a geometric interpretation of the $U(1)_R$ charge, the analysis involves a careful zero-mode counting of five-brane instantons. Remarkably, the agreement with gauge theory is based on subtle aspects of the five-brane world volume dynamics. The flavor contribution to the $U(1)_R$ charge anomaly is results from the effective action of the chiral two-form in the $(2, 0)$ tensor multiplet. We also derive an interesting prediction concerning poles in the superpotential at the origin of the Higgs branch. It would be very interesting to explicitly check this prediction by computing the determinants of non-zero modes in the five-brane effective action. Presumably, the dependence on the complex structure moduli is similar to the dependence of Ray-Singer analytic torsion on the gauge field moduli.

Needless to mention, this article poses a number of questions which we hope to tackle in the future. Aside the superpotential term, there is less understood effective Kähler potential of the gauge theory. Apparently the geometric considerations are powerless against this term, since it is not protected from quantum corrections. At a closer look one can find that holomorphic topological data somehow enters the Kähler potential [16]. This fact is not surprising since the Coulomb moduli space should be *an analog of the special Kähler manifold* for higher-dimensional Calabi-Yau spaces [40] and is expected to serve some constraints on (if not totally define) the Kähler potential. Also at present we lack understanding of what happens when the number of quarks becomes large and equal to special values, e.g. $2N_c$, $3N_c$. This question touches non-abelian duality whose purely geometric nature remains elusive, even though a great step in this direction was done in [7].

Acknowledgments

We are very grateful to Cumrun Vafa and Edward Witten for collaboration and valuable suggestions and to Liviu Nicolaescu for mathematical assistance. We would also like to thank Ofer Aharony, Tom Banks, Jan de Boer, Michael Douglas, Rami Entin, Ori

Ganor, Jaume Gomis, Brian Greene, Barak Kol, Wolfgang Lerche, Nathan Seiberg and Piljin Yi for very helpful discussions and correspondence. S.G also would like to thank Laboratoire de Physique Théorique et Hautes Energies where a part of this work was done, and especially L. Baulieu for kind hospitality and the support of the CNRS grant.

The research of S. G. was supported in part by Merit Fellowship in Natural Sciences and Mathematics, grant RFBR-96-15-96939 and CNRS Foundation.

Appendix A. The superpotential for $SU(N)$ theory with N_f flavors

As stated in the text, we consider only the case when the flavors have equal mass m . The extended Kähler cone is divided in $N - 1$ subcones $K_1 \dots K_{N-1}$ characterized by

$$a_1 > \dots > a_l > m > a_{l+1} > \dots a_N. \quad (\text{A.1})$$

Let $Y_{1,l} \dots Y_{N-1,l}$ denote the corresponding instanton factors satisfying $Y = Y_{1,l} \dots Y_{N-1,l}$. Note that $Y_{i,l}, i \neq l$ have W charge -1 while $Y_{l,l}$ has W charge $N_f - 1$. Semiclassically, we have $Y_{i,l} \sim e^{(A_i - A_{i+1})/g^2}$. These relations can be inverted as follows

$$e^{A_i/g^2} = \left(\frac{Y_{i,l}^{N-i} Y_{i+1,l}^{N-i-1} \dots Y_{N-1,l}}{Y_{1,l} Y_{2,l}^2 \dots Y_{i-1,l}^{i-1}} \right)^{\frac{1}{N}}. \quad (\text{A.2})$$

Note that each boundary between subcones of the form $a_l = m$ introduces a splitting in the Coulomb branch. The perturbative superpotential in that region is of the form

$$W \sim -N_f (V_{+l} V_{-l} \det M)^{\frac{1}{N_f}}. \quad (\text{A.3})$$

where $V_{\pm l}$ are Coulomb branch variables given semiclassically by $V_{\pm l} \sim e^{\pm \alpha A_l/g^2}$ where α is determined such that (A.3) has the correct $U(1)_R$ charge. Using (A.2), we obtain

$$V_{+l} V_{-l} = \left(\frac{Y_{l,l}^{N-l} Y_{l+1,l}^{N-l-1} \dots Y_{N-1,l}}{Y_{1,l} Y_{2,l}^2 \dots Y_{l-1,l}^{l-1}} \right)^{\frac{\alpha}{N}} \left(\frac{Y_{l,l-1}^{N-l} Y_{l+1,l-1}^{N-l-1} \dots Y_{N-1,l-1}}{Y_{1,l-1} Y_{2,l-1}^2 \dots Y_{l-1,l-1}^{l-1}} \right)^{-\frac{\alpha}{N}}. \quad (\text{A.4})$$

The exponent α is determined such that the $U(1)_R$ charge of $(V_{+l} V_{-l})^{\frac{1}{N_f}}$ is 2. We find

$$\alpha = \frac{N-1}{N}. \quad (\text{A.5})$$

Introducing the global instanton factor Y , (A.4) becomes

$$V_{+l}V_{-l} = \left(\frac{Y^{N-1}}{Y_{1,l}^{N-l+1} Y_{1,l-1}^{l-2} Y_{2,l}^{N-l+2} Y_{2,l-1}^{l-3} \cdots Y_{l-1,l}^{N-1} Y_{l,l-1}^{N-1}} \right)^{\frac{1}{N-1}} \left(\frac{1}{Y_{l+1,l} Y_{l+1,l-1}^{N-2} \cdots Y_{N-1,l}^{N-l-1} Y_{N-1,l-1}^l} \right)^{\frac{1}{N-1}}. \quad (\text{A.6})$$

The instanton factors in $Y_{1,l-1} \cdots Y_{l-2,l-1}$ can be analytically continued across the boundary since they are not adjoint to the splitting region. In fact we can choose independent coordinates on the Coulomb branch $Y_{1,N-1} \cdots Y_{N-2,N-1}$, $Y_{21} \cdots Y_{N-1,N-2}$ and Y . Note that there are exactly $2N - 3$ as argued in [11]. Then

$$V_{+l}V_{-l} = \frac{Y}{Y_{1,N-1} Y_{2,N-1} \cdots Y_{l-1,N-1} Y_{l,l-1} Y_{l+1,l} \cdots Y_{N-1,N-2}}. \quad (\text{A.7})$$

We finally obtain the complete superpotential

$$W = -N_f \sum_{l=1}^{N_1} \left(\frac{Y \det M}{Y_{1,N-1} Y_{2,N-1} \cdots Y_{l-1,N-1} Y_{l,l-1} Y_{l+1,l} \cdots Y_{N-1,N-2}} \right)^{\frac{1}{N_f}} + \sum_{l=1}^{N-2} \frac{1}{Y_{l,N-1}} + \sum_{l=1}^{N-2} \frac{1}{Y_{l+1,l}}. \quad (\text{A.8})$$

Appendix B. Intersections

Here we present the details of intersection number computations. Let X denote a smooth Calabi-Yau four-fold and D denote a compact smooth divisor therein. The typical problem encountered in this paper is to compute the intersection of D with a holomorphic curve C embedded in D . According to [41], the relevant intersection number is defined as the degree of the pull-back line bundle $i^* \mathcal{O}(D)$ where $i : C \rightarrow D$ is the inclusion map and $\mathcal{O}(D)$ is the line bundle associated to D on X . Since X is Calabi-Yau, $\mathcal{O}_D(D) \simeq K_D$, therefore the intersection number is simply $\deg(i^* K_D)$. This can be applied in certain particular situations

1. D is rationally ruled over a surface B and C is the class of the P^1 fiber. The adjunction formula

$$K_C = i^* K_D \otimes N_{C/D} \quad (\text{B.1})$$

shows that

$$i^* K_D \simeq K_C \simeq \mathcal{O}_C(-2) \quad (\text{B.2})$$

since the normal bundle of a fiber is trivial. Hence $C \cdot D = -2$.

2. \tilde{D} is the blow-up of a rationally ruled divisor D along a curve Σ in the base and C is the P^1 fiber of the exceptional surface H . Note that H is a P^1 ruling over Σ with normal bundle

$$N_{\Sigma/\tilde{D}} \simeq \mathcal{O}_H(-1). \quad (\text{B.3})$$

Since

$$K_{\tilde{D}} = K_D + H, \quad (\text{B.4})$$

it follows that the restriction

$$i^*(K_{\tilde{D}}) \simeq \mathcal{O}_{P^1}(-1). \quad (\text{B.5})$$

Therefore, $C \cdot \tilde{D} = -1$. The generic rational fiber F splits into two components C, C' satisfying

$$C + C' = F. \quad (\text{B.6})$$

Since $F \cdot \tilde{D} = -2$, it follows that $C' \cdot \tilde{D} = -1$ reproducing the picture in section 1. If D is successively blown-up n times along the curve Σ , the resulting fiber has $n + 1$ components $C_0 \dots C_n$ intersecting transversely as in Fig.

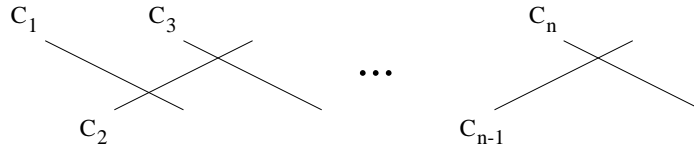


Fig. 6: The reducible fiber after n successive blow-ups.

A local computation shows that the degrees of the normal bundles are $-1, -2 \dots -2, -1$. Therefore the intersection numbers read

$$C_0 \cdot \tilde{D} = -1, \quad C_1 \cdot \tilde{D} = 0 \quad \dots \quad C_{n-1} \cdot \tilde{D} = 0, \quad C_n \cdot \tilde{D} = -1. \quad (\text{B.7})$$

This explains the correspondence between curve classes and representation weights used throughout the paper.

Appendix C. $A_{N-1} + A_{N_f-1}$ collisions

The resolution of $A_{N-1} + A_{N_f-1}$ is best understood for Calabi-Yau threefolds [24,25,26,27,28,29]. The case of interest here involves a four-fold collision when the singularities are fibered over two P^2 components of the singular locus intersecting along an elliptic curve Σ . Let z, t be local coordinates such that Σ is described as $z = 0$ in the first P^2 and $t = 0$ in the second. The local geometry in directions normal to Σ is described by the three-fold singularity

$$xy = z^N t^{N_f}. \quad (\text{C.1})$$

Here t a coordinate on the color P^2 which has finite volume while z is a coordinate on the flavor P^2 which is taken of infinite size.

For concreteness, we consider the particular case $N = 3$, $N_f = 1$. Then the singularity is

$$xy = z^3 t \quad (\text{C.2})$$

and the resolution involves three blow-ups.

1. Blow-up the plane $\{x = z = 0\}$. The resulting space can be covered by two affine coordinate patches (x, y, z_1, t) and (x_1, y, z, t) related by $x_1 z_1 = 1$. They map to the original C^4 by

$$z = x z_1, \quad x = x_1 z \quad (\text{C.3})$$

respectively. The proper transform of X is characterized by the equations

$$y = x^2 z_1^3 t, \quad x_1 y = z^2 t \quad (\text{C.4})$$

in the two coordinate open sets. The hypersurface is still singular in the second coordinate patch, but the degree of the singularity has decreased by one.

2. Blow-up the plane $\{x_1 = z = 0\}$. This leads again to two coordinate open sets (x_1, y, z_2, t) , (x_2, y, z, t) such that $x_2 z_2 = 1$ and

$$z = x_1 z_2, \quad x_1 = x_2 z. \quad (\text{C.5})$$

The blown-up space is described by

$$y = x_1 z_2^2 t, \quad x_2 y = z t. \quad (\text{C.6})$$

This shows that we are left with an ordinary double point (conifold) which can be resolved by a single blow-up

3. Blow-up the plane $\{x_2 = t = 0\}$. The affine coordinate sets are $(x_2, y, z, t) = (x_2, y, z, x_2 t_3) = (x_3, y, z, t)$ with $x_3 t_3 = 1$. The blown-up hypersurface is smooth

$$y = z t_3, \quad y x_3 = z. \quad (\text{C.7})$$

The resolved threefold can be covered by four affine open sets (x, z_1, t) , (x_1, z_2, t) , (x_2, z, t_3) , (x_3, y, t) . The exceptional locus consists of two rational curves $P^1(x_1)$, $P^1(x_2)$ fibered over the t -line yielding two rationally ruled surfaces. Note that the step 3 above is actually an embedded blow-up of the second surface which resolves at the same time the ambient threefold. Therefore, the second ruling has a double fiber localized at the point $x_2 = t = 0$. Since this picture holds locally at any point on the elliptic collision locus Σ , the global picture is the degeneration described in section 1.

The flop of the flavor component can also be understood in this picture. It corresponds to a different resolution in which steps 2 and 3 above are replaced by

2'. Blow-up the plane $\{x_1 = t = 0\}$. The affine open sets are $(x_1, y, z, t) = (x_1, y, z, x_1 t_2) = (x_2 t, y, z, t)$ with $x_2 t_2 = 1$ and

$$y = z^2 t_2, \quad y x_2 = z^2. \quad (\text{C.8})$$

3'. Blow-up the plane $\{x_2 = z = 0\}$. We have $(x_2, y, z, t) = (x_3 z, y, z, t) = (x_2, y, x_2 z_3, t)$ with $x_3 z_3 = 1$ and

$$y = x_2 z_3^2, \quad y x_3 = z. \quad (\text{C.9})$$

The exceptional locus consists again of two ruled surfaces over the t -line with fibers $P^1(x_1)$, $P^1(x_3)$. Step 2' is an embedded blow-up of the first surface at the point $x_1 = t = 0$. Therefore, the singular fiber has been flopped from one surface to the other.

Appendix D. Computation of λ

Here we compute the class $\lambda = \frac{p_1}{2}$ for the total space of the canonical bundle of a rationally ruled divisor D before and after the blow-up. We consider rationally ruled divisors D which can be obtained as a projectivization of the rank two vector bundle $= \mathcal{O} \oplus \mathcal{O}(n)$ over the base $B = P^2$. Let X denote the total space of K_D and let $\pi : X \rightarrow D$ denote the natural projection. The tangent space of X fits in an exact sequence

$$0 \rightarrow \pi^* K_D \rightarrow T_X \rightarrow \pi^* T_D \rightarrow 0. \quad (\text{D.1})$$

Therefore the total Chern class of T_X is given by⁷

$$c(X) = (1 - c_1(D))(1 + c_1(D) + c_2(D)) \quad (\text{D.2})$$

where $c_1(D), c_2(D)$ are Chern classes of D pulled back to X . The class λ of X is then given by

$$\lambda = \frac{1}{2} (c_1(X)^2 - 2c_2(X)) = c_1(D)^2 - c_2(D). \quad (\text{D.3})$$

Therefore the computation reduces to the Chern classes of the divisor D . Before blow-up, the total Chern class of D can be computed by an adjunction formula [43]

$$c(D) = (1 + c_1(B) + c_2(B))(1 + r)(1 + r + t) \quad (\text{D.4})$$

where $r = c_1(\mathcal{O}(1))$ – the $\mathcal{O}(1)$ bundle over the P^1 -bundle D – and $t = c_1(\mathcal{O}(n))$ pulled back from B . In the present situation $H^{1,1}(D)$ is two dimensional and generated by the divisor classes s – the section at infinity – and h – the pull-back of the hyperplane class of $B = P^2$. They satisfy [44]

$$s^3 = n^2, \quad s^2h = n, \quad sh^2 = 1, \quad h^3 = 0. \quad (\text{D.5})$$

Note that the zero section r is $r = s - nh$. It is convenient to introduce the curve classes F, H representing the generic fiber and the hyperplane class of the section s respectively. The intersection ring is then determined by the following tables

$$\begin{array}{ccc} & s & h \\ s & nH & H \\ h & H & F \end{array} \quad \begin{array}{cc} F & H \\ s & 1 & n \\ h & 0 & 1 \end{array}. \quad (\text{D.6})$$

Then, a direct computation shows that

$$c_1(D) = 2s + (3 - n)H, \quad c_2(D) = 3(1 - n)F + 6H. \quad (\text{D.7})$$

Hence

$$\lambda = (n(n - 3) + 6)F + 6H \quad (\text{D.8})$$

is even for any integer value of n .

⁷ Since the degeneration can be thought as embedded in a large compact Calabi-Yau four-fold, we do not use compact vertical cohomology on X . Therefore the Thom class is simply $-s\pi^*c_1$ [42].

Appendix E. Grothendieck-Riemann-Roch for singular varieties

The extension of Grothendieck-Riemann-Roch theorem to singular varieties is due to [45]. Here we are concerned only with applications to singular curves with ordinary double points, following closely [46]. Let $C = \sum_{i=1}^N C_i$ be a reducible curve consisting of N rational components intersecting according to either the ordinary (a) or the affine (b) A_N Dynkin diagram. Let \tilde{C} denote the normalization of C , Note that \tilde{C} consists of N disjoint rational components with a collection of marked points that map pairwise to the double points of C . According to [46], we have

$$\chi(C, \mathcal{O}_C) = \chi(\tilde{C}, \mathcal{O}_{\tilde{C}}) - \sum_P \delta_P \quad (\text{E.1})$$

where the sum is over all double points P and

$$\delta_P = \text{length}(\mathcal{O}_{\tilde{C}}/\mathcal{O}_C)_P. \quad (\text{E.2})$$

For an ordinary double point, $\delta_P = 1$ [47]. Since \tilde{C} is just a collection of N disjoint rational curves we get

$$\chi(C, \mathcal{O}_C) = 1 \quad (\text{E.3})$$

in case (a) and

$$\chi(C, \mathcal{O}_C) = 0 \quad (\text{E.4})$$

in case (b).

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