

# The Coulomb branch of $N=1$ supersymmetric gauge theory with adjoint and fundamental matter \*

Anton Kapustin

*California Institute of Technology, Pasadena, CA 91125*

## Abstract

We consider  $N = 1$   $SU(N_c)$  gauge theory with an adjoint matter field  $\Phi$ ,  $N_f$  flavors of fundamentals  $Q$  and antifundamentals  $\tilde{Q}$ , and tree-level superpotential of the form  $\tilde{Q}\Phi^l Q$ . This superpotential is relevant or marginal for  $lN_f \leq 2N_c$ . The theory has a Coulomb branch which is not lifted by quantum corrections. We find the exact effective gauge coupling on the Coulomb branch in terms of a family of hyperelliptic curves, thus providing a generalization of known results about  $N = 2$  SUSY QCD to  $N = 1$  context. The Coulomb branch has singular points at which mutually nonlocal dyons become massless. These singularities presumably correspond to new  $N = 1$  superconformal fixed points. We discuss them in some detail for  $N_c = 2, N_f = 1$ .

---

\*Work supported in part by the U.S. Dept. of Energy under Grant no. DE-FG03-92-ER 40701.

## I. INTRODUCTION AND SUMMARY

The phenomenon of electric-magnetic duality in supersymmetric gauge theories has attracted a lot of attention recently. N. Seiberg presented strong evidence that certain  $N = 1$  SUSY gauge theories flow to the same theory in the infrared [1]. Soon afterwards many new examples of such “duality” were found, and it became clear that this phenomenon is generic. However, the precise reason why the dual gauge theories are so common is not understood, nor is it clear which theories actually admit a dual description. A particularly mysterious case is that of gauge theories with both fundamental and tensor matter and no tree-level superpotential. Here we will focus on  $SU(N_c)$  gauge theories with one adjoint superfield  $\Phi$ ,  $N_f$  fundamentals  $Q$  and  $N_f$  antifundamentals  $\tilde{Q}$ . A possible line of attack would be to deform the action by adding a suitable relevant or marginal superpotential, so that the moduli space be truncated and the theory become manageable. Kutasov and collaborators argued [2–4] that the operator  $\text{Tr}\Phi^{k+1}$  is relevant in the infrared for all  $k < N_c$  and sufficiently small  $N_f$ , and constructed the dual descriptions of the infrared physics in terms of  $SU(kN_f - N_c)$  gauge theory. In this letter we consider a different deformation, namely by the operator  $\tilde{Q}\Phi^l Q$ . It turns out that this operator is relevant for  $l < 2N_c/N_f$  and marginal for  $l = 2N_c/N_f$ . The case  $l = 1$  is essentially the  $N = 2$  QCD studied by Seiberg, Witten and others. For  $l > 1$  we can only have  $N = 1$  SUSY, and therefore these theories provide an  $N = 1$  generalization of Seiberg-Witten theories. Classically, the theories we consider have Coulomb and Higgs branches. We focus on the Coulomb branch, because unlike the  $N = 2$  case the Higgs branch may be lifted by quantum corrections.

Our results can be summarized as follows: the matrix of the gauge couplings  $\tau$  can be thought of as a normalized period matrix of a genus  $N_c - 1$  algebraic curve. Because of the electric-magnetic duality  $\tau$  is not uniquely defined at any given point in the moduli space, but the curve itself is. We find these curves for arbitrary superpotential of the form

$$W = \sum_{i=0}^l h_{i\beta}^\alpha \tilde{Q}_\alpha \Phi^i Q^\beta \quad (1)$$

and any  $N_f$  and  $N_c$ . The family of curves we obtain is isomorphic to that describing  $N = 2$  SUSY QCD with  $lN_f$  flavors. Thus at nonsingular points of the Coulomb branch the  $N = 1$  theory with the superpotential eq. (1) flows to the same infrared theory as  $N = 2$  SUSY QCD with  $lN_f$  flavors. We can say that these two theories are dual to each other.\*

The curves become singular at the submanifolds of the moduli space where charged particles become massless. Computing the monodromies associated with these singularities allows one to determine their electric and magnetic charges. In general both electric and magnetic charges are nonzero, and we will refer to such particles as dyons. If all charged massless particles are mutually local, then the low-energy theory becomes a free field theory after an appropriate duality transformation. One can also tune the moduli and/or the superpotential so that several singularities corresponding to mutually nonlocal dyons collide. Analogy with Ref. [5] suggests that at such points in the moduli space the theory flows to an interacting  $N = 1$  superconformal field theory. These superconformal theories are manifestly inequivalent to fixed points of  $N = 1$  SUSY QCD discussed in Ref. [1] (they have much smaller flavor symmetry groups).

Though the singularity structure and the monodromies of the curves are the same as for  $N = 2$  SUSY QCD with  $lN_f$  flavors, the interpretation of the singular points appears to be different. For example,  $N = 2$  SUSY QCD has points in the moduli space with  $lN_f$  massless quarks and  $SU(lN_f)$  flavor symmetry. In the theory with superpotential eq. (1), the same points have only  $N_f$  charged quarks and therefore a smaller symmetry group. Similarly, the points corresponding to interacting superconformal field theories presumably have only  $N = 1$  supersymmetry and are not isomorphic to  $N = 2$  superconformal theories of Ref. [6]. As an example, we consider the case of  $N_c = 2$  for arbitrary relevant superpotential of the

---

\* At the microscopic level they have different flavor symmetry groups, but in the infrared, away from the singularities, the massless particles carry no flavor quantum numbers, and therefore the flavor symmetry is effectively absent at low energies.

form eq. (1) and discuss the Argyres-Douglas points in the moduli space.

## II. GENERALITIES

Consider first the N=1  $SU(N_c)$  gauge theory with  $N_f$  fundamentals  $Q$ ,  $N_f$  antifundamentals  $\tilde{Q}$ , an adjoint field  $\Phi$  and no superpotential. It is very likely that this theory is in the nonabelian Coulomb phase at the origin of the moduli space [2–4]. The anomalous dimensions of the fields cannot be computed, but the vanishing of the beta-function [7] imposes one condition on them:

$$N_f\gamma_Q + N_c\gamma_\Phi = N_f - 2N_c. \quad (2)$$

The dimension of the operator

$$M_{l\alpha}^\beta = \tilde{Q}_\alpha \Phi^l Q^\beta \quad (3)$$

is given by

$$\dim M_l = 3 - \frac{(2N_c - lN_f)(\gamma_\Phi + 2)}{2N_f}. \quad (4)$$

Since  $\gamma_\Phi + 2$  is the dimension of a gauge invariant operator  $\text{Tr}\Phi^2$ , it is positive by unitarity. Thus we conclude that  $M_l$  is a relevant operator for  $lN_f < 2N_c$ , marginal for  $lN_f = 2N_c$ , and irrelevant otherwise. Classically, the chiral ring is truncated for  $l < N_c$ . Thus we can have both the situation where the chiral ring is truncated classically but the operator is irrelevant quantum-mechanically, and the situation where the classical chiral ring is not truncated at all, but the operator is still relevant. Note also that here we are in a better position than in the case of the  $\text{Tr}\Phi^{k+1}$  perturbation, since we can determine exactly when our perturbation is relevant.

Interestingly, when  $lN_f = 2N_c$  the operator  $M_{l\alpha}^\alpha$  is not just marginal, it is *exactly* marginal. To see this we note that such a perturbation preserves enough symmetry for all  $Q$ 's and  $\tilde{Q}$ 's to have the same dimension. Then it is an easy matter to check that the beta-functions for the coupling  $h_l$  corresponding to  $M_{l\alpha}^\alpha$  and for the gauge coupling  $g$  are

proportional, and thus in the  $g - h_l$  plane there must be a line emanating from the fixed point  $(g^*, 0)$  on which both beta-functions vanish. For  $l = 1$  the line is actually just  $g = h_1$ . Moreover, if one sets  $g = h_1$  from the very beginning, the theory has  $N = 2$  SUSY and is scale-invariant. For  $l > 1$  the full theory cannot be scale invariant, since eq. (2) requires

$$2\gamma_Q + l\gamma_\Phi = 2(1 - l), \quad (5)$$

and thus both anomalous dimensions cannot vanish simultaneously. Actually, we know that for  $lN_f = 2N_c$  and  $l > 1$  the beta-function for the Wilsonian coupling is nonzero, and thus the theory generates a strong coupling scale  $\Lambda$  (by definition, this is the scale at which the Wilsonian gauge coupling blows up in perturbation theory).

A remark is in order. For  $l > 1$  and  $N_f < 2N_c$  adding  $M_l$  to the Lagrangian renders the theory nonrenormalizable. Therefore one should regard the perturbed theory as an effective field theory valid below some ultraviolet cutoff  $M$  which is of the order  $h_l^{-1/(l-1)}$ . We will assume in what follows that the strong coupling scale  $\Lambda$  is much smaller than  $M$ .

We will consider adding the tree-level superpotential of the form

$$W = \sum_{i=0}^l h_i M_i. \quad (6)$$

The coefficients  $h_i$  transform in the  $(\overline{\mathbf{N}}_{\mathbf{f}}, \mathbf{N}_{\mathbf{f}})$  of the flavor group  $SU(N_f)_L \times SU(N_f)_R$ . Classically, the theory has a moduli space of vacua which has both Coulomb and Higgs branches. We have nothing to say here about the latter (it may not even exist in quantum theory). But it is easy to see that the former cannot be lifted by quantum effects. Indeed, the theory under consideration has an anomaly-free R-symmetry with  $R_\Phi = 0, R_Q = R_{\tilde{Q}} = 1$ . Therefore any dynamically generated superpotential is quadratic in  $Q, \tilde{Q}$  and cannot lift the Coulomb branch. In what follows, when we mention the moduli space, we always mean the quantum moduli space of the Coulomb branch.

### III. THE CURVES

The matrix of the Wilsonian  $U(1)$  gauge couplings  $\tau$  on the Coulomb branch is a complex symmetric  $r \times r$  matrix, where  $r = N_c - 1$  is the rank of the group. It was explained by Seiberg and Witten that any  $Sp(2r, \mathbf{Z})/\mathbf{Z}_2$  transformation of  $\tau$  leaves physics invariant, and therefore  $\tau$  may have nontrivial monodromies as one encircles singularities of the moduli space. Thus  $\tau$  is best thought of as a normalized period matrix of a family of genus  $r$  algebraic curves. Our first task is to find the appropriate family of the curves. The arguments are a straightforward generalization of those in Ref. [8]. We assume that the curve is hyperelliptic, and consider first the case  $lN_f = N_c$ . The symmetries and the requirement that the limit  $\Lambda \rightarrow 0$  give the classical results constrain the curve to be of the form

$$y^2 = P_{N_c}(x, \phi)^2 - \Lambda^{2N_c - N_f} Q(x, \phi, h) - \Lambda^{4N_c - 2N_f} R(x, \phi, h). \quad (7)$$

Here  $P_{N_c}(x, \phi) = \prod_i (x - \phi_i)$ , and  $\phi_i, i = 1, \dots, N_c$  are the eigenvalues of  $\Phi$ . Giving the first  $k$  eigenvalues a vev much larger than  $\Lambda$  (but smaller than the ultraviolet cutoff  $M$ ) breaks the gauge group down to  $SU(k) \times SU(N_c - k) \times U(1)$  at weak coupling. We can also tune the coefficients  $h_i, i = 0, \dots, l$  so that the low-energy theory still has  $N_f$  quarks in the  $(\mathbf{k}, \mathbf{1})$  representation of the gauge group. In this limit the curve should factorize appropriately. In fact, factorization requires that  $R$  be identically zero and  $Q$  be independent of  $\phi_i$ . We conclude that the curve has the form

$$y^2 = P_{N_c}(x, \phi)^2 - \Lambda^{2N_c - N_f} Q(x, h). \quad (8)$$

To determine the form of the polynomial  $Q$  we give all the quarks large masses. The quark mass matrix is  $h_0$ , so the limit we wish to consider corresponds to the eigenvalues of  $h_0$  being much larger than  $\Lambda$ , though still smaller than  $M$ . In this regime the semiclassical analysis is adequate, and we know that the curve must be singular whenever one of the eigenvalues  $\phi_i$  is the root of the polynomial

$$\det \sum_{i=0}^l h_i x^i. \quad (9)$$

This requirement determines that  $Q(x, h)$  must be given by eq. (9).

To obtain the curves for  $lN_f \neq N_c$  we may either integrate out some of the quarks by giving them large masses, or to break the gauge group at weak coupling by large adjoint vevs. The result is that for any  $N_f, N_c$  and  $l$  such that  $lN_f < 2N_c$  the curve is given by

$$y^2 = P_{N_c}(x, \phi)^2 - \Lambda^{2N_c - N_f} \det \sum_{i=0}^l h_i x^i. \quad (10)$$

Note that the “quantum piece” of the curve has degree in  $x$  smaller than the classical piece if  $lN_f < 2N_c$ , which is exactly the condition for the perturbation to be relevant! Thus the curve has genus  $N_c - 1$ , as required, for any  $l$  for which the perturbation is relevant.

The case  $lN_f = 2N_c$  is somewhat special in that the perturbation by  $M_l$  is marginal rather than relevant. The curve may depend then on the dimensionless parameter  $t = \Lambda^{2N_c - lN_f} \det h_l$ . This is in agreement with the discussion in the previous section where it was argued that for  $lN_f = 2N_c$  there exists an exactly marginal perturbation of the theory without the tree-level superpotential. The preceding discussion then does not determine the polynomial  $Q$  uniquely. Instead, we only obtain that it is given by eq. (9) with  $h_i$ 's replaced by

$$\tilde{h}_i = h_i G_i(t), \quad (11)$$

where the functions  $G_i(t)$  are analytic at  $t = 0$  and satisfy  $G_i(0) = 1$  for all  $i$ 's. One may regard this as some nonperturbative renormalization of the coefficients  $h_i$ . We cannot determine the exact form of this renormalization, but this is not important, since the region of interest  $\Lambda \ll M$  corresponds to  $t \ll 1$ , and we can just approximate  $G_i$ 's by 1. Note also that for  $lN_f = 2N_c$  the second term in eq. (10) is of the same degree in  $x$  as the first term, and the curve still has the right genus.

As a check on the curve, one can compute the semiclassical monodromies and compare them with those given by the curve. Here we will do this only for  $N_c = 2$ . It is easy to see that traversing a large circle in the complex plane of  $u = \frac{1}{2} \text{Tr} \Phi^2$  induces the monodromy  $\mathcal{M}_\infty = T^{lN_f/2-2}$ . (The power of  $T$  may be half integer because we normalized the charge

of the quark to be  $1/2$  rather than one. The resulting sign ambiguity has no influence on  $\tau$ , since the duality group is  $SL(2, \mathbf{Z})/\mathbf{Z}_2$  rather than  $SL(2, \mathbf{Z})$ .) On the other hand, at large  $u$  all charged particles acquire large masses, and the gauge coupling stays small all the way down to extreme infrared. Thus the perturbative computation is adequate in this regime. In fact, there are no perturbative corrections to the (Wilsonian) gauge coupling beyond one loop [7]. To do the one loop computation, we notice that the charged particles originating from the adjoint fields acquire masses of the order  $u^{1/2}$  from the  $D$ -terms, while the fundamental matter gets mass only from the  $F$ -terms. This “ $F$ -term” mass is of the order  $h_l u^{l/2} \ll u^{1/2}$ . Thus first we must integrate out the adjoint matter and  $W$ 's, and then the fundamental matter. The resulting expression for the gauge coupling

$$\tau(u) = i \frac{4 - lN_f}{4\pi} \log u + \text{const} \quad (12)$$

produces the right monodromy. In particular, for  $lN_f = 4$ , the effective gauge coupling does not depend on  $u$  in perturbation theory.

One can easily see that the family of curves in eq. (10) is identical to the family of curves describing  $N = 2$   $SU(N_c)$  SUSY QCD with  $lN_f$  fundamental hypermultiplets [8]. Away from singular points the infrared limit is just a free field theory of photons, neutral scalars and their superpartners, and thus automatically has  $N = 2$  SUSY. We may say that our model is dual to  $N = 2$  SUSY QCD with  $lN_f$  flavors. The operators parametrizing the flat directions on the moduli space,  $\text{Tr}\Phi^i, i = 2, \dots, N_c$ , map trivially between the two theories.

For  $l = 1$  the theory actually flows to  $N = 2$  SUSY QCD everywhere in the moduli space, including singularities [9]. For  $l > 1$  this is no longer true, as illustrated in the next section, where we analyze in more detail the case of  $N_c = 2$ .

## IV. EXAMPLES

### A. $N_c = 2, l = 1$

The curve is given by

$$y^2 = (x^2 - u)^2 - \Lambda^{4-N_f} \det(h_0 + h_1 x). \quad (13)$$

This case has been discussed previously in Refs. [9,4]. It was shown in [9] that for  $N_f = 2N_c$  the theory flows to  $N = 2$  SUSY QCD. Giving quarks small masses does not destroy  $N = 2$  SUSY, and since we expect that there is no phase transition between small and large masses, the theory flows to  $N = 2$  SUSY QCD everywhere in the moduli space provided that  $N_f \leq 2N_c$ .

### B. $N_c = 2, N_f = 1, l = 2$

The curve is given by

$$y^2 = (x^2 - u)^2 - \Lambda^3(h_0 + h_1 x + h_2 x^2). \quad (14)$$

It is isomorphic to the curve of  $N = 2$  SU(2) SUSY QCD with two flavors of quarks, and therefore we can take over the results of Ref. [6]. For generic values of  $h_0, \dots, h_2$  there are two quark singularities, <sup>†</sup> one point where the (1,0) dyon becomes massless, and one point where the (1,1) dyon becomes massless. If we set  $h_1^2 = 4h_0h_2$ , the quark singularities collide. This does not mean, however, that there are two massless quarks at this point. This is quite obvious for large  $h_0$ , where the quark singularity occurs in the semiclassical regime, and must be true in the strong coupling regime by continuity. Rather, we still have one massless quark, but its mass vanishes quadratically as one approaches the singular point.

Furthermore, one can make the quark singularity collide with one of the dyonic points by tuning  $h_0$ . The result is a theory with one massless quark and one massless dyon. Such a theory is intrinsically strongly interacting [5], and presents a new example of an  $N = 1$  superconformal fixed point. It has no apparent nonabelian global symmetries, and therefore it is inequivalent to the fixed points of  $N = 1$  SUSY QCD [1].

---

<sup>†</sup>The masses of up and down quarks always vanish simultaneously, and when we refer to a quark, what we mean is the pair of up and down quarks.

We can also compute the scaling dimensions of relevant perturbations around this point up to overall normalization. (Unlike Ref. [6], we cannot determine the normalization, because we do not have dyon mass formulas or any information about the Kähler potential.) In fact, the answer can be read off from section 3.2 of Ref. [6]:

$$D(h_1) : D(\delta u) : D(h_1^2 - 4h_0h_2) = 1 : 2 : 3. \quad (15)$$

**C.**  $N_c = 2, N_f = 1, l = 3$

The curve is given by

$$y^2 = (x^2 - u)^2 - \Lambda^3(h_0 + h_1x + h_2x^2 + h_3x^3). \quad (16)$$

For generic values of  $h_0, \dots, h_3$  there are two dyonic singularities and three quark singularities. Tuning the coefficients of the superpotential one can collide all three quark singularities and produce a point  $u = u_0$  with one massless quark. The mass of the quark vanishes like  $(u - u_0)^3$  as  $u$  approaches  $u_0$ . By further tuning one can collide this point with one of the dyonic singularities, producing an interacting  $N = 1$  superconformal theory. It has no nonabelian flavor symmetries. There are four independent relevant perturbations away from the superconformal point:

$$\delta u, \quad h_2, \quad 3h_1h_3 - h_2^2, \quad 27h_0h_3^2 - 9h_1h_2h_3 + 2h_2^3. \quad (17)$$

Their respective scaling dimensions are in the ratio  $3 : 1 : 4 : 6$ . Thus this superconformal theory is manifestly inequivalent to that discussed in the previous subsection.

**D.**  $N_c = 2, N_f = 1, l = 4$

The curve is given by

$$y^2 = (x^2 - u)^2 - \Lambda^3(h_0 + h_1x + h_2x^2 + h_3x^3 + h_4x^4). \quad (18)$$

Now there are four quark singularities. By tuning the parameters of the superpotential we may collide three of them with one of the dyonic singularities. This occurs at isolated points of the parameter space of the theory and presumably produces the infrared fixed point discussed in the previous subsection. There are also submanifolds in the parameter space where only two quark singularities coincide with the dyonic one. On these submanifolds the theory flows to the fixed point of subsection IV B.

If we set  $h_0 = h_1 = h_2 = h_3 = 0$ , all six singularities collide at the origin of the moduli space. As discussed in section II, the perturbation by the operator  $Q\Phi^4Q$  is truly marginal for  $N_c = 2, N_f = 1$ . In other words, there is a line of interacting superconformal fixed points in the  $g - h_4$  plane emanating from the point  $(g^*, 0)$ , where  $g^*$  is the value of the gauge coupling at the fixed point of the theory without superpotential. Thus the point  $u = 0$  is naturally interpreted as a nonabelian Coulomb phase point. The relative dimensions of the perturbations determined from the curve coincide with those obtained from the microscopic  $R$ -charge assignments.<sup>‡</sup> This is in agreement with the assumption that one can describe the nonabelian Coulomb phase in terms of microscopic degrees of freedom.

## ACKNOWLEDGMENTS

It is a pleasure to thank John Preskill, David Lowe and Eric Westphal for helpful discussions and comments. I am especially grateful to Adam Leibovich for collaboration on some related matters.

---

<sup>‡</sup>In this case there is a one-parameter family of nonanomalous microscopic  $R$ -symmetries.

## REFERENCES

- [1] N. Seiberg, Nucl. Phys. B435 (1995) 129.
- [2] D. Kutasov, Phys. Lett. B351 (1995) 230.
- [3] D. Kutasov and A. Schwimmer, Phys. Lett. B354 (1995) 315.
- [4] D. Kutasov, A. Schwimmer and N. Seiberg, Nucl. Phys. B459 (1996) 455.
- [5] P.C. Argyres and M. Douglas, Nucl. Phys. B448 (1995) 93.
- [6] P.C. Argyres, M.R. Plesser, N. Seiberg and E. Witten, Nucl. Phys. B461 (1996) 71.
- [7] M.A. Shifman and A.I. Vainshtein, Nucl. Phys. B277 (1986) 456, Nucl. Phys. B359 (1991) 571.
- [8] P.C. Argyres, M.R. Plesser and A.D. Shapere, Phys. Rev. Lett. 75 (1995) 1699.
- [9] R.G. Leigh and M.J. Strassler, Nucl. Phys. B447 (1995) 95.