EPAPS

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In this supplementary material, we detail the proof of Observation 4 of the main text. We again label in the Stinespring dilation $T(\rho) = \mathrm{tr}_E(U\rho U^\dagger)$ of the quantum channel T the input by A, the output by B and the environment by E, but also keep a system R holding a purification of the input. We have to show that if $Q(T) = S_{\max}(T)$, then also

$$Q_1(T) = S_{\max}(T). \tag{1}$$

If $Q(T) = S_{\max}(T)$ indeed holds true, then we find for the classical and quantum entanglement-assisted capacities C_E and Q_E, respectively,

$$C_E(T) = 2Q_E(T) \geq 2Q(T) = 2S_{\max}(T). \tag{2}$$

We also know that

$$C_E(T) = \max_\Psi I(\Psi, R:B), \tag{3}$$

where the maximum is taken over pure states Ψ. One way of expressing the right hand side, so the mutual information, in our notation of input A, output B, environment E, and purification R is

$$C_E(T) = \max_\Psi \left(S(\rho_R) + S(\rho_B) - S(\rho)\right), \tag{4}$$

where ρ is a mixed state that is shared between B and R that is obtained as $\rho = (\mathbb{1}_R \otimes T)(\Psi)$. From the subaddivity of the von-Neumann entropy, it follows that for the input state that achieves the maximum in Eq. (4),

$$2S(\rho_B) \geq S(\rho_R) + S(\rho_B) - S(\rho) \geq 2S_{\max}(T). \tag{5}$$

Since at the same time, by definition $S(\rho_B) \leq S_{\max}(T)$, we find that there exists an input pure state $\Psi = |\psi\rangle\langle\psi|$ shared between R, B, and E, with $|\psi\rangle = U|\phi\rangle$, $|\phi\rangle$ being shared between R and A, such that

$$S(\rho_B) = S_{\max}(T), \tag{6}$$

$$S(\rho_R) + S(\rho_B) - S(\rho) = 2S_{\max}(T). \tag{7}$$

The latter equality also implies that

$$S(\rho_{B,E}) = S(\rho_B) + S(\rho_E), \tag{8}$$

which in turn means $\rho_{B,E} = \rho_B \otimes \rho_E$.

In the final step, it is the aim to construct an input to the channel that certifies that the single shot quantum capacity $Q_1(T) = S_{\max}(T)$. Based on the above properties of the channel, this can easily be done. By virtue of the Schmidt decomposition, there exists a unitary V supported on R such that

$$\begin{aligned} V|\psi\rangle &= |\xi\rangle_{R_1,B}|\eta\rangle_{R_2,E} \\ &= \sum_{i,j} \alpha_i\beta_j |i\rangle_{R_1}|e_i\rangle_B|j\rangle_{R_2}|e_j\rangle_E, \end{aligned} \tag{9}$$

where for convenience the tensor factors have been indicated. So the input

$$V|\phi\rangle = \sum_{i,j} \alpha_i\beta_j |i\rangle_{R_1}|j\rangle_{R_2}|d_{i,j}\rangle_A, \tag{10}$$

will give rise to the output as in Eq. (9). From this input to the original problem we can construct a new input which achieves the desired bound: Take

$$|\nu\rangle = \sum_i \alpha_i |i\rangle_{R_1}|0\rangle_{R_2}|d_{i,0}\rangle_A, \tag{11}$$

yielding the output

$$U|\nu\rangle = |\xi\rangle_{R_1,B}|0\rangle_{R_2}|e_0\rangle_E, \tag{12}$$

which is a tensor product between R and E. This means that $Q_1(T) = \max_\Psi(S(\rho_R) - S(\rho)) = S_{\max}(T)$, which is that was to be shown.