

Thermodynamics as a Resource Theory: Supplementary Material

Fernando G.S.L. Brandão,^{1,2} Michal Horodecki,^{3,4}

Jonathan Oppenheim,⁵ Joseph M. Renes,^{6,7} and Robert W. Spekkens⁸

¹*Departamento de Física, Universidade Federal de Minas Gerais, Belo Horizonte, Brazil*

²*Centre for Quantum Technologies, National University of Singapore, 2 Science Drive 3, 117543 Singapore*

³*Institute for Theoretical Physics and Astrophysics, University of Gdansk, 80-952 Gdansk, Poland*

⁴*National Quantum Information Centre of Gdansk, 81-824 Sopot, Poland*

⁵*Department of Applied Mathematics and Theoretical Physics,
University of Cambridge, Cambridge CB3 0WA, U.K.*

⁶*Institut für Angewandte Physik, TU Darmstadt, Darmstadt, Germany*

⁷*Institut für Theoretische Physik, ETH Zurich, Switzerland*

⁸*Perimeter Institute for Theoretical Physics, Waterloo, Canada*

(Dated: September 21, 2013)

This supplementary material contains eight sections. The first shows that the relative entropy distance to the Gibbs state is an asymptotically continuous function. The next four sections discuss in detail the state transformation protocols for the case of two-level systems. Section II presents a distillation protocol for quasi classical states, while section III describes a formation protocol also for quasi classical states. The following two sections extend these protocols to the case of arbitrary nonstationary two-level resources. Then in Section VI we outline how the results can be easily generalized to higher dimensions by considering as an example the distillation protocol for quasi classical states. Section VII discusses some characteristics of the exhaust states produced in these protocols. Finally, Section VIII discusses the equivalence of our formulation to other models of thermodynamics and the degree of control one needs to implement our thermal operations.

I. EXTENSIVITY AND ASYMPTOTIC CONTINUITY

To show that $D(\rho||\gamma)$ is asymptotically-continuous, we make use of the following, from [1]:

Proposition 1 *Suppose a function f satisfies (1) “approximate affinity”*

$$|pf(\rho) + (1-p)f(\sigma) - f(p\rho + (1-p)\sigma)| \leq c, \quad (1)$$

for some constant $c > 0$ and any p such that $0 \leq p \leq 1$, and (2) “subextensivity” $f(\rho) \leq M \log d$, where $M > 0$ is constant and $d = \dim(\mathcal{H})$ for \mathcal{H} the state space on which ρ has support. Then f is asymptotically continuous:

$$|f(\rho_1) - f(\rho_2)| \leq M\|\rho_1 - \rho_2\|_1 \log d + 4c. \quad (2)$$

The entropy relative to the Gibbs state, $f(\rho) := D(\rho||\sigma)$, satisfies both conditions. To see the first, let $\tau = \sum_k p_k \tau_k$ for some arbitrary set of density operators $\{\tau_k\}$ and probability distribution p_k and let ω be another arbitrary density operator. Then

$$D(\tau||\omega) = \text{Tr} \left[\sum_k p_k (-\tau_k \log \omega + \tau_k \log \tau) \right] \quad (3)$$

$$= \text{Tr} \left[\sum_k p_k (\tau_k \log \tau_k - \tau_k \log \omega + \tau_k \log \tau - \tau_k \log \tau_k) \right] \quad (4)$$

$$= \sum_k p_k D(\tau_k||\omega) + \sum_k p_k S(\tau_k) - S(\tau), \quad (5)$$

where in the second line we have added and subtracted $\text{Tr}(\tau \log \tau_k)$. Since $S(\tau) \leq \sum_k p_k S(\tau_k) + H(p_k)$ where $H(p_k)$ denotes the Shannon entropy for the distribution p_k , and since the relative entropy is convex, this implies

$$0 \leq \sum_k p_k D(\tau_k||\omega) - D(\tau||\omega) \leq H(p_k). \quad (6)$$

Finally, letting $\omega = \gamma$ and $\{\tau_k\} = \{\rho, \sigma\}$ with distribution $(p, 1-p)$, we find

$$pD(\rho||\gamma) + (1-p)D(\sigma||\gamma) - D(p\rho + (1-p)\sigma||\gamma) \leq h_2(p) \leq 1, \quad (7)$$

where h_2 is the binary entropy function.

The fact that the second condition is satisfied, i.e., that the entropy relative to the Gibbs state is subextensive, follows from the fact that the maximum energy of the system is extensive. First, note that $D(\rho||\gamma) = \beta F_\beta(\rho) - \beta F_\beta(\gamma)$, where $F_\beta(\rho) = \langle H \rangle_\rho - \frac{1}{\beta} S(\rho)$ is the free energy. Thus, the maximum of $D(\rho||\gamma)$ occurs for $\rho = |E_{\max}\rangle\langle E_{\max}|$ where $|E_{\max}\rangle$ is the eigenstate of maximum energy. Direct calculation shows

$$D(|E_{\max}\rangle\langle E_{\max}||\gamma) = \text{Tr}[|E_{\max}\rangle\langle E_{\max}|(\log |E_{\max}\rangle\langle E_{\max}| - \log \gamma)] = -\langle E_{\max}| \log \gamma |E_{\max}\rangle \quad (8)$$

$$= \beta E_{\max} + \log Z_\beta = \beta E_{\max} + \log \sum_k e^{-\beta E_k} \leq \beta E_{\max} + \log d. \quad (9)$$

Here we have assumed that the energy values $E_j > 0$. When the maximum energy is extensive, i.e. $E_{\max} \leq K \log d$ for some constant K , we obtain $D(|E_{\max}\rangle\langle E_{\max}||\gamma) \leq M \log d$ for $M = \beta K + 1$.

II. DISTILLATION OF QUASICLASSICAL STATES

For simplicity of presentation, we consider qubit systems with the Hamiltonian given by $H = \sum_i |1\rangle_i \langle 1|$, where the sum runs over all involved qubits. We start with l copies of the Gibbs state γ and n copies of the resource state ρ , where

$$\rho = (1-p)|0\rangle\langle 0| + p|1\rangle\langle 1|; \quad \gamma = (1-q)|0\rangle\langle 0| + q|1\rangle\langle 1|, \quad (10)$$

with $q = e^{-\beta}/(1 + e^{-\beta})$ and β the inverse temperature, which we take as a constant parameter. The aim is to obtain the maximal number of copies possible of qubits in the pure excited state $|1\rangle$ by implementing a unitary that commutes with H and taking the partial trace over some subsystem,

$$\gamma^{\otimes l} \otimes \rho^{\otimes n} \rightarrow \sigma^{(k)} \otimes |1\rangle\langle 1|^{\otimes m}. \quad (11)$$

We denote $R = \frac{m}{n}$ (the rate of distillation) and $\epsilon = \frac{n}{l}$ (the ratio between the number of used Gibbs states and the number of resource states). The Gibbs states are free, so we accept that ϵ asymptotically vanishes.

In the protocol we shall use the fact that up to a small error (vanishing for a large number of qubits)

$$\rho^{\otimes n} \approx \sum_t p_t P_t \quad (12)$$

where t run over strongly typical types, i.e. the types containing strings with the number of 1's within the interval $(np - O(\sqrt{n}), np + O(\sqrt{n}))$, and P_t denotes the projector onto type t . Similarly

$$\gamma^{\otimes l} \approx \sum_t q_t Q_t \quad (13)$$

again with $q_t \approx qn$. The errors in both approximations are smaller than $2^{-\sqrt{n}}$ when quantified by the trace norm.

For simplicity we shall first pretend that both $\gamma^{\otimes l}$ and $\rho^{\otimes n}$ consist of a single type. Then further we will show how to extend the argument to a mixture of types.

So we start with a tensor product of two types (one from Gibbs, the other from ρ), i.e. an equal mixture of strings of length $l+n$. The string consist of two substrings: the first has ql 1's and the second has pn 1's:

$$\overbrace{000 \dots 0}^l \underbrace{11 \dots 1}_{lq} \overbrace{00 \dots 0}^n \underbrace{111 \dots 1}_{np}. \quad (14)$$

There are roughly

$$2^{lh(q)} \times 2^{nh(p)} \quad (15)$$

such strings (with the error being a multiplicative $\text{poly}(n)$ factor).

We now apply a unitary transformation to these strings to map them into strings of the same total length which have m 1's to the right:

$$\overbrace{000 \dots 0}^l \underbrace{11 \dots 1}_{lq} \overbrace{00 \dots 0}^n \underbrace{111 \dots 1}_{np} \rightarrow \underbrace{00000 \dots 0000}_{k} \underbrace{111 \dots 11}_{rk} \underbrace{111 \dots 1}_m \quad (16)$$

where

$$k = l + n - m \quad (17)$$

(conservation of dimension), and r and m are about to be determined. First, r is fixed by *conservation of energy*, which requires that the number of 1's is conserved:

$$lq + np = rk + m.. \quad (18)$$

Then *unitarity* requires that there are at least as many strings of length k with rk 1's as the number of initial strings (15):

$$2^{kh(r)} \geq 2^{lh(q)+nh(p)}. \quad (19)$$

Roughly speaking this is *conservation of entropy*. Using (18) and (19) we obtain that our transformation is possible if

$$h(q) + \epsilon h(p) \leq (1 + \epsilon - R\epsilon) h\left(\frac{q + \epsilon p - \epsilon R}{1 + \epsilon - \epsilon R}\right), \quad (20)$$

where recall that ϵ is the ratio of Gibbs states used, and R the ratio of pure excited states obtained. We now expand this with respect to ϵ to first order and let $\epsilon \rightarrow 0$. This means that we take many more Gibbs states than the resource states. In other words, we presume a heat reservoir that is arbitrarily larger than the size of our system. As a result, recalling that $q = e^{-\beta}/(1 + e^{-\beta})$, we obtain that the following rate can be achieved

$$R = \frac{h(q) - h(p) + \beta(p - q)}{h(q) + \beta(1 - q)} = \frac{S(\rho||\gamma)}{S(|1\rangle\langle 1||\gamma)}, \quad (21)$$

i.e., the protocol achieves the upper bound obtained by the monotonicity argument in the main text.

So far we have worked with a single type. But our initial state is actually a mixture of products of types $Q_t \otimes P_{t'}$. We thus apply the protocol separately to each type, with the same number m of required output excited states, for all types, with this number being fitted to the product of less numerous types (i.e. the one with the smallest number of 1's, namely $(np - O(\sqrt{n}))(lq - O(\sqrt{l}))$ 1's). Note, however, that the variations will disappear asymptotically, as we divide the equations by l . Also the approximation to the number of strings in each type by the exponential of the entropy is correct up to a multiplicative polynomial factor, which is also irrelevant asymptotically.

III. FORMATION OF QUASICLASSICAL STATES

We are going to construct the formation protocol in three stages. The first is to show that for a particular type T_q of the l copies of the Gibbs state, we can create a particular type T_p of the state we want to form. We then show that we can do this for all types of the Gibbs state. Finally we show how to correctly get the distribution over types of the target state.

We shall need the following useful lemma:

Lemma 2 (*Birkhoff primitive*) *The following operation can be done by means of thermal operations with arbitrary accuracy:*

$$\rho \rightarrow \sum_k p_k U_k \rho U_k^\dagger \quad (22)$$

where p_k is an arbitrary probability distribution, and the U_k commute with the Hamiltonian. In particular, a random permutation of the systems is a valid operation.

Remark. The accuracy depends on the number of Gibbs states that are used, but in our paradigm they are for free.

Proof. First, let us note that the following unitary transformation preserves energy:

$$\sum_i |i\rangle\langle i| \otimes \tilde{U}_i \quad (23)$$

provided that $|i\rangle$ are eigenvectors of the reservoir Hamiltonian and the \tilde{U}_i commute with the system Hamiltonian. To obtain the required transformation, we take the initial state of the reservoir to be l copies of the Gibbs state. Let q_i denote the probability distribution of single strings. We now divide the set of eigenvectors of the reservoir Hamiltonian into sets, denoted S_k , such that the sum of the probability distribution over i within each set yields

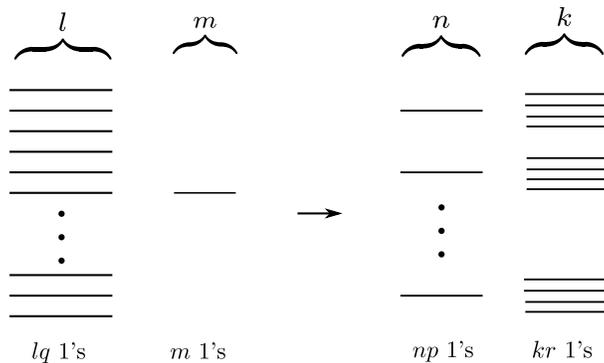


FIG. 1. Strings mapping in formation protocol

approximately the probability p_k from (22), that is, $\sum_{i \in S_k} q_i \approx p_k$. This can be done with arbitrary accuracy by taking l large enough, since $q_i \leq \max\{q, 1 - q\}^l$. Then, for every $i \in S_k$, we set $\tilde{U}_i = U_k$. \square

We want to form n copies of the state $\rho = (1 - p)|0\rangle\langle 0| + p|1\rangle\langle 1|$ from a pure excited state. Let us first show how we can form any of the typical types of the state.

Formation of a maximally mixed state over a fixed type.

Consider a fixed type T_p of the state of n systems that we want to create, and let it be one with np 1's. For other types, the reasoning is the same and the asymptotic rate will be the same. We start with l copies of the Gibbs state γ , and m copies of the excited state $|1\rangle$. We consider a final exhaust system consisting of k two-level systems. Consider a typical type T_q of the Gibbs state; it has lq 1's. In that type, there are $\approx 2^{lh(q)}$ strings up to some $2^{\sqrt{lh(q)}}$ factor. We want to map these initial strings onto the N final strings in the type T_p as follows. Take $\{u_i\}_i$ to be the set of strings in T_p and for each string u_i consider some set $\{v_j^i\}_j$ of strings on the exhaust system. We now map each of the initial strings to some string $u_i v_j^i$. This is illustrated in figure III.

For sets $\{v_j^i\}_j$ corresponding to different values of i , we can take the number of strings in each set to be the same or off by 1, simply by assigning the strings in an order determined by fixing $j = 1$, then incrementing the i register until $i = N$, then incrementing the j register by 1, resetting i to 1 and again incrementing the i register until $i = N$ and repeating. This is all done to ensure that when we trace out the exhaust system, we get an even mixture over permutations within the type class.

We can now use the analogues of equations (17), (18) and (19) to ensure that we can perform the unitary which implements this mapping:

$$m + l = n + k \quad (24)$$

$$lq + m = rk + np \quad (25)$$

$$2^{lh(q)} \leq 2^{kh(r) + nh(p)} \quad (26)$$

We now take $l \propto m^\alpha$ with $1 < \alpha < 2$ and do as before. We take $\alpha > 1$ so that we can take $\epsilon = n/l$ small, and $\alpha < 2$ so that \sqrt{l} is sublinear in m and we can ignore such terms.

This maps type T_q of the l Gibbs states onto the type T_p of $\rho^{\otimes n}$. We can map each of the initial Gibbs types onto T_p in this manner using a unitary U_{pq} . For each such mapping, the above three equations will change, but only by some \sqrt{l} factors which we took to be sublinear in m . We can thus choose m to ensure conservation of energy in the worst case of Eq. (25), and $kh(r)$ is chosen to ensure the inequality Eq. (26) in the worst case.

We now need to implement U_{pq} conditioned on the initial type T_q . Since the typical initial types are on orthogonal and diagonal subspaces, we first do a conditional copying of the type class q onto an initialised register. We then act U_{pq} conditioned on this register. Since the number of types is polynomial in l , the register only needs to be of size $\log l$, and thus this resource does not matter as it is sublinear in m . It is an interesting question whether the formation protocol can be made to work without this sublinear supply of pure states.

We denote by U_p the unitary that creates a particular type T_p . To get the distribution over types, we simply use the Birkoff primitive of Lemma 2 to implement $\sum_p |p\rangle\langle p| \otimes U_p$. This is irreversible, but since the number of typical types is polynomial in n , the rate of entropy that is created by this procedure is negligible, i.e. logarithmic in n . It is not hard to see that Eqs. (24-26) give the required rate (i.e. the inverse of the distillation rate).

IV. DISTILLATION OF ARBITRARY STATES

We now extend distillation to the case where our states are not diagonal in the energy eigenbasis.

Consider a state $\rho = p|\phi_1\rangle\langle\phi_1| + (1-p)|\phi_2\rangle\langle\phi_2|$. The average energy of the state is $\langle E \rangle = p|\langle\phi_1|1\rangle|^2 + (1-p)|\langle\phi_2|1\rangle|^2$. As before, we consider n copies of ρ and l copies of systems in a Gibbs state γ . Regarding $\rho^{\otimes n}$, only the blocks with energy $E \in [n\langle E \rangle - \sqrt{n}, n\langle E \rangle + \sqrt{n}]$ will be relevant, i.e. we have

$$\text{Tr} \left(\sum_E P_E \rho^{\otimes n} \right) \geq 1 - 2^{-O(n)} \quad (27)$$

where the sum runs over $E \in [n\langle E \rangle - O(\sqrt{n}), n\langle E \rangle + O(\sqrt{n})]$, with P_E the projector onto the energy E eigenspace (this follows from Eq. (42), proven in Section V).

Our protocol has two stages:

- (i) unitary rotation within energy blocks of a resource system (consisting of n qubits) solely.
- (ii) drawing work by string permutations on the total system resource (with n qubits) plus heat bath (with l qubits)

We write down the resource state in the energy eigenbasis. As said above, only blocks with energy $\approx nE$ will appear. We will use the fact that the state is, up to exponentially vanishing error, equal to its projection onto the typical subspace, having dimension $2^{nS(\rho) + O(\sqrt{n})}$ where $S(\rho)$ is the von Neumann entropy of ρ . Therefore, within every block the rank of the state is not larger than $\approx 2^{nS}$ (as a projection cannot increase the rank).

Now stage (i) is the following: within each energy block, we apply unitary rotation, which diagonalizes the state restricted to the block in the energy basis. Then there is stage (ii), in which we apply the protocol of distillation of quasiclassical states as in Section VI, i.e. we permute strings in such a way that the output strings have m 1's to the right. Such a protocol produces m systems in a pure excited state (note that all coherences initially present in the state are now left in the garbage). Using the same notation as in Section VI, for a product of two single types, e.g. with lq and $n\langle E \rangle$ 1's, respectively, the constraints now become

$$\begin{aligned} k &= m + m - l \\ lq + n\langle E \rangle &= rk + m \\ 2^{kh(r)} &\geq 2^{lh(q) + nS(\rho)} \end{aligned}$$

As before, taking the limit $\frac{n}{l} \rightarrow 0$ we obtain that any rate R is achievable, provided it satisfies

$$R \leq \frac{h(q) - S(\rho) + \beta(\langle E \rangle - q)}{h(q) + \beta(1 - q)} = \frac{S(\rho||\gamma)}{S(|1\rangle\langle 1||\gamma)}. \quad (28)$$

Thus also for states that are not quasiclassical, we can reach the upper bound given by the relative entropy distance from the Gibbs state.

V. FORMATION OF ARBITRARY STATES

We now show that we can achieve reversibility even in the case of states that are not quasiclassical. To do so, however, we must allow the use of a sublinear amount of states that are a superposition over energy eigenstates. This is a reasonable assumption since the rate at which such states are consumed vanishes in the asymptotic limit. This is very similar to the fact that in entanglement theory, distillation requires no communication but formation requires a sublinear amount of it. Or, instead, formation requires a state which is a superposition over different amounts of entanglement. The superposition over different amount of entanglement (known as entanglement spread [3-6]), is analogous to the superposition over energy eigenstates in the present context. In the athermality context, distillation of work requires no superposition over energy eigenstates, but formation does.

Suppose we want to implement some unitary

$$U = \sum_{ij} u_{ij} |E_i\rangle\langle E_j| \quad (29)$$

that does not conserve energy. We introduce a state that acts as a reference frame for time,

$$|H\rangle = \sum f(h)|h\rangle, \quad (30)$$

where $|h\rangle$ denotes an energy eigenstate, and we implement

$$U^{\text{inv}} = \sum_{ij} u_{ij} |E_i\rangle\langle E_j| \otimes |h - E_i + E_j\rangle\langle h| \quad (31)$$

on the system and reference frame. If we are interested in implementing U on the state

$$|\psi\rangle = \sum_{i \in \mathcal{S}} c_i |E_i\rangle, \quad (32)$$

then we do so by implementing U^{inv} on $|\psi\rangle \otimes |H\rangle$. Note that we must ensure that the reference frame system has energy levels with gaps of size $|E_i - E_j|$ for every transition appearing in U . If the energy spread of $f(h)$ is large compared to the largest value of $|E_i - E_j|$ in an energy transition induced by U , then the state of the reference frame is not disturbed very much in the process.

For the problem in which we are interested, this is indeed the case because on the typical subspace, the variation in energy is sublinear in the number of copies of the state we want to create. To see how this works by way of example, note that if in Eq. (30), we take $f(h)$ to be $1/\sqrt{N}$ for energies $h \in \{1, \dots, N\}$ and $f(h) = 0$ otherwise, then removing a unit of energy and adding it to another system, does not change the state of Eq. (30) much. i.e. the inner product between $|H\rangle$ and $\sum_h |h-1\rangle\langle h|H\rangle$ approaches 1, because

$$\sum_{h=1}^N \frac{1}{\sqrt{N}} \langle h| \sum_{h=0}^{N-1} \frac{1}{\sqrt{N}} |h\rangle = 1 - \frac{1}{N}, \quad (33)$$

which approaches 1 for large N . States like that of Eq. (30) therefore allow us to lift the superselection rule for energy, without being consumed much in the process.

This gives some insight into embezzling states [2]. These are resource states that are often used in entanglement theory in similar situations. For instance, one can use a state similar in form to Eq. (30)

$$|E\rangle = \sum f(k) |\phi\rangle_{AB}^{\otimes k} |00\rangle_{AB}^{\otimes (n-k)} \quad (34)$$

which is a superposition of a different number of entangled EPR pairs $|\phi\rangle_{AB}$. These states can be used to implement operations which need to create superpositions over amounts of entanglement (entanglement spread). Just as removing one unit of energy, doesn't change the state of Eq. (30) much, likewise, removing one EPR pair from the state of Eq. (34) and adding it to another system doesn't change the embezzling state by much. We can embezzle energy, just as one can embezzle entanglement. We therefore see that a superposition over some resource can create an embezzling state for that resource, and will allow us to lift some superselection rule or restriction.

With this small superposition over energy states, let us now show that we can create an arbitrary state at a rate given by the relative entropy distance to the Gibbs state. Let $\rho := p|\phi_1\rangle\langle\phi_1| + (1-p)|\phi_2\rangle\langle\phi_2|$ and

$$\rho^{\otimes n} = \sum_{k,g} p_k |\Psi_{k,g}\rangle\langle\Psi_{k,g}|, \quad (35)$$

with

$$|\Psi_{k,g}\rangle := \pi_g |\phi_1\rangle^{\otimes k} \otimes |\phi_2\rangle^{\otimes n-k}, \quad (36)$$

for π_g a permutation.

The idea of the protocol is as follows: we will first create a diagonal state

$$\varrho_n = \sum p_k |t_k, s_g\rangle\langle t_k, s_g| \quad (37)$$

which has the same spectrum as $\rho^{\otimes n}$ and where each eigenstate has the same average energy as an eigenstate in the typical subspace of $\rho^{\otimes n}$. From the result of the previous section it is not hard to see that this can be done at a rate given by the relative entropy distance of ρ to the Gibbs state, since in the limit of many copies, the regularised relative entropy distance is the same. We would then like to rotate the diagonal basis to the $\Psi_{k,s}$ -basis. This cannot be done by unitaries which commute with the Hamiltonian unless we allow for a reference frame $|H\rangle$ which is a superposition over energy states. We then want to show that the reference frame which allows us to break the energy superselection rule is consumed at a vanishingly small rate. We do so by showing that the reference frame superposition is over a size sublinear in n . This can be understood as coming from the fact that in the typical subspace, the superposition over different types is sublinear.

We consider only typical $|\Psi_{k,g}\rangle$ with $k \in \text{Typ}_\rho := [np - \sqrt{n}, np + \sqrt{n}]$. Then

$$\left\| \rho^{\otimes n} - \sum_{k \in \text{Typ}_\rho, g} p_k |\Psi_{k,g}\rangle \langle \Psi_{k,g}| \right\|_1 \leq 2^{-\Omega(\sqrt{n})} \quad (38)$$

Let also

$$|\Psi_{k,g}\rangle := \sum_{t', s'} c_{t', s'}^{kg} |t', s'\rangle \quad (39)$$

where $|t', s'\rangle$ is an eigenstate of the Hamiltonian with energy t' (s' labels the degeneracy). From Eq. (36) it follows that the sum in Eq. (39) will be peaked around only a few energy values t' . Indeed, with

$$|\phi_1\rangle := a|0\rangle + b|1\rangle, \quad (40)$$

and

$$|\phi_2\rangle := b|0\rangle - a|1\rangle, \quad (41)$$

set $\text{Typ} := [nE_t - \sqrt{n}, nE_t + \sqrt{n}]$, where $E_t := ((n-t)|b|^2 + t|a|^2)/n$. Then

$$\left\| \sum_{t' \notin \text{Typ}_t, s'} c_{t', s'}^{kg} |t', s'\rangle \right\| = 2^{-\Omega(\sqrt{n})} \quad (42)$$

Note that since $E_t := ((n-t)|b|^2 + t|a|^2)/n$ the degeneracy of each energy state $|t_k, g_s\rangle$ is at least as large as the degeneracy of $|\Psi_{k,s}\rangle$.

Now we construct the reference frame. Let $|w\rangle$ be an energy eigenstate with energy $n(p|b|^2 + (1-p)|a|^2) - n^{2/3}$. It is needed to pad the dimension of the reference frame, since although the probability that it happens is vanishingly small, the unitary does connect states with large energy difference. After the protocol, we will see that $|w\rangle$ will hardly be changed, and thus is only used as a catalyst. We define the reference system as follows

$$|H\rangle := \frac{1}{\sqrt{|H|}} \sum_{h \in H} |h\rangle \quad (43)$$

with $|h\rangle := |h'\rangle \otimes |w\rangle$, where $|h'\rangle$ is an energy eigenstate of energy h' and $H := \{0, \dots, 2n^{2/3}\}$.

Consider the energy preserving unitary

$$U := \sum_{h, t, s, t', s'} c_{t', s'}^{kg} |t', s'\rangle \langle t_k, s_g| \otimes |h + t - t'\rangle \langle h|. \quad (44)$$

Then in the sequel we prove that

$$\left\| U \left(\sum_{k \in \text{Typ}_\rho, s} p_k |t_k, s_g\rangle \langle t_k, s_g| \otimes |H\rangle \langle H| \right) U^\dagger - \rho^{\otimes n} \otimes |H\rangle \langle H| \right\|_1 \leq O(n^{-1/6}). \quad (45)$$

We first analyze the action of U in $|s, t\rangle \otimes |H\rangle$:

$$\begin{aligned} U(|t_k, s_g\rangle \otimes |H\rangle) &= \sum_{t', s'} c_{t', s'}^{kg} |t', s'\rangle \otimes \left(\frac{1}{\sqrt{|H|}} \sum_{h \in H} |h + t_k - t'\rangle \right) \\ &= |\nu_1\rangle + |\nu_2\rangle + |\nu_3\rangle \end{aligned} \quad (46)$$

where the non-normalized pure states $|\nu_k\rangle$ are given by

$$|\nu_1\rangle := \sum_{t' \in \text{Typ}_t, s'} c_{t', s'}^{kg} |t', s'\rangle \otimes |H\rangle, \quad (47)$$

$$|\nu_2\rangle := \sum_{t' \in \text{Typ}_t, s'} c_{t's'}^{kg} |t', s'\rangle \otimes |\text{err}_{t'}\rangle \quad (48)$$

with

$$|\text{err}_{t'}\rangle := \frac{1}{\sqrt{|H|}} \sum_{h \in H} |h + t - t'\rangle - |H\rangle, \quad (49)$$

and

$$|\nu_3\rangle := \sum_{t' \notin \text{Typ}_t, s'} c_{t's'}^{kg} |t', s'\rangle \otimes \left(\frac{1}{\sqrt{|H|}} \sum_{h \in H} |h + t - t'\rangle \right). \quad (50)$$

Set $t_k = E_k$ and let us take $s_g = g$. We can do the latter since as we mentioned, the degeneracy of $|t_k, s_g\rangle$ is larger than the degeneracy of $|\Psi_{k,g}\rangle$. Then,

$$\begin{aligned} \|U(|t_k, s_g\rangle \otimes |H\rangle) - |\Psi_{k,g}\rangle \otimes |H\rangle\| &\leq \| |\nu_1\rangle - |\Psi_{k,g}\rangle \otimes |H\rangle \| \\ &\quad + \| |\nu_2\rangle \| + \| |\nu_3\rangle \|. \end{aligned} \quad (51)$$

We now show that the three terms in the R.H.S. are small. For $|\nu_2\rangle$ we first note that for $t' \in \text{Typ}_t$

$$\| |\text{err}_{t'}\rangle \| \leq n^{-1/6}. \quad (52)$$

by taking the worst case. Then

$$\begin{aligned} \| |\nu_2\rangle \|^2 &= \sum_{t' \in \text{Typ}_t, s'} |c_{t's'}^{kg}|^2 \| |\text{err}_{t'}\rangle \|^2 \\ &\leq \max_{t' \in \text{Typ}_t} \| |\text{err}_{t'}\rangle \|^2 \leq n^{-1/3}. \end{aligned} \quad (53)$$

For $|\nu_3\rangle$, in turn, we have

$$\begin{aligned} \| |\nu_3\rangle \|^2 &\leq \sum_{t' \notin \text{Typ}_t, s'} |c_{t's'}^{kg}|^2 \| |\text{err}_{t'}\rangle \|^2 \\ &\leq \sum_{t' \notin \text{Typ}_t, s'} |c_{t's'}^{kg}|^2 \leq 2^{-\Omega(\sqrt{n})}. \end{aligned} \quad (54)$$

Finally, for $|\nu_1\rangle$,

$$\begin{aligned} \| |\nu_1\rangle - |\Psi_{k,g}\rangle \otimes |H\rangle \| &= \left\| \sum_{t' \in \text{Typ}_t, s'} c_{t's'}^{kg} |t', s'\rangle - |\Psi_{k,g}\rangle \right\| \\ &\leq 2^{-\Omega(\sqrt{n})}. \end{aligned} \quad (55)$$

From Eq. (51) it thus follows that

$$\|U(|t_k, s_g\rangle \otimes |H\rangle) - |\Psi_{k,g}\rangle \otimes |H\rangle\| \leq O(n^{-1/6}). \quad (56)$$

Since $\| |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \|_1 \leq \sqrt{2} \| |\psi\rangle - |\phi\rangle \|$ for every two states $|\psi\rangle, |\phi\rangle$, we find

$$\begin{aligned} &\|U(|t_k, s_g\rangle\langle t_k, s_g| \otimes |H\rangle\langle H|)U^\dagger - |\Psi_{k,g}\rangle\langle\Psi_{k,g}| \otimes |H\rangle\langle H|\|_1 \\ &\leq O(n^{-1/6}). \end{aligned} \quad (57)$$

Eq. (45) then follows from the triangle inequality for trace-norm and Eq. (38).

VI. DISTILLATION OF QUASICLASSICAL STATES IN ARBITRARY DIMENSIONS

In this section we present the details of the distillation protocol for quasiclassical states for the general case of d -dimensional systems. This is presented as an example of how the results can be extended to arbitrary dimensions using arguments very similar to those used in the two-dimensional case.

The input to the protocol consists of n copies of the initial resource ρ and ℓ copies of the Gibbs state γ of the same Hamiltonian H . Since the states are quasiclassical, the overall state of the input is fully described by a collection of strings $\mathbf{s} \in \{0, \dots, d-1\}^{n+\ell}$ listing the energy level occupied by each system, each string weighted by its probability of occurrence. Here d is the total number of energy levels of H , and the first n entries of \mathbf{s} correspond to the state of ρ and the remainder correspond to γ . If some of the energy levels are degenerate, we simply work in the eigenbasis of ρ to remove the degeneracy in labeling. Because permutations within each of the two substrings do not change the overall probability, we can therefore instead work with the collection of occupation frequencies \mathbf{f} of the state, which describe the number of systems in the ground state, first excited state, second excited state, and so on (divided by the total number of systems), again each weighted by an appropriate probability.

We would now like to define a protocol which creates as many standard resources in the form of work as possible. In the qubit case the standard resource had a very simple form, namely the excited state of the Hamiltonian. Here, however, the setup is more cumbersome, as there are $d-1$ excited states and no guarantees that their energy differences are in any way commensurate (as, e.g. in the case of a harmonic oscillator, where the energy of the state $|2\rangle$ can be transferred to two instances of the state $|1\rangle$). To handle this issue most simply, we imagine that, in addition to the resource and thermal states, we also have a work system at our convenience. The work system is capable of accepting arbitrary amounts of energy, i.e. it has energy transitions which precisely correspond to those of H , but it cannot accept any entropy. Now the goal of the protocol is to change the occupation of the energy levels so as to transfer as much energy to the work system as possible.

Let us now restrict attention to a fixed occupation frequency \mathbf{f}_ρ of the resource and a fixed \mathbf{f}_γ for the Gibbs state. We will later design the protocol so that it works for every such frequency pair which has appreciable probability. Suppose that we now change the occupation numbers by an amount described by the vector $-n\mathbf{x}$ (whose prefactor is chosen for later convenience). This results in a new occupation vector $\boldsymbol{\nu}$ defined by

$$(n + \ell)\boldsymbol{\nu} \equiv n\mathbf{f}_\rho + \ell\mathbf{f}_\gamma - n\mathbf{x}. \quad (58)$$

For this to be an allowable transformation in our framework, this mapping must satisfy two constraints: energy conservation and unitarity. In contrast to the qubit case, here the input and output dimensions are equal by design.

Energy conservation is simply enforced by requiring the work system to take up the change in energy of the input systems. Using the vector \mathbf{H} to describe the energy of each energy level, the initial energy is given by $E_{\text{in}} = n\mathbf{H} \cdot \mathbf{f}_\rho + \ell\mathbf{H} \cdot \mathbf{f}_\gamma$ while the final energy is $E_{\text{out}} = n\mathbf{H} \cdot \mathbf{f}_\rho + \ell\mathbf{H} \cdot \mathbf{f}_\gamma - n\mathbf{H} \cdot \mathbf{x}$. Energy conservation is then the statement that the work extracted is given by $W = E_{\text{in}} - E_{\text{out}} = n\mathbf{H} \cdot \mathbf{x}$.

Unitarity is enforced by making sure that the total number of configurations (strings) consistent with each occupation vector is conserved by the process. The total number of possible input strings in this case, N_{in} , is just the product of the multinomial coefficients using the frequency vectors:

$$N_{\text{in}} = M(n\mathbf{f}_\rho)M(\ell\mathbf{f}_\gamma) = \frac{n!}{(n(\mathbf{f}_\rho)_0)! \cdots (n(\mathbf{f}_\rho)_{d-1})!} \frac{\ell!}{(\ell(\mathbf{f}_\gamma)_0)! \cdots (\ell(\mathbf{f}_\gamma)_{d-1})!} \quad (59)$$

The maximum number of strings $N_{\text{out}} = M((n + \ell)\boldsymbol{\nu})$ which can be created in the $n + \ell$ systems given the new occupation frequency $\boldsymbol{\nu}$ is the multinomial coefficient of the new occupation frequency vector,

$$N_{\text{out}} = M((n + \ell)\boldsymbol{\nu}) = \frac{(n + \ell)!}{((n + \ell)\nu_0)! \cdots ((n + \ell)\nu_{d-1})!}. \quad (60)$$

Therefore, a sufficient condition for unitarity is $N_{\text{in}} \leq N_{\text{out}}$, or $M(n\mathbf{f}_\rho)M(\ell\mathbf{f}_\gamma) \leq M((n + \ell)\boldsymbol{\nu})$.

It can be shown that the multinomial coefficients obey the bounds

$$\frac{e}{(ne)^d} \frac{1}{f_1 \cdots f_d} 2^{nH(\mathbf{f})} \leq \frac{n!}{(nf_1)! \cdots (nf_d)!} \leq \frac{n}{e^d - 1} 2^{nH(\mathbf{f})}, \quad (61)$$

and therefore $M(n\mathbf{f}) \approx 2^{nH(\mathbf{f}) \pm O(\log n)}$. The unitarity condition then becomes

$$nH(\mathbf{f}_\rho) - O(\log n) + \ell H(\mathbf{f}_\gamma) - O(\log \ell) \leq (n + \ell)H(\boldsymbol{\nu}) + O(\log(n + \ell)) \quad (62)$$

Defining $\epsilon = \frac{n}{\ell}$ we may express this as

$$\epsilon H(\mathbf{f}_\rho) + H(\mathbf{f}_\gamma) \leq (1 + \epsilon)H(\boldsymbol{\nu}) + O\left(\frac{\log \ell}{\ell}\right). \quad (63)$$

Using the expression for $\boldsymbol{\nu}$ from (58) and assuming that $\epsilon \ll 1$ gives

$$H(\boldsymbol{\nu}) = H\left(\frac{n\mathbf{f}_\rho + \ell\mathbf{f}_\gamma - n\mathbf{x}}{n + \ell}\right) \quad (64)$$

$$= H(\mathbf{f}_\gamma) + \epsilon [(\mathbf{f}_\gamma + \mathbf{x} - \mathbf{f}_\rho) \cdot \mathbf{1} - H(\mathbf{f}_\gamma) + (\mathbf{x} - \mathbf{f}_\rho) \cdot \log \mathbf{f}_\gamma] + O(\epsilon^2) \quad (65)$$

$$= H(\mathbf{f}_\gamma) - \epsilon(\mathbf{f}_\rho - \mathbf{x}) \cdot \log \mathbf{f}_\gamma - \epsilon H(\mathbf{f}_\gamma) + O(\epsilon^2). \quad (66)$$

Here $\mathbf{1}$ is the vector of all ones, and we have made use of the fact that $\mathbf{f} \cdot \mathbf{1} = 1$ for any frequency vector \mathbf{f} , which also implies $\mathbf{x} \cdot \mathbf{1} = 0$. Combining this with (62) we obtain the relation

$$-\mathbf{x} \cdot \log \mathbf{f}_\gamma \leq D(\mathbf{f}_\rho || \mathbf{f}_\gamma) + O\left(\frac{\log \ell}{\ell}\right). \quad (67)$$

The next step is to fix the protocol to the worst case among the likely frequency vectors \mathbf{f}_ρ and \mathbf{f}_γ . Their probabilities sharply peaked around the individual distributions $\boldsymbol{\rho}$ and $\boldsymbol{\gamma}$, respectively. Specifically, fixing an error parameter δ , the probability that $\|\mathbf{f}_\rho - \boldsymbol{\rho}\|_1 \geq \delta$ is less than a quantity of order $e^{-n\delta^2}$. The variations of likely \mathbf{f}_ρ from $\boldsymbol{\rho}$ itself are again $O(\frac{1}{\sqrt{n}})$ as in the argument presented in the main text (there the statement was phrased in terms of the number of 1's and not the type class or frequency distribution itself). Thus we may choose $\ell = (Rn^{3/2})$ to ensure that (58) and (62) hold with \mathbf{f}_ρ replaced with $\boldsymbol{\rho}$ and similarly for γ , at least to terms sublinear in n . We conclude that even in the worst, but still probable case, we have

$$-\mathbf{x} \cdot \log \boldsymbol{\gamma} \leq D(\boldsymbol{\rho} || \boldsymbol{\gamma}) - O\left(\frac{1}{\sqrt{n}}\right). \quad (68)$$

Now $\log \boldsymbol{\gamma} = -\beta \mathbf{H} - \mathbf{1} \log Z$, so this condition becomes

$$\beta \mathbf{H} \cdot \mathbf{x} \leq D(\boldsymbol{\rho} || \boldsymbol{\gamma}) - O\left(\frac{1}{\sqrt{n}}\right). \quad (69)$$

This equation gives the minimum amount of extractable work among all the likely frequencies, which is taken to be the target amount for the process. As the extraction is unitary for each frequency, and these correspond to disjoint quantum states, we can thus find a unitary for the entire input capable of generating $\frac{1}{\beta} \left[D(\boldsymbol{\rho} || \boldsymbol{\gamma}) - O\left(\frac{1}{\sqrt{n}}\right) \right]$ units of useful work per input resource state, with probability greater than $1 - O\left(\frac{1}{\sqrt{n}}\right)$.

VII. STRUCTURE OF THE EXHAUST STATE

When doing a transformation of $\rho^{\otimes n}$ into $\sigma^{\otimes m}$, at the end of the protocol we actually obtain $\sigma^{\otimes m} \otimes \pi_k$ and we trace out π_k , which lives in $k = \Omega(n)$ copies of the system. Although π_n is usually far away, in fidelity, from many copies of a Gibbs state, we show that its reductions are very close to a Gibbs state. The main observation is that because π_n should be useless for extracting more copies of σ at a non-zero rate, we must have

$$S(\pi_k || \rho_\beta^{\otimes k}) \leq k^{1-\delta}, \quad (70)$$

for $\delta > 0$. But by subadditivity of the entropy we have

$$\begin{aligned} S(\pi_k || \rho_\beta^{\otimes k}) &= -S(\pi_k) - \sum_{l=1}^k \text{Tr}(\rho_{k,l} \log \rho_\beta) \\ &\geq -\sum_{l=1}^k S(\pi_{k,l}) - \sum_{l=1}^k \text{Tr}(\rho_{k,l} \log \rho_\beta) \\ &= \sum_{l=1}^k S(\pi_{k,l} || \rho_\beta), \end{aligned} \quad (71)$$

where $\pi_{k,l} := \text{Tr}_{\setminus l}(\pi_k)$ is the reduced state of π_k that is obtained by partial tracing all the systems except the l -th one. Let us assume for simplicity that all the $\pi_{k,l}$ are identical. Then

$$S(\pi_{k,1} || \rho_\beta) \leq k^{-\delta}, \quad (72)$$

which by Pinsker's inequality implies

$$\|\pi_{k,1} - \rho_\beta\|_1 \leq \Omega(k^{-2\delta}). \quad (73)$$

More generally, repeating the same argument for larger blocks we get that

$$\|\text{Tr}_{L,L+1,\dots,k}(\pi_k) - \rho_\beta^{\otimes L}\|_1 \leq \Omega(Lk^{-2\delta}). \quad (74)$$

VIII. EQUIVALENCE AND DEGREE OF CONTROL FOR THERMAL OPERATIONS

Here, we address two questions. The first is how our paradigm, where we use unitaries V which commute with the total Hamiltonian H , relates to other approaches. The second is how much control an experimenter needs over the choice of unitaries V . To answer the first question, consider a common approach to thermodynamics, which is to manipulate thermodynamical systems using an external apparatus. In this model, the systems are manipulated using a time-dependent Hamiltonian, $H(t)$. Another approach is to add an interaction term H_{int} between various systems we are trying to manipulate (e.g. the resource, and the heat bath), and then bring these systems into contact with one another. Let us now see that these are equivalent to considering unitaries V which commute with the original Hamiltonian H .

First, observe that in the case of a time-dependent Hamiltonian $H(t)$, we can simply include the clock as one of our systems. Letting τ be the coordinate operator of the clock system and Π_τ such that $[\tau, \Pi_\tau] = -i$, define $H_{\text{indep}} = H(\tau) + \Pi_\tau$. The τ observable faithfully records the time t , as can be seen by solving the Heisenberg equations of motion to get $\tau(t) = \tau(0) + t$. Now consider a joint density matrix for the system plus clock of the form $\xi(t) = \rho(t) \otimes |t\rangle\langle t|$, where $|t\rangle$ are the eigenstates of τ . The time-independent Hamiltonian H_{indep} acting on the state $\xi(t)$ will generate the equation of motions of $H(t)$ acting on $\rho(t)$, but will also conserve energy. To see this, recall that the product rule of derivatives gives

$$\frac{d\xi(t)}{dt} = \frac{d\rho(t)}{dt} \otimes |t\rangle\langle t| + \rho(t) \otimes \frac{d}{dt}|t\rangle\langle t|, \quad (75)$$

while the Heisenberg equation of motion gives

$$\begin{aligned} \frac{d\xi(t)}{dt} &= i[H_{\text{indep}}, \xi(t)] \\ &= i[H(t), \rho(t)] \otimes |t\rangle\langle t| + \rho(t) \otimes [\Pi_\tau, |t\rangle\langle t|] \end{aligned} \quad (76)$$

Comparing the above two equations we have $\dot{\rho}(t) = i[H(t), \rho(t)]$ as claimed. That a system in a pure state stays in a pure state, can be achieved by having the clock have a large coherent superposition over energy levels, thus the change in its state can be made arbitrarily small, as explained in Section V. We thus can go from a picture with a changing Hamiltonian, to one with a fixed one. The model with time-dependent Hamiltonian is therefore equivalent to the one considered here, with fixed Hamiltonian.

Likewise, in the case where an interaction term is added, we can take the total Hamiltonian to be $H_{\text{tot}} = H + H_{\text{int}}$ and assume that initially, $(H + H_{\text{int}})|\psi\rangle \approx H|\psi\rangle$ i.e. the systems are initially far apart. They can then evolve unitarily, such that the systems interact, and then move far enough apart that the interaction terms are negligible again. In such a picture, an eigenstate of the initial Hamiltonian H will evolve into an eigenstate of H with the same energy (by conservation of energy, and the fact that the interaction is negligible at initial and final times). Thus, all that happens here is that eigenstates of fixed energy evolve to other eigenstates of the same energy, and this can be accomplished by means of a fixed Hamiltonian H and a unitary V which commutes with it. We thus see that also the picture of adding interaction terms is equivalent to having a fixed Hamiltonian H , and operations V which commute with it.

Similarly, the application of a unitary during some time period can be made via application of a fixed Hamiltonian. One can include an internal clock τ which merely acts as a catalyst and thus have some fixed Hamiltonian

$$H_{\text{tot}} = H + H_{\text{int}}g(\tau) + \Pi_\tau \quad (77)$$

which effectively implements $e^{-iH_{\text{int}}t}$ over some time interval determined by the function $g(\tau)$. Here Π_τ is conjugate to τ and one can verify via the Heisenberg equations of motion that τ depends linearly on t . Since $[H_{\text{int}}, H] = 0$ one can also verify via the Heisenberg equations of motion for Π_τ that there is no backreaction or energy exchange to the clock at late times provided $g(\tau)$ is chosen such that $\int_{t_i}^{t_f} g'(\tau) = 0$ and $g(\tau)$ and $\tau(t)$ chosen such that $g(\tau) = 0$ before $t = t_i$ and after $t = t_f$.

One might be concerned that if we perturb the Hamiltonian slightly, our work extraction will not be robust. To see this, let us consider the case where we don't succeed in implementing our unitary exactly, but rather some H_{int} which does not completely commute with the original Hamiltonian $[H, H_{\text{int}}] = -i\delta$. Viewed internally, will see that this is equivalent to allowing a violation of conservation of energy by amount δ – something which is interesting to study in its own right. Now the equations of motion for τ are unchanged, and thus the unitary $e^{-iH_{\text{int}}t}$ is still implemented. However there is some backreaction on the clock. Solving the Heisenberg equations of motion for Π , we find that there is a small momentum kick to the clock

$$\begin{aligned}\Pi(t_f) - \Pi(t_i) &= \int_{t_i}^{t_f} g'(t) H_{\text{int}}(t) \\ &= H_{\text{int}}(t_f) - H_{\text{int}}(t_i)\end{aligned}\tag{78}$$

where we have taken $g(\tau)$ to be 1 between t_i and t_f and 0 everywhere else. At each cycle some amount of energy gets stored in the clock, depending on the initial and final states of the system. If we allow these to fluctuate, then the transfer is some δ and the clock undergoes a random walk.

We thus find that if we run the extraction process as a cycle, where we repeat the process over several cycles, then at each cycle we still exactly implement $e^{-iH_{\text{int}}t}$, it's just that it now has some tiny non-commuting part with the original Hamiltonian, resulting in some energetic backreaction to the clock, and thus some entropy being stored in the clock. However, the extracted work grows linearly, and δ can be made arbitrarily small. After n cycles, the moment of the clock has undergone a random walk, of order \sqrt{n} , an amount which is negligible compared to the extractable work in the case of many cycles.

Let us now turn to the second question. It might appear that an experimenter who wished to implement our protocols would need to very carefully manipulate all the many degrees of freedom of the n systems and the heat bath. However, this is not the case, as we will now demonstrate explicitly using the example of work distillation. There, we were mapping eigenstates which had a type lq on the heatbath γ , and pn on the resource ρ to microstates which had type rk on the garbage σ , and m 1's on the work system. However, although for any implementation of the protocol, we need a particular mapping of strings (i.e. microstates) of these initial types, to strings of the final type, **any mapping** will do. The only important thing which is required is just that the unitary operation map the initial types to the final types. Thus an experimenter who wishes to implement the protocol, does not need fine-grained control over the mapping of microstates within one type to microstates within another type. She only needs to know that the unitary maps one type into another. In other words, there are an exponentially large number of possible implementations of our protocols each of which map particular strings within the initial types to particular strings in the final types. However, it doesn't matter which implementation is chosen, and the experimenter thus does not need the fine degree of control that is required to achieve a particular implementation.

We can think of the type as being like a macroscopic variable such as the total magnetisation of a composite system, or its total energy (indeed it is the latter). In the distillation protocol, we map the macroscopic variables of energy on two large systems ($\gamma^{\otimes l}$ and $\rho^{\otimes n}$) to the macroscopic variable of energy on the final system. Any unitary which accomplishes this will successfully implement our protocol. Thus, the experimenter only needs control over the macroscopic variables, not the microscopic ones.

Equivalence of these paradigms is discussed in more detail, and in the case of finite systems, in [7].

-
- [1] K. Horodecki, M. Horodecki, P. Horodecki, and J. Oppenheim, Phys. Rev. Lett. **94**, 200501 (2005).
 - [2] W. Van Dam and P. Hayden, Phys. Rev. A **67**, 060302, (2003).
 - [3] H-K. Lo and S. Popescu. arXiv:quant-ph/9707038.
 - [4] P. Hayden and A. Winter. Phys. Rev. A **67**, 012326 (2003).
 - [5] A. Harrow and H-K. Lo. IEEE Trans. Inf. Theory **50**, 319 (2004).
 - [6] A. Harrow. Proc. of the XVIth Intl. Cong. on Math. Phys., pp. 536–540, 2009. Available at arXiv:0909.1557 [quant-ph].
 - [7] M. Horodecki, J. Oppenheim et al., in preparation.