

# Estimating operator norms using covering nets

Fernando G.S.L. Brandão \*

Aram W. Harrow †

## Abstract

We present several polynomial- and quasipolynomial-time approximation schemes for a large class of generalized operator norms. Special cases include the  $2 \rightarrow q$  norm of matrices for  $q > 2$ , the support function of the set of separable quantum states, finding the least noisy output of entanglement-breaking quantum channels, and approximating the injective tensor norm for a map between two Banach spaces whose factorization norm through  $\ell_1^n$  is bounded.

These reproduce and in some cases improve upon the performance of previous algorithms by Brandão-Christandl-Yard [BCY11] and followup work, which were based on the Sum-of-Squares hierarchy and whose analysis used techniques from quantum information such as the monogamy principle of entanglement. Our algorithms, by contrast, are based on brute force enumeration over carefully chosen covering nets. These have the advantage of using less memory, having much simpler proofs and giving new geometric insights into the problem. Net-based algorithms for similar problems were also presented by Shi-Wu [SW12] and Barak-Kelner-Steurer [BKS13], but in each case with a run-time that is exponential in the rank of some matrix. We achieve polynomial or quasipolynomial runtimes by using the much smaller nets that exist in  $\ell_1$  spaces. This principle has been used in learning theory, where it is known as Maurey's empirical method.

## 1 Introduction

Given a  $n \times m$  matrix  $M$ , its operator norm is given by  $\|M\| = \max_{x \in \mathbb{C}^m} \|Mx\|_2 / \|x\|_2$ , with  $\|x\|_2 = (\sum_i |x_i|^2)^{\frac{1}{2}}$  the Euclidean norm. The operator norm is also given by the square root of the largest eigenvalue of  $M^\dagger M$  and thus can be efficiently computed. There are numerous ways of generalizing the operator norm, e.g. by considering tensors instead of matrices, by changing the Euclidean norm to another norm, or by considering other vector spaces instead of  $\mathbb{C}^m$ . Although such generalizations are very useful in applications, they can be substantially harder to compute than the basic operator norm, and in many cases we still do not have a good grasp of the computational complexity of computing, or even only approximating, them. In some cases quasipolynomial algorithms are known, usually based on semidefinite programming (SDP) hierarchies, and in other cases quasipolynomial hardness results are known. These are partially overlapping so that some problems have sharp bounds on their complexity and for others there are exponential gaps between the best upper and lower bounds. As we will discuss below, the complexity of these problems is not only a basic question in the theory of algorithms, but also is closely related to the unique games conjecture and the power of multiprover quantum proof systems.

In this paper we give new algorithms for several variants of the basic operator norm of interest in quantum information theory, theoretical computer science, and the theory of Banach spaces. Unlike most past work which was based on SDP hierarchies, our algorithms simply enumerate over a carefully chosen net of points. This yields run-times that often match the SDP hierarchies and sometimes improve upon them. Besides improved performance, our algorithms have the advantage of being based on simple geometric properties of spaces we are optimizing over, which may help explain which types of norms are amenable to quasipolynomial optimization. In particular we consider the following four optimization problems in this work:

**Optimization over Separable States:** An important problem in quantum information theory is to optimize a linear function over the set of separable (i.e. non-entangled) states, defined as bipartite

---

\* (a) Quantum Architectures and Computation Group, Microsoft Research, Redmond, WA and (b) Department of Computer Science, University College London WC1E 6BT. email: [fbrandao@microsoft.com](mailto:fbrandao@microsoft.com)

† Center for Theoretical Physics, Massachusetts Institute of Technology. email: [aram@mit.edu](mailto:aram@mit.edu)

density matrices that can be written as a convex combination of tensor product states. This problem is closely related to the task of determining if a given quantum state is entangled or not (called the quantum separability problem) and to the computation of several other quantities of interest in quantum information, including the optimal acceptance probability of quantum Merlin-Arthur games with unentangled proofs, optimal entanglement witnesses, mean-field ground-state energies, and measures of entanglement; see [HM13] for a review of many of these connections.

Given an operator  $M$  acting on the bipartite vector space  $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$  the support function of  $M$  on the set of separable states is given by

$$h_{\text{Sep}(d_1, d_2)}(M) := \max_{\alpha \in \mathcal{D}_{d_1}, \beta \in \mathcal{D}_{d_2}} \text{tr}[M(\alpha \otimes \beta)], \quad (1)$$

with  $\mathcal{D}_d$  the set of density matrices on  $\mathbb{C}^d$  ( $d \times d$  positive semidefinite matrices of unit trace). Our goal is to approximate  $h_{\text{Sep}(d_1, d_2)}(M)$ . For  $M \in L(\mathbb{C}^{d^n})$ , define

$$h_{\text{Sep}^n(d)}(M) = \max_{\alpha_1, \dots, \alpha_n \in \mathcal{D}_d} \text{tr}[M(\alpha_1 \otimes \dots \otimes \alpha_n)]. \quad (2)$$

The first result on the complexity of computing  $h_{\text{Sep}(d_1, d_2)}$  was negative: Gurvits showed that the problem is NP-hard for sufficiently small additive error (inverse polynomial in  $d_1 d_2$ ) [Gur03]. Then [HM13] showed there is no  $\exp(O(\log^{2-\Omega(1)}(d_1 d_2)))$  time algorithm even for a constant error additive approximation of the quantity, assuming the exponential time hypothesis (ETH<sup>1</sup>). This left open the question whether there are quasipolynomial-time algorithms (i.e. of time  $\exp(\text{polylog}(d_1, d_2))$ ).

In [BCY11] it was shown that this is indeed the case at least for a class of linear functions: namely those corresponding to quantum measurements that can be implemented by local operations and one-directional classical communication (one-way LOCC or 1-LOCC). For this particular class of measurements the problem can be solved with error  $\delta$  in time  $\exp(O(\delta^{-2} \log(d_1) \log(d_2)))$ . The proof was based on showing that the hierarchy of semidefinite programs for the problem introduced in 2004 by Doherty, Parrilo and Spedalieri [DPS04] (which is an application of the more general Sum-of-Squares (SoS) hierarchy, also known as the Lasserre hierarchy, to the separability problem) converges quickly. The approach of [BCY11] was to use ideas from quantum information theory (monogamy of entanglement, entanglement measures, hypothesis testing, etc) to find good bounds on the quality of the SoS hierarchy. Since then several follow-up work gave different proofs of the result, but always using quantum information-theoretic ideas [BH13, LW14, BC11, Yan06].

A corollary of [BCY11] and the other results on 1-LOCC  $M$  is that  $h_{\text{Sep}(d_1, d_2)}(M)$  can also be approximated for a different class of operators  $M$ : those with small Hilbert-Schmidt norm  $\|M\|_{\text{HS}} := \text{tr}(M^\dagger M)^{\frac{1}{2}}$ . Ref. [BaCY11] showed that also in this case there is a quasipolynomial-time algorithm for estimating Eq. (1). An interesting subsequent development was the work of Shi and Wu [SW12] (see also [BKS13]), who gave a different algorithm for the problem based on enumerating over nets. It was left as an open question whether a similar approach could be given for the case of one-way LOCC measurements (which is more relevant both physically<sup>2</sup> and in terms of applications; see again [HM13]).

**Estimating the Output Purity of Quantum Channels:** Another important optimization problem in quantum information theory consists of determining how much noise a quantum channel introduces. A quantum channel models a general physical evolution and is given mathematically by a completely positive trace preserving map  $\Lambda : \mathcal{D}_{d_1} \rightarrow \mathcal{D}_{d_2}$ . One way to measure the level of noise of the channel is to compute the maximum over states of the output Schatten- $\alpha$  norm, for a given  $\alpha > 1$ :

$$\|\Lambda\|_{1 \rightarrow \alpha} := \max_{\rho \in \mathcal{D}_{d_1}} \|\Lambda(\rho)\|_\alpha, \quad (3)$$

<sup>1</sup>The ETH is the conjecture that 3-SAT instances of length  $n$  require time  $2^{\Omega(n)}$  to solve. This is a plausible conjecture for deterministic, randomized or quantum computation, and each version yields a corresponding lower bound on the complexity of estimating  $h_{\text{Sep}}$ .

<sup>2</sup>The one-way LOCC norm gives the optimal distinguishability of two multipartite quantum states when only local measurements can be done, and the parties can coordinate by one-directional communication. See [MWW09] for a discussion of its power.

with  $\|Z\|_\alpha = \text{tr}(|Z|^\alpha)^{\frac{1}{\alpha}}$ . The quantity  $\|\Lambda\|_{1 \rightarrow \alpha}$  varies from one, for an ideal channel, to  $d_2^{-1+\alpha^{-1}}$  for the depolarizing channel mapping all states to the maximally mixed state. This optimization problem has been extensively studied, in particular because for  $\alpha \approx 1$  it is related to the Holevo capacity of the channel, whose regularization gives the classical capacity of the channel (i.e. how many reliable bits can be transmitted per use of the channel).

It was shown in [HM13] that, assuming ETH, there is no algorithm that runs in time  $\exp(O(\log^{2-\Omega(1)}(d)))$  and can decide if  $\|\Lambda\|_{1 \rightarrow \alpha}$  is one or smaller than  $\delta$  (for any fixed  $\delta > 0$  and  $\alpha > 1$ ) for a general quantum channel  $\Lambda : \mathcal{D}_d \rightarrow \mathcal{D}_d$ . On the algorithmic side, nothing better than exhaustive search over the input space (taking time  $\exp(\Omega(d_1))$ ) is known.

An interesting subclass of quantum channels, lying somewhere between classical channels and fully quantum channels, are the so-called entanglement-breaking channels, which are the channels that cannot be used to distribute entanglement. Any entanglement-breaking quantum channel  $\Lambda$  can be written as [HSR04]:

$$\Lambda(\rho) := \sum_i \text{tr}(X_i \rho) Y_i, \quad (4)$$

with  $Y_i \geq 0$ ,  $\text{tr}(Y_i) = 1$  quantum states and  $X_i \geq 0$ , and  $\sum_i X_i = I$  a quantum measurement. Because of their simpler form, one can expect that there are more efficient algorithms for computing the maximum output norm of entanglement-breaking channels. However until now no algorithm better than exhaustive search was known either (apart from the case  $\alpha = \infty$  where the Sum-of-Squares hierarchy can be used and analyzed using [BCY11]).

**Computing  $p \rightarrow q$  Norms:** Given a  $d_1 \times d_2$  matrix  $A$  we define its  $p \rightarrow q$  norm by

$$\|A\|_{p \rightarrow q} := \max_{x \in \mathbb{C}^{d_2}} \frac{\|Ax\|_q}{\|x\|_p}, \quad \|x\|_p := \left( \sum_{i=1}^{d_2} |x|^p \right)^{1/p} \quad (5)$$

Such norms have many different applications, such as in hypercontractive inequalities and determining if a graph is a small-set expander [BBH<sup>+</sup>12], to oblivious routing [BV11] and robust optimization [Ste05]. However we do not have a complete understanding of the complexity of computing them. For  $2 < q \leq p$  or  $q \leq p < 2$ , it is NP-hard to approximate them to any constant factor [BV11]. In the regime  $q > p$  (the one relevant for hypercontractivity and small-set expansion) the only known hardness result is that to obtain any (multiplicative) constant-factor approximation for the  $2 \rightarrow 4$  norm of a  $n \times n$  matrix is as hard as solving 3-SAT with  $\tilde{O}(\log^2(n))$  variables [BBH<sup>+</sup>12].

On the algorithmic side, besides the  $2 \rightarrow 2$  and  $2 \rightarrow \infty$  norms being exactly computable in polynomial time, Ref. [BBH<sup>+</sup>12] showed that one can use the Sum-of-Squares hierarchy to compute in time  $\exp(O(\log^2(n)\varepsilon^{-2}))$  a number  $X$  s.t.

$$\|A\|_{2 \rightarrow 4}^4 \leq X \leq \|A\|_{2 \rightarrow 4}^4 + \varepsilon \|A\|_{2 \rightarrow 2}^2 \|A\|_{2 \rightarrow \infty}^2. \quad (6)$$

Whether similar approximations can be obtained for  $2 \rightarrow q$  norms for other values of  $q$  was left as an open problem.

**Computing the Operator Norm between Banach Spaces:** These problems are all special cases of the following general question. Given a map  $T : \mathcal{A} \rightarrow \mathcal{B}$  between Banach spaces  $\mathcal{A}, \mathcal{B}$ , can we approximately compute the following operator norm?

$$\|T\|_{\mathcal{A} \rightarrow \mathcal{B}} := \sup_{x \neq 0} \frac{\|Tx\|_{\mathcal{B}}}{\|x\|_{\mathcal{A}}} \quad (7)$$

## 1.1 Summary of Results

In this paper we give new algorithmic results for the four problems discussed above. They can be summarized as follows.

**Separable-state optimization by covering nets:** We give a different algorithm for optimizing linear functions over separable states (corresponding to one-way LOCC measurements) based on enumerating over covering nets (see Algorithm 1). The complexity of the algorithm matches the time complexity of [BCY11] (see Theorem 2). The proof does not use information theory in any way, nor the SoS hierarchy. Instead the main technical tool is a matrix version of the Hoeffding bound (see Lemma 3). It gives new geometric insight into the problem and gives arguably the simplest and most self-contained proof of the result to date. It also gives an explicit rounding (as does [BH13] but in contrast to [BCY11, LW14, BC11, Yan06]).

For particular subclasses of one-way LOCC measurements our algorithm improves the run time of [BCY11]. One example is the case where Bob’s measurement outcomes are low rank, in which we find a  $\text{poly}(d_2)d_1^{O(\varepsilon^{-2})}$ -time algorithm.

**Generalization to arbitrary operator norms:** Computing  $h_{\text{Sep}}$  is mathematically equivalent to computing the  $1 \rightarrow \infty$  norm of a quantum channel, or more precisely the  $S_1 \rightarrow S_\infty$  norm where  $S_\alpha$  denotes the Schatten- $\alpha$  norm. This perspective will help us generalize the scope of our algorithm, to estimating the  $S_1 \rightarrow \mathcal{B}$  norm for a general Banach space  $\mathcal{B}$ . The analysis of this algorithm is based on tools from asymptotic geometric analysis, and we will see that its efficiency depends on properties of  $\mathcal{B}$  known as the Rademacher type and the modulus of uniform smoothness. Besides generalizing the scope of the algorithm, this also gives more of a geometric explanation of its performance. We focus on two special cases of the problem:

1. **maximum output norm:** A particular case of the generalization is the problem of computing the maximum output purity of a quantum entanglement-breaking channel (measured in the Schatten- $\alpha$  norms). We prove that for any  $\alpha > 1$  one can compute  $\|\Lambda\|_{1 \rightarrow \alpha}$  in time  $\text{poly}(d_2)d_1^{O(\varepsilon^{-2})}$  to within additive error  $\varepsilon$ . (see Corollary 16). In contrast known hardness results [HM10, HM13] show that no such algorithm exists for general quantum channels (under the exponential time hypothesis). Previously the entanglement-breaking case was not known to be easier.
2. **matrix  $2 \rightarrow q$  norms:** As a second particular case of the general framework we extend the approximation of [BBH<sup>+</sup>12] to the  $2 \rightarrow 4$  norm, given in Eq. (6), to the  $2 \rightarrow q$  norms for all  $q \geq 2$  (see Corollary 17).

**Operator norms between Banach spaces:** This framework can be further generalized to estimating the operator norm of any linear map from  $\mathcal{A} \rightarrow \mathcal{B}$  for Banach spaces  $\mathcal{A}, \mathcal{B}$ . Here we have replaced  $S_1$  with any finite-dimensional Banach space  $\mathcal{A}$  whose norm can be computed efficiently. When applied to an operator  $\Lambda$ , the approximation error scales with the  $\mathcal{A} \rightarrow \ell_1^n \rightarrow \mathcal{B}$  factorization norm, which is the minimum of  $\|\Lambda_1\|_{\ell_1^n \rightarrow \mathcal{B}} \|\Lambda_2\|_{\mathcal{A} \rightarrow \ell_1^n}$  such that  $\Lambda = \Lambda_1 \Lambda_2$ . Factorization norms have applications to communication complexity [LS07, LS09], Banach space theory [Pie07], and machine learning [LRS<sup>+</sup>10], and here we argue that they help explain what makes the class of 1-LOCC measurements uniquely tractable for algorithms. In Section 4 we describe an algorithm for this general norm estimation problem, which to our knowledge previously had no efficient algorithms. This problem equivalently can be viewed as computing the injective tensor norm of two Banach spaces.

We remark that this generalization is not completely for free, so we cannot simply derive all our other algorithms from this final one. In the case where  $\mathcal{A} = S_1^d$  (which corresponds to all of our specific applications), we are able to easily sparsify the input; i.e. given  $\sum_{i=1}^n A_i \otimes B_i$ , we can reduce  $n$  to be  $\text{poly}(d)$  without loss of generality. For general input spaces  $\mathcal{A}$  we do not know if this is possible. Also, the case of  $h_{\text{Sep}}$  is much simpler, and so it may be helpful to read it first.

## 1.2 Comparison with prior work

As discussed in the introduction, previous algorithms for separable-state optimization (and as a corollary, the  $2 \rightarrow 4$  norm) have been obtained using SDP hierarchies. Our algorithms generally match or improve upon their parameters, but with the added requirement for the separable-state problem that the input be presented in a more structured form.

Several parallels between LP/SDP hierarchies and net-based algorithms have been developed for other problems. The first example of this was Ref. [DKLP06a] which gave both types of algorithms for the problem of maximizing a polynomial over the simplex, improving on a result implicit in the 1980 proof of the finite de Finetti theorem by Diaconis and Freedman [DF80]. Besides the separable-state approximation problem that we study, hierarchies and nets have been found to have similar performance in finding approximate Nash equilibria [LMM03, Har15] and in estimating the value of free two-prover games [AIM14, BH13]. The state-of-the-art run-time for solving Unique Games and Small Set Expansion have also been achieved using both hierarchies and covering-nets. These parallels are summarized in the table:

Problem	nets	hierarchies/information theory
$\max_{x \in \Delta_n} p(x)$	[DKLP06a]	[DF80, DKLP06a]
approximate Nash	[LMM03, ALSV13a]	[Har15]
free games	[AIM14]	[BH13, Cor 4]
unique games	[ABS10]	[BRS11]
small-set expansion	[ABS10]	[BBH <sup>+</sup> 12, §10]
separable states	[SW12, BKS13], this work	[BaCY11, BH13, BKS13, LW14, LS14]

Table 1: We briefly describe these problems here. Full descriptions can be found in the references in the table. In  $\max_{x \in \Delta_n} p(x)$ ,  $\Delta_n$  is the  $n$ -dimensional probability simplex and  $p(x)$  is a low-degree polynomial. “Approximate Nash” refers to the problem of finding a pair of strategies in a two-player non-cooperative game for which no player can improve their welfare by more than  $\varepsilon$ . “Free games” refers to two-prover one-round proof systems where the questions asked are independent; the computational problem is to estimate the largest possible acceptance probability. “Unique games” describes instead proof systems with “unique” constraints; i.e. for each question pair and each answer given by one of the provers, there is exactly one correct answer possible for the other prover. Small-set expansion asks, given a graph  $G$  and parameters  $\varepsilon, \delta > 0$ , whether all subsets with a  $\delta$  fraction of the vertices have a  $\geq 1 - \varepsilon$  fraction of edges leaving the set or whether there exists one with a  $\leq \varepsilon$  fraction of edges leaving the set. Finally “separable states” refers to estimating  $h_{\text{Sep}(n,n)}$  as we will discuss elsewhere in the paper. It can also be thought of as estimating  $\max_{\|x\|_2=1} p(x)$  for some low-degree polynomial  $p(x)$ .

While this paper focuses on the particular problems where we can improve upon the state-of-the-art algorithms, we hope to be a step towards more generally understanding the connections between these two methods. In almost every case above, the best covering-net algorithms achieve nearly the same complexity as the best analyses of SDP hierarchies. There are a few exceptions. Ref. [BBH<sup>+</sup>12] shows  $O(1)$  rounds of the SoS hierarchy can certify a small value for the Khot-Vishnoi integrality gap instances of the unique games problem, but we do not know how to achieve something similar using nets. A more general example is in [BKS13], which shows that the SoS hierarchy can approximate  $h_{\text{Sep}}(M)$  in quasipolynomial time when  $M$  is entrywise nonnegative.

The closest related paper to this work is [SW12] by Shi and Wu (as well as Appendix A of [BKS13]), which also used enumeration over  $\varepsilon$ -nets to approximate  $h_{\text{Sep}}$ . Here we explain their results in our language.

Shi and Wu [SW12] have two algorithms: one when  $M$  has low Schmidt rank (i.e. factorizes as  $S_1 \rightarrow \ell_2^r \rightarrow S_\infty$  for small  $r$ ) and one where  $M$  has low rank, which we can interpret as a  $\ell_2^r \rightarrow S_\infty^{d_1 \times d_2}$  factorization (here  $S_\infty^{d_1 \times d_2}$  refers to the space of  $d_1 \times d_2$ -dimensional matrices with norm given by the largest singular value). These correspond to their Theorems 5 and 8 respectively. In both cases they construct  $\varepsilon$ -nets for the  $\ell_2^r$  unit ball of size  $\varepsilon^{-O(r)}$  (here,  $\ell_2$  could be replaced with any norm; see Lemma 9.5 of [LT91]). In both cases, their results can be improved to yield multiplicative approximations, using ideas from [BKS13].

Appendix A of Barak, Kelner and Steurer [BKS13] considers fully symmetric 4-index tensors  $M \in (\mathbb{R}^n)^{\otimes 4}$ , so that when viewed as  $n^2 \times n^2$  matrices their rank and Schmidt rank are the same; call them  $r$ . Their algorithm is similar to that of [SW12], although they observe additionally (using different terminology) that for any self-adjoint operator  $T : \mathcal{A}^* \rightarrow \mathcal{A}$  (i.e. satisfying  $\langle T(X), Y \rangle = \langle X, T(Y) \rangle$ ) the  $\mathcal{A}^* \rightarrow \ell_2 \rightarrow \mathcal{A}$  norm is

equal to the  $\mathcal{A}^* \rightarrow \mathcal{A}$  norm. This means that constructing an  $\varepsilon$ -net for  $B(\ell_2)$  actually yields a multiplicative approximation of the  $\mathcal{A}^* \rightarrow \mathcal{A}$  norm (here the  $S_1 \rightarrow S_\infty$  norm).

Achieving a multiplicative approximation is stronger than what our algorithms achieve, but it is at the cost of a runtime that can be exponential in the input size even for a constant-factor approximation. By contrast, our algorithms yield nontrivial approximations in polynomial or quasipolynomial time.

### 1.3 Notation

Define the sets of  $d \times d$  real and complex semidefinite matrices by  $\mathcal{S}_+^d, \mathcal{H}_+^d$  respectively. For complex vector spaces  $V, W$ , define  $\mathcal{L}(V, W)$  to be the set of linear operators from  $V$  to  $W$ ,  $\mathcal{L}(V) := \mathcal{L}(V, V)$  and  $\mathcal{H}(V), \mathcal{H}_+(V)$  to be respectively the Hermitian and positive-semidefinite operators on  $V$ .

For  $\alpha \geq 1$  define the  $\ell_\alpha, S_\alpha$  metrics on vectors and matrices respectively by  $\|x\|_{\ell_\alpha} = (\sum_i |x_i|^\alpha)^{1/\alpha}$  and  $\|X\|_{S_\alpha} = (\text{tr}|X|^\alpha)^{1/\alpha}$ . Denote the corresponding normed spaces by  $\ell_\alpha^d, S_\alpha^d$ . Where it is clear from context we will refer to both norms by  $\|\cdot\|_\alpha$ . We use  $\|\cdot\|$  without subscript to denote the operator norm for matrices (i.e.  $\|X\| = \|X\|_{S_\infty}$ ) and the Euclidean norm for vectors (i.e.  $\|x\| = \|x\|_{\ell_2}$ ).

We use  $O(f(x))$  to mean  $O(f(x) \text{poly log}(f(x)))$  and say that  $f(x)$  is ‘‘quasipolynomial’’ in  $x$  if  $f \leq O(\exp(\text{poly log}(x)))$ .

For a normed space  $V$ , define  $B(V) = \{v \in V : \|v\| \leq 1\}$ . Two important special cases are the probability simplex  $\Delta_n := B(\ell_1^n) \cap \mathbb{R}_{\geq 0}^n$  and the set of density matrices (also called ‘‘quantum states’’)  $\mathcal{D}_d := B(S_1^d) \cap \mathcal{H}_+^d = \text{conv}\{vv^\dagger : v \in B(\ell_2^d)\}$ . Here  $v^\dagger$  is the conjugate transpose of  $v$ . For  $k$  a positive integer, define also

$$\Delta_n(k) := \left\{ \frac{e_{i_1} + \dots + e_{i_k}}{k} : i_1, \dots, i_k \in [n] \right\} \subset \Delta_n, \quad (8)$$

where  $e_i$  is the vector in  $\mathbb{R}^n$  with a 1 in position  $i$  and zeros elsewhere. For a convex set  $K$  define the support function  $h_K(x) := \sup_{y \in K} \langle x, y \rangle$ . For matrices  $\langle, \rangle$  refers to the Hilbert-Schmidt inner product  $\langle X, Y \rangle := \text{tr}(X^\dagger Y)$ .

Banach spaces are normed vector spaces with an additional condition (completeness, i.e. convergence of Cauchy sequences) that is relevant only in the infinite dimensional case. In this work we will consider only finite-dimensional Banach spaces.

## 2 Warmup: algorithm for bipartite separability

In this section we describe a simple version of our algorithm. It contains all the main ideas which we will later generalize. Let  $M = \sum_{i=1}^n X_i \otimes Y_i$ , where  $X_i \in \mathcal{S}_+^{d_1}, Y_i \in \mathcal{S}_+^{d_2}, \sum_i X_i \leq I$ , and each  $Y_i \leq I$ . In quantum information language,  $M$  is a 1-LOCC measurement, meaning it can be implemented with local operations and one-way classical communication<sup>3</sup>. In later sections we will see that  $M$  can also be interpreted in a (mathematically) more natural way as a bounded map from  $S_1$  to  $S_\infty$ . The goal of our algorithm is to approximate  $h_{\text{Sep}(d_1, d_2)}(M)$ , where we define the set of separable states as

$$\text{Sep}(d_1, d_2) := \text{conv}\{\alpha \otimes \beta : \alpha \in \mathcal{D}_{d_1}, \beta \in \mathcal{D}_{d_2}\}. \quad (9)$$

There have been several recent proofs [BCY11, BH13, LW14], each based on quantum information theory, that SDP hierarchies can estimate  $h_{\text{Sep}(d_1, d_2)}(M)$  to error  $\varepsilon\|M\|$  in time  $\exp(O(\log^2(d)/\varepsilon^2))$ . Similar techniques also appeared in [BKS13, LS14] for different classes of operators  $M$ . The role of the 1-LOCC conditions in these proofs was typically not completely obvious, and indeed it entered the proofs of [BCY11, BH13, LW14] in three different ways. We now give another interpretation of it that is arguably more geometrically natural.

<sup>3</sup>Conventionally these have  $\sum_i X_i = I$ , but our formulation is essentially equivalent.

Begin by observing that

$$\begin{aligned}
h_{\text{Sep}(d_1, d_2)}(M) &= \max_{\alpha \in \mathcal{D}_{d_1}, \beta \in \mathcal{D}_{d_2}} \text{tr}[(\alpha \otimes \beta)M], \\
&= \max_{\alpha \in \mathcal{D}_{d_1}, \beta \in \mathcal{D}_{d_2}} \sum_{i=1}^n \text{tr}[\alpha X_i] \text{tr}[\beta Y_i] \\
&= \max_{p \in S_X} \|p\|_Y.
\end{aligned} \tag{10}$$

In the last step we have defined

$$S_X := \{p \in \Delta_n : \exists \alpha \in \mathcal{D}_{d_1}, p_i = \text{tr}[\alpha X_i] \forall i \in [n]\}, \quad \text{and} \quad \|a\|_Y := \left\| \sum_{i=1}^n a_i Y_i \right\|. \tag{11}$$

The basic algorithm is the following:

**Algorithm 1** (Basic algorithm for computing  $h_{\text{Sep}}(M)$  for one-way LOCC  $M = \sum_i X_i \otimes Y_i$ ).

**Input:**  $\{X_i\}_{i=1}^n \subset \mathcal{H}_+^{d_1}, \{Y_i\}_{i=1}^n \subset \mathcal{H}_+^{d_2}$ .

**Output:** States  $\alpha \in \mathcal{D}_{d_1}$  and  $\beta \in \mathcal{D}_{d_2}$ .

1. Enumerate over all  $p \in \Delta_n(k)$ , with  $k = 9 \ln(d_2)/\delta^2$ .
  - (a) For each  $p$ , check (using Lemma 5) whether there exists  $q \in S$  with  $\|p - q\|_Y \leq \delta/2$ .
  - (b) If so, compute  $\|q\|_Y$ .
2. Let  $q$  be such that the  $\|q\|_Y$  is the maximum, and let  $\alpha \in \mathcal{D}_{d_1}$  be the state for which  $q_i = \text{tr}[X_i \alpha]$ . Output this  $\alpha$  and  $\beta$  satisfying  $\text{tr}[\beta \sum_i q_i Y_i] = \|\sum_i q_i Y_i\|$ .

The main result of this section is:

**Theorem 2.** Let  $M = \sum_{i=1}^n X_i \otimes Y_i$  be such that  $\sum_i X_i \leq I$ ,  $X_i \geq 0$ ,  $0 \leq Y_i \leq I$ . Algorithm 1 runs in time  $\text{poly}(d_1, d_2, n) \exp(O(\delta^{-2} \log(n) \log(d_2)))$  and outputs  $\alpha \in \mathcal{D}_{d_1}$  and  $\beta \in \mathcal{D}_{d_2}$  such that

$$h_{\text{Sep}}(M) \geq \text{tr}[M(\alpha \otimes \beta)] \geq h_{\text{Sep}}(M) - \delta, \tag{12}$$

For  $n = \text{poly}(d_1, d_2)$  this is the same running time as found in [BCY11] (while for  $n \ll \text{poly}(d_1, d_2)$  it is an improvement). Later in this section we will show how we can always modify the measurement to have  $n = \text{poly}(d_1, d_2)$  only incurring in a small error. But before that, we now show that Theorem 2 follows easily from two simple lemmas.

One of the lemmas is a consequence of the the well-known matrix Hoeffding bound.

**Lemma 3** (Matrix Hoeffding Bound [Tro10]). Suppose  $Z_1, \dots, Z_k$  are independent random  $d \times d$  Hermitian matrices satisfying  $\mathbb{E}[Z_i] = 0$  and  $\|Z_i\| \leq \lambda$ . Then

$$\Pr \left[ \left\| \frac{1}{k} \sum_{i=1}^k Z_i \right\| \geq \delta \right] \leq d \cdot e^{-\frac{k\delta^2}{8\lambda^2}}. \tag{13}$$

This is a special case of Theorem 2.8 from [Tro10]:

Our first lemma shows that one can restrict the optimization to a net of size  $n^{O(\log(d_2)\delta^{-2})}$ :

**Lemma 4.** For any  $p \in \Delta_n$  there exists  $q \in \Delta_n(k)$  with

$$\|p - q\|_Y \leq \sqrt{\frac{9 \ln(d_2)}{k}}. \tag{14}$$

*Proof.* Sample  $i_1, \dots, i_k$  according to  $p$  and set  $q = (e_{i_1} + \dots + e_{i_k})/k$ . Define  $\bar{Y} := \sum_{i=1}^n p_i Y_i$  and  $Z_j = \bar{Y} - Y_{i_j}$ . Observe that  $\mathbb{E}[Z_j] = 0$  and  $\|Z_j\| \leq 1$ . Then Lemma 3 implies that

$$\|p - q\|_Y = \left\| \frac{1}{k} \sum_{j=1}^k Z_j \right\| \leq \delta. \quad (15)$$

with positive probability if  $k > 8 \ln(d)/\delta^2$ . Setting  $\delta = \sqrt{9 \ln(d_2)/k}$  we find that there exists a choice of  $q \in \Delta_n(k)$  satisfying Eq. (14).  $\square$

The second lemma shows that one can decide efficiently if an element of the net is a valid solution. A similar result is in [SW12].

**Lemma 5.** *Given  $p \in \Delta_n$  and  $\varepsilon > 0$ , we can decide in time  $\text{poly}(d_1, d_2, n)$  whether the following set is nonempty*

$$S_X \cap \{q : \|p - q\|_Y \leq \varepsilon\}. \quad (16)$$

*Proof.* Both are convex sets, defined by semidefinite constraints. So we can test for feasibility with a SDP of size  $\text{poly}(d_1, d_2, n)$ . Indeed this is manifest for  $S_X$  in Eq. (11), while  $\{q : \|p - q\|_Y \leq \varepsilon\}$  can be written as

$$\{q : \|p - q\|_Y \leq \varepsilon\} = \left\{ (q_1, \dots, q_n) : q_i \geq 0, -\varepsilon I \leq \sum_i p_i Y_i - \sum_i q_i Y_i \leq \varepsilon I \right\}. \quad (17)$$

$\square$

We are ready to prove Theorem 2:

*Proof of Theorem 2.* Whatever the output  $x$  is,  $x \leq h_{\text{Sep}}(M) + \delta/2$ . On the other hand, let  $q = \arg \max_{q \in S} \|q\|_Y$ , so that  $\|q\|_Y = h_{\text{Sep}}(M')$ . By Lemma 4, there exists  $p \in \Delta_n(k)$  with  $\|p - q\|_Y \leq \delta/2$ . Thus our algorithm will output a value that is  $\geq h_{\text{Sep}}(M) - \delta$ . We conclude that the algorithm achieves an additive error of  $\delta$  in time  $\text{poly}(d_1, d_2) n^{O(\log(d_2)/\delta^2)}$ .  $\square$

## 2.1 Sparsification

We now consider the case where  $n \gg \text{poly}(d_1, d_2)$ . It turns out that we can modify the algorithm such that its running time is polynomial in  $n$  by first sparsifying the number of local terms of the measurement. This results in the following theorem.

**Theorem 6.** *Let  $M = \sum_{i=1}^n X_i \otimes Y_i$  be such that  $\sum_i X_i \leq I$ ,  $X_i \geq 0$ ,  $0 \leq Y_i \leq I$ . Algorithm 8 runs in time  $\text{poly}(n) \exp(O(\delta^{-2} \log d_1 \log(d_1 d_2)))$  and outputs  $\alpha \in \mathcal{D}_{d_1}$  and  $\beta \in \mathcal{D}_{d_2}$  such that*

$$h_{\text{Sep}}(M) \geq \text{tr}(M(\alpha \otimes \beta)) \geq h_{\text{Sep}}(M) - \delta, \quad (18)$$

The key element of the theorem is the following Lemma.

**Lemma 7.** *Given a 1-LOCC measurement  $M = \sum_{i=1}^n X_i \otimes Y_i$  and some  $\varepsilon > 0$  there exists a 1-LOCC measurement  $M' = \sum_{j=1}^{n'} X'_j \otimes Y'_j$  with  $\|M - M'\| \leq \varepsilon$  and  $n' \leq \text{poly}(d_1, d_2)/\varepsilon^2$ . If the decomposition of  $M$  is explicitly given then  $M'$  and its decomposition can be found in time  $\text{poly}(d_1, d_2, n)$  using a randomized algorithm.*

The modified algorithm is the following:

**Algorithm 8** (Algorithm for computing  $h_{\text{Sep}}(M)$  for one-way LOCC  $M = \sum_i X_i \otimes Y_i$ ).

**Input:**  $\{X_i\}_{i=1}^n, \{Y_i\}_{i=1}^n$ .

**Output:** States  $\alpha \in \mathcal{D}_{d_1}$  and  $\beta \in \mathcal{D}_{d_2}$ .

1. Use Lemma 7 to replace  $M = \sum_{i=1}^n X_i \otimes Y_i$  with  $M' = \sum_{i=1}^{n'} X'_i \otimes Y'_i$  satisfying  $\|M - M'\| \leq \delta/2$ .
2. Run Algorithm 1 on  $M'$ .

The proof of correctness is straightforward.

*Proof of Theorem 6.* Whatever the output  $x$  is,  $x \leq h_{\text{Sep}}(M') \leq h_{\text{Sep}}(M) + \delta/2$ . On the other hand, let  $q = \arg \max_{q \in \mathcal{S}} \|q\|_Y$ , so that  $\|q\|_Y = h_{\text{Sep}}(M')$ . By Lemma 4, there exists  $p \in \Delta_n(k)$  with  $\|p - q\|_Y \leq \delta/2$ . Thus our algorithm will output a value that is  $\geq h_{\text{Sep}}(M') - \delta/2 \geq h_{\text{Sep}}(M) - \delta$ . We conclude that the algorithm achieves an additive error of  $\delta$  in time  $\text{poly}(n)(d_1 d_2)^{O(\log(d_2)/\delta^2)}$ .  $\square$

It remains only to prove Lemma 7. This requires a careful use of the matrix Hoeffding bound (Lemma ??). The details are in Appendix A.

## 2.2 Multipartite

We now consider the generalization of the problem to the multipartite case. We consider measurements on a  $l$ -partite vector space  $\mathbb{C}^{d_1} \otimes \dots \otimes \mathbb{C}^{d_l}$ . Following Li and Smith [LS14], we define the class of fully one-way LOCC measurements on  $\mathbb{C}^{d_1} \otimes \dots \otimes \mathbb{C}^{d_l}$  recursively as all measurements  $M = \sum_i X_i \otimes M_i$ , where  $X_i \in \mathcal{H}_+^{d_1}$ ,  $\sum_i X_i \leq I$ ,  $M_i \in \mathcal{H}_+^{d_2 \dots d_l}$ , and each  $M_i$  is a fully one-way LOCC measurement in  $\mathbb{C}^{d_2} \otimes \dots \otimes \mathbb{C}^{d_l}$ .

Ref. [LS14] recently strengthened the result of [BH13] (from parallel one-way LOCC to fully one-way LOCC measurement) and proved that the SoS hierarchy approximates

$$h_{\text{Sep}(d_1, \dots, d_l)}(M) := \max_{\alpha_1 \in \mathcal{D}_{d_1}, \dots, \alpha_l \in \mathcal{D}_{d_l}} \text{tr}[(\alpha_1 \otimes \dots \otimes \alpha_l)M] \quad (19)$$

to within additive error  $\delta$  in time  $\exp(O(\log^2(d)l^3/\delta^2))$ , with  $d := \max_{i \in [l]} d_i$ . Here we show that our previous algorithm for the bipartite case can be extended to the multipartite setting to give the same run time.

**Theorem 9.** *Algorithm 10 above runs in time  $\exp(O(l^3 \ln^2(d)/\delta^2))$  and outputs states  $\alpha_i, i \in [l]$ , satisfying*

$$h_{\text{Sep}(d_1, \dots, d_l)}(M) \geq \text{tr}[M(\alpha_1 \otimes \dots \otimes \alpha_l)] \geq h_{\text{Sep}(d_1, \dots, d_l)}(M) - \delta. \quad (20)$$

**Algorithm 10** (Algorithm for computing  $h_{\text{Sep}(d_1, \dots, d_l)}(M)$  for fully one-way LOCC  $M$ ).

**Input:**  $\{X_{i_1, \dots, i_m}^{(m)} : m \in [l], i_1 \in [n_1], \dots, i_m \in [n_m]\} \subset \mathcal{H}_+^{d_m}$  such that

$$M = \sum_{i_1=1}^{n_1} X_{i_1}^{(1)} \otimes \sum_{i_2=1}^{n_2} X_{i_1, i_2}^{(2)} \otimes \dots \otimes \sum_{i_m=1}^{n_m} X_{i_1, i_2, \dots, i_m}^{(m)}$$

**Output:** States  $\alpha_i \in \mathcal{D}_{d_i}$ ,  $i \in [l]$ .

1. Use Lemma 7 to replace  $M = \sum_{i_1=1}^{n_1} X_{i_1}^{(1)} \otimes M_{i_1}$  with  $M' = \sum_{i_1=1}^{n'_1} (X_{i_1}^{(1)})' \otimes M'_{i_1}$  satisfying  $\|M - M'\| \leq \delta/2l$ . Here  $M_{i_1}$  is a shorthand for the collection  $\{X_{i_1, \dots, i_m}^{(m)}\}$  for  $m \geq 2$  and likewise for  $M'_{i_1}$ . Redefine  $M, \{X_{i_1, \dots, i_m}^{(m)}\}, \{n_i\}$  appropriately.
2. Initialize the variables  $\alpha_1, \dots, \alpha_l$  to  $\emptyset$ .
3. Enumerate over all  $p \in \Delta_n(k)$ , with  $k = 9l^2 \ln(d)/\delta^2$ . For each  $p$ ,
  - (a) Check (using Lemma 5) whether there exists  $q \in S_{X^{(1)}}$  with  $\|\sum_i (p_i - q_i) M_i\| \leq \delta/2l$ .
  - (b) If no such  $q$  exists then do not evaluate this value of  $p$  any further. Otherwise let  $\beta_1$  be the density matrix found in the SDP in Lemma 5 satisfying  $q_i = \text{tr}[\beta_1 X_i^{(1)}]$ .
  - (c) For  $m' \in \{2, \dots, m\}, i_2 \in [n_2], \dots, i_{m'} \in [n_{m'}]$ , define  $\tilde{X}_{i_2, \dots, i_{m'}}^{(m'-1)} := \sum_{i_1} q_{i_1} X_{i_1, i_2, \dots, i_{m'}}^{(m')}$ .
  - (d) Recursively call Algorithm 10 on input  $\{\tilde{X}_{i_1, \dots, i_{m'}}^{(m')}\}$ . Denote the output by  $\beta_2, \dots, \beta_l$ .
  - (e) If  $\text{tr}[M(\beta_1 \otimes \dots \otimes \beta_l)] > \text{tr}[M(\alpha_1 \otimes \dots \otimes \alpha_l)]$  then replace  $\alpha_1, \dots, \alpha_l$  with  $\beta_1, \dots, \beta_l$ .

### 2.3 The need for an explicit decomposition

The input to our algorithm is not only a 1-LOCC measurement  $M$  but an explicit decomposition of the form  $M = \sum_i X_i \otimes Y_i$  with each  $X_i \geq 0$ . Previous algorithms for  $h_{\text{Sep}}$  were mostly based on the SoS hierarchy (or its restriction to the separability problem also known as  $k$ -extendible hierarchy) [DPS04]. Running these requires only knowledge of  $M$  and not its decomposition. The decomposition appears in the analysis of [BCY11, LW14, BC11, BH13, Yan06], but not the algorithm.

On the other hand, previous algorithms did not yield an explicit rounding, i.e. a separable state  $\sigma$  with  $\text{tr} M \sigma \approx h_{\text{Sep}}(M)$ . The only exception to this [BH13] also required an explicit decomposition in order to produce a rounding.

In general any bipartite measurement  $M$  can be written in the form  $\sum_i X_i \otimes Y_i$ , with individual terms that are not necessarily positive semidefinite. Finding *some* such decomposition is straightforward, e.g. using the operator Schmidt decomposition or even writing  $M = \sum_{ijkl} M_{ijkl} |i\rangle\langle j| \otimes |k\rangle\langle l|$ . Our algorithm can be readily modified to incorporate non-positive  $X_i$  (along the lines of Section 4), but the run-time will then include a factor of  $\sum_i \|X_i\|_1$  in the exponent. In general this will be  $O(1)$  only if  $M$  is close to 1-LOCC and the decomposition is close to the correct one.

This raises an interesting open question: given  $M$ , find a decomposition  $M = \sum_i X_i \otimes Y_i$  that (approximately) minimizes  $\sum_i \|X_i\|_1$ . We are not aware of nontrivial algorithms or hardness results for this problem.

### 3 Generalized algorithm for arbitrary norms

An important step in the algorithm of the previous section was the identity,

$$h_{\text{Sep}(d_1, d_2)}(M) = \max_{p \in S_X} \|p\|_Y, \quad (21)$$

valid for any one-way LOCC  $M = \sum_i X_i \otimes Y_i$ . This equation suggests ways of generalizing the algorithm. In this section we consider the setting where the operators  $\{Y_1, \dots, Y_n\}$  belong to some Banach space  $\mathcal{B}$  with norm  $\|\cdot\|_{\mathcal{B}}$ . In analogy with Eq. (11), given  $Y = \{Y_1, \dots, Y_n\}$  we define the  $(\mathcal{B}, Y)$  norm in  $\mathbb{R}^n$  as

$$\|a\|_{\mathcal{B}, Y} := \left\| \sum_{i=1}^n a_i Y_i \right\|_{\mathcal{B}}. \quad (22)$$

The goal is then to estimate

$$\max_{p \in S_X} \|p\|_{\mathcal{B}, Y}, \quad (23)$$

where, as before,  $S_X$  is given by Eq. (11).

Also this generalization is of interest in quantum information theory. As we discuss more in the next subsection, it includes as a particular case the well-studied problem of computing the maximum output  $\alpha$ -norms of an entanglement-breaking channel. Consider a general entanglement-breaking quantum channel  $\Lambda : \mathcal{D}_{d_1} \rightarrow \mathcal{D}_{d_2}$  given by [HSR04]:

$$\Lambda(\rho) := \sum_i \text{tr}(X_i \rho) Y_i, \quad (24)$$

with  $Y_i \geq 0$ ,  $\text{tr}(Y_i) = 1$ ,  $X_i \geq 0$ , and  $\sum_i X_i = I$ . Then

$$\max_{\rho \in \mathcal{D}_{d_1}} \|\Lambda(\rho)\|_{\alpha} = \max_{p \in S_X} \|p\|_{S_{\alpha}, Y}. \quad (25)$$

In order to find an algorithm for computing Eq. (23), we need to replace the quantum Hoeffding bound (Lemma 3) by more sophisticated concentration bounds. Since in Lemma 5 all we needed was a bound in expectation, the right concept will turn out to be the Rademacher type- $\gamma$  constant of the space  $\mathcal{B}$ , which we now define:

**Definition 11.** We say a Banach space  $\mathcal{B}$  has Rademacher type- $\gamma$  constant  $C$  if for every  $Z_1, \dots, Z_k \in \mathcal{B}$  and Rademacher random variables  $\varepsilon_1, \dots, \varepsilon_k$  (i.e. independent and uniformly distributed on  $\pm 1$ ),

$$\mathbb{E}_{\varepsilon_1, \dots, \varepsilon_k} \left\| \sum_{i=1}^k \varepsilon_i Z_i \right\|_{\mathcal{B}}^{\gamma} \leq C^{\gamma} \sum_{i=1}^k \|Z_i\|_{\mathcal{B}}^{\gamma}. \quad (26)$$

It is known that Schatten- $\alpha$  spaces with norm  $\|X\|_{\alpha} := \text{tr}(|X|^{\alpha})^{1/\alpha}$  have type-2 constant  $\sqrt{\alpha - 1}$  for  $\alpha \geq 2$  [BCL94], and type- $\alpha$  constant 1 for every  $\alpha \in [1, 2]$  [KPT00, Thm 3.3].

For a reader unfamiliar with the type- $\gamma$  constant, we suggest verifying that the type-2 constant of  $\ell_2$  is 1. A more nontrivial calculation is using the Hoeffding bound or its operator version to verify that the type-2 constant of  $\ell_{\infty}^n$  or  $S_{\infty}^n$  is  $O(\sqrt{\log n})$ . (This also follows from the fact that the  $S_{\infty}$  and  $S_{\log(n)}$  norms are within a constant multiple of each other on the space of  $n$ -dimensional matrices.)

For sparsification (the analogue of Lemma 7) we will actually need a slightly stronger condition than a bound on the type- $\gamma$  constant:

**Definition 12.** The modulus of uniform smoothness of a Banach space  $\mathcal{B}$  is defined to be the function

$$\rho_{\mathcal{B}}(\tau) := \sup \left\{ \frac{\|x + \tau y\|_{\mathcal{B}} + \|x - \tau y\|_{\mathcal{B}}}{2} - 1 : \|x\|_{\mathcal{B}} = \|y\|_{\mathcal{B}} = 1 \right\}. \quad (27)$$

By the triangle inequality,  $\rho_{\mathcal{B}}(\tau) \leq \tau$  for all  $\mathcal{B}$ . But when  $\lim_{\tau \rightarrow 0} \frac{\rho_{\mathcal{B}}(\tau)}{\tau} = 0$  then we say that  $\mathcal{B}$  is uniformly smooth. For example, if  $\mathcal{B} = \ell_2$  then  $\rho_{\mathcal{B}}(\tau) = \tau^2/2$ , whereas  $\rho_{\ell_1}(\tau) = \tau$ . More generally [BCL94] (building on [TJ74]) proved that  $\rho_{S_\alpha}(\tau) \leq \frac{\alpha-1}{2}\tau^2$  for  $\alpha > 1$ . We say that  $\mathcal{B}$  has modulus of smoothness of power type  $\gamma$  if  $\rho_{\mathcal{B}}(\tau) \leq C\tau^\gamma$  for some constant  $C$ . This implies (using an easy induction on  $k$ ) that the type- $\gamma$  constant is  $\leq C$ , and indeed this was how the type- $\gamma$  constant was bounded in [TJ74, BCL94].

The algorithm for approximating the optimization problem given by Eq. (23) is the following:

**Algorithm 13** (Algorithm for computing  $\max_{p \in S_X} \|p\|_{\mathcal{B}, Y}$  for  $\mathcal{B}$  of type- $\gamma$  constant  $C$  and modulus of uniform smoothness  $\rho_{\mathcal{B}}(\tau) \leq s\tau^2$ , with  $X := \{X_i\}$  and  $Y := \{Y_i\}$ ).

**Input:**  $\{X_i\}_{i=1}^n, \{Y_i\}_{i=1}^n$

**Output:**  $p \in S$

1. Use Lemma 22 to replace  $X = \{X_i\}$  and  $Y = \{Y_i\}$  with  $X' := \{X'_i\}$  and  $Y' := \{Y'_i\}$ .
2. Enumerate over all  $p \in \Delta_n(k)$ , with  $k = \left(\frac{2C^\gamma}{\delta^\gamma} \max_i \|Y_i\|_{\mathcal{B}}^\gamma\right)^{1/(\gamma-1)}$ .
  - (a) For each  $p$ , check (using Lemma 21) whether there exists  $q \in S$  with  $\|p - q\|_{\mathcal{B}, Y} \leq \delta$ .
  - (b) If so, compute  $\|p\|_Y$ .
3. Output  $p$  such that  $\|p\|_{\mathcal{B}, Y}$  is the maximum.

We have:

**Theorem 14.** Let  $\mathcal{B}$  be a Banach space with norm  $\|\cdot\|_{\mathcal{B}}$ . Suppose the type- $\gamma$  constant of  $\mathcal{B}$  is  $C$  and that there is  $s > 0$  such that the modulus of uniform smoothness satisfies  $\rho_{\mathcal{B}}(\tau) \leq s\tau^2$ . Suppose one can compute  $\|\cdot\|_{\mathcal{B}}$  in time  $T$ . Consider  $\{X_i\}_{i=1}^n$  with  $X_i$   $d \times d$  matrices satisfying  $X_i \geq 0$ ,  $\sum X_i \leq I$ , and  $\{Y_i\}_{i=1}^n$  with  $Y_i \in \mathcal{B}$ . Algorithm 13 runs in time

$$\text{poly}(T, d, s) \exp\left(O\left(\left(C\delta^{-1} \max_i \|Y_i\|_{\mathcal{B}}\right)^{\frac{\gamma}{\gamma-1}} \log(d)\right)\right) \quad (28)$$

and outputs  $p$  such that

$$\max_{p \in S_X} \|p\|_{\mathcal{B}, Y} \geq \|p\|_{\mathcal{B}, Y} \geq \max_{p \in S_X} \|p\|_{\mathcal{B}, Y} - \delta, \quad (29)$$

As an example, suppose  $\mathcal{B}$  is  $\mathcal{S}_\infty^{d_2}$ . Then the type-2 constant is  $O(\sqrt{\log(d_2)})$ ,  $\max_i \|Y_i\| \leq 1$ , and Theorem 2 shows one can compute  $\max_{p \in S_X} \|p\|_Y$  in time  $\exp(O(\delta^{-2} \log(d_1) \log(d_1 d_2)))$ .

In the next subsection we discuss a few particular cases of the theorem worth emphasizing. Then we prove the theorem.

### 3.1 Consequences of Theorem 14

#### 3.1.1 Restricted one-way LOCC measurements

The next lemma shows that for subclasses of one-way LOCC measurements one has a PTAS for computing  $h_{\text{Sep}}$ . The class include in particular one-way LOCC measurements in which Bob's measurements are low rank.

**Corollary 15.** Let  $M = \sum_i X_i \otimes Y_i$  be such that  $X_i \geq 0$ ,  $\sum_i X_i \leq I$  and  $\|Y_i\|_2 \leq r$ . Then one can compute  $\alpha \in \mathcal{D}_{d_1}$  and  $\beta \in \mathcal{D}_{d_2}$  such that

$$h_{\text{Sep}}(M) \geq \text{tr}(M(\alpha \otimes \beta)) \geq h_{\text{Sep}}(M) - \delta \quad (30)$$

in time  $d_1^{O(\delta^{-2}r)}$ .

*Proof.* We use Theorem 14 and Algorithm 13 to estimate the optimal  $p$  and then find  $\alpha$  and  $\beta$  by semidefinite programming.  $\square$

If instead we use the multipartite version of the algorithm (see Algorithm 10), we find that for  $M = \sum_i X_i \otimes Y_{i_1} \otimes \dots \otimes Y_{i_l}$ , with  $X_i \geq 0$ ,  $\sum_i X_i \leq I$  and  $\|Y_i\|_2 \leq r$ , we can compute  $\alpha \in \mathcal{D}_d$  and  $\beta_1, \dots, \beta_l \in \mathcal{D}_d$  such that

$$h_{Sep}(M) \geq \text{tr}(M\alpha \otimes \beta_1 \otimes \dots \otimes \beta_l) \geq h_{Sep}(M) - \delta \quad (31)$$

in time  $d^{O(\delta^{-2}l^3r)}$ .

### 3.1.2 Maximum output norm of entanglement-breaking channels

The next corollary shows that for all  $\alpha > 1$ , there is a PTAS for computing the maximum output Schatten- $\alpha$  norm of an entanglement-breaking channel.

**Corollary 16.** *Let  $\Lambda : \mathcal{D}_{d_1} \rightarrow \mathcal{D}_{d_2}$  be an entanglement-breaking channel with decomposition  $\Lambda(\rho) := \sum_i \text{tr}(X_i \rho) Y_i$  (where  $X_i \geq 0$ ,  $\sum_i X_i = I$ ,  $Y_i \in \mathcal{D}_{d_2}$ ).*

1. For every  $\alpha \geq 2$  one can compute in time  $\text{poly}(d_2)d_1^{O(\delta^{-2\alpha})}$  a number  $r$  such that

$$\max_{\rho \in \mathcal{D}_{d_1}} \|\Lambda(\rho)\|_\alpha \geq r \geq \max_{\rho \in \mathcal{D}_{d_1}} \|\Lambda(\rho)\|_\alpha - \delta, \quad (32)$$

2. For every  $1 < \alpha \leq 2$  one can compute in time  $\text{poly}(d_2)d_1^{O\left(\left(\alpha\delta^{-\alpha}\right)^{\frac{1}{\alpha-1}}\right)}$  a number  $r$  such that

$$\max_{\rho \in \mathcal{D}_{d_1}} \|\Lambda(\rho)\|_\alpha \geq r \geq \max_{\rho \in \mathcal{D}_{d_1}} \|\Lambda(\rho)\|_\alpha - \delta, \quad (33)$$

*Proof.* Part 1 follows from Theorem 14 and the fact that  $S_\alpha$ , with  $\alpha \geq 2$ , has type-2 constant  $\sqrt{\alpha-1}$  [BCL94] and  $\rho_{S_\alpha}(\tau) \leq \frac{\alpha-1}{2}\tau^2$  for  $\alpha > 1$ . Part 2, in turn, follows from Theorem 14 and the fact that for  $S_\alpha$ , with  $\alpha \geq 2$ , has type- $\alpha$  constant one [KPT00, Thm 3.3].  $\square$

We note that computing maximum output  $\alpha$ -norms for general quantum channels is harder. In particular it was shown in [HM10, HM13] that there is no algorithm that run in time  $\exp(O(\log^{2-\varepsilon} d))$  for any  $\varepsilon > 0$  and can decide if  $\max_\rho \|\Lambda(\rho)\|$  is one or smaller than  $\delta$  (for any fixed  $\delta > 0$ ) for a general quantum channel  $\Lambda : \mathcal{D}_d \rightarrow \mathcal{D}_d$ , unless the exponential time hypothesis (ETH) is wrong (meaning there is a subexponential time algorithm for 3-SAT).

The result of [HM10] is one example of many that found  $d^{\tilde{O}(\log d)}$  upper or lower bounds for related optimization problems [LMM03, BKW14, HM10]. In a few cases [ALSV13b, SW12, DKLP06b] poly-time approximate schemes (PTASs) are known. Our results here fall into this second class. We hope that the geometric perspective from our paper can lead to a better understanding of what distinguishes these cases.

What is known about hardness results for entanglement-breaking channels? Using the results of [BBH<sup>+</sup>12] one can show that to determine if  $\max_\rho \|\Lambda(\rho)\|$  is  $\geq C/d$  or  $\leq c/d$  (for any two constants  $C > c > 0$ ) cannot be done in time  $\exp(O(\log^{2-\varepsilon} d))$  assuming ETH. So one cannot hope to find a polynomial-time algorithm for a *multiplicative* approximation of the maximum output norm.

Note that the complexity of the algorithm blows up when  $\alpha \rightarrow 1$ . This is not only an artifact of the proof. Computing the quantity for  $\alpha$  close to one allow us to estimate the von Neumann minimum output entropy of the channel. However to estimate it we need a number of samples of order  $O(d)$  and so the net-based approach we explore in this paper does not lead to efficient algorithms.

### 3.1.3 Hypercontractive norms

Our third corollary concerns the problem of computing hypercontractive norms, in particular computing the  $2 \rightarrow s$  norm of a  $d \times d$  matrix  $A$ , for  $s > 2$ , defined as

$$\|A\|_{2 \rightarrow s} := \max_{\|x\|_2=1} \|Ax\|_s. \quad (34)$$

This norms are important in several applications, e.g. bounding the mixing time of Markov chains and determining if a graph is a small-set expander [BBH<sup>+</sup>12]. In [BBH<sup>+</sup>12] it was also shown that to compute any constant-factor multiplicative approximation to the  $2 \rightarrow 4$  norm of a  $n \times n$  matrix is as hard as solving 3-SAT with  $O(\log^2(n))$  variables. In Appendix B we extend the approach of [BBH<sup>+</sup>12] to show hardness results for multiplicatively approximating all  $2 \rightarrow q$  norms, for even  $q \geq 4$ .

In [BBH<sup>+</sup>12] it was shown that the result of [BCY11] implies that for any  $d \times d$  matrix  $A$  the Sum-of-Squares hierarchy computes in time  $d^{O(\log(d)\delta^{-2})}$  an additive approximation  $x$  s.t.

$$\|A\|_{2 \rightarrow 4}^4 \leq x \leq \|A\|_{2 \rightarrow 4}^4 + \delta \|A\|_{2 \rightarrow 2}^2 \|A\|_{2 \rightarrow \infty}^2, \quad (35)$$

where  $\|A\|_{2 \rightarrow 2}$  is the largest singular value of  $A$  and  $\|A\|_{2 \rightarrow \infty}$  the largest 2-norm of any row of  $A$ .

Using Theorem 14 we can improve this algorithm in two ways: First we can compute an approximation to  $\|\cdot\|_{2 \rightarrow s}$  for any  $s > 2$ . Second the running time for fixed error is polynomial, instead of quasipolynomial.

**Corollary 17.** *For any  $s \geq 2$  one can compute in time  $d^{O(s\delta^{-2})}$  a number  $x$  such that*

$$\|A\|_{2 \rightarrow s}^2 \geq x \geq \|A\|_{2 \rightarrow s}^2 - \delta \|A\|_{2 \rightarrow 2}^2. \quad (36)$$

*Proof.* Let  $X_i := A^\dagger |i\rangle \langle i| A / \|A\|_{2 \rightarrow 2}^2$ . Note  $X_i \geq 0$  and  $\sum_i X_i \leq I$ . We can write

$$\|A\|_{2 \rightarrow s}^s = \max_{|\psi\rangle \in \ell_2} \sum_i \langle \psi | A^\dagger |i\rangle \langle i| A | \psi \rangle^{s/2} = \|A\|_{2 \rightarrow 2}^s \max_{p \in S_X} \|p\|_{s/2}^{s/2}. \quad (37)$$

Since  $\ell_s$  has type-2 constant  $\sqrt{s-1}$ , by Theorem 14 we can estimate

$$\max_{p \in S_X} \|p\|_{s/2} = \frac{\|A\|_{2 \rightarrow s}^2}{\|A\|_{2 \rightarrow 2}^2} \quad (38)$$

in time  $\exp(O(s\delta^{-2} \log(d)))$  with additive error  $\delta$ .  $\square$

Although the corollary above gives an approximation for every  $s > 2$  that can be computed in polynomial time for every fixed error, it gives a worse approximation to the  $2 \rightarrow 4$  than [BBH<sup>+</sup>12] (given by Eq. (35)). We now show a second corollary that strictly improves the result of [BBH<sup>+</sup>12] for  $2 \rightarrow 4$  and generalizes it to  $2 \rightarrow s$  norms for every even  $\geq 4$ .

**Corollary 18.** *For any even  $s \geq 4$  one can compute in time  $d^{O(s\delta^{-2})}$  a number  $x$  such that*

$$\|A\|_{2 \rightarrow s}^s \geq x \geq \|A\|_{2 \rightarrow s}^s - \delta \|A\|_{2 \rightarrow 2}^2 \|A\|_{2 \rightarrow \infty}^{s-2}. \quad (39)$$

*Proof.* Define

$$X_i := \frac{A^\dagger |i\rangle \langle i| A}{\|A\|_{2 \rightarrow 2}^2} \quad \text{and} \quad Y_i := \left( \frac{A^\dagger |i\rangle \langle i| A}{\|A\|_{2 \rightarrow \infty}^2} \right)^{\otimes \frac{s}{2}-1}. \quad (40)$$

Observe that  $X_i, Y_i \geq 0$ ,  $\sum_i X_i \leq I$  and  $Y_i \leq I$ . Additionally

$$\|A\|_{2 \rightarrow s}^s = \|A\|_{2 \rightarrow 2}^2 \|A\|_{2 \rightarrow \infty}^{s-2} h_{\text{Sep}^{s/2}(n)} \left( \sum_i X_i \otimes Y_i \right) \quad (41)$$

This last term can be approximated to additive error  $\varepsilon$  in time

$$\exp(O(s^3/\varepsilon^2))$$

using the multipartite results of Section 3.1.1.  $\square$

### 3.2 Proof of Theorem 14

The proof of Theorem 14 will follow from three lemmas, the first showing that it is enough to search over a net of small size, the second showing that one can decide membership of  $\{q : \|p - q\|_{\mathcal{B}, Y} \leq \delta\}$  efficiently (assuming that  $\|\cdot\|_{\mathcal{B}}$  can be computed efficiently), and the third giving a sparsification for the number of  $\{X_i\}_i^n$  and  $\{Y_i\}_{i=1}^n$ .

We first show how the type- $\gamma$  constant gives a concentration bound. This uses a standard argument.

**Lemma 19** (Symmetrization Lemma). *Suppose we are given  $p \in \Delta_n$ ,  $Z_i, \dots, Z_n$  elements of a Banach space  $\mathcal{B}$  with norm  $\|\cdot\|_{\mathcal{B}}$ , and  $\varepsilon_1, \dots, \varepsilon_k$  Rademacher distributed random variables. Then for every  $\gamma \geq 1$*

$$\mathbb{E}_{i_1, \dots, i_k \sim p^{\otimes k}} \left\| \frac{1}{k} \sum_{j=1}^k Z_{i_j} - \mathbb{E}_{i \sim p} Z_i \right\|_{\mathcal{B}}^{\gamma} \leq 2 \mathbb{E}_{i_1, \dots, i_k \sim p^{\otimes k}} \mathbb{E}_{\varepsilon_1, \dots, \varepsilon_k} \left\| \frac{1}{k} \sum_{j=1}^k \varepsilon_j Z_{i_j} \right\|_{\mathcal{B}}^{\gamma}. \quad (42)$$

*Proof.*

$$\mathbb{E}_{i_1, \dots, i_k \sim p^{\otimes k}} \left\| \frac{1}{k} \sum_{j=1}^k (Z_{i_j} - \mathbb{E}_{i \sim p} Z_i) \right\|_{\mathcal{B}}^{\gamma} = \mathbb{E}_{i_1, \dots, i_k \sim p^{\otimes k}} \left\| \frac{1}{k} \sum_{j=1}^k (Z_{i_j} - \mathbb{E}_{i'_j \sim p} [Z_{i'_j}]) \right\|_{\mathcal{B}}^{\gamma} \quad (43)$$

$$\leq \mathbb{E}_{i_1, \dots, i_k \sim p^{\otimes k}} \mathbb{E}_{i'_1, \dots, i'_k \sim p^{\otimes k}} \left\| \frac{1}{k} \sum_{j=1}^k (Z_{i_j} - Z_{i'_j}) \right\|_{\mathcal{B}}^{\gamma} \quad (44)$$

$$= \mathbb{E}_{i_1, \dots, i_k \sim p^{\otimes k}} \mathbb{E}_{i'_1, \dots, i'_k \sim p^{\otimes k}} \mathbb{E}_{\varepsilon_1, \dots, \varepsilon_k} \left\| \frac{1}{k} \sum_{j=1}^k \varepsilon_j (Z_{i_j} - Z_{i'_j}) \right\|_{\mathcal{B}}^{\gamma} \quad (45)$$

$$\leq 2 \mathbb{E}_{i_1, \dots, i_k \sim p^{\otimes k}} \mathbb{E}_{\varepsilon_1, \dots, \varepsilon_k} \left\| \frac{1}{k} \sum_{j=1}^k \varepsilon_j Z_{i_j} \right\|_{\mathcal{B}}^{\gamma}. \quad (46)$$

$$(47)$$

□

Then we have the following generalization of Lemma 5:

**Lemma 20.** *Let the Banach space  $\mathcal{B}$  have type- $\gamma$  constant  $C$ . Then for any  $p \in \Delta_n$  there exists  $q \in N_k$  with*

$$\|p - q\|_{\mathcal{B}, Y} \leq \left( \frac{2C^{\gamma}}{k^{\gamma-1}} \mathbb{E}_{i \sim p} \|Y_i\|_{\mathcal{B}}^{\gamma} \right)^{1/\gamma}. \quad (48)$$

*Proof.* Sample  $i_1, \dots, i_k$  according to  $p$  and set  $q = (e_{i_1} + \dots + e_{i_k})/k$ . Then Definition 11 and Lemma 20

give

$$\begin{aligned}
\left( \mathbb{E}_{i_1, \dots, i_k \sim p^{\otimes k}} \|p - q\|_{\mathcal{B}, Y} \right)^\gamma &\leq \mathbb{E}_{i_1, \dots, i_k \sim p^{\otimes k}} \|p - q\|_{\mathcal{B}, Y}^\gamma \\
&= \mathbb{E}_{i_1, \dots, i_k \sim p^{\otimes k}} \left\| \frac{1}{k} \sum_{j=1}^k Y_{i_j} - \mathbb{E}_{i \sim p} Y_i \right\|_{\mathcal{B}}^\gamma \\
&\leq 2 \mathbb{E}_{i_1, \dots, i_k \sim p^{\otimes k}} \mathbb{E}_{\varepsilon_1, \dots, \varepsilon_k} \left\| \frac{1}{k} \sum_{j=1}^k \varepsilon_j Y_{i_j} \right\|_{\mathcal{B}}^\gamma \\
&\leq \frac{2C^\gamma}{k^\gamma} \mathbb{E}_{i_1, \dots, i_k \sim p^{\otimes k}} \sum_{i=1}^k \|Y_{i_j}\|_{\mathcal{B}}^\gamma \\
&= \frac{2C^\gamma}{k^{\gamma-1}} \mathbb{E}_{i \sim p} \|Y_i\|_{\mathcal{B}}^\gamma. \tag{49}
\end{aligned}$$

The first inequality follows from the convexity of  $x \mapsto x^\gamma$ , the second inequality from Lemma 19, and the third from the fact that  $\mathcal{B}$  has type- $\gamma$  constant  $C$ .  $\square$

The next lemma is an analogue of Lemma 5:

**Lemma 21.** *Let the Banach space  $\mathcal{B}$  be such that  $\|\cdot\|_{\mathcal{B}}$  can be computed in time  $T$ . Given  $p \in \Delta_n$  and  $\varepsilon > 0$ , we can decide in time  $\text{poly}(T, d, n)$  whether the following set is nonempty*

$$S_X \cap \{q : \|p - q\|_{\mathcal{B}, Y} \leq \varepsilon\} \tag{50}$$

*Proof.* Since we have an efficient algorithm for  $\|\cdot\|_{\mathcal{B}}$  we can efficiently test membership in the set  $\{q : \|p - q\|_{\mathcal{B}, Y} \leq \varepsilon\}$ . Thus we can determine if Eq. (50) is nonempty using the ellipsoid algorithm [GLS93].  $\square$

We now state an analogous sparsification result of Lemma 7 for the more general case we consider in this section. The proof is in Appendix A.

**Lemma 22.** *Suppose  $\Lambda$  is a map from  $d \times d$  Hermitian matrices to a Banach space  $\mathcal{B}$  and is given by  $\Lambda(\rho) = \sum_{i=1}^n \langle X_i, \rho \rangle Y_i$  where each  $X_i \geq 0$ ,  $\sum_{i=1}^n X_i \leq I$  and each  $\|Y_i\|_{\mathcal{B}} \leq 1$ . Suppose that  $\mathcal{B}$  has modulus of smoothness  $\rho_{\mathcal{B}}(\tau) \leq s\tau^2$ . Then there exists  $\Lambda'$  such that  $\Lambda'(\rho) = \sum_{i=1}^k \langle X'_i, \rho \rangle Y'_i$  where each  $X'_i \geq 0$ ,  $\sum_{i=1}^k X'_i \leq I$  and each  $\|Y'_i\|_{\mathcal{B}} \leq 1$ . Additionally  $k \leq cd^2(d+s)/\delta^2$  for some constant  $c > 0$ ,*

$$\max_{\rho \in \mathcal{D}_d} \|(\Lambda' - \Lambda)(\rho)\|_{\mathcal{B}} \leq \delta \tag{51}$$

and  $\Lambda'$  can be found efficiently.

With the lemmas in hand the proof of Theorem 14 follows along the same lines as Theorem 6.

## 4 Algorithm for injective tensor norm

In this section we present one further generalization, this time on the input space. While this final generalization does not have natural applications in quantum information (to our knowledge), it does give perspective on why it is natural to consider 1-LOCC measurements and entanglement-breaking channels.

First, we introduce some more definitions. Suppose that  $\|\cdot\|_{\mathcal{A}}$  and  $\|\cdot\|_{\mathcal{B}}$  are two norms. For  $\Lambda$  an operator from  $\mathcal{A} \rightarrow \mathcal{B}$  define the operator norm

$$\|\Lambda\|_{\mathcal{A} \rightarrow \mathcal{B}} := \sup_{a \in B(\mathcal{A})} \|\Lambda(a)\|_{\mathcal{B}}. \tag{52}$$

Define the injective tensor norm  $\mathcal{A} \otimes_{\text{inj}} \mathcal{B}$  by

$$\|x\|_{\mathcal{A} \otimes_{\text{inj}} \mathcal{B}} = \sup_{\substack{a \in B(\mathcal{A}^*) \\ b \in B(\mathcal{B}^*)}} \langle a \otimes b, x \rangle. \quad (53)$$

Here  $\mathcal{A}^*$  is the space of functions from  $\mathcal{A}$  to  $\mathbb{R}$ , and  $\|\tilde{a}\|_{\mathcal{A}^*} := \sup_{a \in B(\mathcal{A})} \tilde{a}(a)$ . For example, if  $\mathcal{A} = \mathcal{B} = \ell_2$  then  $\mathcal{A} \otimes_{\text{inj}} \mathcal{B}$  is the usual operator norm for matrices, i.e. largest singular value. More generally  $\mathcal{A}^* \otimes_{\text{inj}} \mathcal{B}$  is isomorphic to the operator norm on maps from  $\mathcal{A} \rightarrow \mathcal{B}$ . Finally if  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  are Banach spaces then define the factorization norm  $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$  for  $x \in \mathcal{L}(\mathcal{A}, \mathcal{C})$  by

$$\|\Lambda\|_{\mathcal{A} \rightarrow \mathcal{C} \rightarrow \mathcal{B}} = \inf_{\substack{\Lambda_1 \in \mathcal{L}(\mathcal{C}, \mathcal{B}) \\ \Lambda_2 \in \mathcal{L}(\mathcal{A}, \mathcal{C}) \\ \Lambda = \Lambda_1 \Lambda_2}} \|\Lambda_1\|_{\mathcal{C} \rightarrow \mathcal{B}} \|\Lambda_2\|_{\mathcal{A} \rightarrow \mathcal{C}}. \quad (54)$$

We can now (informally) state our generalized estimation theorem. Given an operator  $\Lambda \in B(\mathcal{A} \rightarrow \ell_1 \rightarrow \mathcal{B})$  we can estimate  $\|\Lambda\|_{\mathcal{A} \rightarrow \mathcal{B}}$  efficiently.

For example, consider  $h_{\text{Sep}}$ , which we considered in Section 2. In our new notation

$$h_{\text{Sep}}(M) = \|M\|_{S_\infty \otimes_{\text{inj}} S_\infty} = \|\hat{M}\|_{S_1 \rightarrow S_\infty}, \quad (55)$$

where  $\hat{M}$  is the map defined by  $\hat{M}(X) = \text{tr}_A[M(X \otimes I)]$ . The requirement that  $M$  is 1-LOCC is roughly equivalent to the requirement that

$$\|\hat{M}\|_{S_1 \rightarrow \ell_1 \rightarrow S_\infty} \leq 1. \quad (56)$$

**Theorem 23.** *Suppose  $\mathcal{A}, \mathcal{B}$  are  $d$ -dimensional Banach spaces. Suppose  $\|\Lambda\|_{\mathcal{A} \rightarrow \ell_1^n \rightarrow \mathcal{B}} \leq 1$  and that a good factorization is known; i.e.  $x_1^*, \dots, x_n^* \in \mathcal{A}^*$  and  $y_1, \dots, y_n \in \mathcal{B}$  are given such that  $\Lambda = \sum_{i=1}^n y_i x_i^*$ ,  $\sup_{a \in \mathcal{A}} \sum_{i=1}^n |x_i^*(a)| \leq 1$  and  $\max_i \|y_i\|_{\mathcal{B}} \leq 1$ . Suppose further that algorithms exist for computing the  $\mathcal{A}$  and  $\mathcal{B}$  norms running in times  $T_{\mathcal{A}}, T_{\mathcal{B}}$  respectively. Let  $\lambda$  denote the type- $\gamma$  constant of  $\mathcal{B}$ . Then we can estimate  $\|\Lambda\|_{\mathcal{A} \rightarrow \mathcal{B}}$  to accuracy  $\varepsilon$  in time*

$$T_{\mathcal{A}} T_{\mathcal{B}} \text{poly}(d) n^{c(\lambda/\delta)^{\frac{\gamma}{\gamma-1}}}. \quad (57)$$

The algorithm follows similar lines to the earlier algorithms. It lacks only the sparsification step since we do not know how to extend Lemma 22 to this case.

**Algorithm 24** (Algorithm for computing  $\|\Lambda\|_{\mathcal{A} \rightarrow \mathcal{B}}$ ).

**Input:**  $\{x_i\}_{i=1}^n, \{y_i\}_{i=1}^n$

**Output:**  $p \in \mathcal{S}$

1. Enumerate over all  $p \in N_k$ , with  $k = (2\lambda/\delta)^{\frac{\gamma}{\gamma-1}}$ .
  - (a) For each  $p$ , check whether there exists  $q \in \mathcal{S}$  with  $\|p - q\|_{\mathcal{B}, Y} \leq \delta$ .
  - (b) If so, compute  $\|p\|_Y$ .
2. Output  $p$  such that  $\|p\|_{\mathcal{B}, Y}$  is the maximum.

The proof of Theorem 23 is almost the same as that of Theorem 14. The only new ingredient is checking whether  $p \in S_X$ . This is equivalent to asking whether  $\exists a \in B(\mathcal{A})$  such that  $p_i = x_i^*(a)$ . This is a convex program which can be decided in time  $\text{poly}(d)T_{\mathcal{A}}$  using the ellipsoid algorithm along with our assumption that  $\|\cdot\|_{\mathcal{A}}$  can be computed in time  $T_{\mathcal{A}}$ .

## Acknowledgments

We thank Jop Briet, Pablo Parrilo and Ben Recht for interesting discussions. FGSLB is supported by EPSRC. AWH was funded by NSF grants CCF-1111382 and CCF-1452616, ARO contract W911NF-12-1-0486 and a Leverhulme Trust Visiting Professorship VP2-2013-041. Part of this work was done while A.W. was visiting UCL.

## A Sparsification

In this appendix we prove two Lemmas about sparsification: one (Lemma 7) for the problem of  $h_{\text{Sep}}$  and the second (Lemma 22) for the estimate  $S_1 \rightarrow \mathcal{B}$  norms. While the former is a special case of the latter, it is also far more self-contained (requiring only the operator Hoeffding bound), so we recommend reading it first.

*Proof of Lemma 7.* Assume initially that each  $\|Y_i\| = 1$ . This is possible because we can always drop terms with  $Y_i = 0$  and then rewrite  $M$  as

$$\sum_{i=1}^n \|Y_i\| X_i \otimes \frac{Y_i}{\|Y_i\|}. \quad (58)$$

Redefining  $X_i$  appropriately we see that  $\sum_i X_i \leq I$  still holds.

Now write  $M = \sum_{i=1}^n p_i W_i$  with  $p_i = \frac{\text{tr}(X_i \otimes Y_i)}{\text{tr}(M)}$  and  $W_i = \frac{X_i \otimes Y_i}{p_i}$ . Sample  $i_1, \dots, i_{n'}$  according to  $p$  and define

$$A = \sum_{j=1}^{n'} \frac{W_{i_j}}{n'} \quad \text{and} \quad B = \sum_{j=1}^{n'} \frac{X_{i_j}/p_{i_j}}{n'}. \quad (59)$$

We would like to guarantee that

$$\|A - M\| \leq \delta \quad (60a)$$

$$\|B - I_{d_1}\| \leq \delta \quad (60b)$$

for some  $\delta$  to be chosen later. We can use Lemma 3 here. To do so, note that

$$\|W_i\| \leq \text{tr } W_i \leq \text{tr } M \leq d_1 d_2 \quad (61a)$$

$$\|X_i/p_i\| \leq \frac{\text{tr } M}{\text{tr } Y_i} \leq \text{tr } M \leq d_1 d_2, \text{ using the assumption that } \|Y_i\| = 1 \quad (61b)$$

Now we find that the probability that Eq. (60) fails to hold is

$$\leq d_1 d_2 \exp\left(-\frac{n' \delta^2}{8d_1^2 d_2^2}\right) + d_1 \exp\left(-\frac{n' \delta^2}{8d_1^2 d_2^2}\right). \quad (62)$$

Taking  $n' = 8d_1^2 d_2^2 \log(2d_1 d_2)/\delta^2$  we have that (60) holds with positive probability. Fix the corresponding  $i_1, \dots, i_{n'}$ . Choose  $M' = A/(1 + \delta)$ . Together with Eq. (60) this means that  $M'$  is a valid 1-LOCC measurement. By Eq. (60a) we can achieve our result by choosing  $\delta = \varepsilon/3$ . Indeed

$$\begin{aligned} \|M' - M\| &= \left\| \frac{A}{1 + \delta} - M \right\| \\ &\leq \left\| \frac{A}{1 + \delta} - A \right\| + \|A - M\| \\ &\leq \left(1 - \frac{1}{1 + \delta}\right) (1 + \delta) + \delta \\ &\leq 2\delta + \delta^2 \leq \varepsilon. \end{aligned} \quad (63)$$

□

We now turn to the proof of Lemma 22, covering the case of general Banach spaces with bounded modulus of smoothness.

We will need the following Azuma-type inequality from Naor [Nao12], who attributes it to Pisier. We will state a weaker Hoeffding-type formulation that suffices for our purposes.

**Lemma 25** (Theorem 1.5 of [Nao12]). *Suppose  $X_1, \dots, X_k$  are independent random variables on  $B(\mathcal{B})$  for  $\mathcal{B}$  a Banach space with  $\rho_{\mathcal{B}}(\tau) \leq s\tau^2$ . Then*

$$\Pr \left[ \left\| \frac{1}{k} \sum_{i=1}^k X_i \right\| \geq \delta \right] \leq e^{s+2-ck\delta^2}. \quad (64)$$

*Proof of Lemma 22.* First we introduce notation. For a matrix  $X$ , define the map  $\hat{X}$  by  $\hat{X}(A) := \langle X, A \rangle$ . Thus  $\Lambda = \sum_{i=1}^n Y_i \hat{X}_i$ .

As in Lemma 7 we first drop terms with  $Y_i = 0$  and rewrite  $\Lambda = \sum_{i=1}^n \frac{Y_i}{\|Y_i\|_{\mathcal{B}}} \cdot \|Y_i\|_{\mathcal{B}} \hat{X}_i$ . Redefine  $X_i, Y_i$  appropriately and assume from now on that each  $\|Y_i\|_{\mathcal{B}} = 1$ .

Define  $p_i = \text{tr}[X_i]/d$ . Note that  $p \in \mathbb{R}_+^n$  and  $\|p\|_1 \leq 1$ . Sample  $i_1, \dots, i_k$  according to  $p$  and let them take value 0 with probability  $1 - \sum_i p_i$ . Set  $\Lambda'' = \sum_{j=1}^k Y_j' \hat{X}_j'$  where  $Y_j' = Y_{i_j}$  and  $X_j' = \frac{X_{i_j}}{kp_{i_j}}$ . (Set  $X_j' = Y_j' = 0$  if  $i_j = 0$ .) These choices mean that  $\mathbb{E}[\Lambda''] = \Lambda$ .

Let  $\bar{X} := \sum_{i=1}^n X_i$  and observe that  $0 \leq \bar{X} \leq I$ . Additionally  $\mathbb{E}[X_j'] = \bar{X}/k$ . Thus if we define  $Z_j := kX_j' - \bar{X}$  then  $\mathbb{E}[Z_j] = 0$  and  $\|Z_j\| \leq d$ . The operator Hoeffding bound (Lemma 3) implies that  $\|\frac{1}{k} \sum_{j=1}^k Z_j\| \leq \delta$  with probability  $\geq 1 - d \exp(-k\delta^2/8d^2)$ . When this occurs we have  $\|\sum_{j=1}^k X_j' - \bar{X}\| \leq \delta$  and thus

$$\sum_{j=1}^k X_j' \leq (1 + \delta)I. \quad (65)$$

Next we attempt to bound the LHS of Eq. (51). First we can relax  $\mathcal{D}_d$  to  $B(S_1)$  and obtain

$$\max_{\rho \in \mathcal{D}_d} \|(\Lambda'' - \Lambda)(\rho)\|_{\mathcal{B}} \leq \|\Lambda'' - \Lambda\|_{S_1 \rightarrow \mathcal{B}} = \|\Lambda'' - \mathbb{E}[\Lambda'']\|_{S_1 \rightarrow \mathcal{B}}. \quad (66)$$

This formulation allows to apply the symmetrization trick (Lemma 19) to obtain

$$\mathbb{E}_{i_1, \dots, i_k \sim p^{\otimes k}} \max_{\rho \in \mathcal{D}_d} \|(\Lambda'' - \Lambda)(\rho)\|_{\mathcal{B}} \leq 2 \mathbb{E}_{i_1, \dots, i_k \sim p^{\otimes k}} \mathbb{E}_{\varepsilon_1, \dots, \varepsilon_k} \left\| \frac{1}{k} \sum_{j=1}^k \varepsilon_j Y_{i_j} \frac{\hat{X}_{i_j}}{p_{i_j}} \right\|_{S_1 \rightarrow \mathcal{B}} \quad (67)$$

We will bound this last quantity for any fixed  $i_1, \dots, i_k$ . For  $\rho \in B(S_1)$ , define  $q_j := \langle X_{i_j}, \rho \rangle / kp_{i_j}$ . Denote the set of feasible  $q$  by  $S_{X, \vec{i}}$  where this notation emphasizes the dependence on both  $X$  and  $i_1, \dots, i_k$ . Then  $\sum_j |q_j| \leq 1$ , each  $|q_j| \leq d/k$  and

$$\left\| \underbrace{\frac{1}{k} \sum_{j=1}^k \varepsilon_j Y_{i_j} \frac{\hat{X}_{i_j}}{p_{i_j}}}_{=: \Lambda'_\varepsilon} \right\|_{S_1 \rightarrow \mathcal{B}} = \max_{q \in S_{X, \vec{i}}} \left\| \sum_{j=1}^k \varepsilon_j Y_{i_j} q_j \right\|_{\mathcal{B}}. \quad (68)$$

Now let us fix  $\rho$  (or equivalently  $q$ ). Observe that  $\|Y_{i_j} q_j\|_{\mathcal{B}} \leq d/k$ . Then Lemma 25 implies that

$$\Pr_{\varepsilon_1, \dots, \varepsilon_k} \left[ \left\| \sum_{j=1}^k \varepsilon_j Y_{i_j} q_j \right\|_{\mathcal{B}} \geq \delta \right] \leq e^{s+2 - \frac{ck\delta^2}{d^2}}. \quad (69)$$

According to Lemma II.2 of [HLSW04] there exists a net of pure states  $\rho_1, \dots, \rho_m \in \mathcal{D}_d$  such that  $m \leq 10^{2d}$  and for any pure state  $\rho$ , we have  $\min_l \|\rho - \rho_l\|_1 \leq 1/2$ . Say that  $\varepsilon_1, \dots, \varepsilon_k$  is a good sequence if  $\|\sum_j \varepsilon_j Y_{i_j} \langle X_{i_j}, \rho_l \rangle / k p_{i_j}\| \leq \delta$  for all  $l \in [m]$ . By Eq. (69) and the union bound the probability that  $\varepsilon_1, \dots, \varepsilon_k$  is bad (i.e. not good) is  $\leq 10^{2d} e^{s+2-ck\delta^2/d^2}$ . For a bad sequence we still have that Eq. (68) is  $\leq d$  by the triangle inequality. For a good sequence, let  $\alpha$  denote Eq. (68) and let  $\beta$  be the corresponding maximum with  $\rho$  restricted to the set  $\{\rho_1, \dots, \rho_m\}$ . By our assumption that the sequence is good we have  $\beta \leq \delta$ . Observe that  $\alpha = \max_{\rho \in B(S_1)} \|\Lambda'_\varepsilon(\rho)\|_{\mathcal{B}}$  and by convexity (and symmetry of the  $\|\cdot\|_{\mathcal{B}}$  norm) this max is achieved for  $\rho$  a pure state. Let  $\rho_l$  satisfy  $\|\rho - \rho_l\|_1 \leq 1/2$ . Then

$$\|\Lambda'_\varepsilon(\rho)\|_{\mathcal{B}} \leq \|\Lambda'_\varepsilon(\rho_l)\|_{\mathcal{B}} + \|\Lambda'_\varepsilon(\rho - \rho_l)\|_{\mathcal{B}} \leq \beta + \alpha \cdot \frac{1}{2}. \quad (70)$$

Maximizing the LHS over  $\rho$  we obtain  $\alpha \leq \beta + \alpha/2$ , or equivalently  $\alpha \leq 2\beta \leq 2\delta$ . Thus Eq. (67) is

$$\leq 4\delta + 2d10^{2d} e^{s+2-\frac{ck\delta^2}{d^2}}. \quad (71)$$

Redefining  $c$ , this is  $\leq 5\delta$  when  $k \geq cd^3/\delta^2$ .

Since Eq. (67) controls the expectation with respect to  $i_1, \dots, i_k$ , we conclude that for at least half of the  $i_1, \dots, i_k$ , the LHS of Eq. (51) is  $\leq 10\delta$ . Since Eq. (65) holds with high probability ( $\geq 1 - d \exp(-cd/8)$ ) it follows that there exists a sequence of  $i_1, \dots, i_k$  that simultaneously fulfills both criteria. Fix this choice. Finally we choose  $\Lambda' = \Lambda''/(1 + \delta)$  so that the normalization condition on  $\sum_j X_j$  is satisfied. This increases the error by at most a further factor of  $\delta$ . We conclude the proof by redefining  $\delta$  to be  $11\delta$ .  $\square$

## B Hardness of computing $2 \rightarrow q$ norms

In this section we extend the hardness results of [BBH<sup>+</sup>12] (Theorem 9.4, part 2) for estimating the  $2 \rightarrow 4$  norm to general  $2 \rightarrow q$  norms for even  $q \geq 4$ .

The next lemma is an extension from Lemma 9.5 from [BBH<sup>+</sup>12].

**Lemma 26.** *Let  $M \in L(\mathbb{C}^d \otimes \mathbb{C}^d)$  satisfy  $0 \leq M \leq I$ . Assume that either (case Y)  $h_{\text{Sep}(d,d)}(M) = 1$  or (case N)  $h_{\text{Sep}(d,d)}(M) \leq 1 - \delta$ . Let  $k$  be a positive integer and  $q \geq 4$  an even positive integer. Then there exists a matrix  $A$  of size  $d^{4kq} \times d^{2kq}$  such that in case Y,  $\|A\|_{2 \rightarrow q} = 1$ , and in case N,  $\|A\|_{2 \rightarrow q} \leq (1 - \delta/2)^k$ . Moreover,  $A$  can be constructed efficiently from  $M$ .*

*Proof.* Consider the following operator

$$N := (M_{A_1 B_1}^{1/2} \otimes \dots \otimes M_{A_{q/2} B_{q/2}}^{1/2}) P_{A_1, \dots, A_{q/2}} \otimes P_{B_1, \dots, B_{q/2}} (M_{A_1 B_1}^{1/2} \otimes \dots \otimes M_{A_{q/2} B_{q/2}}^{1/2}), \quad (72)$$

with  $P_{A_1, \dots, A_{q/2}}$  the projector onto the symmetric subspace over  $A_1, \dots, A_{q/2}$ . We will first relate  $h_{\text{Sep}^{q/2}(d^2)}(N)$  to  $h_{\text{Sep}(d,d)}(M)$ , and then relate  $h_{\text{Sep}^{q/2}(d^2)}(N)$  to  $\|A\|_{2 \rightarrow q}$  for a matrix  $A$  of size  $d^{4kq} \times d^{2kq}$ .

First we show that in case Y,  $h_{\text{Sep}^{q/2}(d^2)}(N) = 1$ . Indeed since there are unit vectors  $x, y \in \mathbb{C}^d$  satisfying  $M_{AB}(x \otimes y) = x \otimes y$ , we have

$$\begin{aligned} h_{\text{Sep}^{q/2}(d^2)}(N) &= \max_{v_1, \dots, v_{q/2} \in \mathbb{C}^{d^2}} (v_1 \otimes \dots \otimes v_{q/2})^* N (v_1 \otimes \dots \otimes v_{q/2}) \\ &\geq (x^{\otimes q/2} \otimes y^{\otimes q/2})^* N (x^{\otimes q/2} \otimes y^{\otimes q/2}) \\ &= (x^{\otimes q/2} \otimes y^{\otimes q/2})^* P_{A_1, \dots, A_{q/2}} \otimes P_{B_1, \dots, B_{q/2}} (x^{\otimes q/2} \otimes y^{\otimes q/2}) = 1 \end{aligned}$$

In case N we show that  $h_{\text{Sep}^{q/2}(d^2)}(N) \leq 1 - \delta/2$ . Note that

$$P_{A_1, \dots, A_{q/2}} \leq P_{A_1 A_2} \otimes I_{A_3 \dots A_{q/2}}. \quad (73)$$

Then

$$\begin{aligned}
h_{\text{Sep}^{q/2}(d^2)}(N) &= \max_{v_1, \dots, v_{q/2} \in \mathbb{C}^{d^2}} (v_1 \otimes \dots \otimes v_{q/2})^* N (v_1 \otimes \dots \otimes v_{q/2}) \\
&\leq \max_{v_1, v_2 \in \mathbb{C}^{d^2}} (v_1 \otimes v_2)^* (M_{A_1 B_1}^{1/2} \otimes M_{A_2 B_2}^{1/2}) P_{A_1 A_2} \otimes P_{B_1 B_2} (M_{A_1 B_1}^{1/2} \otimes M_{A_2 B_2}^{1/2}) (v_1 \otimes v_2) \\
&\leq 1 - \delta/2,
\end{aligned} \tag{74}$$

where the last inequality follows from Lemma 9.6 of [BBH<sup>+</sup>12].

To construct a matrix  $A$  of size  $d^{4kq} \times d^{2kq}$  s.t.  $\|A\|_{2 \rightarrow q} = h_{\text{Sep}^{q/2}(d^2)}(N)$  we follow the proof of Lemma 9.5 of [BBH<sup>+</sup>12], the only difference being that we apply Wick's theorem to  $P_{A_1, \dots, A_{q/2}}$ , i.e. there is a measure  $\mu$  over unit vectors s.t.

$$P_{A_1, \dots, A_{q/2}} = \binom{d + q/2 - 1}{q/2} \int \mu(dv) (vv^*)^{\otimes q/2}. \tag{75}$$

□

The basic idea of the Lemma is to use the product test of [HM10] to force  $v_1, \dots, v_{q/2}$  to be product states. Our proof can be summarized as saying that  $q/2$  copies can enforce this more effectively than 2 copies (assuming  $q/2 \geq 2$ ), and therefore we obtain soundness at least as sharp as in [BBH<sup>+</sup>12]. This analysis may be wasteful, since using more copies should *improve* the effectiveness of the product test.

The main result of this section is the following analogue of Theorem 9.4, part 2, of [BBH<sup>+</sup>12]:

**Theorem 27.** *Let  $\phi$  be a 3-SAT instance with  $n$  variables and  $O(n)$  clauses and  $q \geq 4$  an even integer. Determining whether  $\phi$  is satisfiable can be reduced in polynomial time to determining whether  $\|A\|_{2 \rightarrow q} \geq C$  or  $\|A\|_{2 \rightarrow q} \leq c$  where  $0 \leq c < C$  and  $A$  is an  $m \times m$  matrix, where  $m = \exp(q\sqrt{n} \text{polylog}(n) \log(C/c))$ .*

This gives nontrivial hardness for super-constant  $q$ , in fact up to  $\tilde{O}(\sqrt{\log d})$ , but not yet all the way up to  $O(\log d)$ , where multiplicative approximations are known to be easy.

*Proof.* Corollary 14 of [HM10] gives a reduction from determining satisfiability of  $\phi$  to distinguishing between  $h_{\text{Sep}(d,d)}(M) = 1$  and  $h_{\text{Sep}(d,d)}(M) \leq 1/2$ , with  $0 \leq M \leq I$  that can be constructed in time  $\text{poly}(d)$  from  $\phi$  with  $d = \exp(\sqrt{n} \text{polylog}(n))$ . Applying Lemma 26 gives the result. □

## References

- [ABS10] Sanjeev Arora, Boaz Barak, and David Steurer. Subexponential algorithms for unique games and related problems. In *FOCS*, pages 563–572, 2010.
- [AIM14] S. Aaronson, R. Impagliazzo, and D. Moshkovitz. AM with multiple Merlins. In *Computational Complexity (CCC), 2014 IEEE 29th Conference on*, pages 44–55, June 2014, [arXiv:1401.6848](#).
- [ALSV13a] Noga Alon, Troy Lee, Adi Shraibman, and Santosh Vempala. The approximate rank of a matrix and its algorithmic applications: Approximate rank. In *Proceedings of the 45th Annual ACM Symposium on Theory of Computing*, STOC '13, pages 675–684, 2013.
- [ALSV13b] Noga Alon, Troy Lee, Adi Shraibman, and Santosh Vempala. The approximate rank of a matrix and its algorithmic applications: Approximate rank. In *Proceedings of the Forty-fifth Annual ACM Symposium on Theory of Computing*, STOC '13, pages 675–684. ACM, 2013.
- [BaCY11] Fernando G.S.L. Brandão, Matthias Christandl, and Jon Yard. A quasipolynomial-time algorithm for the quantum separability problem. In *Proceedings of the 43rd annual ACM symposium on Theory of computing*, STOC '11, pages 343–352, 2011, [arXiv:1011.2751](#).

- [BBH<sup>+</sup>12] Boaz Barak, Fernando G.S.L. Brandão, Aram W. Harrow, Jonathan Kelner, David Steurer, and Yuan Zhou. Hypercontractivity, sum-of-squares proofs, and their applications. In *STOC '12*, STOC '12, pages 307–326, 2012, [arXiv:1205.4484](#).
- [BC11] F.G.S.L. Brandão and M. Christandl. Detection of multiparticle entanglement: Quantifying the search for symmetric extensions, 2011, [arXiv:1105.5720](#).
- [BCL94] Keith Ball, Eric A. Carlen, and Elliott H. Lieb. Sharp uniform convexity and smoothness inequalities for trace norms. *Inventiones mathematicae*, 115(1):463–482, 1994.
- [BCY11] F. G. S. L. Brandão, M. Christandl, and J. Yard. Faithful squashed entanglement. *Commun. Math. Phys.*, 306(3):805–830, 2011, [arXiv:1010.1750](#).
- [BH13] Fernando G. S. L. Brandão and Aram W. Harrow. Quantum de Finetti theorems under local measurements with applications. In *Proceedings of the 45th annual ACM Symposium on theory of computing*, STOC '13, pages 861–870, 2013, [arXiv:1210.6367](#).
- [BKS13] Boaz Barak, Jonathan Kelner, and David Steurer. Rounding sum-of-squares relaxations. In *STOC '14*, STOC '14, 2013, [arXiv:1312.6652](#).
- [BKW14] M. Braverman, Y. K. Ko, and Omri Weinstein. Approximating the best Nash equilibrium in  $n^{o(\log(n))}$ -time breaks the Exponential Time Hypothesis, 2014. ECCC TR14-092.
- [BRS11] Boaz Barak, Prasad Raghavendra, and David Steurer. Rounding semidefinite programming hierarchies via global correlation. In *FOCS*, 2011, [arXiv:1104.4680](#).
- [BV11] Aditya Bhaskara and Aravindan Vijayaraghavan. Approximating matrix p-norms. In *Proceedings of the twenty-second annual ACM-SIAM symposium on Discrete Algorithms*, pages 497–511. SIAM, 2011, [arXiv:1001.2613](#).
- [DF80] P. Diaconis and D. Freedman. Finite exchangeable sequences. *Annals of Probability*, 8:745–764, 1980.
- [DKLP06a] Etienne De Klerk, Monique Laurent, and Pablo A Parrilo. A PTAS for the minimization of polynomials of fixed degree over the simplex. *Theoretical Computer Science*, 361(2):210–225, 2006.
- [DKLP06b] Etienne De Klerk, Monique Laurent, and Pablo A Parrilo. A PTAS for the minimization of polynomials of fixed degree over the simplex. *Theoretical Computer Science*, 361(2):210–225, 2006.
- [DPS04] Andrew C. Doherty, Pablo A. Parrilo, and Federico M. Spedalieri. Complete family of separability criteria. *Phys. Rev. A*, 69:022308, Feb 2004, [arXiv:quant-ph/0308032](#).
- [GLS93] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric algorithms and combinatorial optimization*. Springer-Verlag, 1993.
- [Gur03] Leonid Gurvits. Classical deterministic complexity of Edmonds’ problem and quantum entanglement. In *STOC '03*, pages 10–19, 2003, [arXiv:quant-ph/0303055](#).
- [Har15] Aram W. Harrow. Finding approximate Nash equilibria using LP hierarchies, 2015. in preparation.
- [HLSW04] P. Hayden, D. W. Leung, P. W. Shor, and A. J. Winter. Randomizing quantum states: Constructions and applications. *Commun. Math. Phys.*, 250:371–391, 2004, [arXiv:quant-ph/0307104](#).
- [HM10] Aram W. Harrow and Ashley Montanaro. An efficient test for product states, with applications to quantum Merlin-Arthur games. In *FOCS '10*, pages 633–642, 2010, [arXiv:1001.0017](#).

- [HM13] Aram W. Harrow and Ashley Montanaro. Testing product states, quantum Merlin-Arthur games and tensor optimization. *J. ACM*, 60(1):3:1–3:43, February 2013, [arXiv:1001.0017](#).
- [HSR04] M. Horodecki, P.W. Shor, and M.B. Ruskai. General entanglement breaking channels. *Rev. Math. Phys.*, 15(3):629–641, 2004, [arXiv:quant-ph/0012127](#).
- [KPT00] Mikio Kato, Lars-Erik Persson, and Yasuji Takahashi. Clarkson type inequalities and their relations to the concepts of type and cotype. *Collectanea Mathematica*, 51(3):327–346, 2000.
- [LMM03] Richard J. Lipton, Evangelos Markakis, and Aranyak Mehta. Playing large games using simple strategies. In *Proceedings of the 4th ACM Conference on Electronic Commerce, EC '03*, pages 36–41. ACM, 2003.
- [LRS<sup>+</sup>10] Jason D Lee, Ben Recht, Ruslan R Salakhutdinov, Nathan Srebro, and Joel Tropp. Practical large-scale optimization for max-norm regularization. In J.D. Lafferty, C.K.I. Williams, J. Shawe-Taylor, R.S. Zemel, and A. Culotta, editors, *Advances in Neural Information Processing Systems 23*, pages 1297–1305. Curran Associates, Inc., 2010.
- [LS07] Nati Linial and Adi Shraibman. Lower bounds in communication complexity based on factorization norms. In *In Proc. of the 39th Symposium on Theory of Computing (STOC)*, pages 699–708, 2007.
- [LS09] Nati Linial and Adi Shraibman. Lower bounds in communication complexity based on factorization norms. *Random Structures & Algorithms*, 34(3):368–394, 2009.
- [LS14] Ke Li and Graeme Smith. Quantum de Finetti theorem measured with fully one-way LOCC norm, 2014, [arXiv:1408.6829](#).
- [LT91] Michel Ledoux and Michel Talagrand. *Probability in Banach spaces. Isoperimetry and processes*. Springer-Verlag, 1991.
- [LW14] Ke Li and Andreas Winter. Relative entropy and squashed entanglement. *Commun. Math. Phys.*, 326(1):63–80, 2014, [arXiv:1210.3181](#).
- [MWW09] William Matthews, Stephanie Wehner, and Andreas Winter. Distinguishability of quantum states under restricted families of measurements with an application to quantum data hiding. *Commun. Math. Phys.*, 291(3):813–843, 2009, [arXiv:0810.2327](#).
- [Nao12] Assaf Naor. On the Banach-space-valued Azuma inequality and small-set isoperimetry of Alon-Roichman graphs. *Combinatorics, Probability and Computing*, 21:623–634, 7 2012, [arXiv:1009.5695](#).
- [Pie07] A. Pietsch. *History of Banach Spaces and Linear Operators*. Birkhäuser, 2007.
- [Ste05] Daureen Steinberg. Computation of matrix norms with applications to robust optimization. Master’s thesis, Technion, 2005. Available on A. Nemirovski’s website <http://www2.isye.gatech.edu/~nemirovs/>.
- [SW12] Yaoyun Shi and Xiaodi Wu. Epsilon-net method for optimizations over separable states. In *Proceedings of the 39th International Colloquium Conference on Automata, Languages, and Programming - Volume Part I, ICALP’12*, pages 798–809, Berlin, Heidelberg, 2012. Springer-Verlag, [arXiv:1112.0808](#).
- [TJ74] Nicole Tomczak-Jaegermann. The moduli of smoothness and convexity and the Rademacher averages of the trace classes  $s\{p\}$  ( $1 \leq p < \infty$ ). *Studia Mathematica*, 50(2):163–182, 1974.
- [Tro10] J. A. Tropp. User-friendly tail bounds for sums of random matrices, 2010, [arXiv:1004.4389](#).

[Yan06] D. Yang. A simple proof of monogamy of entanglement. *Phys. Lett.*, 360(1):249, 2006, arXiv:quant-ph/0604168.