Quantum voting and violation of Arrow's Impossibility Theorem

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Abstract

We propose a quantum voting system in the spirit of quantum games such as the quantum Prisoner's Dilemma. Our scheme violates a quantum analogue of Arrow's Impossibility Theorem, which states that every (classical) constitution endowed with three innocuous-seeming properties is a dictatorship. Superpositions, interference, and entanglement of votes feature in voting tactics available to quantum voters but not to classical.

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Voting schemes used today are classical: Each voter submits one preference, and preferences combine deterministically into society's preference. What if citizens could superpose, interfere, and entangle votes? The quantization of voting schemes offers an opportunity to explore the power of quantum information theory. This investigation furthers the tradition of quantum games such as the quantum Prisoner's Dilemma [1], in which players superpose classical strategies and share entanglement [2].

We propose a quantum voting system that violates a quantum analog of Arrow's Impossibility Theorem [3] and that accommodates entanglement and superpositions. According to Arrow's Theorem, every (classical) dictatorship that has three innocuous-seeming properties—that respects transitivity, unanimity, and independence of irrelevant alternatives—is a dictatorship. Arrow's Theorem is surprisingly deep and has spawned interpretations of its mathematical formulation and of its implications about fair elections [4]. The theorem has fundamentally impacted game theory and voting theory. Yet Arrow's Theorem derives from classical logic, and a quantum extension turns out to be false.

Our approach differs from earlier quantum and classical work as follows. Quantum voting theory has focused on privacy and cryptography [5–7], whereas we draw inspiration from game theory. Classical voting is known to violate Arrow's Theorem if voters' opinions are restricted to *single-peaked preferences* [8]. Rather than introduce an assumption into classical elections, we recast Arrow's scheme in quantum mechanical terms. Another classical violation attempt involves probabilistic mixtures of votes. This attempt, however, violates one of Arrow's postulates [9].

The paper is organized as follows. We first define quantum votes, constitutions, and constitutional properties. After introducing and disproving a quantum analog of Arrow's Theorem, we explore entanglement- and superposition-dependent voting tactics.

I. SOCIETIES, VOTERS, AND QUANTUM PREFERENCES

Let $\mathcal{S}$ denote a *society* that consists of N *voters*. Voter $i = 1, 2, \ldots, N$ is associated with a finite Hilbert space $\mathcal{H}_i$. By $\mathcal{L}(\mathcal{H})$, we denote the space of linear positive semidefinite operators defined on the Hilbert space $\mathcal{H}$. Society is associated with a *joint state* $\sigma_{\text{soc}} \in \mathcal{P}(\mathcal{H}_1) \times \ldots \times \mathcal{P}(\mathcal{H}_N)$ that represents the votes.

Voter i has a *quantum preference* ρ_i that results from tracing out every subsystem except the i^{th} from society's joint state: $\rho_i := \text{Tr}'(\sigma_{\text{soc}})$, wherein Tr' denotes a trace over all voters except i. We will sometimes denote a pure quantum preference by $|\psi_i\rangle$. The set of all voters' quantum preferences forms society's *quantum profile* $\mathcal{P} := \{\rho_1, \ldots, \rho_N\}$, just as the set of all voters' classical preferences forms a profile in a classical election.

Let $a, b, \ldots, m$ denote the M candidates ranked by voters. Society must form a transitive ordered list, which we term a *classical preference*, of the candidates. In each classical preference, each candidate is ranked above, ranked below, or tied with each other candidate: $a > b$, $a < b$, or $a = b$. With each classical preference, we associate one pure state in each of $\mathcal{H}_1, \ldots, \mathcal{H}_N$. For example, $c > a = b > d$ corresponds to $|c > a = b > d\rangle$. We denote by $|\gamma\rangle$ the γ^{th} classical-preference state and by χ_i^γ the associated density operator on $\mathcal{H}_i$: $\chi_i^\gamma := |\gamma\rangle\langle\gamma|$. The set $\{|\gamma\rangle\}$ forms the *preference basis* $\mathcal{B}_\mathcal{H}$ for the Hilbert space $\mathcal{H}$.

For each pair (a, b) of candidates, each Hilbert space $\mathcal{H}$ can be decomposed into subspaces associated with the possible relationships between a and b. By $\mathcal{G}_\mathcal{H}^{a>b}$, we denote the subspace spanned by the $\mathcal{B}_{\mathcal{H}}$ elements associated with $a > b$.[1] The subspaces $\mathcal{G}_{\mathcal{H}}^{b>a}$ and $\mathcal{G}_{\mathcal{H}}^{a=b}$ are defined analogously. For example, $|a > b > c\rangle$ occupies the intersection of three subspaces: $|a > b > c\rangle \in \mathcal{G}_{\mathcal{H}}^{a>b} \cap \mathcal{G}_{\mathcal{H}}^{a>c} \cap \mathcal{G}_{\mathcal{H}}^{b>c}$. The $a > b$, $b > a$, and $a = b$ subspaces are disjoint, e.g., $\mathcal{G}_{\mathcal{H}}^{a>b} \cap \mathcal{G}_{\mathcal{H}}^{b>a} = \emptyset$.

Projectively measuring a quantum preference with $\mathcal{B}_{\mathcal{H}}$ yields a classical preference. If ρ_i is a nontrivial linear combination or mixture of $\mathcal{B}_i$ elements, the measurement is nondeterministic. A voter's ability to superpose classical preferences resembles a prisoner's ability to superpose classical tactics in the quantum Prisoner's Dilemma [1].

Elections proceed as follows: Every voter sends a quantum preference to an election committee. The ρ_i's enter a quantum circuit that implements a constitution (defined in Sec. II). The circuit's output is measured with $\mathcal{B}_{\text{soc}}$, yielding society's classical preference.

II. QUANTUM CONSTITUTIONS

A *classical constitution* is a map from a classical profile to a classical preference. We will define quantum constitutions, then review properties of classical constitutions and introduce quantum analogs. Our quantization scheme is justified in Appendix A.

Definition 1 (Quantum constitution)**.** *A* quantum constitution *is a convex-linear completely positive trace-preserving (CPTP) map*[2]

$$\mathcal{E} : \mathcal{P}(\mathcal{H}_1) \times \ldots \times \mathcal{P}(\mathcal{H}_N) \to \mathcal{P}(\mathcal{H}_{\text{soc}})$$

that transforms society's joint state σ_{soc} into society's quantum preference *ρ_{soc}:*

$$\mathcal{E}(\sigma_{\text{soc}} \otimes |0\rangle\langle 0|) = \rho_{\text{soc}}, \tag{2}$$

wherein $|0\rangle\langle 0|$ denotes a fiducial state to which society's quantum preference is initialized.

A. Four properties in Arrow's Theorem

A classical constitution can have properties such as transitivity, respecting of unanimity, respecting of independence of irrelevant alternatives, being a dictatorship, and respecting of majority rule. The first four properties feature in Arrow's Theorem. Let us review these properties and define quantum analogs.

A classical constitution is *transitive* if every classical preference in its range is transitive. A classical preference is transitive if $a \geq b$ and $b \geq c$, together, imply $a \geq c$.

Definition 2 (Quantum transitivity)**.** *A quantum constitution $\mathcal{E}$ respects* quantum transitivity *if every possible output ρ_{soc}, upon being measured in the preference basis $\mathcal{B}_{\text{soc}}$, collapses to a state $|a \ldots m\rangle$ associated with a transitive classical preference $(a \ldots m)$.*

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[1] We will sometimes condense the subscript to $\mathcal{G}_i^{a>b} := \mathcal{G}_{\mathcal{H}_i}^{a>b}$ and $\mathcal{G}_{\text{soc}}^{a>b} := \mathcal{G}_{\mathcal{H}_{\text{soc}}}^{a>b}$. These abbreviations will be used elsewhere, as in notations for projectors and preference bases.

[2] A convex-linear map $\mathcal{E}$ satisfies

$$\mathcal{E}\left(\sum_i p_i \rho_i\right) = \sum_i p_i \mathcal{E}(\rho_i), \quad \text{wherein} \quad p_i \geq 0 \ \forall i \quad \text{and} \quad \sum_i p_i = 1. \tag{1}$$

Every $\mathcal{E}$ obeys quantum transitivity by definition: Given any input, $\mathcal{E}$ outputs a ρ_{soc} that is a linear combination or a mixture of preference-basis elements. A $\mathcal{B}_{\text{soc}}$ measurement of ρ_{soc} yields a $\mathcal{B}_{\text{soc}}$ element, which corresponds to a transitive classical preference.

A classical constitution that respects *unanimity* ranks $a > b$ if every voter ranks $a > b$.

Definition 3 (Quantum unanimity)**.** *A quantum constitution respects* quantum unanimity *if it has the following two properties: (i) Suppose that every voter's quantum preference has support on the $a > b$ subspace: If $\Pi_{\mathcal{H}}^{a>b}$ denotes the projector onto $\mathcal{G}_{\mathcal{H}}^{a>b}$, then* $\text{Tr}\left(\Pi_i^{a>b}\,\rho_i\right) > 0\ \forall i$. *Society's quantum preference has support on the $a > b$ subspace:* $\text{Tr}\left(\Pi_{\text{soc}}^{a>b}\,\rho_{\text{soc}}\right) > 0$. *(ii) If every voter's quantum preference lacks support on the $a > b$ subspace, so does society's quantum preference:* $\text{Tr}\left(\Pi_i^{a>b}\,\rho_i\right) = 0\ \forall i \quad \Rightarrow \quad \text{Tr}\left(\Pi_{\text{soc}}^{a>b}\,\rho_{\text{soc}}\right) = 0$.

A classical constitution respects *independence of irrelevant alternatives* (IIA) if society's relative ranking of a and b depends only on the relative ranking of a and b by every voter. How voters rank c does not affect whether society ranks $a > b$, $a < b$, or $a = b$.

Definition 4 (Quantum independence of irrelevant alternatives)**.** *A quantum constitution respects* quantum independence of irrelevant alternatives *(QIIA) if whether ρ_{soc} has support on $\mathcal{G}_{\text{soc}}^{a>b}$, on $\mathcal{G}_{\text{soc}}^{a<b}$, and/or on $\mathcal{G}_{\text{soc}}^{a=b}$ depends only on whether each ρ_i has support on $\mathcal{G}_i^{a>b}$, on $\mathcal{G}_i^{a<b}$, and/or on $\mathcal{G}_i^{a=b}$.*

A classical constitution is a *classical dictatorship* if there exists a voter i such that, for all pairs (a, b), society ranks $a > b$ if and only if Voter i ranks $a > b$.

Definition 5 (Quantum dictatorship)**.** *A quantum constitution is a* quantum dictatorship *if there exists a voter i who has the following two characteristics: (i) If Voter i's quantum preference has support on the $a > b$ subspace, so does society's:*

$$\text{Tr}\left(\Pi_i^{a>b}\rho_i\right) > 0 \quad \Rightarrow \quad \text{Tr}\left(\Pi_{\text{soc}}^{a>b}\rho_{\text{soc}}\right) > 0. \tag{3}$$

(ii) If Voter i's quantum preference lacks support on the $a > b$ subspace, so does society's:

$$\text{Tr}\left(\Pi_i^{a>b}\rho_i\right) = 0 \quad \Rightarrow \quad \text{Tr}\left(\Pi_{\text{soc}}^{a>b}\rho_{\text{soc}}\right) = 0. \tag{4}$$

B. Majority rule

We will disprove a quantum analog of Arrow's Theorem with a quantum analog of majority rule. The majority-rule classical constitution ranks a relative to b as most voters do. A subtlety arises if society's profile $\mathcal{P}$ is cyclic. A set $T = \{a, b, \ldots, k\}$ of candidates forms a cycle if every $c \in T$ participates in pairwise preferences (e.g., $b > c, c > d$) that appear in $\mathcal{P}$ and that violate transitivity. Every $c \in T$, furthermore, participates in such a transitivity-violating pair with each other $d \in T$.

For example, consider $\mathcal{P} = \{(a > b > c), (c > a > b), (b > c > a)\}$. A naïve application of majority rule implies $a > b$ and $b > c$, whereupon transitivity implies $a > c$. But a naïve application of majority rule implies also $c > a$, so society's classical preference must respect $c > a$, which violates transitivity. The constitution may be defined as outputting $a = b = c$ or an error message. Because of cycles, classical majority rule fails to satisfy IIA and transitivity simultaneously.[3]

[3] One profile can contain multiple cycles. For example, $\{(a > b > c), (b > a > c), (a > c > b)\}$ contains a cycle over (a, b) (because Voters 1 and 3 rank $a > b$, whereas Voter 2 ranks $b > a$) and a cycle over (b, c) (because Voters 1 and 2 rank $b > c$, whereas Voter 3 ranks $c > b$).

A quantum extension of majority rule can have both properties. Before defining the extension, we introduce quantum cycles. Let $\sigma_{\text{soc}} = \chi_1^\alpha \otimes \ldots \otimes \chi_N^\mu$ denote a product of preference-basis elements. Suppose that at least two χ_i^γ's correspond to classical preferences that form a classical cycle. We will say that σ_{soc} contains a cycle.

The Quantum Majority-Rule (QMR) constitution $\mathcal{E}$ sends each ρ_i through the phase-damping channel Φ:

$$\Phi(\rho_i) := \sum_\gamma |\gamma\rangle\langle\gamma|\rho_i|\gamma\rangle\langle\gamma| = \sum_\gamma p_i^\gamma \chi_i^\gamma =: \rho_i', \quad \text{wherein} \quad \sum_\gamma p_i^\gamma = 1. \tag{5}$$

QMR then processes the ρ_i''s as would the constitution $\mathcal{E}'$, defined as follows. By linearity,

$$\mathcal{E}'\left(\rho_1' \otimes \ldots \otimes \rho_N'\right) = \sum_{\alpha,\ldots,\mu} \left(p_1^\alpha \ldots p_N^\mu\right) \mathcal{E}'(\chi_1^\alpha \otimes \ldots \otimes \chi_N^\mu). \tag{6}$$

From $\chi_1^\alpha \otimes \ldots \otimes \chi_N^\mu$, $\mathcal{E}'$ constructs a directed graph, or digraph. Each candidate is associated with one vertex. Edges are formed from the χ_i^γ's. If more γ's correspond to $a > b$ than to $b > a$, an edge points from a to b. If precisely as many γ's correspond to $a > b$ as to $b > a$, then one edge points from a to b, and one edge points from b to a.

$\mathcal{E}'$ inputs the graph into *Tarjan's algorithm* for finding the graph's strongly connected components [10]. A *strongly connected component* (SCC) is a subgraph in which every node can be accessed from each other node via edges. Every vertex appears in exactly one SCC. Every SCC in the QMR graph represents a cycle or a set of interlinked cycles. For example, suppose $\mathcal{E}'$ acts on $|bacd\rangle|acbd\rangle$, whose $>$ signs we have dropped to condense notation. Candidates a and b participate in a cycle, as do b and c. The a, b, and c vertices form one SCC; and the d vertex forms another, as shown in Fig. 1.

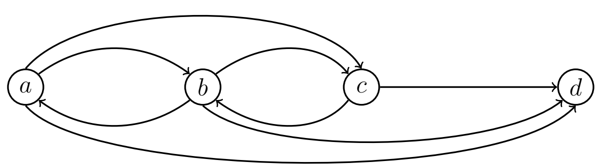

FIG. 1: Digraph formed from $|bacd\rangle|acbd\rangle$ and inputted into Tarjan's algorithm. Because a and b form a cycle, while b and c form another, (a, b, c) forms an SCC. So does d.

Tarjan's algorithm returns a list of the SCCs. Every vertex in the j^{th} SCC is preferred to every vertex in the i^{th}, if $i < j$. For example, Tarjan's algorithm maps Fig. 1 to $(\{d\}, \{a, b, c\})$. $\mathcal{E}'$ forms a maximally mixed state over the classical preferences formable from the candidates in each SCC. Combining the mixtures, $\mathcal{E}'$ orders the SCCs according to preference (reverses the ordering outputted by Tarjan's algorithm):

$$\begin{aligned} &C[|abcd\rangle\langle abcd| + |cabd\rangle\langle cabd| + |bcad\rangle\langle bcad| + |cbad\rangle\langle cbad| \\ &\quad + |bacd\rangle\langle bacd| + |acbd\rangle\langle acbd| + (\text{terms that contain } = \text{ signs})], \end{aligned} \tag{7}$$

wherein C denotes a normalization factor.

Expression (7) contains terms associated with $c > a$ and $c = a$, but every voter ranks $a > c$. To respect quantum unanimity, $\mathcal{E}'$ projects Expression (7) onto the $a > c$ subspace:

$$\mathcal{E}'(|bacd\rangle\langle bacd| \otimes |acbd\rangle\langle acbd|) = \frac{1}{3}(|abcd\rangle\langle abcd| + |bacd\rangle\langle bacd| + |acbd\rangle\langle acbd|). \tag{8}$$

Formally, QMR is defined as follows.

Definition 6 (Quantum Majority Rule). *The* Quantum Majority-Rule *constitution* $\mathcal{E}$ *maps* σ_{soc} *to the same* ρ_{soc} *as the following algorithm:*

1. *Calculate* ρ'_i *for all* i.
2. *Evaluate* $\mathcal{E}'(\rho'_1 \otimes \ldots \otimes \rho'_N) = \sum_{\alpha,\ldots,\mu} (p_1^\alpha \ldots p_N^\mu)\, \mathcal{E}'(\chi_1^\alpha \otimes \ldots \otimes \chi_N^\mu)$, *wherein the quantum constitution* $\mathcal{E}'$ *acts as follows.*
   (a) *Form a digraph in which each node corresponds to one candidate and vice versa.*
   (b) *For each pair* (a, b) *of candidates, count the* χ_i^γ*'s associated with* $a > b$*, the* χ_i^γ*'s associated with* $b > a$*, and the* χ_i^γ*'s associated with* $a = b$.
   (c) *Construct the graph's edges as follows: If more* χ_i^γ*'s correspond to* $a > b$ *than to* $b > a$*, an edge points from* a *to* b*. If precisely as many* χ_i^γ*'s correspond to* $a > b$ *as to* $b > a$*, then one edge points from* a *to* b*, and one edge points from* b *to* a.
   (d) *Input the graph to Tarjan's algorithm, which returns a list of the SCCs.*
   (e) *Form an intermediate state* ξ *as follows: Associate each SCC with the maximally mixed state over the classical preferences formable from the candidates in the SCC. Combine the mixtures, to form a* ξ *defined on* $\mathcal{H}_{\text{soc}}$*, as follows: If* a *is in the* j^{th} *SCC and* b *is in the* i^{th} *SCC, for any* $i < j$*,* ξ *is defined on* $\mathcal{G}_{\text{soc}}^{a>b}$.
   (f) *For each pair* (a, b)*, if every* χ_i^γ *corresponds to* $a > b$*, project* ξ *onto* $\mathcal{G}_{\text{soc}}^{a>b}$.

To learn society's classical preference r_{soc}, one measures the constitution's output with $\mathcal{B}_{\text{soc}}$. Due to this decohering measurement and to the dephasing channel Φ, r_{soc} can be obtained by a random choice from among classical preferences, rather than from a quantum circuit. But QMR is not equivalent to the classical majority-rule constitution $\mathcal{F}$. Suppose that $\mathcal{F}$, given a cycle over at least three candidates, chose society's preference randomly. $\mathcal{F}$ would violate transitivity. QMR constitutions respect quantum transitivity by definition. (As proved in Appendix B, QMR respects also quantum unanimity and QIIA.) This discrepancy between classical and quantum majority rule enables QMR to violate a quantum analog of Arrow's Theorem.

III. ARROW'S IMPOSSIBILITY THEOREM

Arrow's Impossibility Theorem states that every classical constitution endowed with three innocent-seeming properties is a dictatorship [3, 11].

Theorem 1 (Arrow's Impossibility Theorem). *Every classical constitution that respects transitivity, unanimity, and independence of irrelevant alternatives is a dictatorship.*

We quantize Arrow's Theorem in the following conjecture, which we will disprove.

Conjecture 1 (Quantum Arrow Conjecture). *Every quantum constitution that respects quantum transitivity, quantum unanimity, and QIIA is a quantum dictatorship.*

Theorem 2. *The Quantum Arrow Conjecture is false.*

Proof. To simplify notation, we focus on strict preferences and drop binary-relation signs: $|abc\rangle := |a > b > c\rangle$. To prove the theorem by contradiction, we suppose that

$$\mathcal{P} = \{|abc\rangle, |cab\rangle, |bca\rangle)\} \tag{9}$$

The QMR constitution $\mathcal{E}$ constructs a digraph in which one edge points from a to b (because two voters prefer $a > b$, whereas one prefers $b > a$), one edge points from b to c, and one edge points from c to a (Fig. 2). The digraph consists of one SCC, and no candidate is preferred unanimously to any other. Hence

$$\rho_{\text{soc}} = C[|abc\rangle\langle abc| + |cab\rangle\langle cab| + |bca\rangle\langle bca| + |cba\rangle\langle cba| + |bac\rangle\langle bac| + |acb\rangle\langle acb| \\ + (\text{terms that contain } = \text{ signs})], \tag{10}$$

wherein C normalizes the state.

ρ_{soc} has a larger support than each of the votes. By Definition 5, no voter is a dictator. Yet $\mathcal{E}$ respects quantum transitivity, quantum unanimity, and QIIA, by Lemma 1 in Appendix B. Hence $\mathcal{E}$ violates the Quantum Arrow Conjecture. □

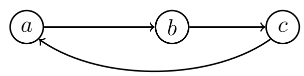

FIG. 2: Digraph formed from $\{|abc\rangle, |cab\rangle, |bca\rangle\}$. All the candidates form one cycle.

One can understand as follows why our voting system violates the Arrow conjecture. First, consider the motivations for quantizing elections as in Sections I and II. Given the successes of quantum game theory, an election that accommodates superposed and entangled preferences merits construction. To introduce superpositions and entanglement, one must define a quantum election in terms of a general quantum process: a preparation procedure, an evolution, and a measurement [12]. One must translate the definitions of "dictatorship," "transitivity," etc. as faithfully as possible into properties of quantum systems. These quantum definitions combine with QMR and a cyclic profile into a violation of the Arrow conjecture.

Simpler disproofs exist, though the disproof above offers interpretational advantages. For instance, a quantum constitution $\mathcal{K}$ that outputs a superposition over all inputs violates the conjecture. But imposing $\mathcal{K}$ on society—choosing society's classical preference randomly—makes little economic sense. Also, disproving the conjecture with a quantum analog of classical majority rule, which does not violate Arrow's Theorem, demonstrates how quantum mechanics invalidates the theorem.

IV. QUANTUM VOTING TACTICS

How one should vote, to secure the most desirable election outcome possible, is not always clear. *Strategic voting* is the submission of a preference other than one's opinion in an election amongst at least three candidates, to secure an unobjectionable outcome.

For example, imagine that Alice, Bob, and Charlie vie for the presidency of the American Physical Society. Suppose that Alice and Bob have greater chances of winning than

Charlie, and that Charlie agrees more with Alice than with Bob. Charlie's supporters might strategically vote for Alice, to elect a president whom they neither prefer most nor mind.

Quantum strategic voting and other quantum voting tactics rely on superpositions, interference, and entanglement. For simplicity, we assume that σ_{soc} is pure and focus on strict preferences $a > b$. We will denote the classical preference $a > b > \ldots > m$ and its even permutations by $\alpha, \ldots, \mu$, as in Sec. I. Each anticycle $m > \ldots > b > a$ will be denoted by a bar: $\bar{\alpha}$. Pure quantum preferences have the form

$$\sum_{\gamma}(c_{\gamma}|\gamma\rangle + c_{\bar{\gamma}}|\bar{\gamma}\rangle), \quad \text{wherein} \quad \sum_{\gamma}(c_{\gamma} + c_{\bar{\gamma}}) = 1. \tag{11}$$

Society's joint quantum state has the form

$$|\sigma_{\text{soc}}\rangle = (c_{\alpha_1} \ldots c_{\alpha_N})|\alpha \ldots \alpha\rangle + \ldots + (c_{\bar{\mu}_1} \ldots c_{\bar{\mu}_N})|\bar{\mu} \ldots \bar{\mu}\rangle. \tag{12}$$

A. Quantum strategic voting via interference

Relative phases and interference facilitate quantum strategic voting. Consider a society $\mathcal{S}$ whose voters submit pure states and that uses the following variation, dubbed *QMR2*, on QMR.[4] As an example, suppose that

$$\mathcal{P} = \left\{ |abc\rangle, \frac{1}{\sqrt{2}}(|bac\rangle + |acb\rangle), \frac{1}{\sqrt{2}}(|bac\rangle + |cba\rangle) \right\}, \tag{13}$$

such that

$$|\sigma_{\text{soc}}\rangle = \frac{1}{2}(|abc\rangle|bac\rangle|bac\rangle + |abc\rangle|bac\rangle|cba\rangle + |abc\rangle|acb\rangle|bac\rangle + |abc\rangle|acb\rangle|cba\rangle). \tag{14}$$

Like the $\mathcal{E}'$ in the QMR definition, the QMR2 constitution $\mathcal{E}_2$ forms a digraph from each $|\sigma_{\text{soc}}\rangle$ term. Each graph is inputted into Tarjan's algorithm, which returns a list of the SCCs. Just as $\mathcal{E}'$ maps each list to a mixed state ξ, $\mathcal{E}_2$ maps the i^{th} list to a superposition $|\xi_i\rangle$. In the example,

$$\sum_{i=1}^{4}|\xi_i\rangle = \frac{1}{2}(|bac\rangle + |bac\rangle + |abc\rangle + |acb\rangle) = |bac\rangle + \frac{1}{2}(|abc\rangle + |acb\rangle). \tag{15}$$

If $\langle\Xi|\Xi\rangle \neq 0$, $\mathcal{E}_2$ normalizes $|\Xi\rangle$: $|\rho_{\text{soc}}\rangle := \frac{|\Xi\rangle}{\langle\Xi|\Xi\rangle}$, and $\rho_{\text{soc}} = |\rho_{\text{soc}}\rangle\langle\rho_{\text{soc}}|$. If $\langle\Xi|\Xi\rangle = 0$ (no quantum system emerges from the constitution circuit), society can hold a revote.

Suppose that Voter 3 wishes to eliminate *bac* from society's possible classical preferences. Eliminating $|bac\rangle$ from $|\psi_3\rangle$ will not suffice. Voter 3 can introduce a relative phase of -1, such that society's quantum profile becomes

$$\mathcal{P}' = \left\{ |abc\rangle, \frac{1}{\sqrt{2}}(|bac\rangle + |acb\rangle), \frac{1}{\sqrt{2}}(-|bac\rangle + |cba\rangle) \right\}. \tag{16}$$

Tarjan's algorithm leads to $|\Xi\rangle = \frac{1}{2}(-|bac\rangle + |bac\rangle - |abc\rangle + |acb\rangle)$, so $|\rho_{\text{soc}}\rangle = \frac{1}{\sqrt{2}}(|abc\rangle - |acb\rangle)$. Though keeping the undesired $|bac\rangle$ in $|\psi_3\rangle$ contradicts our intuitions, interfering this superposition with the other votes eliminates *bac* from society's possible classical preferences.[5]

[4] Because QMR2 is defined on just pure states and does not preserve all inputs' norms, QMR2 does not satisfy Definition 1. QMR2 can be thought of as an extension of quantum constitutions.

[5] Alternatively, Voter 3 could submit a superposition of $|abc\rangle$ and $|acb\rangle$.

B. Three entanglement-dependent voting tactics

As the QMR algorithm eliminates entanglement, we define the entanglement-preserving variation *QMR3*. First, every ρ_i is measured in the preference basis. The outcomes form a list L of classical preferences. If most of the preferences are identical—say, if most equal γ—γ becomes society's classical preference. If no majority favors any γ, the constitution randomly chooses from amongst the classical preferences that appear with the highest frequency in L.

Entanglement can help one voter obstruct another. Suppose that the Supreme Court justices vote via QMR3. Suppose that Justice Alice wants to diminish Justice Bob's influence. However Bob votes, Alice should vote oppositely. Alice should entangle her quantum preference with Bob's; if Bob votes as in Eq. (11), Alice should form[6]

$$\sum_{\gamma}(c_\gamma|\gamma\,\bar{\gamma}\rangle + c_{\bar{\gamma}}|\bar{\gamma}\,\gamma\rangle). \tag{17}$$

Insofar as γ represents Bob's preference, Alice votes oppositely, with $\bar{\gamma}$. Even if Bob changes his mind seconds before everyone votes, Alice need not scramble to alter her vote.

Entanglement also facilitates party-line voting, if society uses QMR3. Suppose that Alice leads the Scientists' Party, to which Bob and Charlie belong. However Alice votes, Bob and Charlie wish to vote identically. The voters should form the entangled state

$$\sum_{\gamma}(c_\gamma|\gamma\,\gamma\,\gamma\rangle + c_{\bar{\gamma}}|\bar{\gamma}\,\bar{\gamma}\,\bar{\gamma}\rangle), \tag{18}$$

whose weights Alice chooses. This state generalizes the GHZ state: If the weights equal each other and only two candidates run, Expression (18) reduces to $\frac{1}{\sqrt{2}}(|\alpha\alpha\alpha\rangle + |\bar{\alpha}\bar{\alpha}\bar{\alpha}\rangle)$.

Finally, entangling voters' quantum preferences can pare down society's possible classical preferences. Suppose that Alice, Bob, and Charlie separately favor α twice as much as they prefer β. Each voter plans to submit $\sqrt{\frac{2}{3}}\,|\alpha\rangle + \sqrt{\frac{1}{3}}\,|\beta\rangle$, so society's joint state is

$$\begin{aligned}|\sigma_{\text{soc}}\rangle_{\text{prod}} = \left(\frac{2}{3}\right)^{3/2}|\alpha\alpha\alpha\rangle + \left(\frac{1}{3}\right)^{3/2}|\beta\beta\beta\rangle + \frac{2}{3^{3/2}}(|\beta\alpha\alpha\rangle + |\alpha\beta\alpha\rangle + |\alpha\alpha\beta\rangle)\\ + \frac{\sqrt{2}}{3^{3/2}}(|\alpha\beta\beta\rangle + |\beta\alpha\beta\rangle + |\beta\beta\alpha\rangle).\end{aligned} \tag{19}$$

If the constitution is QMR3, society might adopt α or β as its classical preference.

Suppose that Alice, Bob, and Charlie misunderstand entanglement. Eve can take advantage of their ignorance to eliminate β from society's possible classical preferences. Suppose that Eve convinces the three citizens to submit

$$|\sigma_{\text{soc}}\rangle_{\text{ent}} = \frac{1}{\sqrt{3}}(|\beta\alpha\alpha\rangle + |\alpha\beta\alpha\rangle + |\alpha\alpha\beta\rangle). \tag{20}$$

This entangled analog of $|\sigma_{\text{soc}}\rangle_{\text{prod}}$, Eve might claim, represents the voters' opinion by containing twice as many α's as β's. But QMR3 cannot map $|\sigma_{\text{soc}}\rangle_{\text{ent}}$ to β. Entangled states lead to different possible election outcomes than product states. Equation (20), we note, generalizes the W state: If only two candidates run, $|\sigma_{\text{soc}}\rangle_{\text{ent}}$ reduces to $\frac{1}{\sqrt{3}}(|\beta\alpha\alpha\rangle + |\alpha\beta\alpha\rangle + |\alpha\alpha\beta\rangle)$.

[6] Given how opinionated Supreme Court justices are, Bob might not mind broadcasting his quantum preference to Alice.

V. CONCLUSIONS

We have quantized elections in the tradition of quantum game theory. The quantization obviates a quantum analog of Arrow's Theorem about the impossibility of a nondictatorship's having three common properties. Superpositions, interference, and entanglement expand voters' arsenals of manipulation tactics. Whether other quantum tactics unavailable to ordinary voters exist merits investigation. So does whether monogamy of entanglement [13] limits one voter's influence on others' quantum preferences. If creating entanglement is difficult (as in many labs), the resource theory of multipartite entanglement [14] might illuminate how voters can optimize their influence. Other voting schemes could be quantized, such as proportional representation (in which the percentage of voters who favor Party a dictates the number of government seats Party a wins) and cardinal voting (in which voters grade candidates).

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Appendix A: Classical limit

Our quantum voting scheme differs from classical schemes for the following reasons. Quantum voters can superpose voting tactics, like prisoners who superpose tactics in the quantum Prisoner's Dilemma. Quantum voters can submit mixtures of classical preferences, whereas classical voters effectively submit $\mathcal{B}_i$ elements. (Superpositions and mixtures may appeal to classical readers who agonize over choosing single classical preferences on Voting Day.) Voters can entangle quantum preferences, as discussed in Sec. IV. Quantum constitutions can output superpositions and mixtures of classical preferences. Finally, preference-basis measurements replace the determinism of classical voting with probabilistic maps.

Elections could be quantized in many ways. The definitions in Sections I and II have merit because they depart minimally from their classical analogs. They depart, furthermore, only because (i) the classical definitions lack meaning in the context of superpositions and entanglement and (ii) quantum voting is modeled by a general quantum process, which consists of a preparation procedure, an evolution, and a measurement.

We define as the *classical limit* any context in which every ρ_i is an element of tje [refer-emce basos $\mathcal{B}_i$. If ρ_{soc} is a $\mathcal{B}_{\text{soc}}$ element, society's classical preference is determined (is not probabilistic). Even in the classical limit, ρ_{soc} may be a superposition of $\mathcal{B}_{\text{soc}}$ elements or may be mixed. QMR on cyclic inputs does not reduce to classical majority rule. For this reason, QMR can violate the Quantum Arrow Conjecture.

Appendix B: Properties of Quantum Majority Rule

QMR satisfies the three postulates in the Quantum Arrow Conjecture.

Lemma 1. *The QMR constitution $\mathcal{E}$ respects quantum transitivity, quantum unanimity, and QIIA.*

Proof. By the definition of "quantum constitution," $\mathcal{E}$ outputs a ρ_{soc} that collapses to a preference-basis element under a measurement of $\mathcal{B}_{\text{soc}}$. The $\mathcal{B}_{\text{soc}}$ element corresponds to a classical preference, which obeys classical transitivity by definition. Hence $\mathcal{E}$ obeys quantum transitivity. Step (2f) of Definition 6 ensures that QMR respects quantum unanimity. Because $\mathcal{E}$ constructs ρ_{soc} from the pairwise preferences calculated in Step (2b), QMR respects QIIA. □

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