# Some Results in the Theory of Crosstalk-Free Transmultiplexers

R. David Koilpillai, *Member, IEEE*, Truong Q. Nguyen, *Member, IEEE*, and P. P. Vaidyanathan, *Fellow, IEEE*

***Abstract*—The theory of transmultiplexers involves the design of filters for time domain multiplexing (TDM) ↔ frequency division multiplexing (FDM) interconversion such that the undesirable crosstalk is minimized. The crosstalk-free transmultiplexer (CF-TMUX) focuses on crosstalk cancellation (CC) rather than on suppressing it. In this paper, we present an analysis of the CF-TMUX based on the polyphase component matrices of the filter banks used in TDM → FDM and FDM → TDM conversions, respectively. Thus a necessary and sufficient condition for complete CC is obtained. It is shown that the filters for a CF-TMUX are the same as the filters for a 1-skewed alias-free QMF bank. In addition, if the QMF bank satisfies the perfect reconstruction (PR) property, then the TMUX also satisfies PR. The relation between CF-TMUX filters and alias-free QMF banks is used to obtain a direct design procedure for CF-TMUX filters (both FIR and IIR). It is also shown that we can obtain approximately crosstalk-free TMUX filters from any approximately alias-free QMF bank. Design examples and comparison tables are included.**

## I. Introduction

Transmultiplexers are used for interconversion between the time division multiplexing (TDM) format and the frequency division multiplexing (FDM) format. This topic has received widespread attention and, hence, there exists a considerable amount of literature [1]–[10] covering the theory, design, and implementation details of transmultiplexers. The main problem in transmultiplexers is the leakage of signal from one channel to another in the TDM → FDM → TDM conversion, which is known as crosstalk [1]. The focus of transmultiplexer designs is to minimize the crosstalk.

A schematic of the digital transmultiplexer system is presented in Fig. 1(a). The $M$ input signals are $[x_0(n), x_1(n), \cdots x_{M-1}(n)]$ (which are also the $M$ components of the TDM signal). $[F_0(z), F_1(z), \cdots F_{M-1}(z)]$ are the filters used in TDM → FDM conversion and will be called synthesis filters. The $M$ input signals are interpolated and passed through the synthesis filter bank and combined to

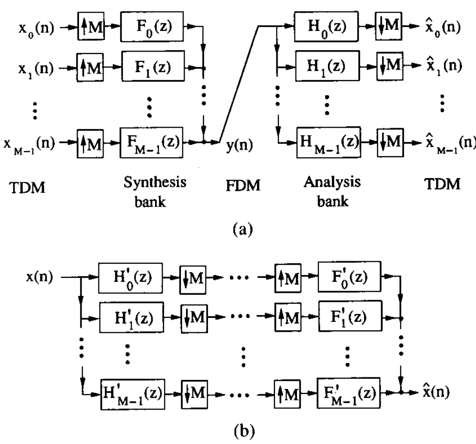


Fig. 1. (a) The transmultiplexer system. (b) The $M$-channel maximally decimated QMF circuit.

produce the FDM signal. At the other end, $[H_0(z), H_1(z), \cdots H_{M-1}(z)]$ are the filters used in FDM → TDM conversion and will be called analysis filters. The FDM signal $y(n)$ is passed through the analysis filter bank and then decimated to get back the TDM signals, $\hat{x}_i(n)$, $0 \leq i \leq M - 1$.

Figs. 2(a), (b) show the frequency spectra of a typical input $x_i(n)$ and the FDM signal $y(n)$, respectively. The voiceband channels are placed adjacent to one another and hence the bandwidth of the FDM signal is equal to the sum of the bandwidths of the component signals. Since the synthesis and analysis filters are nonideal, crosstalk occurs between the different channels of the transmultiplexer. The crosstalk decreases as the transition bandwidth ($\Delta f$) of the channel filters decreases and as their stopband attenuation ($A_s$) increases. These are the only handles on the crosstalk that are available in the traditional transmultiplexer designs approaches.

The aim of this paper is to present new results on crosstalk-free transmultiplexers. This novel method, originally presented in [2], [3], focuses on crosstalk cancellation (CC) rather than suppressing it. Using this approach, the crosstalk can be reduced to very low values (in some cases, crosstalk can be completely eliminated) even with filters having nominal values of $\Delta f$ and $A_s$. This will be elaborated more quantitatively in the later sections. As an illustration, consider design 4 in Table I. For filters having the same specifications (and same length), the cross-

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R. D. Koilpillai was with the Department of Electrical Engineering, California Institute of Technology, Pasadena, CA 91125. He is now with the General Electric Corporate R&D Center, Schenectady, NY.

T. Q. Nguyen is with M.I.T. Lincoln Laboratories, Lexington, MA 01801.

P. P. Vaidyanathan is with the Department of Electrical Engineering, California Institute of Technology, Pasadena, CA 91125.

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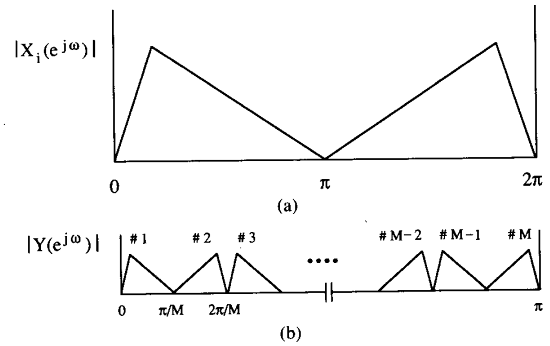


Fig. 2. (a) Spectrum of an input signal $x_i(n)$. (b) Spectrum of the FDM signal $y(n)$.

TABLE I
COMPARISON OF PERFORMANCE OF TRANSMULTIPLEXERS DESIGNED BY THE CC METHOD AND THE TRADITIONAL METHOD (EACH DESIGN WITH DIFFERENT $N$, $A_s$)

| 24-Channel TMUX | | | | |
|---|---|---|---|---|
| Prototype | | | $e_{max}$ | |
| Length $N$ | $A_s$ (dB) | $\omega_s$ (rads) | New CC Method | Traditional Method |
| 48 | 19.64 | $0.0448\pi$ | 6.582 E-06 | 3.895 E-03 |
| 96 | 27.69 | $0.0430\pi$ | 1.329 E-06 | 2.846 E-03 |
| 144 | 38.87 | $0.0406\pi$ | 1.115 E-07 | 2.114 E-03 |
| 192 | 47.00 | $0.0400\pi$ | 3.338 E-08 | 1.932 E-03 |

talk error (defined in Section IV) using the traditional design approach is 1.932 E-03 (54.3 dB) whereas with the crosstalk cancellation (CC) approach, it is 3.338 E-08 (149.5 dB), which is an improvement of 95.2 dB. In concept, the CC approach may be compared to the QMF solution to the subband coding problem, which focuses on aliasing cancellation rather than on suppressing it.

In [3], a necessary and sufficient condition for crosstalk-free transmultiplexers was presented. The condition was obtained in terms of the analysis and synthesis filters of the transmultiplexer. In this paper we present the derivation of an equivalent necessary and sufficient condition, based on the polyphase component matrices of the analysis and synthesis filters. This approach will throw additional light on the understanding of the problem. It also provides a direct method of designing crosstalk-free transmultiplexer filters by starting from an arbitrary, alias-free QMF bank. We will also show that approximately crosstalk-free transmultiplexer filters can be obtained from a QMF bank in which the alias cancellation condition is only approximately satisfied.

The main focus of this approach to the transmultiplexer problem is 1) crosstalk cancellation (CC), and 2) elimination of amplitude and phase distortions, i.e., exact recovery of the signals in TDM → FDM → TDM conversion.

If both the above conditions are satisfied, it will be called perfect reconstruction transmultiplexer (PR-TMUX). If only the first condition is satisfied, then it will be called crosstalk-free transmultiplexer (CF-TMUX). In both of the above cases, crosstalk is canceled completely. This should, however, be contrasted with the traditional approaches to transmultiplexer design, which aim to suppress crosstalk and, hence, there is always residual crosstalk. We will often refer to the following QMF abbreviations, viz., PR-QMF for perfect reconstruction QMF and AF-QMF for alias-free QMF. Since both the transmultiplexer circuit and the QMF circuit involve their respective analysis and synthesis filters, the filters and matrices associated with the QMF circuit always have a prime notation associated with them (as in $H'(z)$), while the filters and matrices associated with the transmultiplexer do not. Further, $\{H_i(z), F_i(z)\}$ is an abbreviation for "transmultiplexer with $M$ analysis filters $H_i(z)$ and $M$ synthesis filters $F_i(z)$, $0 \leq i \leq M-1$." Similarly, $\{H_i'(z), F_i'(z)\}$ refer to the analysis and synthesis filters of the QMF circuit of Fig. 1(b).

In Section II, an analysis of the transmultiplexer circuit, based on the polyphase component matrices of the analysis (FDM → TDM) and synthesis (TDM → FDM) filters, is presented. This formulation and subsequent simplifications help to bring out the fact that the above transmultiplexer circuit, if considered as a MIMO system, is linear time invariant (LTI), even though there are time-varying components such as decimators and interpolators in the circuit. Based on this framework, Lemma 1, which gives a necessary and sufficient condition for CC, is presented in Section II-B. In the next subsection, it is shown in Lemma 2 that we can always obtain a CF-TMUX from a 1-skewed AF-QMF bank. This gives a design procedure for CF-TMUX filters based on the design of AF-QMF banks, thereby utilizing the extensive results available in the areas of AF-QMF and PR-QMF designs. Then, the main result of this section is presented in Lemma 3, which establishes the relation between CF-TMUX filters and 1-skewed AF-QMF banks and, hence, is a stronger result than Lemma 2.

Section II-D contains a brief derivation of the necessary and sufficient condition for CC obtained in [3]. Then, in Section II-E, fact 3 is used to show the equivalence between this result and the necessary and sufficient condition given in Lemma 3. Further, in [2] it was observed that filters (designed by using the pseudo-QMF theory) which satisfy the AC condition approximately, can be used in the design of a TMUX that is approximately crosstalk free. A formal justification for the exact condition under which this result holds is presented in Section III. It is also shown that we can obtain approximately CF-TMUX filters from any approximately AF-QMF bank. In Section IV, a detailed comparison of the performance of transmultiplexers (designed by both methods—the CC approach and the traditional method) is given.

Bold faced letters indicate vectors and matrices. Superscript $T$ and $^\dagger$ denote transposition and transposed conjugation, respectively. The tilde accent on a function $\boldsymbol{F}(z)$ is defined such that, $\tilde{\boldsymbol{F}}(z) = \boldsymbol{F}_*^T(z^{-1})$, $\forall z$, where the subscript asterisk (*) denotes the conjugation of coefficients.

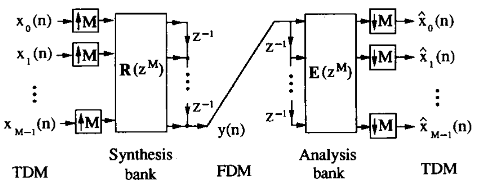

Fig. 3. The polyphase representation of the transmultiplexer system in Fig. 1.

## II. Transmultiplexer Analysis

### A. Simplified Equivalent of the Transmultiplexer

A schematic of the digital transmultiplexer is given in Fig. 1(a). Using the polyphase decompositions of types 1 and 2 [11], [12], we can express the analysis and synthesis filter of the transmultiplexer circuit as

$$H_k(z) = \sum_{l=0}^{M-1} z^{-l} E_{k,l}(z^M), \qquad 0 \le k, l \le M-1 \tag{1}$$

$$F_k(z) = \sum_{l=0}^{M-1} z^{-(M-1-l)} R_{l,k}(z^M), \qquad 0 \le k, l \le M-1. \tag{2}$$

Let $\boldsymbol{E}(z)$ and $\boldsymbol{R}(z)$ be the polyphase component matrices of the analysis and synthesis filter banks, respectively. The elements of the $M \times M$ matrices $\boldsymbol{E}(z)$ and $\boldsymbol{R}(z)$ are given by $[\boldsymbol{E}(z)]_{k,l} = E_{k,l}(z)$ and $[\boldsymbol{R}(z)]_{k,l} = R_{k,l}(z)$, where $0 \le k, l \le M-1$. Representing the analysis and synthesis filter banks in terms of their respective polyphase component matrices, we get Fig. 3. Then, applying the standard identities of multirate signal processing [13], the interpolators and decimators can be moved appropriately to yield Fig. 4. This structure can be further simplified in view of the readily verifiable fact given below.

*Fact 1:* If an input signal $u(n)$ is passed through an interpolator, a delay of $k$ units and a decimator as shown in Fig. 5, then in the Z-transform domain the output can be expressed in terms of the input as

$$V(z) = \begin{cases} 0, & \text{if } k \ne \text{multiple of } M \\ z^{-k/M} U(z), & \text{if } k \text{ is a multiple of } M. \end{cases} \tag{3}$$

Applying this result in Fig. 4, we obtain Fig. 6 which is a simplified equivalent representation of the transmultiplexer system. It is important to note that this is a linear time invariant (but multi-input, multioutput) system, even though time-varying components such as decimators and interpolators are present in the original representation (Fig. 1(a)). This equivalent circuit will be used throughout this paper.

It adds insight to note that the FDM signal $y(n)$ can be considered as a "time-multiplexed version" of the signals $y_i(n)$ in Fig. 4, since the interpolators and the delay chain on the synthesis side implement a time domain multiplexer. On the other hand, the delay chain and decimators on the analysis side implement a time domain demultiplexer.

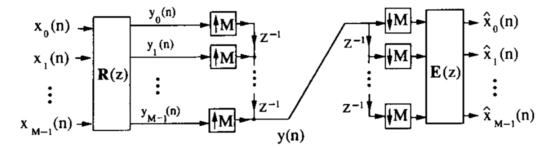
Fig. 4. Equivalent structure for the transmultiplexer system in Fig. 3.

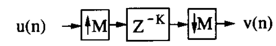
Fig. 5. A circuit with an interpolator, a delay, and a decimator.

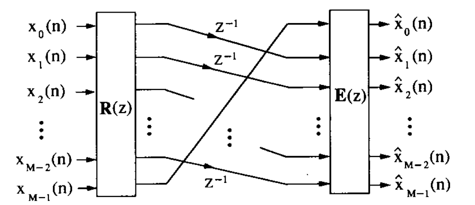
Fig. 6. The simplified equivalent representation of the transmultiplexer system in Fig. 1.

### B. Necessary and Sufficient Condition for CC

From Fig. 6, we can write

$$\begin{bmatrix} \hat{X}_0(z) \\ \hat{X}_1(z) \\ \vdots \\ \hat{X}_{M-1}(z) \end{bmatrix} = \boldsymbol{E}(z) \begin{bmatrix} 0 & 1 \\ z^{-1}\boldsymbol{I}_{M-1} & 0 \end{bmatrix} \boldsymbol{R}(z) \begin{bmatrix} X_0(z) \\ X_1(z) \\ \vdots \\ X_{M-1}(z) \end{bmatrix}. \tag{4}$$

To cancel crosstalk, it is evidently necessary and sufficient that

$$\boldsymbol{E}(z) \begin{bmatrix} \boldsymbol{0} & 1 \\ z^{-1}\boldsymbol{I}_{M-1} & \boldsymbol{0} \end{bmatrix} \boldsymbol{R}(z) = \boldsymbol{C}\boldsymbol{T}(z) \tag{5}$$

where $\boldsymbol{T}(z) = \text{diag}\,[T_0(z), T_1(z), \cdots T_{M-1}(z)]$ (with $T_i(z)$, $0 \le i \le M-1$, being stable transfer functions) and $\boldsymbol{C}$ is an arbitrary permutation matrix. If (5) is satisfied, we get a CF-TMUX. To simplify the notation, we shall restrict our attention to the case when $\boldsymbol{C} = \boldsymbol{I}_M$, the identity matrix.

A special case of CF-TMUX (which has all $T_i(z)$ equal) is obtained when $\boldsymbol{T}(z) = S(z)\boldsymbol{I}_M$. In this case (5) becomes

$$\boldsymbol{E}(z) \begin{bmatrix} \boldsymbol{0} & 1 \\ z^{-1}\boldsymbol{I}_{M-1} & \boldsymbol{0} \end{bmatrix} \boldsymbol{R}(z) = S(z)\boldsymbol{I}_M. \tag{6}$$

The condition on $\boldsymbol{R}(z)$ in terms of $\boldsymbol{E}(z)$ in order to achieve CC is

$$\boldsymbol{R}(z) = S(z) \begin{bmatrix} \boldsymbol{0} & z\boldsymbol{I}_{M-1} \\ 1 & \boldsymbol{0} \end{bmatrix} \boldsymbol{E}^{-1}(z) \tag{7}$$

provided that $\boldsymbol{E}^{-1}(z)$ is stable. From (7) we see that the elements of $\boldsymbol{R}(z)$ may not be FIR even if the elements of

$E(z)$ are FIR, unless the determinant of $E(z)$ is a delay.[1] Multiplying both sides of (7) by $E(z)$, we get a necessary and sufficient condition for a CF-TMUX (which has all $T_i(z)$ equal)

$$P(z) \triangleq R(z)E(z) = S(z)\begin{bmatrix} 0 & zI_{M-1} \\ 1 & 0 \end{bmatrix} = zS(z)\begin{bmatrix} 0 & I_{M-1} \\ z^{-1} & 0 \end{bmatrix}. \quad (8)$$

If $S(z)$ is a pure delay, then the CF-TMUX achieves perfect reconstruction (PR). So from (8) we can write the necessary and sufficient condition for a PR-TMUX as

$$P(z) = z^{-k}\begin{bmatrix} 0 & zI_{M-1} \\ 1 & 0 \end{bmatrix} = z^{-(k-1)}\begin{bmatrix} 0 & I_{M-1} \\ z^{-1} & 0 \end{bmatrix} \quad (9)$$

where $k$ is a nonnegative integer. We can summarize these results in the following lemma.

*Lemma 1:* Let $E(z)$ and $R(z)$ represent the polyphase component matrices of the transmultiplexer filters $\{H_i(z), F_i(z)\}$ (obtained according to (1), (2)). The necessary and sufficient condition for $\{H_i(z), F_i(z)\}$ to yield a CF-TMUX, with all $T_i(z)$ equal, is that the matrix $P(z)[= R(z)E(z)]$ should satisfy (8) for some stable, scalar system $S(z)$. The same filters yield a PR-TMUX if $S(z)$ in (8) is a pure delay.

### *C. Relation Between CF-TMUX Filters and AF-QMF Banks*

Let $E'(z)$ and $R'(z)$ be the polyphase component matrices of the analysis and synthesis filter banks of an $M$-channel, maximally decimated QMF circuit $\{H'_i(z), F'_i(z)\}$, shown in Fig. 1(b). From the results in [12], [16], we know that the QMF circuit is alias free (AF) if and only if the matrix $P'(z)$, defined as $P'(z) \triangleq R'(z)E'(z)$, is a pseudocirculant matrix. A special case of a pseudocirculant matrix is obtained if $P'(z)$ has the following form:

$$P'(z) = S'(z)\begin{bmatrix} 0 & I_{M-n} \\ z^{-1}I_n & 0 \end{bmatrix}, \qquad 0 \le n < M \quad (10)$$

where $S'(z)$ is a stable, scalar system. The overall transfer function $T(z)$ (also called the distortion function) of the AF-QMF bank $\{H'_i(z), F'_i(z)\}$ is given by

$$T(z) = z^{-(M-1)}z^{-n}S'(z^M). \quad (11)$$

If $S'(z)$ in (11) is a pure delay, then the AF-QMF bank satisfies perfect reconstruction (PR). In that case it is called a PR-QMF bank. This result is also contained in Lemma 3.2 [14], which states that the necessary and sufficient condition for a PR-QMF bank is that the matrix $P'(z)$ should have the form

$$P'(z) = z^{-k_1}\begin{bmatrix} 0 & I_{M-n} \\ z^{-1}I_n & 0 \end{bmatrix} \quad (12)$$

where $k_1$ is a nonnegative integer and $0 \le n \le M - 1$.

If a QMF bank satisfies (10) with $n = 0$, it will be called a standard AF-QMF bank, whereas if it satisfies (10) with $n \neq 0$, it will be called an $n$-skewed AF-QMF bank. From (11), the distortion function of a standard AF-QMF bank is given by $T(z) = z^{-(M-1)}S'(z^M)$. The following fact relates standard AF-QMF banks and $n$-skewed AF-QMF banks.

*Fact 2:* An $n$-skewed AF-QMF bank $\{H''_i(z), F''_i(z)\}$ is always obtainable from a standard AF-QMF bank $\{H'_i(z), F'_i(z)\}$ by choosing the filters as $H''_i(z) = H'_i(z)$ and $F''_i(z) = z^{-n}F'_i(z)$, $0 \le i \le M - 1$.

*Proof:* Since $\{H'_i(z), F'_i(z)\}$ is a standard AF-QMF bank, we know that $P'(z) = R'(z)E'(z) = S'(z)I_M$. We want to obtain $\{H''_i(z), F''_i(z)\}$ such that

$$P''(z) = R''(z)E''(z) = S'(z)\begin{bmatrix} 0 & I_{M-n} \\ z^{-1}I_n & 0 \end{bmatrix}. \quad (13)$$

Since $H''_i(z) = H'_i(z)$, $\forall i$, we have $E''(z) = E'(z)$. Comparing the expressions for $R'(z)$ and $R''(z)$, we get

$$R''(z) = \begin{bmatrix} 0 & I_{M-n} \\ z^{-1}I_n & 0 \end{bmatrix} R'(z). \quad (14)$$

The synthesis filter bank $f'(z)$ corresponding to $R'(z)$ is

$$f'(z) \triangleq [F'_0(z) \quad F'_1(z) \quad \cdots \quad F'_{M-1}(z)]^T = R'^T(z^M)e(z) \quad (15)$$

where $e(z) = [z^{-(M-1)} \quad z^{-(M-2)} \quad \cdots \quad 1]^T$. Similarly we can write

$$f''(z) = R''^T(z^M)e(z) \quad (16)$$

$$= R'^T(z^M)\begin{bmatrix} 0 & z^{-M}I_n \\ I_{M-n} & 0 \end{bmatrix} e(z)$$

$$= z^{-n}R'^T(z^M)e(z) = z^{-n}f'(z) \quad (17)$$

proving that $F''_i(z) = z^{-n}F'_i(z)$. □

Note that to obtain the above result, we used the choice $H''_i(z) = H'_i(z)$, $F''_i(z) = z^{-n}F'_i(z)$. It can be readily verified that the choice $H''_i(z) = z^{-n}H'_i(z)$, $F''_i(z) = F'_i(z)$ will also enable us to obtain an $n$-skewed AF-QMF bank from a standard AF-QMF bank.

Comparing (8) and (10), we see that the CF-TMUX filters and the filters of a 1-skewed AF-QMF bank satisfy the same condition. This enables us to establish the relation between them, as summarized in the following lemma.

*Lemma 2:* Let $\{H'_i(z), F'_i(z)\}$ represent a 1-skewed AF-QMF bank. Define $H_i(z) = H'_i(z)$, $F_i(z) = F'_i(z)$, $\forall i$. Then $\{H_i(z), F_i(z)\}$ represents a CF-TMUX.

*Proof:* By definition, the matrix $P'(z)$ of the 1-skewed AF-QMF bank $\{H'_i(z), F'_i(z)\}$ satisfies

[1] A special case of this type of polynomial matrices (which have been extensively studied in the QMF context [14], [15]) is when $E(z)$ satisfies the property $\tilde{E}(z)E(z) = cI_M$, where $c > 0$. Then, $E(z)$ is called paraunitary and we can express (7) as

$$R(z) = \frac{1}{c}S(z)\begin{bmatrix} 0 & zI_{M-1} \\ 1 & 0 \end{bmatrix}\tilde{E}(z).$$

$$\boldsymbol{P}'(z) = \boldsymbol{R}'(z)\boldsymbol{E}'(z) = S'(z)\begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_{M-1} \\ z^{-1} & \boldsymbol{0} \end{bmatrix} \tag{18}$$

So, for the TMUX filters, the matrix $\boldsymbol{P}(z)$ satisfies $\boldsymbol{P}(z) = \boldsymbol{R}(z)\boldsymbol{E}(z) = \boldsymbol{P}'(z)$. Hence, from (18), we can write

$$\boldsymbol{R}(z) = S'(z)\begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_{M-1} \\ z^{-1} & \boldsymbol{0} \end{bmatrix}\boldsymbol{E}^{-1}(z). \tag{19}$$

Using (19) in (4), we get $\hat{X}_i(z) = z^{-1}S'(z)X_i(z)$, $\forall i$. Thus, $\{H_i(z), F_i(z)\}$ represents a CF-TMUX in which all the $T_i(z)$ are equal and are given by $T_i(z) = z^{-1}S'(z)$, $\forall i$. □

*1) Design Procedure for CF-TMUX Filters:* The above result highlights the close relation between CF-TMUX filters and the filters of 1-skewed AF-QMF banks. Hence, it yields a design procedure for CF-TMUX filters, starting with an arbitrary AF-QMF bank. This is an advantage because the design of AF-QMF banks is well known. The design steps are as follows:

1) Design an AF-QMF bank $\{H_i'(z), F_i'(z)\}$.

2) Using fact 2, obtain a 1-skewed AF-QMF bank (by inserting appropriate delays).

3) Choose the CF-TMUX filters to be the same as the 1-skewed AF-QMF bank (as given in Lemma 2).

Since PR-QMF banks are a subset of the class of AF-QMF banks, the above results are valid for PR-QMF banks also. In particular, if we start the design with a PR-QMF bank, then we obtain a PR-TMUX. Extensive work has been done in the area of designing PR-QMF banks [14]–[19] and these results can be fully used in the design of PR-TMUX filters.

*2) Comment on IIR Designs:* From Lemma 2 we see that if we start from a 1-skewed AF-QMF bank with distortion function $T(z) = z^{-M}S'(z^M)$, then we can obtain the filters for a CF-TMUX such that the distortion function in each channel is $T_i(z) = z^{-1}S'(z)$, $\forall i$. In particular, this means that if the QMF bank is free from amplitude distortion, then, so is the transmultiplexer. For example, the techniques described in [18] and [20] show two methods to obtain IIR QMF banks that are free from aliasing and amplitude distortions. (In both cases, $T(z)$ is an all-pass function). By Lemma 2, we can obtain IIR analysis and synthesis filters for a CF-TMUX with no amplitude distortion. This emphasizes the fact that the CC results derived earlier are valid both in the FIR case and in the IIR case.

Lemma 2 shows that if we have a 1-skewed AF-QMF bank, then we can always obtain a CF-TMUX. However, this result is further strengthened by the next lemma, which establishes the relation between CF-TMUX filters and the filters of a AF-QMF bank.

*Lemma 3:* Let $\{H_i'(z), F_i'(z)\}$ represent an AF-QMF bank. If the TMUX filters, $\{H_i(z), F_i(z)\}$, are chosen such that $H_i(z) = H_i'(z)$ and $F_i(z) = F_i'(z)$, then the TMUX is crosstalk free with all $T_i(z)$ equal if and only if $\{H_i'(z), F_i'(z)\}$ is a 1-skewed AF-QMF bank.

*Proof:* Let $\boldsymbol{E}(z)$, $\boldsymbol{R}(z)$ be the polyphase component matrices of the TMUX and $\boldsymbol{E}'(z)$, $\boldsymbol{R}'(z)$ be the polyphase components of the AF-QMF bank. By choice of TMUX filters

$$\boldsymbol{E}(z) = \boldsymbol{E}'(z) \quad \text{and } \boldsymbol{R}(z) = \boldsymbol{R}'(z). \tag{20}$$

From (8) we have

"TMUX is crosstalk free and all $T_i(z)$ are equal"

$$\Leftrightarrow \boldsymbol{P}(z) = \boldsymbol{R}(z)\boldsymbol{E}(z) = S'(z)\begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_{M-1} \\ z^{-1} & \boldsymbol{0} \end{bmatrix} \tag{21}$$

$$\Leftrightarrow \boldsymbol{P}'(z) = \boldsymbol{R}'(z)\boldsymbol{E}'(z) = S'(z)\begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_{M-1} \\ z^{-1} & \boldsymbol{0} \end{bmatrix}, \quad \text{(by using (20))} \tag{22}$$

$$\Leftrightarrow \{H_i'(z), F_i'(z)\} \text{ is a 1-skewed AF-QMF bank, (from (10), by definition).} \tag{23}$$

□

### D. Alternate Derivation of the CC Condition (Vetterli)

The main result on transmultiplexers in [2], [3] is the necessary and sufficient condition under which a given set of filters can be used to obtain a AF-QMF bank as well as a CF-TMUX (i.e., the filters will simultaneously satisfy AC and CC). This result is derived in terms of the analysis and synthesis filters. In this subsection, we rederive the same result for two reasons—first, to prove (in fact 3) the equivalence between this result and the result of Lemma 3 and, second, to develop the framework based on the analysis/synthesis filters, which will be used to extend the CC results to the case of approximate CC, which is presented in Section III.

Let $\{H_i(z), F_i(z)\}$ be a set of TMUX filters. In Fig. 1(a), the FDM signal $y(n)$ can be expressed as

$$Y(z) = \sum_{i=0}^{M-1} F_i(z)X_i(z^M). \tag{24}$$

After the FDM → TDM conversion, we have

$$\hat{X}_i(z) = \frac{1}{M}\sum_{l=0}^{M-1} H_i(z^{1/M}W^l)Y(z^{1/M}W^l), \quad 0 \le i \le M-1 \tag{25}$$

where $W = e^{-j(2\pi/M)}$. Rewriting (25) in matrix form we get

$$\begin{bmatrix} \hat{X}_0(z) \\ \hat{X}_1(z) \\ \vdots \\ \hat{X}_{M-1}(z) \end{bmatrix} = \frac{1}{M}\begin{bmatrix} H_0(z^{1/M}) & H_0(z^{1/M}W) & \cdots & H_0(z^{1/M}W^{M-1}) \\ H_1(z^{1/M}) & H_1(z^{1/M}W) & \cdots & H_1(z^{1/M}W^{M-1}) \\ \vdots & \vdots & \ddots & \vdots \\ H_{M-1}(z^{1/M}) & H_{M-1}(z^{1/M}W) & \cdots & H_{M-1}(z^{1/M}W^{M-1}) \end{bmatrix}\begin{bmatrix} Y(z^{1/M}) \\ Y(z^{1/M}W) \\ \vdots \\ Y(z^{1/M}W^{M-1}) \end{bmatrix}. \tag{26}$$

Expressing the FDM signal $u(n)$ in terms of the inputs

$$\begin{bmatrix} \hat{X}_0(z) \\ \hat{X}_1(z) \\ \vdots \\ \hat{X}_{M-1}(z) \end{bmatrix} = \frac{1}{M} \boldsymbol{H}^T(z^{1/M})\boldsymbol{F}^T(z^{1/M}) \begin{bmatrix} X_0(z) \\ X_1(z) \\ \vdots \\ X_{M-1}(z) \end{bmatrix} \quad (27)$$

where

$$\boldsymbol{H}^T(z^{1/M}) = \begin{bmatrix} H_0(z^{1/M}) & H_0(z^{1/M}W) & \cdots & H_0(z^{1/M}W^{M-1}) \\ H_1(z^{1/M}) & H_1(z^{1/M}W) & \cdots & H_1(z^{1/M}W^{M-1}) \\ \vdots & \vdots & \ddots & \vdots \\ H_{M-1}(z^{1/M}) & H_{M-1}(z^{1/M}W) & \cdots & H_{M-1}(z^{1/M}W^{M-1}) \end{bmatrix}$$

and

$$\boldsymbol{F}^T(z^{1/M}) = \begin{bmatrix} F_0(z^{1/M}) & F_1(z^{1/M}) & \cdots & F_{M-1}(z^{1/M}) \\ F_0(z^{1/M}W) & F_1(z^{1/M}W) & \cdots & F_{M-1}(z^{1/M}W) \\ \vdots & \vdots & \ddots & \vdots \\ F_0(z^{1/M}W^{M-1}) & F_1(z^{1/M}W^{M-1}) & \cdots & F_{M-1}(z^{1/M}W^{M-1}) \end{bmatrix}.$$

In (27), for CC we need

$$\boldsymbol{H}^T(z^{1/M})\boldsymbol{F}^T(z^{1/M}) = \text{diag}\,[T_0(z) \quad T_1(z) \quad \cdots \quad T_{M-1}(z)] \quad (28)$$

where $T_i(z)$, $0 \le i \le M-1$, are stable transfer functions. Equation (28) can be rewritten as

$$\boldsymbol{F}(z)\boldsymbol{H}(z) = \text{diag}\,[T_0(z^M) \quad T_1(z^M) \quad \cdots \quad T_{M-1}(z^M)]. \quad (29)$$

This is the necessary and sufficient condition for CC [2], [3] in terms of the analysis and synthesis filters. From QMF theory [3], [14] we know that the aliasing cancellation (AC) equations (for the choice $H_i'(z) = H_i(z)$ and $F_i'(z) = F_i(z)$, $\forall i$) can be written as

$$\begin{aligned} \boldsymbol{H}(z)\boldsymbol{F}(z) &= \boldsymbol{H}'(z)\boldsymbol{F}'(z) \\ &= \text{diag}\,[T(z) \quad T(zW) \quad \cdots \quad T(zW^{M-1})]. \end{aligned} \quad (30)$$

So, from (29) and (30) AC and CC are simultaneously satisfied if and only if

$$\boldsymbol{F}^{-1}(z) \begin{bmatrix} T_0(z^M) & & & \\ & T_1(z^M) & & \\ & & \ddots & \\ & & & T_{M-1}(z^M) \end{bmatrix} = \begin{bmatrix} T(z) & & & \\ & T(zW) & & \\ & & \ddots & \\ & & & T(zW^{M-1}) \end{bmatrix} \boldsymbol{F}^{-1}(z) \quad (31)$$

i.e., if and only if

$$\begin{bmatrix} T_0(z^M) & & & \\ & T_1(z^M) & & \\ & & \ddots & \\ & & & T_{M-1}(z^M) \end{bmatrix} \boldsymbol{F}(z) = \boldsymbol{F}(z) \begin{bmatrix} T(z) & & & \\ & T(zW) & & \\ & & \ddots & \\ & & & T(zW^{M-1}) \end{bmatrix}. \quad (32)$$

Comparing $j$th column of LHS and RHS of (32), we get the equivalent condition

$$T_i(z^M) = T(z), \qquad 0 \le i \le M-1 \qquad \text{unless } F_{i,j}(z) \equiv 0, \ \forall j \quad (33)$$

where $F_{i,j}(z)$ are the elements of $\boldsymbol{F}(z)$. In summary, "CC and AC are simultaneously satisfied iff $T_i(z^M) = T(z)$, $\forall i$." This is the main result on transmultiplexers which is presented in [2], [3]. The equivalence between the above result and the result of Lemma 3 is shown next.

### *E. Relation Between Vetterli's Result and Lemma 3*

*Fact 3:* Let $\{H_i'(z), F_i'(z)\}$ be an AF-QMF bank. Then the overall transfer function $T(z)$ of the AF-QMF system is a (rational) function of $z^M$ if and only if $\{H_i'(z), F_i'(z)\}$ form a 1-skewed AF-QMF bank.

*Proof:* Since $\{H_i'(z), F_i'(z)\}$ is an AF-QMF bank, the corresponding $\boldsymbol{P}'(z)$ is necessarily a pseudocirculant matrix. From the results in [12], [16], we know that the transfer function of the AF-QMF system (in terms of $P_{i,j}'(z)$, the elements of $\boldsymbol{P}'(z)$) is given by

$$T(z) = \frac{\hat{X}(z)}{X(z)} = z^{-(M-1)} \sum_{j=0}^{M-1} z^{-j} P_{0,j}'(z^M). \quad (34)$$

From (34), we see that "$T(z)$ is a function of $z^M$ iff $P_{0,j}'(z) = 0$, for all $j \neq 1$," and hence

$$P'(z) = P'_{0,1}(z)\begin{bmatrix} \mathbf{0} & I_{M-1} \\ z^{-1} & \mathbf{0} \end{bmatrix} \tag{35}$$

which is of the form in (10) with $n = 1$, thereby proving that $\{H'_i(z), F'_i(z)\}$ is a 1-skewed AF-QMF bank. □

Hence, it can be readily seen that the necessary and sufficient condition in Lemma 3 is equivalent to the one presented in [2], [3].

## III. Approximate Crosstalk Cancellation

A number of papers [21]–[26] deal with the problem of approximate aliasing cancellation (AC) in the subband coding problem. In Section II-C, we presented the relation between AF-QMF banks and CF-TMUX filters. We will now show that there is a similar relation between filters that satisfy approximate aliasing cancellation and TMUX filters that achieve approximate CC. First, we consider pseudo-QMF banks [21]–[25] and then generalize the result to cover all approximately AF-QMF banks.

For the pseudo-QMF bank $\{H'_k(z), F'_k(z)\}$, the approximate AC condition can be expressed as

$$\begin{bmatrix} H'_0(z) & H'_1(z) & \cdots & H'_{M-1}(z) \\ H'_0(zW) & H'_1(zW) & \cdots & H'_{M-1}(zW) \\ \vdots & \vdots & \ddots & \vdots \\ H'_0(zW^{M-1}) & H'_1(zW^{M-1}) & \cdots & H'_{M-1}(zW^{M-1}) \end{bmatrix} \cdot \begin{bmatrix} F'_0(z) \\ F'_1(z) \\ \vdots \\ F'_{M-1}(z) \end{bmatrix} = \begin{bmatrix} T(z) \\ \simeq 0 \\ \vdots \\ \simeq 0 \end{bmatrix}. \tag{36}$$

The aliasing terms are small ($\simeq 0$) but not exactly zero. (36) can also be expressed as

$$H'(z)F'(z) \simeq \text{diag}\,[T(z) \quad T(zW) \quad \cdots \quad T(zW^{M-1})] \tag{37}$$

where the nondiagonal entries in (37) are small but not necessarily zero and the matrices $H'(z)$, $F'(z)$ are as defined in (27).

The pseudo-QMF bank is derived from a low-pass prototype $H'(z)$ (length $= N$) as follows: first, we define $U_k(z)$ and $V_k(z)$, which are complex-modulated versions of $H'(z)$

$$U_k(z) \triangleq c_k H'\left(z \exp\left[-j(2k+1)\frac{\pi}{2M}\right]\right)$$

$$V_k(z) \triangleq c_k^* H'\left(z \exp\left[j(2k+1)\frac{\pi}{2M}\right]\right)$$

$$0 \le k \le M-1 \tag{38}$$

where $c_k = \exp\,[-j(2k+1)(\pi/2M)((N-1)/2)]$. The analysis and synthesis filters are obtained as

$$H_k(z) \triangleq \tfrac{1}{2}[a_k U_k(z) + a_k^* V_k(z)], \qquad 0 \le k \le M-1 \tag{39}$$

$$F_k(z) \triangleq \tfrac{1}{2}[b_k U_k(z) + b_k^* V_k(z)], \qquad 0 \le k \le M-1 \tag{40}$$

where $a_k$'s are unit-magnitude complex constants. After going through the various steps [10], [21], we can express the overall transfer function $T(z)$ of the approximately alias-free system as

$$T(z) = \frac{1}{M}\sum_{k=0}^{M-1} H'_k(z)F'_k(z) \simeq \frac{1}{4M}\sum_{k=0}^{M-1}[U_k^2(z) + V_k^2(z)]. \tag{41}$$

It is shown in the Appendix that $\{z^{(N-1)}T(z)\}$ is approximately a function of $z^{2M}$. Let $p_0$, $p_1$ be defined as $p_0 \triangleq (N-1)_{\text{modulo}\,M}$ and $p_1 \triangleq M - p_0$. If we choose $H''_k = H'_k$ and $F''_k = z^{-p_1}F'_k$, then the overall transfer function $T_1(z) = 1/M \sum_{k=0}^{M-1} H''_k(z)F''_k(z) \simeq$ a function of $z^M$. So the approximate AC condition (37) can be written as

$$H''(z)F''(z) \simeq T_1(z)I_M \tag{42}$$

since $T_1(z)$ is approximately a function of $z^M$. Using fact 3, we can conclude that $\{H''_i(z), F''_i(z)\}$ yield an "approximately" 1-skewed AF-QMF bank. The immediate question that arises is, can we obtain an approximately crosstalk-free transmultiplexer by choosing $H_i(z) = H''_i(z)$ and $F_i(z) = F''_i(z)$?

The answer is in the affirmative as shown next. The term "approximately crosstalk-free" is also made more quantitative.

*Fact 4:* If $H''(z)F''(z) \simeq T_1(z)I_M$, the $F''(z)H''(z) \simeq T_1(z)I_M$.

*Proof:* We will prove this by a continuity argument for the inverse of complex matrices. Let

$$H''(z)F''(z) = T_1(z)I_M + \Delta(z) \tag{43}$$

$$F''(z)H''(z) = T_1(z)I_M + \Gamma(z). \tag{44}$$

Given any $\epsilon > 0$, however small, we will show how to find a $\delta > 0$ such that

$$\text{if } |\Delta_{i,j}| \le \delta, \quad \forall i, j, \quad \text{then, } |\Gamma_{i,j}| \le \epsilon, \quad \forall i, j. \tag{45}$$

From (43), we get

$$F''(z) = T_1(z)[H''(z)]^{-1} + [H''(z)]^{-1}\Delta(z). \tag{46}$$

Let $J(z) = [H''(z)]^{-1}$. So we can write

$$F''(z)H''(z) = T_1(z)I_M + \underbrace{J(z)\Delta(z)H''(z)}_{\Gamma(z)}. \tag{47}$$

From (47)

$$\Gamma_{i,j}(z) = \sum_{k,l} J_{i,k}(z)\Delta_{k,l}(z)H''_{l,j}(z) \tag{48}$$

$$|\Gamma_{i,j}(z)| \le \delta \sum_{k,l} |J_{i,k}(z)|\,|H''_{l,j}(z)|. \tag{49}$$

So, given $\epsilon$, choose

$$\delta \leq \frac{\epsilon}{\sum_{k,l} |J_{i,k}(z)|\, |H''_{l,j}(z)|}. \tag{50}$$

Then, $|\Gamma_{i,j}(z)| \leq \epsilon$, $\forall i, j$, thereby proving the stated fact. □

In summary, the above fact shows that since the QMF bank $\{H''_k(z), F''_k(z)\}$ (obtained from the pseudo-QMF bank $\{H'_k(z), F'_k(z)\}$) satisfies (42), then the transmultiplexer filters chosen as $H_k(z) = H''_k(z)$, $F_k(z) = F''_k(z)$, $\forall k$, satisfy the condition

$$F(z)H(z) = F''(z)H''(z) \simeq T(z)I_M. \tag{51}$$

So we conclude that $\{H_i(z), F_i(z)\}$ yield a transmultiplexer in which the crosstalk terms are negligibly small. So we say that the TMUX is approximately crosstalk free.

The above result pertains to approximately AF-QMF banks designed according to pseudo-QMF theory. This can be generalized to cover all filter banks that satisfy the approximate AC condition as follows: Let $\{H'_i(z), F'_i(z)\}$ be any approximately AF-QMF bank in standard form i.e., $P'(z) = R'(z)E'(z) \simeq S'(z)I_M$. Then we can get approximately CF-TMUX filters by choosing them according to the design procedure in Section II-C.

## IV. Design Comparison

In this section, we compare the performance of 24-channel transmultiplexers designed by the crosstalk cancellation (CC) method and the traditional approach. Let $\{H'_k(z), F'_k(z)\}$ be a 24-channel pseudo-QMF bank derived from a prototype of length $N$. From the definitions in Section III, $p_1 = M - (N-1)_{\text{modulo} M}$. And the choice of transmultiplexer filters $\{F_k(z), H_k(z)\}$ given by

$$H_k(z) = H'_k(z)$$

and

$$F_k(z) = z^{-p_1}F'_k(z), \qquad 0 \leq k \leq 23 \tag{52}$$

yields an approximate CF-TMUX. On the other hand, in traditional TMUX designs (where crosstalk is not canceled using multirate techniques), the analysis and synthesis filters are chosen to have similar specifications. For example, in the choice

$$H_k(z) = H'_k(z)$$

and

$$F_k(z) = F'_k(z) = z^{-(N-1)}\tilde{H}_k(z), \qquad 0 \leq k \leq 23 \tag{53}$$

the analysis and synthesis filters of the TMUX have identical magnitude responses. Here, the crosstalk depends only on the sharpness of the filters (the transition bandwidth) and their stopband attenuation $A_s$.

In order to do the comparison, we define a quantitative measure of the performance of the transmultiplexer. Equation (27) relates the inputs and outputs of the transmultiplexer. We define the transfer function matrix $C(z^M)$ $\triangleq H^T(z)F^T(z)$ where

$$[C(z^M)]_{k,l} = C_{k,l}(z^M) = \frac{\hat{X}_k(z^M)}{X_l(z^M)}, \qquad 0 \leq k, l \leq M-1. \tag{54}$$

For $l \neq k$, $C_{k,l}(z^M)$ gives the crosstalk transfer functions. The total crosstalk error for the $k$th channel is defined as

$$e_k \triangleq \int_0^{\pi/M} \sum_{\substack{l=0 \\ l\neq k}}^{M-1} |C_{k,l}(e^{jM\omega})|^2\, d\omega. \tag{55}$$

The integration is done in the interval $[0, \pi/M]$ since the transfer functions are functions of $z^M$. The maximum crosstalk error is

$$e_{\max} \triangleq \max_{0 \leq k \leq M-1} e_k. \tag{56}$$

Next, we design a 24-channel pseudo-QMF bank with prototype length $N = 96$. Hence, $p_1 = 24 - (95)_{\text{modulo} 24} = 1$ and the approximate CF-TMUX is obtained by choosing the TMUX filters as in (52) (with $p_1 = 1$). For the traditional TMUX approach, the filters are chosen as in (53). For both designs, the maximum crosstalk error (56) is computed. The above steps are repeated with 24-channel TMUX designs with filter lengths $N = 48, 144$, and 192. The maximum crosstalk error for each design (along with the stopband attenuation ($A_s$) and the stopband edge ($\omega_s$) of the respective prototypes) is shown in Table I. In these four design examples, it can be seen that as the filter length increases, the $A_s$ of the filters increases while the transition bandwidth ($\Delta f$) remains approximately constant. In the traditional method, $e_{\max}$ stays relatively same (since it depends on $\Delta f$) while with the CC method, $e_{\max}$ decreases (since it depends mainly on $A_s$).

Next, we have another comparison of TMUX designs with filters of the same length ($N = 192$) but whose prototypes had different $A_s$ and $\Delta f$. These results are shown in Table II. In these design examples, as $A_s$ increases, $\Delta f$ also increases. In the traditional method, as $\Delta f$ increases, $e_{\max}$ also increases (even though $A_s$ is higher) while on the other hand, for the CC method, $e_{\max}$ decreases. From these two tables, it can be seen that the transmultiplexers designed by the CC approach perform consistently better than those designed by the traditional method.

The traditional TMUX designs [1], [5] differ from the approaches discussed in this paper. To elaborate this point, consider the TMUX design in [5]. For the 60 channel TMUX, the voice channels are each restricted to be in the 0.3–3.6 kHz band (by bandpass filtering) and then multiplexed into frequency slots which are 4 kHz wide as shown in Fig. 7 (and the spectral gaps are then utilized for transmitting the signaling information). This, however, differs from the multiplexing scheme considered in Fig. 2(b). Because of this difference, a direct comparison between the method in this paper and the traditional designs is not applicable.

TABLE II
COMPARISON OF PERFORMANCE OF TRANSMULTIPLEXERS DESIGNED BY THE CC METHOD AND THE TRADITIONAL METHOD (EACH DESIGN WITH DIFFERENT $A_s$, $\Delta f$)

| 24-Channel TMUX | | | | |
|---|---|---|---|---|
| Prototype | | | $e_{\max}$ | |
| Length $N$ | $A_s$ (dB) | $\omega_s$ (rads) | New CC Method | Traditional Method |
| 192 | 40.38 | $0.0374\pi$ | 6.173 E-08 | 1.808 E-03 |
| 192 | 47.00 | $0.0400\pi$ | 3.338 E-08 | 1.932 E-03 |
| 192 | 50.32 | $0.0414\pi$ | 1.947 E-08 | 2.002 E-03 |

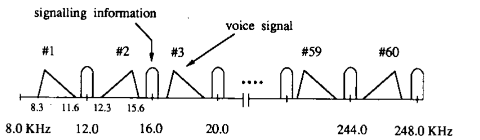


Fig. 7. Conventional 60 channel FDM signal with both voice and signaling information (refer to [5]).

## V. CONCLUSIONS

In this paper, we have presented new results in the theory of crosstalk-free transmultiplexers (CF-TMUX). A necessary and sufficient condition for complete crosstalk cancellation is derived. It is shown that the filters for a CF-TMUX are the same as those for a 1-skewed AF-QMF bank. In addition, if the QMF bank satisfies the perfect reconstruction (PR) property, the TMUX also satisfies PR. The relation between AF-QMF banks and CF-TMUX filters yields a design procedure for CF-TMUX filters. It is also shown that pseudo-QMF banks and other approximately AF-QMF banks can be used to obtain approximately CF-TMUX filters. Lastly, examples to demonstrate the improved performance of transmultiplexers designed by the CC method, over the traditional TMUX designs, are included.

## APPENDIX

To prove that in (41)

$$z^{(N-1)}T(z) = \frac{z^{(N-1)}}{4M} \sum_{k=0}^{M-1} [U_k^2(z) + V_k^2(z)] = \text{a function of } z^{2M}. \tag{A.1}$$

*Proof:* Using (38) in (A.1) we get

$$z^{(N-1)}T(z) = \frac{z^{(N-1)}}{4M} \sum_{k=0}^{M-1} \left[\exp\left[-j(2k+1)\frac{\pi}{2M}(N-1)\right] \cdot H'^2\left(z \exp\left[-j(2k+1)\frac{\pi}{2M}\right]\right) + \exp\left[j(2k+1)\frac{\pi}{2M}(N-1)\right] \cdot H'^2\left(z \exp\left[j(2k+1)\frac{\pi}{2M}\right]\right)\right]. \tag{A.2}$$

Substituting $W_{2M} = e^{-j(2\pi/2M)}$ in (A.2)

$$z^{(N-1)}T(z) = \frac{1}{4M} \sum_{k=0}^{M-1} [(zW_{2M}^{(k+1/2)})^{N-1} H'^2(zW_{2M}^{(k+1/2)}) + (zW_{2M}^{-(k+1/2)})^{N-1} H'^2(zW_{2M}^{-(k+1/2)})]$$
$$= \frac{1}{4M} \sum_{k=0}^{2M-1} (zW_{2M}^{(k+1/2)})^{N-1} H'^2(zW_{2M}^{(k+1/2)}) \tag{A.3}$$

since $W_{2M}^{-(k+1/2)} = W_{2M}^{2M-(k+1/2)} = W_{2M}^{(2M-1-k)+1/2}$. Let

$$A(z) \triangleq (zW_{2M}^{1/2})^{N-1} H'^2(zW_{2M}^{1/2}). \tag{A.4}$$

Expressing (A.3) in terms of $A(z)$ and using the property of $W_{2M}$, we get

$$z^{(N-1)}T(z) = \frac{1}{4M} \sum_{k=0}^{2M-1} A(zW_{2M}^k) = \text{a function of } z^{2M}. \tag{A.5}$$

□

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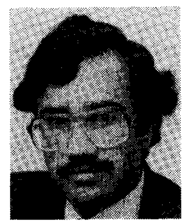
**R. David Koilpillai** (S'83-M'90) was born in Madras, India, on February 18, 1962. He received the B.Tech. degree in electrical engineering from the Indian Institute of Technology, Madras, in 1984 and the M.S. and Ph.D. degrees in electrical engineering from the California Institute of Technology in 1985 and 1991, respectively.

His main research interests are DSP applications in communications, multirate systems, and the theory and design of digital filter banks. He is currently on the staff of the Communications Program at the General Electric Corporate R&D Center, Schenectady, NY.

**Truong Q. Nguyen** (S'85-M'85) was born in Saigon, Vietnam, on November 2, 1962. He received the B.S. (Hons.), M.S. and Ph.D. degrees from the California Institute of Technology in electrical engineering in 1985, 1986, and 1989, respectively.

During the academic year 1986-1987, he was a recipient of a fellowship from Aerojet Dynamics, for advanced studies. He is currently with M.I.T. Lincoln Laboratory. His main research interests are in digital signal processing, multirate filter bank, adaptive array processing, and filter design.

Dr. Nguyen is a member of Tau Beta Pi, Eta Kappa Nu, and Sigma Xi.



**P. P. Vaidyanathan** (S'80-M'83-SM'88-F'91) was born in Calcutta, India, on October 16, 1954. He received the B.Sc. (Hons.) degree in physics and the B.Tech. and M.Tech. degrees in radiophysics and electronics, all from the University of Calcutta, India, in 1974, 1977, and 1979, respectively, and the Ph.D. degree in electrical and computer engineering from the University of California, Santa Barbara, in 1982.

He was a postdoctoral fellow at the University of California, Santa Barbara, from September 1982 to March 1983. In March 1983 he joined the Department of Electrical Engineering of the California Institute of Technology as an Assistant Professor, and since 1988 has been an Associate Professor of electrical engineering there. His main research interests are in digital signal processing, multirate systems, and adaptive filtering.

Dr. Vaidyanathan served as Vice Chairman of the Technical Program Committee for the 1983 IEEE International Symposium on Circuits and Systems, and as an Associate Editor for the IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS for the period 1985-1987. He was a recipient of the Award for Excellence in Teaching at the California Institute of Technology for the year 1983-1984. He also received the NSF's Presidential Young Investigator Award in 1986. In 1989 he received the IEEE ASSP Senior Award for his paper on multirate perfect-reconstruction filter banks. In 1990 he was the recipient of the S. K. Mitra Memorial Award from the Institute of Electronics and Telecommunications Engineers, India, for his joint paper in the *Journal of the IETE*.