

Time reversal invariant gapped boundaries of the double semion state

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The boundary of a fractionalized topological phase can be gapped by condensing a proper set of bosonic quasiparticles. Interestingly, in the presence of a global symmetry, such a boundary can have different symmetry transformation properties. Here we present an explicit example of this kind, in the double semion state with time reversal symmetry. We find two distinct cases where the semionic excitations on the boundary can transform either as time reversal singlets or as time reversal (Kramers) doublets, depending on the coherent phase factor of the Bose condensate. The existence of these two possibilities are demonstrated using both field theory argument and exactly solvable lattice models. Furthermore, we study the domain walls between these two types of gapped boundaries and find that the application of time reversal symmetry tunnels a semion between them.

I. INTRODUCTION

The interplay of topology and symmetry can lead to interesting phenomena in quantum many-body systems. In particular, in the presence of global symmetries, one topological phases can divide into several different phases with the fractional excitations in the system transforming under symmetry in different ways. Much recent effort has been devoted to the classification of such ‘Symmetry Enriched Topological’ (SET) phases by identifying possible ways for the symmetry to act on the fractional excitations^{1–11}. One possibility is for the fractional excitations to carry fractional quantum numbers of the global symmetry. For example, in an electronic system composed of charge e electrons, the fractional excitations in the $\nu = 1/3$ fractional quantum Hall state can carry charge $e/3$. A systematic counting exists for this class of SET phases³ (although it is not completely clear which of these phases exist in pure two dimension and which exist as the surface of a three dimensional system).

Another possibility in SET phases is for the symmetry to map one type of fractional excitation to another. For example, the double semion topological order, which exists in for example a double layer fractional quantum Hall system with $\nu = \pm 1/2$, is time reversal invariant. Time reversal symmetry maps between the semion (with topological spin i) and the anti-semion (with topological spin $-i$) while keeping their combination – a bosonic quasiparticle – invariant.

Even though the semion / anti-semion are not individually invariant under time reversal symmetry, one might wonder if we can still make sense of the ‘quantum number’ or local symmetry transformation on each of them. This question becomes more concrete when we consider the boundary of the topological state. The boundary of the double semion state can be gapped, as shown in Fig.1, by condensing the bosonic quasiparticle, which is the combination of the semion and the anti-semion. Because both the semion and the anti-semion obtain non-

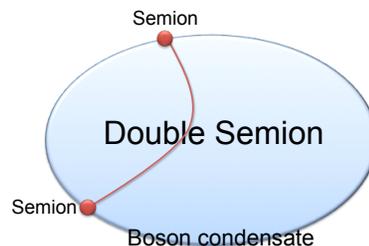


FIG. 1. A double semion state whose boundary is gapped by condensing the bosonic quasiparticle.

trivial phase factors (-1) when braiding around the boson, they become confined in the condensate. Moreover, because the bosonic quasiparticle is condensed, i.e. they can appear and disappear freely, a semion becomes indistinguishable from an anti-semion. Now it becomes reasonable to ask about the local time reversal symmetry transformation of the semion (anti-semion) and there could be two options: $\mathcal{T}^2 = 1$ and $\mathcal{T}^2 = -1$. Such a distinction leads to measurable physical effect in the system. Imagine creating a pair of semions in the bulk, separating them from each other and bringing them to the gapped boundary. When the semion transforms as $\mathcal{T}^2 = -1$, a local Kramer degeneracy appears at the location of each semion while with $\mathcal{T}^2 = 1$ semion no local degeneracy is expected.

So what does this ‘time reversal quantum number’ of the semion imply? Does it label different bulk SET phases or does it correspond to different boundary conditions for the same bulk phase? In this paper we show that the latter is true. In particular, we use both field theory arguments and exactly solvable models to show that there are two types of Bose condensates in the double semion state, with different coherent phase factors of the condensed boson. In one of them the semion transforms as $\mathcal{T}^2 = 1$ while in the other it transforms as $\mathcal{T}^2 = -1$.

The paper is organized as follows: In section II,

we present a simple field theory argument for the result, which is supported by exactly solvable models constructed in section III. In section IV, we consider the situation where different segments of the boundary of the double semion state is gapped in the two different ways and ask what happens at the domain wall between the segments. We find that the domain wall carries extra degeneracy protected by time reversal symmetry and the symmetry action tunnels a semion between pairs of domain walls. In section V, we discuss how this is all related to the symmetry protected topological phase with Z_2 and time reversal symmetry, which becomes the double semion state under study by gauging the Z_2 symmetry. Finally, we conclude in section VI and compare this example with similar models studied previously.

II. FIELD THEORY ANALYSIS

The double semion state contains an abelian topological order with three types of fractional excitations: the semion s , the anti-semion s' and their combination – the boson $b = ss'$. The topological spins of the three are i , $-i$ and 1 respectively and the mutual statistics between s and s' is trivial. In field theory language, the double semion topological order can be described as a $U(1) \times U(1)$ Chern-Simons theory:

$$L = \frac{2}{4\pi} \epsilon^{\lambda\mu\nu} a_\lambda^1 \partial_\mu a_\nu^1 - \frac{2}{4\pi} \epsilon^{\lambda\mu\nu} a_\lambda^2 \partial_\mu a_\nu^2 \quad (1)$$

whose edge state can be described as

$$L_e = \frac{2}{4\pi} \partial_x \phi_1 \partial_t \phi_1 - \frac{2}{4\pi} \partial_x \phi_2 \partial_t \phi_2 \quad (2)$$

where only the topological term in the Lagrangian is shown.

Time reversal symmetry action on the edge fields ϕ_1 and ϕ_2 can be written in two ways:

$$\mathcal{T}_1 : \phi_1 \rightarrow \phi_2, \phi_2 \rightarrow \phi_1 \quad (3)$$

Or equivalently,

$$\mathcal{T}_2 : \phi_1 \rightarrow \phi_2, \phi_2 \rightarrow \phi_1 + \pi \quad (4)$$

\mathcal{T}_1 and \mathcal{T}_2 differ by a gauge transformation¹²

$$g : \phi_1 \rightarrow \phi_1 + \pi, \phi_2 \rightarrow \phi_2 \quad (5)$$

g acts as $g = (-1)^{N_s}$ where N_s is the number of semions on the edge and the action is trivial on all local operators of the form $e^{i2n\phi_1 + i2m\phi_2}$ with integer n and m .

It appears that a semion, generated by $e^{i\phi_1}$ (or the anti-semion generated by $e^{i\phi_2}$) transforms as $\mathcal{T}^2 = 1$ under \mathcal{T}_1 and as $\mathcal{T}^2 = -1$ under \mathcal{T}_2 . So which time reversal transformation should we use? That depends on the boundary condition we choose for the edge theory. In particular, the edge state described by Eq.2 can be

gapped out by adding a Higgs term

$$\Delta L = -\lambda \cos(2\phi_1 - 2\phi_2 + \alpha) \quad (6)$$

with $\lambda > 0$. When λ is large enough, the bosonic quasiparticle, generated by $e^{i(\phi_1 - \phi_2)}$, is condensed on the edge. However, there are two types of condensates which preserve time reversal symmetry, one with $\alpha = 0$ and the other with $\alpha = \pi$.

When $\alpha = 0$, the term ΔL has two classical minima:

$$\phi_1 - \phi_2 = 0, \quad \phi_1 - \phi_2 = \pi \quad (7)$$

The two minima are related by g and hence are physically identical (no local observable distinguishes them). Each minimum is invariant under \mathcal{T}_1 . Therefore this is the form of time reversal transformation that we should consider and the semion $e^{i\phi_1}$ (or the anti-semion $e^{i\phi_2}$) transforms as time reversal singlets $\mathcal{T}^2 = 1$ on the boundary.

When $\alpha = \pi$, the term ΔL has two classical minima as well:

$$\phi_1 - \phi_2 = \pi/2, \quad \phi_1 - \phi_2 = -\pi/2 \quad (8)$$

The two minima are related by g again but neither of them is invariant under \mathcal{T}_1 . Instead they are preserved by \mathcal{T}_2 . Therefore, \mathcal{T}_2 is the manifest time reversal operation on the edge when boson is condensed with $\alpha = \pi$ and the semion $e^{i\phi_1}$ (or the anti-semion $e^{i\phi_2}$) transforms as time reversal doublets $\mathcal{T}^2 = -1$ on the boundary.

Let us look more carefully at how the time reversal transformation of the semion / anti-semion depends on the coherent phase factor of the Bose condensate. Note that in the condensates, the process of creating or annihilating a boson pair is associated with a phase factor of

$$\langle e^{i(\phi_1(x) - \phi_2(x))} e^{i(\phi_1(x') - \phi_2(x'))} \rangle = e^{i\alpha} = \pm 1 \quad (9)$$

while the process of boson hopping always comes with a phase factor of 1

$$\langle e^{i(\phi_1(x) - \phi_2(x))} e^{-i(\phi_1(x') - \phi_2(x'))} \rangle = 1 \quad (10)$$

Therefore, the total wave function of the condensate reads

$$|\psi_{bc}\rangle = \sum_N (e^{i\alpha})^N \sum_{x_1, \dots, x_{2N}} |x_1, \dots, x_{2N}\rangle \quad (11)$$

where the inner sum is over all possible position configurations of $2N$ bosons and the outer sum is over all integer N . Note that there are always an even number of bosons because it is a quasiparticle (self) boson and can only be created in pairs.

The relation between the condensed phase and the time reversal transformation of the semions can be understood as follows: when applying time reversal to a semion, it is mapped to an anti-semion and hence a boson is created; when applying time reversal again and mapping the anti-

semion back to a semion, another boson is created. That is, the process of applying \mathcal{T}^2 to a semion is accompanied by the creation of a boson pair and hence

$$\mathcal{T}^2 = e^{i\alpha} \quad (12)$$

on each semion.

III. EXACTLY SOLVABLE MODEL CONSTRUCTION

Using the exactly solvable model of double semion introduced by Ref.13, we can construct the two types of condensates corresponding to $\alpha = 0$ and $\alpha = \pi$ and show explicitly that the semion excitations transform respectively as $\mathcal{T}^2 = 1$ and $\mathcal{T}^2 = -1$ under time reversal.

A. $\alpha = 0$ condensate and $\mathcal{T}^2 = 1$ semion

Consider the double semion model on Honeycomb lattice, with one spin 1/2 living on each link. To simplify notation, we write spin operators $\sigma_x, \sigma_y, \sigma_z$ as X, Y, Z . The Hamiltonian in the topological bulk contains vertex terms A_v and plaquette terms B_p ¹³.

$$H_T = - \sum_v A_v + \sum_p B_p \quad (13)$$

where

$$\begin{aligned} A_v &= \prod_{i \in v} Z_i \\ B_p &= \prod_{i \in p} X_i \prod_{j \in p} (-)^{n_j(1-n_{j+1})} \end{aligned} \quad (14)$$

$i \in v$ labels links attached to a vertex v and $j \in p$ labels links around a plaquette p . $n_j = 0$ if $Z_j = 1$ and $n_j = 1$ if $Z_j = -1$.

Time reversal \mathcal{T} acts as complex conjugation in the Z basis. Then the Hamiltonian is time reversal invariant and so is the ground state. Note that the form of the plaquette term B_p used here is slightly different from that used in Ref.13, but they are equivalent when all vertex terms A_v are satisfied. The form used here is simpler for our purpose because it is explicitly time reversal invariant.

It is easy to check that 1. all A_v 's commute, all A_v 's commute with all B_p 's and B_p 's commute with each other when the A_v constraints are satisfied 2. $(B_p)^2 = 1$.

Now let's condense the bosonic quasiparticle in the double semion state and gap out the boundary. In the simplest form, the boson condensation can be achieved by enforcing a Z_k term on the link variables. That is

$$H_{C0} = - \sum_k Z_k - \sum_v A_v \quad (15)$$

Obviously the second term is redundant. We include it here just for comparison with later cases. In the dou-

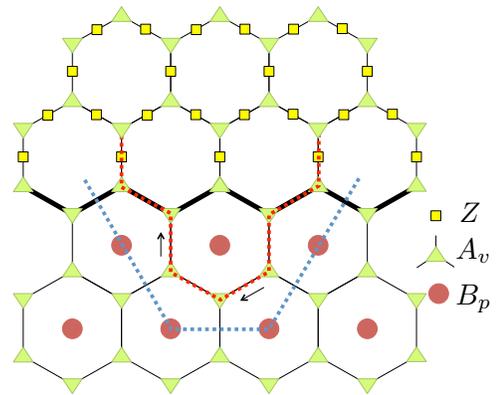


FIG. 2. Boundary (thick black line) of the double semion state (lower half plane) gapped by $\alpha = 0$ Bose condensate (upper half plane) with the semion excitation transforming as $\mathcal{T}^2 = 1$ under time reversal.

ble semion state, the Z_k term creates / annihilates boson pairs and also hops bosons around. Therefore, when the H_{C0} term dominates over the H_T term, the boson is condensed. Here we consider the exactly solvable situation as shown in Fig.2 where the Hamiltonian in the region below the thick black boundary line is H_T and that above the boundary line is H_{C0} . All Hamiltonian terms commute with each other.

From simple counting, we see that the boundary between the condensate and the topological region is totally gapped. Moreover, this condensate does not break time reversal symmetry. A boson-boson pair is created by string operator

$$W_1^b = \prod_{k \in L'} Z_k \quad (16)$$

with L' being a string in the dual lattice. W_1^b obviously has eigenvalue 1 everywhere in the condensate. Boson hopping is also generated with this term and also comes with a phase factor of 1. Therefore, the condensate generated with H_{C0} is the $\alpha = 0$ condensate discussed in the previous section.

Now let's see what happens when we create a semion on the boundary. A semion-semion pair is created, as shown in Fig.2 along the red dotted line L . We have chosen a particular direction for this string operator. The string operator acts as¹³

$$W^s = \prod_{k \in L} X_k \prod_{v_l} \alpha_l \prod_{v_r} \alpha_r \quad (17)$$

where α_l and α_r are phase factors in the Z basis at vertices where the string turns left (v_l) and right (v_r) respectively. $\alpha_l = \pm 1$ is always real. $\alpha_r = (i)^{n_r}$ acts on the leg to the left side of L at this vertex. Obviously in the condensate, the string operator costs linear energy and the semion is confined.

Under time reversal symmetry, this string operator

changes. In particular, the α_r phase factors change to α_r^{-1} and the difference is

$$W_1^b = \prod_{k \in L'} Z_k \quad (18)$$

where L' is the string in the dual lattice to the left side of L , as shown in Eq.2 with the dotted blue line. This is nothing but the string operator for creating a pair of bosons on the boundary, which has eigenvalue 1. Therefore, the state with a pair of semions on the boundary is invariant under time reversal and each semion transforms as a time reversal singlet with $\mathcal{T}^2 = 1$.

B. $\alpha = \pi$ condensate and $\mathcal{T}^2 = -1$ semion

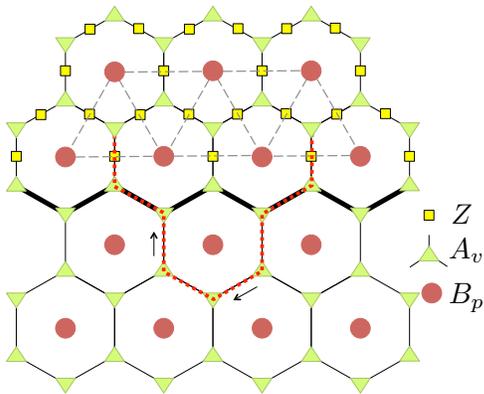


FIG. 3. Boundary (thick black line) of the double semion state (lower half plane) gapped by $\alpha = \pi$ Bose condensate (upper half plane) with the semion excitation transforming as $\mathcal{T}^2 = -1$ under time reversal. The Hamiltonian terms in the condensate $B_{p_a} Z_{ab} B_{p_b}$ are represented as two red dots (B_{p_a} and B_{p_b}) connected by a grey dashed line via a yellow triangle (Z_{ab}).

Now let's define a second type of condensate and see how the semion on the boundary can transform as a time reversal doublet. The Hamiltonian realizing the second type of condensate is

$$H_{C\pi} = - \sum_{\langle ab \rangle} B_{p_a} Z_{ab} B_{p_b} - \sum_v A_v \quad (19)$$

where B_{p_a} and B_{p_b} are neighboring plaquette operators sharing a link ab and the sum is over all such pairs. In the subspace where all vertex constraints A_v are satisfied, it is easy to see that all terms in $H_{C\pi}$ commute with each other (Z_{ab} anti-commutes with B_{p_a} or B_{p_b}). Moreover

$$(B_{p_a} Z_{ab} B_{p_b})^2 = 1 \quad (20)$$

The relative signs in Eq.19 are fixed by the requirement that the Hamiltonian is not frustrated. With our choice

of signs it is not, since

$$1 = A_v = (B_{p_1} Z_{12} B_{p_2})(B_{p_2} Z_{23} B_{p_3})(B_{p_3} Z_{31} B_{p_1}) \quad (21)$$

when both $B_{p_a} Z_{ab} B_{p_b} = 1$ and $A_v = 1$.

The $B_{p_i} Z_{ij} B_{p_j}$ terms create boson pairs or hop bosons around in this condensate. When this term acts on a pair of plaquettes with either 2 or 0 bosons ($B_{p_1} B_{p_2} = 1$), there is an extra sign relative to its action on a pair of plaquettes with only one boson ($B_{p_1} B_{p_2} = -1$). Counting the phase factors at both ends, pair creation / annihilation is accompanied by a phase factor of -1 while boson hopping has a phase factor of $+1$. Note that this is true even when we put the small pieces of boson string operators together and make longer strings.

$$(B_{p_1} Z_{12} B_{p_2})(B_{p_2} Z_{23} B_{p_3}) \dots (B_{p_{m-1}} Z_{(m-1)m} B_{p_m}) \\ = B_{p_1} Z_{12} \dots Z_{(m-1)m} B_{p_m} \quad (22)$$

Hence in the ground state we have

$$B_{p_1} Z_{12} \dots Z_{(m-1)m} B_{p_m} = 1 \quad (23)$$

Therefore, the condensate generated with $H_{C\pi}$ corresponds to the $\alpha = \pi$ condensate discussed in section II. Even though an extra phase factor is present in the wave function of this condensate, the condensate is still time reversal invariant.

Now let's create a semion pair on the boundary along the red dotted line in Fig.3 using the same string operator W^s as in the previous section. Similar to the previous case, the string operator W^s violates the vertex A_v terms at its end points, but this has extra consequences in this new type of condensate. In particular, when A_v is violated, the plaquette operators B_p around this vertex no longer commute with each other. To restore exact solvability, certain terms need to be removed, introducing local degeneracies into the low energy Hilbert space. As we will show below, this local degeneracy indicates the presence of a local Kramers pair with $\mathcal{T}^2 = -1$ under time reversal.

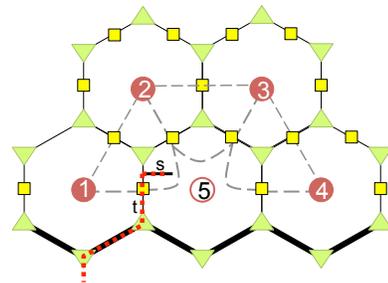


FIG. 4. Semion excitation on the boundary with $\alpha = \pi$ condensate transforms as a Kramer doublet with $\mathcal{T}^2 = -1$.

Let us zoom in on the semion as shown in Fig.4. To expose the two fold Kramer degeneracy related to the semion, we redefine the Hamiltonian locally as follows: imagine breaking the link between plaquette 1 and 5 into

two parts and adding a link s sticking into plaquette 5 as shown in Fig.4. The state of the link s is initially set to be $n_s = 0$. We can choose the semion string operator as going into the condensate and ending on link s . The state of the link s is hence flipped to $n_s = 1$. By doing so, we have moved the vertex violation to the end of link s which does not affect plaquette 1 and 2 but only plaquette 5.

Due to the existence of link s , B_{p_5} needs to be redefined. According to the string-net rule given in Ref.13, B_{p_5} can be obtained by merging a semion loop into plaquette 5. Now with the link s occupied by a semion string, direct calculation shows that

$$B_{p_5} = i \prod_{k \in p_5} X_k \prod_{j \in p_5} (-)^{n_j \bar{n}_{j+1}} Z_t \quad (24)$$

where t is the lower half of the link between plaquette 1 and 5. We can explicitly check that B_{p_5} is Hermitian, $(B_{p_5})^2 = I$, it commutes with all other plaquette terms already in the Hamiltonian (in the sector where all A_v constraints are satisfied), but is NOT time reversal invariant. Indeed, we find

$$\mathcal{T} B_{p_5} \mathcal{T}^{-1} = -B_{p_5} \quad (25)$$

To indicate the time reversal violation, we denote this term with an empty circle in Fig.4.

To allow the semion string to end without violating time-reversal symmetry, we are therefore forced to remove terms containing B_{p_5} from the Hamiltonian. This suggests that four terms need to be removed: $B_{p_i} Z_{i5} B_{p_5}$, $i = 1, 2, 3, 4$. In fact, we can recombine the Hamiltonian terms and add some terms back (see Fig.4)

$$\sum_{i=1,2,3} B_{p_i} Z_{i5} Z_{(i+1)5} B_{p_{i+1}} \quad (26)$$

These terms are time reversal symmetric, commute with all other terms in the original Hamiltonian and take eigenvalue 1 in the ground state. Now counting the number of terms we find that we are missing one term and hence have a local two-fold degeneracy.

Is this the local Kramer degeneracy we are looking for? Obviously this two fold degeneracy corresponds to eigenstates of B_{p_5} (with eigenvalue ± 1) which commutes with all other terms in the Hamiltonian but is also independent of them. Notice that B_{p_5} anti-commutes with \mathcal{T} , hence \mathcal{T} interchanges the two states with $B_{p_5} = \pm 1$. Therefore, to determine the \mathcal{T}^2 value on this local degeneracy, we need to find a local operator Q which maps between these two states and check its transformation under time reversal². That is, Q needs to commute with all terms in the Hamiltonian but anti-commute with B_{p_5} . One possible choice is

$$Q = B_{p_1} Z_t \quad (27)$$

Because

$$(\mathcal{T} Q \mathcal{T}^{-1}) Q = B_{p_1} Z_t B_{p_1} Z_t = -1 \quad (28)$$

we see that the degenerate states form a local Kramer pair under time reversal. Or in other words, on the boundary with $\alpha = \pi$ condensate, semions transform as Kramer doublets under time reversal symmetry.

IV. DOMAIN WALL BETWEEN TWO TYPES OF BOUNDARIES

Although the two types of condensates, with $\alpha = 0$ and $\alpha = \pi$, give rise to different boundaries with the double semion state, they do not correspond to different phases. Indeed, both $\alpha = 0$ and $\alpha = \pi$ condensates are short range entangled states with time reversal symmetry. As we know that there is no nontrivial time reversal symmetry protected topological phase in 2D¹⁴, both condensates belong to the same phase. Therefore, the interface between these two types of condensates can be gapped out without breaking time reversal.

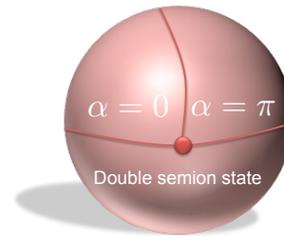


FIG. 5. Domain wall (red dot) between the two types of boundary between the double semion state and the $\alpha = 0$ and $\alpha = \pi$ condensates respectively.

Now we can ask the question of what happens on the domain wall between the two types of boundaries. Imagine a situation as shown in Fig.5 where a 2D sphere is partitioned into three parts, occupied by the double semion state, $\alpha = 0$ condensate and $\alpha = \pi$ condensate respectively. The interface between any two parts is gapped with time reversal symmetry being preserved and now we can investigate the property of the two domain walls (red dots in Fig.5). In this section, we are going to see whether there are degeneracies associated with the domain walls, and how they transform under time reversal symmetry.

A. Field theory analysis

From the field theory analysis in section II, we see that the boundary with the $\alpha = 0$ condensate is in state $|A\rangle = |\phi_1 - \phi_2 = 0\rangle$ (or equivalently $|\bar{A}\rangle = |\phi_1 - \phi_2 = \pi\rangle$) and the boundary with the $\alpha = \pi$ condensate is in state $|B\rangle = |\phi_1 - \phi_2 = \pi/2\rangle$ (or equivalently $|\bar{B}\rangle = |\phi_1 - \phi_2 = -\pi/2\rangle$). The difference between $|A\rangle$ and $|\bar{A}\rangle$ (or $|B\rangle$ and

$|\bar{B}\rangle\rangle$) is an artificial one as they are related by the gauge transformation

$$g : \phi_1 \rightarrow \phi_1 + \pi, \phi_2 \rightarrow \phi_2 \quad (29)$$

In reality, the gauge symmetry is not broken so we need to restore the symmetry and write the two boundary states as

$$|A\rangle + |\bar{A}\rangle \quad (30)$$

and

$$|B\rangle + |\bar{B}\rangle \quad (31)$$

When the boundary contains both parts, there are two possible configurations,

$$|\psi_1\rangle = |AB\rangle + |\bar{A}\bar{B}\rangle \quad (32)$$

and

$$|\psi_2\rangle = |A\bar{B}\rangle + |\bar{A}B\rangle \quad (33)$$

which form a two fold degeneracy on the boundary as long as time reversal symmetry is not broken. This can be shown as follows. Suppose the two domain walls between type A and type B boundaries are at L and $-L$ respectively, then the local operator near L

$$O_1 = ie^{i(\phi_1(L-\epsilon)-\phi_2(L-\epsilon))} e^{-i(\phi_1(L+\epsilon)-\phi_2(L+\epsilon))} \quad (34)$$

(ϵ is small and finite) tunnels a boson across the domain wall at L and takes ± 1 eigenvalues in the two states.

$$O_1|\psi_1\rangle = |\psi_1\rangle, \quad O_1|\psi_2\rangle = -|\psi_2\rangle \quad (35)$$

However, O_1 cannot be added to the Hamiltonian because it breaks time reversal symmetry, either in the form \mathcal{T}_1 or \mathcal{T}_2 .

$$\mathcal{T}_1^{-1}O_1\mathcal{T}_1 = -O_1, \quad \mathcal{T}_2^{-1}O_1\mathcal{T}_2 = -O_1 \quad (36)$$

On the other hand, the operator O_2 which tunnels a semion from $-L$ to L

$$O_2 = e^{-i\phi_1(-L)} e^{i\phi_1(L)} \quad (37)$$

maps between $|\psi_1\rangle$ and $|\psi_2\rangle$ and anti-commutes with O_1 . O_2 is a nonlocal operator and cannot be added to the Hamiltonian to split the degeneracy. Moreover, as O_1 and O_2 generate the full operator algebra of the two dimensional space spanned by $|\psi_1\rangle$ and $|\psi_2\rangle$, we see that $|\psi_1\rangle$ and $|\psi_2\rangle$ are necessarily degenerate if time reversal symmetry is preserved.

We can take either form of the time reversal action, \mathcal{T}_1 or \mathcal{T}_2 , and we find that their action on these two states is the same as O_2

$$\mathcal{T}|\psi_1\rangle = |\psi_2\rangle \quad (38)$$

Therefore, the domain walls between the two types of boundaries carry a two fold degeneracy and time reversal action in this degenerate subspace is equivalent to the tunneling of semions between the two domain walls.

More generally, if the boundary is divided into $2N$ alternating segments with $2N$ domain walls in between, similar analysis shows that there is a 2^{2N-1} fold degeneracy protected by time reversal symmetry.

We are going to confirm this conclusion with exactly solvable models in the next section.

B. Interface between the two types of condensates

First we need to show that the interface between the two types of condensates can be gapped. This is expected because the two condensates are both time reversal invariant short range entangled states. Because there are no nontrivial symmetry protected topological phases with time reversal symmetry in two dimensions, the two condensates are in the same phase and should be able to connect to each other in a gapped way without breaking the symmetry

Consider an interface between the two condensates as shown in Fig.6

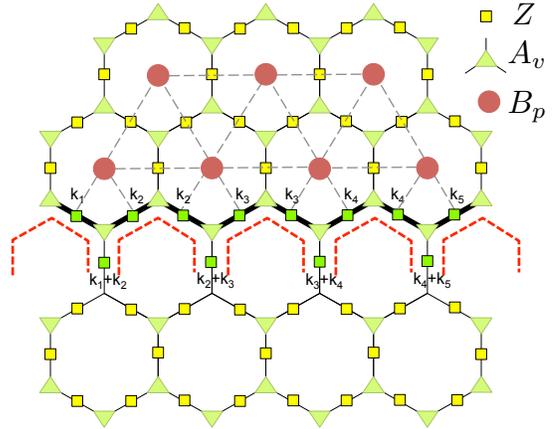


FIG. 6. Interface between condensate I and II.

The upper half plane is in the $\alpha = \pi$ condensate with Hamiltonian terms $B_{p_a} Z_{ab} B_{p_b}$ (red dot - yellow square - red dot) and the lower half plane is in the $\alpha = 0$ condensate with Hamiltonian terms Z_k (yellow square). The green squares label the degrees of freedom on the interface and form the interface Hilbert space.

Let's be more explicit about this. First, notice that the $\alpha = \pi$ condensate can be mapped to the $\alpha = 0$ condensate by unitary transformation

$$U = \prod_{p \in C_\pi} \frac{(I + iB_p)}{1 + i} \quad (39)$$

where the product is over all placates in the condensate.

We can see this explicitly from

$$\begin{aligned} & \frac{(I+iB_{p_a})}{1+i} \frac{(I+iB_{p_b})}{1+i} Z_{ab} \frac{(I-iB_{p_a})}{1-i} \frac{(I-iB_{p_b})}{1-i} \\ & = B_{p_a} Z_{ab} B_{p_b} \end{aligned} \quad (40)$$

Therefore, after applying U , all links inside the two condensates (yellow squares) are in the state $|n_i = 0\rangle$ and $U \prod |0\rangle$ is the ground state of these regions.

Next, we enforce the closed loop constraints

$$A_v = \prod_{i \in v} Z_i \quad (41)$$

along the interface (thick black line) as indicated by the green triangles.

Now we are left with one 2 dimensional degree of freedom per plaquette, labelled by $k_s = 0, 1$ as shown in Fig.6. $|k_s = 0\rangle$ and $|k_s = 1\rangle$ are eigenvalue 1 and -1 eigenstates of Z_s respectively. Each state in the interface Hilbert space can be then be written as

$$|\psi_{\{k_s\}}\rangle = U \prod_{\nabla} |k_s\rangle |k_{s+1}\rangle |k_s + k_{s+1}\rangle \prod_C |0\rangle \quad (42)$$

where the first product is over all downward pointing vertices on the interface and the second product is over all the links inside the two condensates (the ones with the yellow squares). Note that the links on the interface are not all independent due to the A_v constraints. There is one free degree of freedom per plaquette on the interface. As given in Eq.42, each state $|\psi_{\{k_s\}}\rangle$ is the eigenvalue $e^{i\pi k_s}$ eigenstate of $iZ_s B_{p_s}$, where s labels links along the interface and p_s labels the plaquette on the $\alpha = \pi$ condensate side of link s . Under time reversal, which acts as complex conjugation, $iZ_s B_{p_s}$ is mapped to $-iZ_s B_{p_s}$. Therefore, each $|\psi_{\{k_s\}}\rangle$ is not time reversal invariant.

To find a time reversal symmetric interface, we must add some terms to the boundary that mix the $|\psi_{k_s}\rangle$ states, respect \mathcal{T} , and gap the interface out. While it may be hard to directly find such an operator, we can apply the unitary transformation $U = \prod_{p \in C_\pi} \frac{(I+iB_p)}{1+i}$ and map the interface Hilbert space to that spanned by

$$|\psi_{\{k_s\}}^0\rangle = \prod_{\nabla} |k_s\rangle |k_{s+1}\rangle |k_s + k_{s+1}\rangle \prod_C |0\rangle \quad (43)$$

The transformed interface Hilbert space now takes a simple direct product form of local degrees of freedom labeled by k_s and allows simpler analysis of possible Hamiltonian terms. Even though the unitary U is not local, it preserves the spectrum and hence a gapped edge in this basis is also gapped in the original basis.

There is one complication though: The action of time reversal is also transformed under U . Before we can write down time reversal invariant Hamiltonians, we need to find the correct time reversal transformation in this new

basis.

$$\tilde{\mathcal{T}} = U^\dagger \mathcal{T} U = \prod_{p \in C_\pi} \left(\frac{1+iB_p}{1+i} \right)^2 = \prod_{p \in C_\pi} B_p \quad (44)$$

Therefore, the effective time reversal action $\tilde{\mathcal{T}}$ on the interface Hilbert space is complex conjugation in the $|\psi_{\{k_s\}}^0\rangle$ basis and

$$\prod X_s \prod (-)^{n_s(1-n_{s+1})} \quad (45)$$

Although it looks complicated and non-onsite, we know that it should have a short range entangled ground state. Indeed we find that under local unitaries

$$V = \prod_s (i)^{n_s(1-n_{s+1})} \quad (46)$$

The effective time reversal action is mapped to

$$V \tilde{\mathcal{T}} V^\dagger = \prod X_s K \quad (47)$$

which has a gapped symmetric Hamiltonian $\sum_s X_s$. Note that in Fig.6 X_s acts on four links, the two links labeled by k_s and also the ones labeled by $k_{s-1}+k_s$ and k_s+k_{s+1} . This is illustrated in Fig.6 with red dotted lines.

Therefore, the transformed interface Hilbert space spanned by $|\psi_{\{k_s\}}^0\rangle$ can be gapped in a time reversal invariant way by effective Hamiltonians

$$\sum_s V^\dagger X_s V \quad (48)$$

In the original interface Hilbert space spanned by $|\psi_{\{k_s\}}\rangle$, the Hamiltonian reads

$$\sum_s UV^\dagger X_s VU^\dagger \quad (49)$$

In appendix A, we explicitly confirm that the term $UV^\dagger X_s VU^\dagger$ is indeed real.

C. Tri-junction between $\alpha = 0$, $\alpha = \pi$ condensates and double semion

Now we have found a gapped interface between the $\alpha = 0$ and the $\alpha = \pi$ condensates, we can study the domain wall between the two types of boundaries as the tri-junction between the two condensates and the topological state (double semion).

As shown in Fig. 7, topological region is on the outside and the two condensates are in the middle. Following previous discussion, Hamiltonian in the topological region is given by (red dots and green triangles)

$$H_T = \sum_v A_v + \sum_p B_p \quad (50)$$

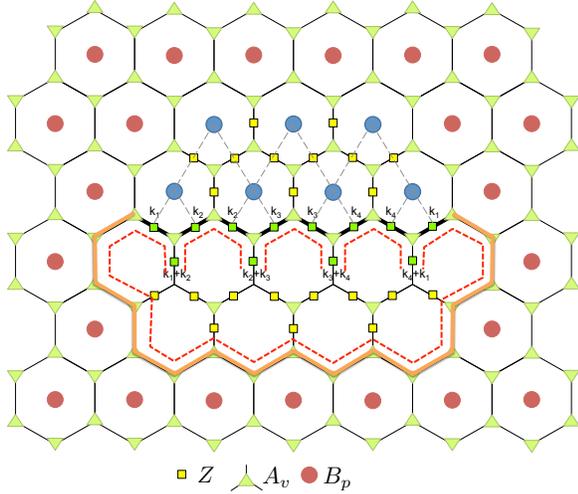


FIG. 7. Tri-junction between $\alpha = 0$, $\alpha = \pi$ condensates and the double semion state.

Hamiltonian in the $\alpha = 0$ condensate is given by (yellow squares)

$$H_{C_0} = - \sum_k Z_k - \sum_v A_v \quad (51)$$

and that in the $\alpha = \pi$ condensate is given by (blue dot – yellow square – blue dot)

$$H_{C_\pi} = - \sum_{\langle ab \rangle} B_{p_a} Z_{ab} B_{p_b} - \sum_v A_v \quad (52)$$

Moreover, we enforce the closed loop constraints A_v along the interface between the two condensates (thick black line).

Now we can see what the low energy Hilbert space is composed of. Apply again the unitary transformation $U = \prod_{p \in C_\pi} \frac{1+iB_p}{1+i}$. Note that U commutes with all the terms in H_T and the A_v constraints on the interface, maps the $B_{p_a} Z_{ab} B_{p_b}$ terms in H_{C_π} to Z_{ab} and leaves H_{C_0} untouched. Therefore, states in the low energy Hilbert space can be written as

$$|\phi_{\{k_i\}}\rangle = \prod_{p \in C_{I1}} \frac{1+iB_p}{1+i} |\phi_{\{k_i\}}^0\rangle \quad (53)$$

where

$$|\phi_{\{k_i\}}^0\rangle = |\phi_{DS}\rangle \prod_{\nabla} |k_i\rangle |k_{i+1}\rangle |k_i + k_{i+1}\rangle \prod_C |0\rangle \quad (54)$$

The first product is over all downward pointing triangles on the interface between the two condensates and the second product is over all links in the two condensates. Note that the two k_1 's near the two tri-junctions are the same, due to the closed loop constraints over the whole state. $|\phi_{DS}\rangle$ is the wave function in the topological region (including all the links around red dots as shown

in Fig.7). The exact form of $|\phi_{DS}\rangle$ depends on whether $k_1 = 0$ or $k_1 = 1$. The two can be mapped into each other by running a semion string operator from one tri-junction to another along, for example, the orange line shown in Fig.7.

Following a similar line of reasoning as discussed in the previous section, we find that the interface can be gapped with time reversal invariant terms

$$UV^\dagger X_s VU^\dagger \quad (55)$$

where U is again a product over plaquettes in the $\alpha = \pi$ condensate $U = \prod_{p \in C_\pi} \frac{1+iB_p}{1+i}$ and $V = \prod_s (i)^{n_s(1-n_{s+1})}$.

There is one major difference though from the situation discussed in the previous section. X_1 is now a non-local operator. Not only does X_1 change k_1 to $1 - k_1$, it also changes the form of $|\phi_{DS}\rangle$ by running a semion string from one tri-junction to another. Therefore, if we require time reversal symmetry and locality, we are left with a two fold degeneracy with $k_1 = 0$ or 1 respectively. Time reversal symmetry maps k_1 to $1 - k_1$ which can be equivalently accomplished by X_1 in this degenerate Hilbert space. Therefore, time reversal symmetry acts as semion tunneling in the low energy Hilbert space of tri-junctions.

V. RELATION TO $Z_2 \times Z_2^T$ SYMMETRY PROTECTED TOPOLOGICAL PHASE

The discussions in the previous sections tell us that there is one symmetry enriched topological phase for double semion topological order with time reversal symmetry. However, there can be two different gapped boundaries with time reversal symmetry. The double semion topological order is a twisted Z_2 gauge theory and by ungauging the Z_2 symmetry we can get $Z_2 \times Z_2^T$ symmetry protected topological phases. So how is our conclusion about double semion SET consistent with what we know about $Z_2 \times Z_2^T$ SPT?

From the classification of SPT we know that there are four phases with $Z_2 \times Z_2^T$ symmetry¹⁴. In particular, there are two root phases: phase (1,0) which has non-trivial Z_2 SPT order and trivial action of time reversal and phase (0,1) with trivial Z_2 SPT order and projective action of time reversal on Z_2 twist defects¹⁵. Phase (1,1) is their combination with both nontrivial Z_2 SPT order and projective Z_2 twist defects under time reversal.

After gauging the Z_2 symmetry, we get SET phases. Phase (0,0) and (0,1) lead to the usual non-twisted Z_2 gauge theory (toric code order) and the resulting states differ in the way Z_2 fluxes transform under time reversal (as singlet or doublet). Therefore, after gauging, phase (0,0) and (0,1) results in two different SET phases.

On the other hand, phase (1,0) and (1,1) gauge into the twisted Z_2 gauge theory (double semion order). As there is only one time reversal SET with double semion order, the two SPT phases have to merge into one upon

gauging. To understand how this happens we notice that phase (1,0) and (1,1) can be mapped into each other by relabeling the group element. In particular, in the symmetry group $Z_2 \times Z_2^T = \{I, g, \mathcal{T}, g\mathcal{T}\}$, there are two anti-unitary operators \mathcal{T} and $g\mathcal{T}$. If we exchange the two, the group is still a $Z_2 \times Z_2^T$ group and phase (1,0) is mapped into phase (1,1) and vice versa. Therefore, upon gauging the Z_2 symmetry, phase (1,0) and (1,1) become the same SET.

More explicitly, the edge state of the $Z_2 \times Z_2^T$ SPTs can be described in general with

$$L_e = \frac{1}{2\pi} \partial_x \phi_1 \partial_t \phi_2 \quad (56)$$

Different phases correspond to different ways symmetry act on ϕ_1 and ϕ_2 . In both phase (1,0) and (1,1), Z_2 acts as

$$g : \phi_1 \rightarrow \phi_1 + \pi, \phi_2 \rightarrow \phi_2 + \pi \quad (57)$$

Time reversal acts respectively as

$$\begin{aligned} \mathcal{T}_1 : \phi_1 \rightarrow \phi_1, \phi_2 \rightarrow -\phi_2 \\ \mathcal{T}_2 : \phi_1 \rightarrow \phi_1 + \pi, \phi_2 \rightarrow -\phi_2 \end{aligned} \quad (58)$$

$e^{i(\phi_1+\phi_2)/2}$ (or $e^{i(\phi_1-\phi_2)/2}$) creates a Z_2 twist defect on the edge and transforms as $\mathcal{T}_1^2 = 1$ and $\mathcal{T}_2^2 = -1$.

Combining g with \mathcal{T}_1 we get

$$g\mathcal{T}_1 : \phi_1 \rightarrow \phi_1 + \pi, \phi_2 \rightarrow -\phi_2 - \pi \quad (59)$$

Redefine $\tilde{\phi}_2 = \phi_2 + \pi/2$, we find

$$g\mathcal{T}_1 : \phi_1 \rightarrow \phi_1 + \pi, \tilde{\phi}_2 \rightarrow -\tilde{\phi}_2 \quad (60)$$

which is exactly the same as the action of \mathcal{T}_2 . Hence after gauging \mathcal{T}_1 and \mathcal{T}_2 are equivalent, and both correspond to the same SET phase. However a memory of these two distinct SPT phases is retained at the boundary of our SET, where depending on boundary conditions the semion can transform with either $\mathcal{T}^2 = 1$ or -1 .

VI. CONCLUSION AND DISCUSSION

To summarize, in this paper we have learned the following things about the double semion topological order:

1. There are two different ways to condense boson and gap out the boundary while preserving time reversal symmetry in the double semion state. One corresponds to a Bose condensate with a coherent phase factor 1 and the semion excitations on the boundary transform as $\mathcal{T}^2 = 1$. The other corresponds to a Bose condensate with a coherent phase factor of -1 and the semion excitations on the boundary transform as $\mathcal{T}^2 = -1$.
2. A pair of domain walls between the two types of

boundaries carry a two fold degeneracy. Time reversal symmetry acts in this two dimensional space by tunneling a semion from one domain wall to another.

3. There is only one SET phase with time reversal symmetry and double semion topological order. The different transformation properties of the semions under time reversal symmetry is a pure boundary effect. In the bulk, semions are mapped to anti-semions and it is not meaningful to talk about the time reversal representation carried by semion itself.
4. Different SPT phases can become the same SET phase after partly gauging the unitary symmetry of the system. Examples of this kind have been pointed out in Ref.16.

These results can be generalized to other SET or SPT phases. In appendix B, we present another example of SPTs coalescing upon gauging with unitary symmetry. We leave the study of more general cases to the future.

We want to comment briefly on the relation between this double semion example and some previous studies of gapped boundaries of topological states.

A simple yet very interesting case was the boundary state of toric code topological order. Even in the absence of symmetry, there are two types of boundaries corresponding to the two types of (self) bosons in the toric code¹⁷. Each domain wall between the two types of boundaries carry a Majorana mode, giving rise to a 2^N fold degeneracy for $N+1$ pairs of domain walls (with fixed fermion parity).

Similarly, degeneracies arise with domain walls in our double semion example. With N pairs of domain walls, there is a 2^{2N-1} fold degeneracy. Of course, this degeneracy requires the protection of time reversal symmetry and can be completely removed by adding time reversal symmetry breaking local terms. If time reversal symmetry is preserved, the degenerate states can only be mapped to each other through non-local operators which tunnel semionic excitations from one domain wall to another.

The existence of such domain wall degeneracy has also been noticed in (fractional) topological insulators where domain walls between ferromagnetic gapped edges and superconducting gapped edges carry Majorana (parafermion) zero modes¹⁸⁻²². Such models are different from the double semion example studied here in that symmetry is broken in order to gap out the edge.

Finally, Wang and Levin studied different ways to gap out the edge of a ‘strong pairing insulator’²³, either with an interface with a topological insulator or with an interface with a trivial insulator. Symmetry is preserved in these two kinds of edges, but the interface between the two condensates – the topological insulator and the trivial insulator – is always gapless when symmetry is preserved.

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Appendix A: Explicit calculation of \mathcal{T} -invariance of the gapping terms between the two condensates

In this section, we are going to explicitly verify that the $UV^\dagger X_s VU^\dagger$ term given in section IV B is indeed real.

First, let's write the X_s term in full as $\sigma_\gamma^x \sigma_\alpha^x \sigma_\beta^x \sigma_\delta^x$, as shown in Fig.8 Then

$$\begin{aligned} & V^\dagger X_s V \\ &= S_{\alpha-1}^\dagger S_\beta^\dagger \sigma_\gamma^x \sigma_\alpha^x \sigma_\beta^x \sigma_\delta^x S_{\alpha-1} S_\beta \\ &= \sigma_\gamma^x \sigma_\alpha^x \sigma_\beta^x \sigma_\delta^x V_\phi \end{aligned} \quad (\text{A1})$$

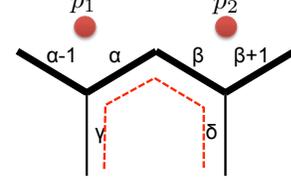


FIG. 8. Each X_s term flips four links α , β , γ and δ .

where

$$S_\alpha = i^{n_\alpha(1-n_{\alpha+1})} \quad (\text{A2})$$

and

$$V_\phi = i^{n_{\alpha-1}+1-n_{\beta+1}} (-)^{n_{\alpha-1}n_\alpha+(1-n_\beta)(1-n_{\beta+1})} \quad (\text{A3})$$

Complex conjugating $UV^\dagger X_s VU^\dagger$ then gives

$$\begin{aligned} & (UV^\dagger X_s VU^\dagger)^* \\ &= \frac{1-iB_{p_1}}{1-i} \frac{1-iB_{p_2}}{1-i} \sigma_\gamma^x \sigma_\alpha^x \sigma_\beta^x \sigma_\delta^x V_\phi^* \frac{1+iB_{p_1}}{1+i} \frac{1+iB_{p_2}}{1+i} \\ &= UB_{p_1} B_{p_2} \sigma_\gamma^x \sigma_\alpha^x \sigma_\beta^x \sigma_\delta^x V_\phi^* B_{p_1} B_{p_2} U^\dagger \end{aligned} \quad (\text{A4})$$

where p_1 and p_2 are the plaquettes above link $\alpha - 1$, α and β , $\beta + 1$ respectively, as shown in Fig.8.

Therefore, $UV^\dagger X_s VU^\dagger$ is real if

$$B_{p_1} B_{p_2} \sigma_\gamma^x \sigma_\alpha^x \sigma_\beta^x \sigma_\delta^x V_\phi^* B_{p_1} B_{p_2} = \sigma_\gamma^x \sigma_\alpha^x \sigma_\beta^x \sigma_\delta^x V_\phi \quad (\text{A5})$$

This can be explicitly checked as

$$\begin{aligned} B_{p_1} B_{p_2} \prod \sigma^x &= \prod \sigma^x (-)^{n_{\alpha-1}+n_{\beta+1}+1} \\ (-)^{n_{\alpha-1}+n_{\beta+1}+1} V_\phi^* &= V_\phi \end{aligned} \quad (\text{A6})$$

and $B_{p_1} B_{p_2}$ commutes with V_ϕ . Therefore,

$$(UV^\dagger X_s VU^\dagger)^* = UV^\dagger X_s VU^\dagger \quad (\text{A7})$$

Appendix B: Example of SPTs coalescing upon gauging with unitary symmetry

Consider SPTs with $Z_2^{(1)} \times Z_2^{(2)}$ (unitary) group in 2d. The classification is Z_2^3 . The root phases have the edge structure,

$$L = \frac{i}{2\pi} \partial_x \phi \partial_\tau \theta \quad (\text{B1})$$

and transformation properties:

$$\begin{aligned} \text{Phase } (1, 0, 0) : Z_2^{(1)} : & \phi \rightarrow \phi + \pi, \quad \theta \rightarrow \theta + \pi, \\ Z_2^{(2)} : & \text{trivial} \end{aligned}$$

$$\begin{aligned} \text{Phase } (0, 1, 0) : Z_2^{(1)} : & \text{trivial} \\ Z_2^{(2)} : & \phi \rightarrow \phi + \pi, \quad \theta \rightarrow \theta + \pi \end{aligned}$$

$$\begin{aligned} \text{Phase } (0, 0, 1) : Z_2^{(1)} : & \phi \rightarrow \phi + \pi, \quad \theta \rightarrow \theta \\ Z_2^{(2)} : & \phi \rightarrow \phi, \quad \theta \rightarrow \theta + \pi \end{aligned}$$

Combining phases (0, 0, 1) and (0, 1, 0) we obtain

$$\begin{aligned} \text{Phase } (0, 1, 1) : Z_2^{(1)} : & \phi \rightarrow \phi + \pi, \quad \theta \rightarrow \theta \\ Z_2^{(2)} : & \phi \rightarrow \phi + \pi, \quad \theta \rightarrow \theta + \pi \end{aligned} \quad (\text{B2})$$

Indeed, denote the edge modes of (0, 0, 1) as ϕ_1, θ_1 and edge-modes of (0, 1, 0) as ϕ_2, θ_2 . Then the combined edge is described by,

$$L = \frac{i}{2\pi} \partial_x \phi_1 \partial_\tau \theta_1 + \frac{i}{2\pi} \partial_x \phi_2 \partial_\tau \theta_2 \quad (\text{B3})$$

Let's define $\phi'_1 = \phi_1, \theta'_1 = \theta_1 - \theta_2, \phi'_2 = \phi_1 + \phi_2, \theta'_2 = \theta_2$. The action has the same form in the primed variables as in the unprimed. The transformation properties of the primed variables are,

$$\begin{aligned} Z_2^{(1)} : & \phi'_1 \rightarrow \phi'_1 + \pi, \quad \theta'_1 \rightarrow \theta'_1 \\ & \phi'_2 \rightarrow \phi'_2 + \pi, \quad \theta'_2 \rightarrow \theta'_2 \\ Z_2^{(2)} : & \phi'_1 \rightarrow \phi'_1, \quad \theta'_1 \rightarrow \theta'_1 \\ & \phi'_2 \rightarrow \phi'_2 + \pi, \quad \theta'_2 \rightarrow \theta'_2 + \pi \end{aligned} \quad (\text{B4})$$

Adding a term $-\lambda \cos(\theta'_1)$ we gap out the ϕ'_1, θ'_1 modes. The transformation properties of ϕ'_2, θ'_2 are then exactly the same as in Eq. (B2).]

Let us denote the generator of $Z_2^{(1)}$ as g_1 , the generator of $Z_2^{(2)}$ as g_2 and $g_3 = g_1 g_2$. Then, in phase (0, 1, 1) under g_3 we have

$$g_3 : \phi \rightarrow \phi, \quad \theta \rightarrow \theta + \pi \quad (\text{B5})$$

So the phase (0, 1, 1) is like the phase (0, 0, 1) but with the

actions of g_2 and g_3 interchanged. Now imagine gauging the $Z_2^{(1)}$ group in the two phases (0, 0, 1) and (0, 1, 1). The action of g_1 in both cases is identical and the twist defect is given by $e^{i\tilde{\theta}} = e^{i\theta/2}$. In terms of $\tilde{\theta}$ the edge theory is,

$$L = \frac{2i}{2\pi} \partial_x \phi \partial_\tau \tilde{\theta} \quad (\text{B6})$$

i.e. after gauging we get a toric code topological order. Now, $Z_2^{(2)}$ remains a global symmetry in the resulting SET,

$$\begin{aligned} (0, 0, 1) \text{ with } Z_2^{(1)} \text{ gauged; } Z_2^{(2)} : & \phi \rightarrow \phi, \quad \tilde{\theta} \rightarrow \tilde{\theta} + \pi/2 \\ (0, 1, 1) \text{ with } Z_2^{(1)} \text{ gauged; } Z_2^{(2)} : & \phi \rightarrow \phi + \pi, \quad \tilde{\theta} \rightarrow \tilde{\theta} + \pi/2 \end{aligned} \quad (\text{B7})$$

Clearly, as SETs the two phases are the same since all local degrees of freedom transform in the same way (the transformation properties differ by a pure ‘‘gauge’’ transformation $\phi \rightarrow \phi + \pi$).

We can also think about gauging the remaining $Z_2^{(2)}$ symmetry. In the case of (0, 0, 1) the twist defect of g_2 is $e^{i\tilde{\phi}} = e^{i\phi/2}$. Thus, we get an overall Z_4 topological order,

$$L = \frac{4i}{2\pi} \partial_x \tilde{\phi} \partial_\tau \tilde{\theta} \quad (\text{B8})$$

where $e^{i\tilde{\phi}}$ is the twist defect of $Z_2^{(2)}$ and $e^{i\tilde{\theta}}$ is the twist defect of $Z_2^{(1)}$.

Now, in the case of (0, 1, 1) the twist defect of g_2 is $e^{i\phi/2} e^{i\theta/2} = e^{i\tilde{\phi}} e^{i\tilde{\theta}}$. If we are interested in the ‘‘overall’’ topological order, we can still use the $e^{i\tilde{\phi}}, e^{i\tilde{\theta}}$ basis, obtaining a Z_4 topological order. However, the way that the twist defect of g_2 is ‘‘embedded’’ within this topological order is different. In (0, 0, 1) it is just $e^{i\tilde{\phi}}$ (which is a boson), whereas in (0, 1, 1) it is $e^{i\tilde{\phi}} e^{i\tilde{\theta}}$ (which is a semion). If we are treating the two phases as SET phases (with $Z_2^{(1)}$ ‘‘fully gauged’’), then the difference of an extra factor of $e^{i\tilde{\theta}}$ is irrelevant, since the twist defect of the global symmetry $Z_2^{(2)}$ can always trap an extra anyon (vison) of the SET, $e^{i\tilde{\theta}}$. Thus, we cannot distinguish the two SET phases. However, if we are working with SPT phases and putting the twist defects in ‘‘by hand’’, then we know whether the twist defect of g_1 is present or not, so can distinguish the two SPT phases.