THE VERLINDE FORMULA FOR HIGGS BUNDLES

JØRGEN ELLEGAARD ANDERSEN, SERGEI GUKOV, AND DU PEI

Abstract. We propose and prove the Verlinde formula for the quantization of the Higgs bundle moduli spaces and stacks for any simple and simply-connected group. This generalizes the equivariant Verlinde formula for the case of $\mathrm{SU}(n)$ proposed in [GP]. We further establish a Verlinde formula for the quantization of parabolic Higgs bundle moduli spaces and stacks.

1. Introduction

Let G be a simple and simply-connected compact Lie group. Let M be the moduli space of semi-stable holomorphic $G^{\mathbb{C}}$-bundles on Σ, where Σ is a smooth algebraic curve over complex numbers. Let L be the determinant bundle over M. In order to recall the Verlinde formula for the dimension of the space of holomorphic sections of L^k over M for any non-negative integer level k, we specify some notations.

Let $T \subset G$ be the maximal torus and W the Weyl group. Then the space of conjugacy classes is given by

$$C_G = G/\mathrm{Ad}G = T/W.$$

We now consider the killing form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g} = \mathrm{Lie(G)}$, normalized so that the longest root has norm squared 2. Let $\mathfrak{t} = \mathrm{Lie(T)}$. Pick a level $k \in \mathbb{Z}_+$ and consider the map

$$i_k : \mathfrak{t} \to \mathfrak{t}^*,$$

given by

$$i_k(\xi)(\eta) = (k+h)\langle \xi, \eta \rangle,$$

where h is the dual Coxeter number. This descends to a map

$$\chi : T \to T^*.$$

Let

$$F = \ker \chi.$$

Let $\rho = \frac{1}{2}\sum_{\alpha \in \mathfrak{R}_+} \alpha$, where $\mathfrak{R}_+$ is the set of positive roots of $\mathfrak{g}$. Put

$$F_\rho = \chi^{-1}(e^{2\pi i \rho}).$$

As usual we denote by

$$\Delta = \prod_{\alpha \in \mathfrak{R}_+} 2 \sin\left(\frac{i\alpha}{2}\right)$$

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the Weyl denominator. Define the following function

$$\theta(f) = \frac{\Delta(f)^2}{|F|}.$$

Theorem 1 (Thaddeus, Bertran-Szenes, Donaldson, Faltings, Teleman)**.** *For all non-negative integers k we have that*

$$\dim H^0(M, L^k) = \sum_{f \in F_\rho^{\mathrm{reg}}/W} \theta(f)^{1-g}.$$

In order to generalize this formula to the case of Higgs bundle moduli spaces, we consider the $\mathbb{C}^*$-action this moduli space has, which provides a grading on the infinite-dimensional quantization of this moduli space.

Let M_H be the moduli space of semi-stable $G^{\mathbb{C}}$-Higgs bundles on Σ. By abuse of notation, we also let L be the determinant line bundle over M_H. The $\mathbb{C}^*$-action on M_H naturally lifts to a $\mathbb{C}^*$-action on L. We therefore get a decomposition

$$H^0(M_H, L^k) \cong \bigoplus_{n=0}^{\infty} H^0_n(M_H, L^k),$$

into the finite-dimensional subspace on which $\mathbb{C}^*$ acts by the character of the n tensor power of the identity representation of $\mathbb{C}^*$ on $\mathbb{C}$. We can therefore define the formal sum in the variable t by

$$\dim_t H^0(M_H, L^k) = \sum_{n=0}^{\infty} \dim H^0_n(M_H, L^k) t^n.$$

Following now [TW], consider the following deformation

$$\chi_t = \chi \prod_{\alpha \in \mathfrak{R}_+} \left(\frac{1 - te^{\alpha}}{1 - te^{-\alpha}} \right)^{\alpha} : T \to T^*.$$

Let

$$F_{\rho,t} = \chi_t^{-1}(e^{2\pi i \rho}) \subset T.$$

Further consider the function

$$\theta_t = (1-t)^{\mathrm{rk} G} \frac{1}{|F|} \frac{\prod_\alpha (1-e^{\alpha})(1-te^{\alpha})}{\det H_t^\dagger},$$

where the product is taken over all of the roots of $\mathfrak{g} = \mathrm{Lie}(G)^{\mathbb{C}}$ and $H_t^\dagger$ is the endomorphism of Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ corresponding to the Hessian H_t of the function

$$D_t(\xi) = \frac{1}{2}(k+h)\langle \xi, \xi \rangle - \mathrm{tr}_{\,\mathfrak{g}}(\mathrm{Li}_2(te^{\xi})) + \langle \rho, \xi \rangle. \tag{1}$$

In this paper we prove the following *Verlinde formula for Higgs bundles*

Theorem 2. *For all $g > 1$ and all non-negative integers k we have that*

$$\dim_t H^0(M_H, L^k) = \sum_{f \in F_{\rho,t}^{\mathrm{reg}}/W} \theta_t(f)^{1-g}. \tag{2}$$

We want to stress that our proof relies strongly on the index formula of Teleman and Woodward [TW] and on a series of beautiful results by Teleman that provide deep understanding of natural sheaf cohomology groups over the moduli spaces and stacks of G-bundles on curves [T1, T3, T4, T5], as well as further developments by him with Fishel and Grojnowski [FGT] and with Frenkel [FT].

When $g > 1$ then a codimension argument shows that

$$H^0(M_H, L^k) = H^0(\mathfrak{M}_H, \mathfrak{L}^k) \tag{3}$$

where $\mathfrak{M}_H$ is the stack of $G^{\mathbb{C}}$-Higgs bundles and we denoted the determinant line bundle over the stack by $\mathfrak{L}$. In fact, it is for the stack that the Verlinde formula (2) is deduced and then the above identification of vector spaces gives the theorem. We remark that the right hand side of (2) is, in fact, for all genus g, equal to the index of $\mathfrak{L}^k$ over the stack $\mathfrak{M}_H$, but for genus zero and one, $\mathfrak{M}_H$ is a derived stack and (3) is no longer valid. The proof of Theorem 2 is presented in Section 3, where we also establish the Verlinde formula for the R-twisted Higgs bundles. Please see Section 3 for definition and details.

For genus $g \leq 1$, the formula (2) remains true provided one replaces the left-hand side with the index of the k-th power of the determinant line bundle $\mathfrak{L}$ over the stack of $G^{\mathbb{C}}$-Higgs bundles $\mathfrak{M}_H$. This is also detailed in Section 3.

Let us now consider the parabolic generalization. Let $x_1, \ldots, x_m$ be distinct points on Σ and $D = \sum_i x_i$ be a divisor on Σ. Let λ be a level k integrable dominant weight for G, *e.g.*

$$\langle \lambda, \vartheta \rangle \leq k,$$

where ϑ is the highest root of G. We denote by Λ_k the set of these level-k integrable dominant weights.

Let P be the moduli space of parabolic $G^{\mathbb{C}}$ bundles for a choice of parabolic structures, determined by a parabolic sub-group $P_i \subset G^{\mathbb{C}}$ at x_i. We let P_H be the corresponding moduli space of parabolic $G^{\mathbb{C}}$-Higgs bundles with respect to the same weight vectors. Let L_k be the the level k determinant line bundle over P and P_H determined by weight $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ assigned to each of the marked point $(x_1, \ldots, x_n)$ respectively. The compatibility condition between λ_i and P_i we must require, is that P_i leaves λ_i invariant, for the line bundle L_k to exist. Again, the natural $\mathbb{C}^*$-action on P_H then lifts to L_k and we have that

$$H^0(P_H, L_k) \cong \bigoplus_{n=0}^{\infty} H^0_n(P_H, L_k)$$

and we can again define the dimension to be the formal generating series, in a variable t say, for the dimensions of these finite-dimensional $\mathbb{C}^*$-weight spaces as in the non-parabolic case. We obtain the following *Verlinde formula for parabolic Higgs bundles.*

Theorem 3. *For $g > 1$ we have that*

$$\dim_t H^0(P_H, L_k) = \sum_{f \in F^{\mathrm{reg}}_{\rho,t}/W} \theta_t(f)^{1-g} \prod_i \Theta_{\lambda_i,t}(f), \tag{4}$$

where $\Theta_{\lambda,t}$ is defined for a pair of a level-k highest weight λ of $G^{\mathbb{C}}$ and a parabolic subgroup P' as follows

$$\Theta_{\lambda,t} = \sum_{w\in W} \frac{(-1)^{l(w)}e^{w(\lambda+\rho)}}{\Delta \prod_{\alpha\in\mathfrak{R}'_+}(1-te^{w(\alpha)})}$$

and $\mathfrak{R}'_+$ is the subset of positive roots of $\mathfrak{g}$ determined by P'.

The same remarks concerning the proof and its R-twisted generalizations of Theorem 2 also applies to the proof of this Theorem and its R-twisted generalizations. Please see in Section 4.

We further can show that these dimension formulae have the following $1+1$-dimensional TQFT like properties (in fact, it is the indices over the *stack* that truly has the TQFT behavior, reaffirming the advantage of favoring the moduli stack over the moduli space.) We define $d_{g,n}(\vec{\lambda})$ to be the right-hand side of (4), which is also equal to the index of the k-th power of the determinant bundle over the moduli stack of $G^{\mathbb{C}}$-Higgs bundles as we will see in Section 6, and let

$$d_\lambda = P_{t^{1/2}}(BG_\lambda)$$

where $P_\bullet$ is the Poincare polynomial.

Theorem 4. *Assume that all P_i's used are Borel subgroups of $G^{\mathbb{C}}$. For any g, g_1, g_2 and n, n_1, n_2 non-negative integers and all vectors $\vec{\lambda}, \vec{\lambda}_1, \vec{\lambda}_2$ of level k integrable dominant weights, we have that*

$$d_{g+1,n-2}(\vec{\lambda}) = \sum_{\lambda\in\Lambda_k} d_{g,n}(\vec{\lambda},\lambda,\lambda^*)d_\lambda$$

and

$$d_{g_1+g_2,n_1+n_2}(\vec{\lambda}_1,\vec{\lambda}_2) = \sum_{\lambda\in\Lambda_k} d_{g_1,n_1+1}(\vec{\lambda}_1,\lambda)d_{g_2,n_2+1}(\vec{\lambda}_2,\lambda^*)d_\lambda.$$

In section 6 we provide a proof of this Theorem and we further discuss its generalization to arbitrary choices of P_i's.

This theorem strongly suggest that there should be a kind of deformed Cohomological Field Theory associated to the corresponding bundles, by the pushforward from the universal $G^{\mathbb{C}}$-Higgs bundle stack which fibers over the the moduli space of curves. In fact, one can expect that they will satisfy nice factorization properties (indicated by the above recursion relations for the d's) over the Deligne-Mumford compactifications of the moduli space of curves as was establish by Tsuchiya, Ueno and Yamada in [TUY] when one considers just moduli space of $G^{\mathbb{C}}$-bundles. If this would be possible, most likely one can deform the constructions of [AGO] and provides a kind of deformed CohFT description of the Chern classes of the resulting bundles and compute them by Topological Recursion.

From a physical point of view one should in fact expect that one can push these constructions all the way to the construction of a kind of deformed modular functor, as was done by Andersen and Ueno in [AU1, AU2] in the case of $G^{\mathbb{C}}$-bundles. In particular, this would require the construction of a projectively flat TUY connection in this Higgs bundle case. Pushing this further, one should be able to use this to provide a full geometric construction of Complex Quantum Chern-Simons theory for the group $G^{\mathbb{C}}$ and possibly connect it with the theory construction by Andersen and Kashaev in [AK1, AK2, AK3]. This would then require the development of the isomorphisms provided by Andersen and Ueno for $G = SU(n)$ in [AU3, AU4].

This paper is organized as follows. We recall the Teleman-Woodward index formula in section 2. In Section 3 we provide the proof of Theorem 2 concerning moduli spaces of Higgs bundles and we discuss a possible generalization to the R-twisted case. We give the proof of Theorem 3 regarding the parabolic case in Section 4, where we also discuss R-twisting. In particular, we also propose the relevant deformation of the usual character formulae relevant in this Higgs bundles case. In section 5, we discuss the special cases of g being zero and one, together with some discussion of the rank one case in genus 2. In Section 6, we provide a 2D TQFT viewpoint on these formulae and prove Theorem 4.

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While the paper was being completed, we learned about the independent work from Daniel Halpern-Leistner on the same subject. After communicating with him, we decided to coordinate the arXiv submission. We would like to thank him for his kindness and understanding in this coordination process. Our work was motivated by the TQFT point of view, which drove us to considering parabolic Higgs bundles and to establish a gluing formula for these dimensions. On the other hand, Halpern-Leistner's paper [HL] considered Higgs fields that are twisted by arbitrary line bundles for arbitrary reductive G (with trivial fundamental group).

2. The Teleman-Woodward Index formula

We now briefly review the results of [TW], setting up the stage for the proofs of the main theorems in later sections. An important lesson from [TW] is that one should not work with the moduli space, but rather with the *stack*, as the index theory on the latter behaves much more nicely. Therefore, let $\mathfrak{M}$ be the moduli stack of $G^{\mathbb{C}}$-bundles over Σ. By abuse of notation, we also denote the determinant line bundle over $\mathfrak{M}$ by L.

The K^0-group of $\mathfrak{M}$ is generated by the even Atiyah-Bott generators, which we now recall following [AB] and [TW]. Recall that the there is a universal principle $G^{\mathbb{C}}$-bundle E over $\mathfrak{M}\times\Sigma$, and for a $G^{\mathbb{C}}$ representation V, we denote the associated bundle $E(V)$. For a point $x\in\Sigma$, we denote by E_xV the restriction of $E(V)$ to $\{x\}\times\mathfrak{M}$, and $E_\Sigma V$ the total direct image of $E(V)\otimes K_\Sigma^{1/2}$ along Σ.

We have the following index theorem from [TW], which we only state for the even classes needed in this paper.

Theorem 5 (Teleman and Woodward)**.** *The index formula for even classes is given by*

$$\mathrm{Ind}\left(\mathfrak{M}; L^k\otimes\exp\left[s_1E_\Sigma V_1+\ldots+s_nE_\Sigma V_n\right]\otimes E_xV\right)=\sum_{f\in F_\rho^{\mathrm{reg}}/W}\theta_{\vec{s}}(f_{\vec{s}})^{1-g}\cdot\mathrm{Tr}_V\left(f_{\vec{s}}\right).$$

Here $\vec{s} = (s_1, \ldots, s_n)$, while $\theta_{\vec{s}}$ and $f_{\vec{s}}$ is defined as follows. Consider the multi-parameter transformations on $G^{\mathbb{C}}$ given by

$$g \mapsto m_{\vec{s}}(g) = g \cdot \exp\left[\sum_i s_i \nabla \operatorname{Tr}_{V_i}(g)\right],$$

with the gradient in the bilinear form given by $k + h$ times our normalized Killing form, which is used through the rest of this section as the metric on T. For small $\vec{s}$, there are unique solutions $f_{\vec{s}} \in T$, continuous in $\vec{s}$, to the equations

$$m_{\vec{s}}(f_{\vec{s}}) = f \tag{5}$$

for $f \in F_\rho^{\text{reg}}$. Let $H_{\vec{s}}$ be the Hessian of the function

$$\xi \mapsto \frac{k+h}{2}\langle \xi, \xi\rangle + \sum_i s_i \operatorname{Tr}_{V_i}(e^{\xi}) + \langle \rho, \xi\rangle,$$

which can be converted into an endomorphism $H_{\vec{s}}^{\dagger}$ on $\mathfrak{t}$. Now define the function $\theta_{\vec{s}}$ on T by

$$\theta_{\vec{s}}(f) = \det{}^{-1}\left[H_{\vec{s}}(f)^{\dagger}\right] \cdot \frac{\Delta(f)^2}{|F|}. \tag{6}$$

Of particular interest to us is the series $\sum_p s_p E_\Sigma^* V_p$ given by

$$\lambda_{-t}(E_\Sigma^* \mathfrak{g}) = \exp\left[-\sum_{p>0} \frac{t^p E_\Sigma^* \psi^p(\mathfrak{g})}{p}\right],$$

where the ψ^p are the Adams operations.

3. Proof of the Verlinde formula for Higgs bundles

We recall that a principal $G^{\mathbb{C}}$-Higgs bundle over Σ is the pair (E, Φ), where E is a holomorphic principal $G^{\mathbb{C}}$-bundle and $\Phi \in H^0(\Sigma, \operatorname{ad}(E) \otimes K)$. For moduli space of such we have used the notation M_H, and for the stack, $\mathfrak{M}_H$.

We will also consider the moduli space (stack) of R-twisted Higgs bundles $M_H^{(R)}$ ($\mathfrak{M}_H^{(R)}$), *e.g.* pairs of a $G^{\mathbb{C}}$-bundle E and an R-twisted Higgs field in

$$\Phi \in H^0(\Sigma, \operatorname{ad}(E) \otimes K^{R/2}),$$

where R is an integer. One gets the usual Higgs bundle for $R = 2$ and the "co-Higgs bundle" for $R = -2$. See *e.g.* [SR] and references therein for more discussion about the latter.

Let $N_{\mathfrak{M}}^{(R)}$ be the normal bundle of $\mathfrak{M} \subset \mathfrak{M}_H^{(R)}$. Then the stack $\mathfrak{M}_H^{(R)}$ can be identified as

$$\mathfrak{M}_H^{(R)} = \operatorname{Spec} \operatorname{Sym}(N_{\mathfrak{M}}^{(R)*}). \tag{7}$$

By abuse of notation we also denote the determinant line bundle over both $\mathfrak{M}$ and $\mathfrak{M}_H^{(R)}$ simply by $\mathfrak{L}$. When we consider these bundles and their cohomology groups, it should be clear, from the notation usage and the context, which bundle is being referred to.

Proposition 1. *We have the following identification of cohomology groups*

$$H^i(\mathfrak{M}_H^{(R)}, \mathfrak{L}^k) = H^i(\mathfrak{M}, S^*(N_{\mathfrak{M}}^{(R)*}) \otimes \mathfrak{L}^k)$$

for all i.

Proof. We consider the spectral sequence for the fibration $\mathfrak{M}_H^{(R)} \to \mathfrak{M}$ and use (7) to provide a description of the fibers. From this, it is clear that the spectral sequence collapses at the E_2-page, and since the cohomology groups along the fibers concentrate in degree zero, we get the claimed result. □

The right-hand side of formula (2) is very reminiscent of the above index formula of Teleman and Woodward. In particular, if one sets $R = 0$, as one can explicitly check, the formula coincides with the index of $\lambda_{-t}(\Omega) \otimes L^k$, where

$$\Omega = R\pi_*(E(\mathfrak{g}) \otimes K) \tag{8}$$

is the cotangent complex of $\mathfrak{M}$.

In fact the following theorem for general R can be proved simply by a straightforward computation.

Theorem 6. *The index of the K-theory class of $S_t(N_{\mathfrak{M}}^{(R)*}) \otimes \mathfrak{L}^k$ on the stack $\mathfrak{M}$ is given by*

$$\operatorname{Ind}\left(\mathfrak{M}, S_t(N_{\mathfrak{M}}^{(R)*}) \otimes \mathfrak{L}^k\right) = \sum_{f \in F_{\rho,t}^{\mathrm{reg}}/W} \theta_{t,R}(f)^{1-g}, \tag{9}$$

with $\theta_{t,R}$ now given by

$$\theta_{t,R} = (1-t)^{(R-1)\mathrm{rk}G} \frac{1}{|F|} \frac{\prod_\alpha (1-e^\alpha)(1-te^\alpha)^{R-1}}{\det H_t^\dagger}.$$

Proof. Define the K-theory class

$$\Omega_R = R\pi_*(E(\mathfrak{g}) \otimes K^{1-R/2}) \tag{10}$$

on $\mathfrak{M}$ and observe that

$$\Omega_R[1] = N_{\mathfrak{M}}^{(R)*}$$

at the level of K-theory. This follows from the exact sequence in Remark 2.7 in [BR]. Now we rewrite the class Ω_R in terms of the standard Atiyah-Bott generators using the family index theorem as follows.

Let x^* be the generator of $H^2(\Sigma)$, Poincare dual to the point x, *e.g.* $x^* \cap [\Sigma] = 1$.

$$\begin{aligned}
\mathrm{ch}(\Omega_R) &= \mathrm{Td}(\Sigma) \cup \mathrm{ch}(E(\mathfrak{g}) \otimes K^{1-\frac{R}{2}}) \cap [\Sigma] \\
&= (1 + (1-g)x^*) \cup \mathrm{ch}(E(\mathfrak{g})) \cup (1 - (1-g)(2-R)x^*) \cap [\Sigma] \\
&= (1 + (g-1)(R-1)x^*) \cup \mathrm{ch}(E(\mathfrak{g})) \cap [\Sigma] \\
&= \mathrm{ch}(E_\Sigma \mathfrak{g}) + (1-R)(g-1)\mathrm{ch}(E_x \mathfrak{g}).
\end{aligned}$$

From this we conclude that, at the level of K-theory, we have

$$\Omega_R = (E_\Sigma^* \mathfrak{g}) \oplus (1-R)(g-1)(E_x^* \mathfrak{g}). \tag{11}$$

Now, it is easy to compute the index of $\lambda_{-t}(\Omega_R) \otimes L^k$. Since the coefficients of $E_\Sigma \mathfrak{g}$ in (11) has no R-dependence, the equation (5) has no dependence on R and the solution set just becomes $F_{\rho,t}$. The place where R enters is through the factor

$$\mathrm{Tr}_{\lambda_{-t}(\mathfrak{g})^{(1-R)(g-1)}} = (1-t)^{(1-R)(g-1)\,\mathrm{rk}\,G} \prod_\alpha (1-te^\alpha)^{(1-R)(g-1)} \tag{12}$$

in the summand of the formula in Theorem 6. □

Theorem 7. *For $g > 1$, $R \geq 2$ and all non-negative integers i and k we have that*

$$H^i(\mathfrak{M}, S^*(N_{\mathfrak{M}}^{(R)*}) \otimes \mathfrak{L}^k) = 0.$$

We present here a proof of Theorem 7. The key idea of using the arguments presented in Section 4 of [FT], but now twisted by $\mathcal{O}(k)$ over the thick flag variety, is due to Constantin Teleman, and was past on to us by Nigel Hitchin. Hence, our arguments rely entirely on the beautiful, substantial and deep work by first of all by Teleman and further his collaboration with in the first instance Fishel and Grojnowski [FGT] and second with Frenkel [FT]. Below we present our attempt to fill in the details of Teleman's suggestion. We strongly advice the reader to consult Teleman's appendix to [HL].

Proof of Theorem 7. We present here the proof for the case where $R = 2$, noting that the proof for $R > 2$ is completely similar, one just have to twist with the appropriate power of K, which by the remarks in the second last paragraph on page 12 in [FGT] does not cause any essential change in the arguments.

First we pick a point on p on Σ and let Σ° be the complement of $\{p\}$ on Σ. We shall further fix a formal coordinate z centred at p, in which we do all Laurent expansions. We use the presentation of the stack $\mathfrak{M}$ as the double quotient $\mathfrak{G}_0\backslash\mathbf{X}$, where $\mathbf{X} = \mathfrak{G}/G^{\mathbb{C}}[\Sigma^\circ]$, $\mathfrak{G} = G^{\mathbb{C}}((z))$ and $\mathfrak{G}_0 = G^{\mathbb{C}}[[z]]$. The space $\mathbf{X}$ is be the thick flag variety as in [FT] and [FGT]. Over $\mathbf{X}$ we consider the bundle

$$\mathfrak{g}(\Sigma^\circ)_b = \left(\mathfrak{G} \times \mathfrak{g}(\Sigma^\circ)\right)/G^{\mathbb{C}}[\Sigma^\circ]$$

and the trivial bundle

$$\widetilde{\mathfrak{g}}_\tau = \mathbf{X} \times \widetilde{\mathfrak{g}},$$

where $\widetilde{\mathfrak{g}} = \mathfrak{g}((z))/\mathfrak{g}[[z]]$. Further we present the tangent complex of $\mathfrak{M}$ as a two-step resolution

$$\partial : \mathfrak{g}[\Sigma^\circ]_b \to \widetilde{\mathfrak{g}}_\tau$$

where $\partial_\varphi = \mathrm{Ad}_\varphi \circ \mathcal{E}_p$ at $\varphi \in \mathfrak{G}_0$ and $\mathcal{E}_p$ is Laurent expansion at the p. For each non-negative integer r, we get the following complex

$$0 \to \Lambda^r\mathfrak{g}[\Sigma^\circ]_b \to \Lambda^{r-1}\mathfrak{g}[\Sigma^\circ]_b \otimes S^1(\widetilde{\mathfrak{g}})_\tau \to \ldots \to S^r(\widetilde{\mathfrak{g}})_\tau \to 0. \tag{13}$$

In parallel to (4.3) of [FT] we get that the $\mathfrak{G}_0$-equivariant hyper-cohomology of this differential graded complex with $\mathfrak{g}[\Sigma^\circ]$ in degree -1 gives

$$H^q(\mathfrak{M}, S^rT \otimes \mathcal{L}^k) = \bigotimes_{s+t=r} \mathbb{H}^q_{\mathfrak{G}_0}\left(\mathbf{X}, \Lambda^s\mathfrak{g}[\Sigma^\circ]_b \otimes S^t(\widetilde{\mathfrak{g}}^{(n)})_\tau \otimes \mathcal{O}(k)\right).$$

As in [FT], we filter by s-degree to get a spectral sequence, whose E_1-page is

$$E_1^{-s,q} = H^q_{\mathfrak{G}_0}\left(\mathbf{X}, \Lambda^s\mathfrak{g}[\Sigma^\circ]_b \otimes S^{r-s}(\widetilde{\mathfrak{g}}^{(n)})_\tau \otimes \mathcal{O}(k)\right). \tag{14}$$

In order to compute this equivariant cohomology, we consider the Leray spectral sequence for the fibration $\mathfrak{M} \to BG[[z]]$, which on the E_2-page

$$\widetilde{E}_2^{p,q} = H^p(B\mathfrak{G}_0, H^q\left(\mathbf{X}, \Lambda^s\mathfrak{g}[\Sigma^\circ]_b \otimes \mathcal{O}(k)\right) \otimes S^{r-s}(\widetilde{\mathfrak{g}}^{(n)}).$$

By remark (8.10) in [T3], we know that Theorem (0,5') also of [T3] applies to $H^q\left(\mathbf{X}, \Lambda^s\mathfrak{g}[\Sigma^\circ]_b \otimes \mathcal{O}(k)\right)$ we therefore get that

$$\widetilde{E}_2^{p,q} = \bigoplus_{\lambda\in\Lambda_k} \langle H^q\left(\mathbf{X}, \Lambda^s\mathfrak{g}[\Sigma^\circ]_b \otimes \mathcal{O}(k)\right) \mid \mathbf{H}_\lambda^{(k)}\rangle \otimes H^p_{\mathfrak{G}}(\mathbf{H}_\lambda^{(k)} \otimes S^{r-s}(\widetilde{\mathfrak{g}})),$$

where the multiplicity space

$$\langle H^q\left(\mathbf{X}, \Lambda^s\mathfrak{g}[\Sigma^\circ]_b \otimes \mathcal{O}(k)\right) \mid \mathbf{H}_\lambda^{(k)}\rangle = H^q(\mathfrak{M}, \Lambda^s_{\mathbf{X}} \otimes V_\lambda \otimes L^k), \tag{15}$$

where $\Lambda^s_{\mathbf{X}}$ denotes the descended bundle from $\mathbf{X}$ of $\Lambda^s\mathfrak{g}[\Sigma^\circ]_b$. We now use Theorem 3 of [T2] combined with Remark (8.10) from the same reference to conclude that

$$H^q(\mathfrak{M}, \Lambda^s_{\mathbf{X}} \otimes V_\lambda \otimes L^k) = H^q_{G[\Sigma^0]}(\mathbf{H}_0^{(k)} \otimes \Lambda^s\mathfrak{g}[\Sigma^\circ] \otimes V_\lambda). \tag{16}$$

Now observe that there is a perfect pairing

$$(\mathfrak{g}((z))/\mathfrak{g}[[z]]) \times \mathfrak{g}[[z]] \to \mathbb{C},$$

using a combination of the invariant inner product in $\mathfrak{g}$, the product of Laurent series and the residue of the resulting complex value Laurent series. But then we conclude that

$$H^*_{G[[z]]}\left(S^p(\mathfrak{g}((z))/\mathfrak{g}[[z]]) \otimes \mathbf{H}_\lambda^{(k)}\right) = H^*_{G[[z]]}\left(S^p(\mathfrak{g}[[z]]^*_{\mathrm{res}}) \otimes \mathbf{H}_\lambda^{(k)}\right).$$

Using the argument in (4.11) of [FGT], we get that

$$H^*_{\mathrm{res}}(\mathfrak{g}[z], \mathfrak{g}; S\mathfrak{g}[[z]]^*_{\mathrm{res}} \otimes \mathbf{H}_\lambda^{(k)}) = H^*_{G[[z]]}\left(S^p(\mathfrak{g}[[z]]^*_{\mathrm{res}}) \otimes \mathbf{H}_\lambda^{(k)}\right).$$

But by Theorem E of [FGT] we get that

$$H^p_{\mathrm{res}}(\mathfrak{g}[z], \mathfrak{g}; S\mathfrak{g}[[z]]^*_{\mathrm{res}} \otimes \mathbf{H}_\lambda^{(k)}) = H^p\left(\mathfrak{g}[[z]], \mathfrak{g}; S^p(\mathfrak{g}[[z]])^*) \otimes \mathbf{H}_\lambda^{(k)}\right) = 0 \tag{17}$$

for $p > 0$. We conclude thus that

$$E_1^{-s,q} = \bigoplus_{\lambda \in \Lambda_k} H^q_{G[\Sigma^0]}(\mathbf{H}_0^{(k)} \otimes \Lambda^s\mathfrak{g}[\Sigma^\circ] \otimes V_\lambda) \otimes \left(\mathbf{H}_\lambda^{(k)} \otimes S^{r-s}(\widetilde{\mathfrak{g}}^{(n)})\right)^{\mathfrak{G}_0}. \tag{18}$$

We see now by the above and the following proposition that $E_1^{-s,q}$ is concentrated in the region below the diagonal with $q \leq s$, ensuring the vanishing of all positive degree cohomologies in the abutment.

The vanishing of the cohomology in negative degrees also follows from the fact that, for $g > 1$, $\mathfrak{M}$ is "very good", as established in [BD].

□

We propose the following generalization of Teleman's group cohomology results in [T3].

Proposition 1. *We have the following vanishing of the group cohomology*

$$H^q_{G[\Sigma^\circ]}(\mathbf{H}_0^{(k)} \otimes \Lambda^s\mathfrak{g}[\Sigma^\circ] \otimes V_\lambda) = 0 \tag{19}$$

for all $q > s$.

We will follow the lines of arguments presented in [T1] and [T2] in order to establish the vanishing of this cohomology by a combination of relations between the group cohomology and the Lie-algebra cohomology, invariance of the Lie-algebra cohomology under degeneration of the curve Σ° to a nodal curve, followed by the behavior of the Lie algebra cohomology under resolution of the nodal curve and finally a control of the cohomology for once-punctured $\mathbb{P}^1$. However, these steps are best performed in reverse order.

Let Σ' be a smooth curve obtained from a compact curve by removing one point. We start by recalling that Proposition 10.5 from [FGT] applies to Σ', but as remarked in [FT] on page 12, one can twist with a line bundle, which we in this case choose to be $T\Sigma'$ to obtain that

Theorem 8 (Fishel, Grojnowski, Teleman). *We have a natural isomorphism*

$$H^*(\mathfrak{g}[\Sigma'],\Lambda^s(\mathfrak{g}[\Sigma'])) = H^*(\mathbf{X}_{\Sigma'},\Lambda^s\mathfrak{g}[\Sigma']_b)\otimes H^*(G(\Sigma),\mathbb{C}).$$

Further Theorem D,(i) form the same reference applied to Σ' gives that

Theorem 9 (Fishel, Grojnowski, Teleman). *The ring $H^*(\mathbf{X}_{\Sigma'},\Lambda^*\mathfrak{g}[\Sigma']_b)$ is the free skew-commutative algebra generated by copies of $\Omega^0[T\Sigma']$ and $\Omega^0[\Sigma']$ in*

$$H^m(\mathbf{X}_{\Sigma'},\Lambda^m\mathfrak{g}[\Sigma']_b) \text{ and } H^m(\mathbf{X}_{\Sigma'},\Lambda^{m+1}\mathfrak{g}[\Sigma']_b)$$

respectively.

We now propose the following generalization of Theorem 2.4 in [T2].

Theorem 10. *We have that*

$$H^*(\mathfrak{g}[\Sigma'],\mathbf{H}_0^{(k)}\otimes\Lambda^s\mathfrak{g}[\Sigma']\otimes V_\lambda)\cong H^0(\mathfrak{g}[\Sigma'],\mathbf{H}_0^{(k)}\otimes V_\lambda)\otimes H^*(\mathfrak{g}[\Sigma'],\Lambda^s(\mathfrak{g}[\Sigma'])).$$

We will now assume Theorem 10 and following [T2], consider the van Est spectral sequence which relates the cohomology of $G[\Sigma']$ and that of the Lie algebra $\mathfrak{g}[\Sigma']$. We recall Propostion 6.1 from this reference

Proposition 2 (Teleman). *There is a spectral sequence with*

$$E_2^{p,q} = H^p_{G[\Sigma']}(\mathbf{H}_0^{(k)}\otimes\Lambda^s\mathfrak{g}[\Sigma']\otimes V_\lambda\otimes H^q_s(G[\Sigma']))$$

which converges to $H^*(\mathfrak{g}[\Sigma'],\mathbf{H}_0^{(k)}\otimes\Lambda^s\mathfrak{g}[\Sigma']\otimes V_\lambda)$ and is compatible with products.

It is now simple to argue as in section VII.1 and VII.2 in [T2] leading up to (7.3), that the edge homomorphism in the van Est spectral sequence from Proposition 2

$$H^p(\mathfrak{g}[\Sigma'],\mathbf{H}_0^{(k)}\otimes\Lambda^s\mathfrak{g}[\Sigma']\otimes V_\lambda)\to H^p_{G[\Sigma']}(\mathbf{H}_0^{(k)}\otimes\Lambda^s\mathfrak{g}[\Sigma']\otimes V_\lambda))\otimes H^q_s(G[\Sigma'])$$

will be surjective for $p > s$, hence we will get the vanishing of the group cohomology in Proposition 1.

Proof of Theorem 10. This proof proceeds as the proof of theorem 2.4. in [T2]. We can place $\overline{\Sigma}'$ in a family of curves over some parameter space, with a sections through $\overline{\Sigma}'-\Sigma'$ over some point in the parameter space, in such a way that the fiber over some other point, say Σ_0, is a nodal degeneration of $\overline{\Sigma}'$, whose normalization $\widetilde{\Sigma}_0$ is a copy of $\mathbb{P}^1$. Let Σ_0° denote the complement of the value of the section in Σ_0. We assume this point is not one of the g nodes of Σ_0. We can now apply Theorem 2.3 in [T2] to obtain

$$\dim H^q(\mathfrak{g}[\Sigma^\circ],\mathfrak{g};\mathbf{H}_0^{(k)}\otimes\Lambda^s\mathfrak{g}[\Sigma_0^\circ]\otimes V_\lambda) = \dim H^q(\mathfrak{g}[\Sigma_0^\circ],\mathfrak{g};\mathbf{H}_0^{(k)}\otimes\Lambda^s\mathfrak{g}[\Sigma_0^\circ]\otimes V_\lambda).$$

Theorem 2.3 in [T2] applies directly in the case where $s = 0$ and the proof applies to the case $s > 0$ as well, as we will now argue. We consider the following filtration on $\mathfrak{g}(\Sigma_t^\circ)$

$$\mathcal{F}^d(\mathfrak{g}(\Sigma_t^\circ)) = \{\xi\in\mathfrak{g}(\Sigma_t^\circ)\mid \mathrm{ord}_{p_i}(\xi)\le d\},$$

where t runs through the parameter space. We observe that the action of $\mathfrak{g}[\Sigma_t^\circ]$ on $\mathbf{H}_0^{(k)}\otimes\Lambda^s\mathfrak{g}[\Sigma_0^\circ]\otimes V_\lambda$ is filtration-preserving, where we use the filtration on $\mathbf{H}_0^{(k)}$ induced by the energy grading used in [TUY]. Furthermore we see that the associated graded space to this filtrated space $\mathcal{F}^d(\mathfrak{g}[\Sigma_t^\circ])$, is finite-dimensional in each piece. The discussion in Section 4 and 5 of [T2] can now be applied as is for the module $\mathfrak{g}[\Sigma_t^\circ]$ on $\mathbf{H}_0^{(k)}\otimes\Lambda^s\mathfrak{g}[\Sigma_0^\circ]\otimes V_\lambda$, to establish that the homology sheaves of our family of pointed curves of the affine line are first coherent and secondly locally free, which gives the desired result.

Continuing with the argument in [T2], now entering the part contained in Section 3 of that reference, we define the subalgebra $\mathfrak{h} \subset \mathfrak{g}[\Sigma_0^\circ]$ to be the kernel of the evaluation map at the nodes. We then have the following commutative diagram of short exact sequences of Lie-algebras

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathfrak{h} & \longrightarrow & \mathfrak{h} & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathfrak{g}[\Sigma_0^o] & \longrightarrow & \mathfrak{g}[\widetilde{\Sigma}_0^o] & \longrightarrow & \mathfrak{g}^g & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathfrak{g}^g & \longrightarrow & \mathfrak{g}^{2g} & \longrightarrow & \mathfrak{g}^g & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0.
\end{array} \tag{20}$$

We now consider the spectral sequences E and $\widetilde{E}$ for the second and third columns respectively, with coefficients in the relevant modules, together with the induced map between them from the above diagram. Now the argument proceeds exactly as on page 256 and 257 in [T2] , again using the obvious filtration $\mathcal{F}$ from above, to get semi-simplicity in the same way as is done at this place in [T2]. For the once-punctured $\mathbb{P}^1$, we need a replacement of Theorem 3.1 in [T2].

To this end, we can simply apply the proof of Theorem F in [FGT]. Let $X = G^{\mathbb{C}}((z))/G^{\mathbb{C}}[z^{-1}]$. Without the assumption of G being simply-laced, we can still consider the same spectral sequence, which converges to $H^*(\mathfrak{M}(\mathbb{P}^1), \Omega^*(k)) = H^{*,*}(BG, \mathbb{C})$ and whose E_1-page is given by (18), with $\mathbf{X}$ replaced by X, by the same arguments as given until that equation in the above proof. But we know $H^{*,*}(BG, \mathbb{C})$ thus we get Theorem 10 in this simple $\mathbb{P}^1$ case with one marked point, which completes the proof. □

Our other approach to a proof of Theorem 7 is to use the pushforward of $\mathcal{L}^k$ from $\mathfrak{M}$ along the Hitchin map to the Hitchin base. Over the open of the base, where the Hitchin map is a fibration, positivity of $\mathcal{L}^k$ gives vanishing of all non-zero direct images. However, an argument is need to extend this to all fibers of the Hitchin map. Serre duality might actually do this along all fibers, however there could be problems due to very singular and/or non-reduced fibers. We hope to be able to complete this argument at some point.

We expect that the concentration of $H^*_{G[[z]]}\left(S^p(\mathfrak{g}((z))/\mathfrak{g}[[z]]) \otimes \mathbf{H}_\lambda^{(k)}\right)$ in zero degree can also be obtained by an explicit Laplace type argument à la Theorem E directly based on a positivity result for the Laplacian in higher cohomological degrees.

The arguments given in the upper part of page 13 in [FT], which again refer back to [FGT] Theorem A, should provide us with a description of

$$H^0_{G[[z]]}\left(S^p(\mathfrak{g}((z))/\mathfrak{g}[[z]])\otimes \mathbf{H}^{(k)}_\lambda\right)$$

generated over $H^0_{G[[z]]}\left(\mathbf{H}^{(k)}_\lambda\right) = H^0(\mathfrak{M}, \mathcal{L}^k\otimes V_\lambda)$ by generators in $p = m$, where m runs through the exponents of $\mathfrak{g}$, of copies of $(\mathbb{C}((z))/\mathbb{C}[[z]])\otimes(\partial/\partial z)^{\otimes m}$. We expect that combining such generators with arguments along the line of the arguments presented in the proof of Theorem 4.2., regarding the leading differentials action on the generators should allow for a complete description of $H^0(\mathfrak{M}, S^*T\otimes L^k)$.

Our other attempt of a proof of Theorem 7 is to use the pushforward of $\mathfrak{L}^k$ from $\mathfrak{M}$ along the Hitchin map to the Hitchin base. Over the open of the base, where the Hitchin map is a fibration, positivity of $\mathfrak{L}^k$ gives vanishing of all non-zero direct images. However, an argument is need to extend this to all fibers of the Hitchin map. Serre vanishing might actually do this along all fibers, however there could be problems due to very singular and/or non-reduced fibers. We hope to be able to complete this argument at some point.

In any case, by this vanishing of higher cohomology groups, we now get the Complex Verlinde formulae in its R-twisted generalization for the stack

Theorem 11. *For $R, g\geq 2$ and we have the following graded dimension formula*

$$\dim_t H^0\left(\mathfrak{M}, S_t(N^{(R)*}_{\mathfrak{M}})\otimes \mathfrak{L}^k\right) = \sum_{f\in F^{\text{reg}}_{\rho,t}/W} \theta_{t,R}(f)^{1-g}. \tag{21}$$

Since for $R, g\geq 2$, the stack $(\mathfrak{M}^{(R)}_H)^{\text{ss}}$ of semi-stable $G^{\mathbb{C}}$-Higgs bundles has codimension more than two in $\mathfrak{M}^{(R)}_H$, we observe that

$$H^0(\mathfrak{M}^{(R)}_H, \mathfrak{L}^k) = H^0((\mathfrak{M}^{(R)}_H)^{\text{ss}}, \mathfrak{L}^k) = H^0(M^{(R)}_H, L^k).$$

we get the following Verlinde formula for Higgs bundles in the R-twisted case

Theorem 12. *For $R, g\geq 2$ we have the following graded dimension formula*

$$\dim_t H^0\left(M^{(R)}_H, L^k\right) = \sum_{f\in F^{\text{reg}}_{\rho,t}/W} \theta_{t,R}(f)^{1-g}. \tag{22}$$

The complex Verlinde formula, *e.g.* Theorem 2, stated in the introduction now follows by taking $R = 2$.

For the case of $R\leq 0$, we will in general not have the vanishing of non-zero cohomology groups, neither would we be able to directly relate quantization of the stack to that of the space, as $\mathfrak{M}_H$ is a derived stack for $g > 1$. However, we can offer the following alternative interpretation of our index formula in this case, simply by noting that

$$N^{(R)*}_{\mathfrak{M}} = -N^{(2-R)}_{\mathfrak{M}},$$

and that the non-zero cohomology of $\lambda_{-t}(N^{(R)}_{\mathfrak{M}})\otimes\mathfrak{L}^k$ vanishes, thus we conclude

Theorem 13. *For $R\leq 0$ and $g\geq 2$ we get that*

$$\dim H^0\left(\mathfrak{M}, \lambda_{-t}(N^{(R)}_{\mathfrak{M}})\otimes \mathfrak{L}^k\right) = \sum_{f\in F^{\text{reg}}_{\rho,t}/W} \theta_{t,2-R}(f)^{1-g}. \tag{23}$$

For $R = 0$ we recover Theorem 6.4 in [TW], where U is set to be a trivial representation.

4. Proof of the Verlinde formula for parabolic Higgs bundles

We recall that a parabolic Higgs bundle is a pair (B, Φ) where B is a parabolic bundle with reduction of the structure group to a parabolic subgroup P_i over each parabolic point $\{x_1, \ldots, x_n\}$ of B on Σ. Fix the divisor $D = \sum_i x_i$. We assign integral dominant weights $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$, one for each point, such that G_i preserves λ_i, where G_i is defined as $P_i \cap G$. As for the Higgs field, we require that

$$\Phi \in H^0(\Sigma, \mathrm{ad}_s(B) \otimes K^{R/2}(D)),$$

in the R-twisted case, which we will consider. Further, the notation $\mathrm{ad}_s(B)$ refers to the adjoint bundle of B with the reduction of the structure group corresponding to P_i over x_i. We let P be the corresponding moduli space of semi-stable parabolic $G^{\mathbb{C}}$-bundles and P_H the moduli space of such $G^{\mathbb{C}}$-Higgs bundles. The corresponding stacks we denote $\mathfrak{P}$ and $\mathfrak{P}_H^{(R)}$ respectively. By slight abuse of notation, we also use B to denote the universal bundle over $\Sigma \times \mathfrak{P}$, and $B(\mathfrak{g})$ the associated adjoint bundle. Let $N_{\mathfrak{P}}^{(R)}$ denote the normal bundle to $\mathfrak{P}$ in $\mathfrak{P}_H^{(R)}$.

As discussed in the introduction, each weight is required to be of level k and we then get a level-k determinant line bundle L_k over the parabolic moduli space P and $\mathfrak{L}_k$ over the stack $\mathfrak{P}_H^{(R)}$.

Since

$$\mathfrak{P}_H^{(R)} = \mathrm{Spec}\,\mathrm{Sym}(N_{\mathfrak{P}}^{(R)*}), \tag{24}$$

we can use the same techniques as in the non-parabolic case to prove the following proposition.

Proposition 2. *We have the following identification of cohomology groups*

$$H^i(\mathfrak{P}_H^{(R)}, \mathfrak{L}_k) = H^i(\mathfrak{P}, S^*(N_{\mathfrak{P}}^{(R)*}) \otimes \mathfrak{L}_k)$$

for all i.

Using this in cohomological degree zero and $\mathbb{C}^*$-degree n, we in particular get that

$$H^0_n(\mathfrak{P}_H, \mathfrak{L}_k) = H^0(\mathfrak{P}, S^n(N_{\mathfrak{P}}^{(R)*}) \otimes \mathfrak{L}_k).$$

The stack $\mathfrak{P}$ can be viewed as the total space of a fibration over $\mathfrak{M}$

$$G^{\mathbb{C}}/P_1 \times \ldots \times G^{\mathbb{C}}/P_n \longrightarrow \mathfrak{P} \xrightarrow{\pi} \mathfrak{M}. \tag{25}$$

The pullback of L_k to $G^{\mathbb{C}}/P_i$ becomes L_{λ_i}, whose holomorphic sections give the representation V_{λ_i} associated to λ_i.

We consider the diagram of inclusions of stacks for each point $p \in \mathfrak{M}$

$$\begin{array}{ccccc} (\pi^{(R)})^{-1}(p) & \longrightarrow & \mathfrak{P}_H^{(R)} & \xrightarrow{\pi^{(R)}} & \mathfrak{M}_H^{(R)} \\ \uparrow & & \uparrow & & \uparrow \\ \pi^{-1}(p) & \longrightarrow & \mathfrak{P} & \xrightarrow{\pi} & \mathfrak{M}. \end{array}$$

Along the last square in this diagram, we consider the the exact sequence of the tangent complexes and normals

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_{\pi^{(R)}}\mathfrak{P}_H^{(R)} & \longrightarrow & T\mathfrak{P}_H^{(R)} & \xrightarrow{d\pi_H} & \pi^* T\mathfrak{M}_H^{(R)} & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & T_\pi \mathfrak{P} & \longrightarrow & T\mathfrak{P} & \xrightarrow{d\pi} & \pi^* T\mathfrak{M} & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & T_{\pi^{(R)}}\mathfrak{P}_H / T_\pi \mathfrak{P} & \longrightarrow & N_{\mathfrak{P}}^{(R)} & \xrightarrow{d\pi} & \pi^* N_{\mathfrak{M}}^{(R)} & \longrightarrow & 0. \end{array} \tag{26}$$

We have the following fibered product presentation

$$\mathfrak{P} = E_1/P_i \times_{\mathfrak{M}} \ldots \times_{\mathfrak{M}} E_n/P_n, \tag{27}$$

where E_i is the pull back of the universal bundle E under the inclusion map

$$\{x_i\} \times \mathfrak{M} \subset \Sigma \times \mathfrak{M}.$$

From this we immediately conclude that

Lemma 1. *We have the following isomorphism of bundles over* $\mathfrak{P}$

$$T_{\pi^{(R)}}\mathfrak{P}_H / T_\pi \mathfrak{P} = \bigoplus_{i=1}^{n} p_i^*(E_{x_i} \times \mathfrak{g})/P_i \tag{28}$$

where p_i *is projection on the i'th factor in* (27)*.*

Similar to the non-parabolic case, we get an identification

$$N_{\mathfrak{P}}{}^* = \Omega_{R,D}[1]$$

where

$$\Omega_{R,D} = R\pi_* \left(B(\mathfrak{g}) \otimes K^{1-R/2}(-D) \right).$$

This follows again from infinitesimal deformation considerations presented in [BR] for parabolic bundles in Section 6 of that paper.

For $R \le 0$ the stack $\mathfrak{P}$ is entirely derived, but similar to what we did in the non-parabolic case, we can also consider the following K-theory class

$$\Omega'_{R,D} = R\pi_* \left(B(\mathfrak{g}) \otimes K^{1-R/2}(D) \right).$$

We now compute the index of $S_t(N^*_{\mathfrak{P}}) \otimes \mathfrak{L}_k$ over $\mathfrak{P}$ by first pushing forward along π to $\mathfrak{M}$ followed by an application of the Teleman-Woodward index formula over $\mathfrak{M}$. For the classes that we are interested in, we see (details given in the proof of Theorem 14 below) from Lemma 1, that we in particular need to evaluate the index of the classes $L_{\lambda_i} \otimes S_t(T(G/G_i))$ and $L_{\lambda_i} \otimes \lambda_{-t}(T^*(G/G_i))$ over the flag variety G/G_i. These indices, which we will call the "deformed characters", can be computed using a fix-point formula as we will now argue.

Definition 1. *For a weight* λ *left invariant by parabolic subgroup* G'*, we define the deformed characters for the pair* (λ, G')

$$\Theta_{\lambda,t} = \sum_{w \in W} \frac{e^{w(\lambda)}}{\prod_{\alpha>0}\left(1 - e^{-w(\alpha)}\right) \prod_{\alpha \in \mathfrak{R}'_+}\left(1 - t e^{w(\alpha)}\right)} \tag{29}$$

and

$$\Theta'_{\lambda,t} = \sum_{w\in W} \frac{e^{w(\lambda)}\prod_{\alpha\in\mathfrak{R}'_+}\left(1-te^{-w(\alpha)}\right)}{\prod_{\alpha>0}\left(1-e^{-w(\alpha)}\right)}. \tag{30}$$

where $\mathfrak{R}'_+$ is the subset of the positive roots of G corresponding to G'.

For $t=0$, $\Theta_{\lambda,t}$ and $\Theta'_{\lambda,t}$ indeed reproduces the character of the representation R_λ associated with λ. When $G'=T$, $\Theta'_{\lambda,t}$ are, modulo a normalization factor, the Hall-Littlewood polynomials.

Lemma 2. *The deform characters gives the following T-equivariant indices*

$$\mathrm{Ind}_T(G/G', L^k_\lambda \otimes S_tT(G/G')) = \Theta_{\lambda,t} \tag{31}$$

and

$$\mathrm{Ind}_T(G/G', L^k_\lambda \otimes \lambda_{-t}T(G/G')) = \Theta'_{\lambda,t}. \tag{32}$$

Proof. The left action of T on G/T has fixed points labeled by $w\in W$, since by the very definition, the Weyl group is $W=N(T)/T$. Now the the action of a $\xi\in T$ on $T_w(G/T)$ is specified by $e^{w(\alpha)}(\xi)$, where α runs through $\mathfrak{R}_+$, and hence we get that

$$\mathrm{tr}\ (\xi : (L_\lambda)_w\otimes S_t(T_w(G/T)) \to (L_\lambda)_w\otimes S_t(T_w(G/T))) = \frac{e^{w(\lambda)}}{\prod_{\alpha\in\mathfrak{R}_+}\left(1-te^{w(\alpha)}\right)}(\xi).$$

and further by multiplying by the determinant of one minus the cotangential action at w, and summing over W, we get exactly formula (31). For the case where we consider $\lambda_{-t}T(G/G')$, we get from the trace

$$\mathrm{tr}\ (\xi : (L_\lambda)_w\otimes \lambda_{-t}(T_w(G/T)) \to (L_\lambda)_w\otimes S_t(T_w(G/T))) = e^{w(\lambda)}\prod_{\alpha\in\mathfrak{R}_+}\left(1-te^{-w(\alpha)}\right)(\xi).$$

Further, for the case of $T\subset G'\subset G$, we can consider the projection from G/T to G/G' to get the desired result. Namely, $L_\lambda\otimes S^*T_{G/G'}$ is the pushforward of the sheave $L_\lambda\otimes S^*\left((\mathfrak{g}/\mathfrak{g}_\lambda)^+\times_G G/T\right)$, whose T-equivariant index can be computed in exactly the same way but with the replacement of $\prod_{\alpha\in\mathfrak{R}_+}$ by $\prod_{\alpha\in\mathfrak{R}'_+}$. □

Theorem 14. *The index over $\mathfrak{P}$ of*

$$\mathrm{Ind}\left(\mathfrak{P}, \mathcal{L}^k\otimes \mathrm{S}_t(N^{(R)*}_{\mathfrak{P}})\right) = \sum_{f\in F^{\mathrm{reg}}_{\rho,t}/W} \theta_t(f)^{1-g}\prod_i \Theta_{\lambda_i,t}(f),$$

where $\Theta_{\lambda,t}$ is given by (29)*, which also equals*

$$\Theta_{\lambda,t} = \sum_{w\in W}\frac{(-1)^{l(w)}e^{w(\lambda+\rho)}}{\Delta\prod_{\alpha>0}(1-te^{w(\alpha)})}.$$

Proof. We consider the fibration

$$\pi : \mathfrak{P}\to\mathfrak{M},$$

and consider the presentation (27). By (26) we get following equation in K-theory on $\mathfrak{P}$

$$N^{(R)}_{\mathfrak{P}} = T_{\pi^{(R)}}\mathfrak{P}_H/T_\pi\mathfrak{P} + \pi^*N^{(R)}_{\mathfrak{M}}.$$

So

$$S_t(N_{\mathfrak{P}}^{(R)}) = S_t(T_{\pi^{(R)}}\mathfrak{P}_H/T_\pi\mathfrak{P}) \otimes S_t(\pi^* N_{\mathfrak{M}}^{(R)})$$

and we further get the following expression for L_k

$$L_k = \pi^*(L^k) \otimes \bigotimes_i p_i^* L_{\lambda_i}^P,$$

where $L_{\lambda_i}^P$ is the line bundle over E_i/P_i associated to the character λ_i of the bundle E_i. and by the push-pull formula we get that

$$\pi_*(S_t(N_{\mathfrak{P}}^{(R)}) \otimes L_k) = \pi_* \left(S_t(T_{\pi^{(R)}}\mathfrak{P}_H/T_\pi\mathfrak{P}) \otimes \bigotimes_i p_i^* L_{\lambda_i}^P \right) \otimes S_t(N_{\mathfrak{M}}^{(R)}) \otimes L_k.$$

By combining Lemma 2 with Lemma 1 to conclude the following formula in the K-theory of $\mathfrak{M}$

$$\pi_* \left(S_t(T_{\pi^{(R)}}\mathfrak{P}_H/T_\pi\mathfrak{P}) \otimes \bigotimes_i p_i^* L_{\lambda_i}^P \right) = \bigoplus_{i=1}^n E_{x_i} \Theta_{\lambda_i,t}.$$

But now the theorem follows from using the expression for $S_t(N_{\mathfrak{M}}^{(R)}) \otimes L_k$ in K-theory of $\mathfrak{M}$ from Section 3 and the Teleman-Woodward index theorem. □

Theorem 15. *For $g > 1$, $R \geq 2$ and all non-negative integers i and k we have that*

$$H^i(\mathfrak{P}, S^*(N_{\mathfrak{P}}^{(R)*}) \otimes \mathfrak{L}_k) = 0.$$

The proof of this theorem is completely analogous to the proof of Theorem 7, by the arguments presented in the first paragraph of section VII.4 in [T2]. Combined with vanishing theorems over $\mathfrak{M}_H^{(R)}$, we have the *Verlinde formula for parabolic Higgs bundles*:

Theorem 16. *For $R, g \geq 2$ we have that*

$$\dim_t H^0(P_H^{(R)}, L_k) = \mathrm{Ind}(\mathfrak{P}, L_k \otimes \mathrm{S}_t N_{\mathfrak{P}}^{(R)*}) = \sum_{f \in F_{\rho,t}^{\mathrm{reg}}/W} \theta_t(f)^{1-g} \prod_i \Theta_{\lambda_i,t}(f).$$

Further, we also get an analog of Theorem 4 in the parabolic case

Theorem 17. *Provided $g > 2$ and $R \geq 2$, we have for all non-negative integers n a natural isomorphism*

$$H^0(\mathfrak{P}, S^n N_P^{(R)} \otimes \mathcal{L}_R^k) \cong H^0(P, S^n N_P^{(R)} \otimes \mathcal{L}_R^k)$$

where $\mathfrak{P}$ is the stack of parabolic bundles.

Again, for $R \leq 0$, one can relate the index with another index over the moduli stack of $(2-R)$-twisted Higgs bundles. For that, one considers instead

$$\Omega'_{2-R,D} = R\pi_*(B(\mathfrak{g}) \otimes K^{R/2}(D))[1].$$

Just as for the non-parabolic case, we have

Theorem 18. *For $R \leq 0$, we have*

$$\dim H^0(P_H, L_k \otimes \lambda_{-t} N_M^{(2-R)}) = \sum_{f \in F_{\rho,t}^{\mathrm{reg}}/W} \theta_{t,R}(f)^{1-g} \sum_i \Theta'_{\lambda_i,t}.$$

5. Some exceptional cases

When the Riemann surface Σ has small genus, the moduli stack of (parabolic) Higgs bundles $\operatorname{Spec}\mathrm{S}(N_{\mathfrak{M}}^{(R)*})$ with $R \geq 2$ can be an honest derived stack with a non-trivial derived structure. As a consequence, not only is there a drastic difference between the moduli space and the moduli stack, but also $H_n^*(\mathfrak{M}_H^{(R)}, \mathfrak{L}^k)$ can have negative degree cohomology groups or negative dimensions for certain n. One may attempt to stay away and regard them as pathological. But we will do the opposite, because simple Riemann surfaces serve as the building blocks of the 2D TQFT that we will discuss in the next section, making the indices associated with them among the most interesting ones. In this section, we will look at several special cases with small genera. For simplicity, we will henceforth set $R = 2$, and generalizations to $R > 2$ are often straightforward.

When $g = 0$, the moduli stack of $G^{\mathbb{C}}$-bundles contains the classifying stack BG as the substack of semi-stable bundles. The tangent complex is $\mathfrak{g}[1]$ and $\mathfrak{L}^k$ becomes trivial. Then,

$$\mathrm{S}^r(T\mathfrak{M}) = \wedge^r \mathfrak{g}$$

placed in degree $-r$. As $H^0_G(\lambda^\bullet \mathfrak{g})$ is a skew-symmetric algebra with generators in degree $2m_i + 1$, where m_i are the exponents of G, we have:

Proposition 3. *For $g = 0$,*

$$\dim H^0(\mathfrak{M}, S_t T\mathfrak{M}) = \prod_{i=1}^{\mathrm{rk} G} (1 - t^{2m_i+1}) = P_{-t}(G),$$

where $P_{-t}(G)$ is the Poincaré polynomial of the group G in the variable $-t$.

As cohomology groups in non-zero degrees still vanish in this case, $P_{-t}(G)$ also gives the index. This agrees with results in [GP] and [GPYY], where it was found that the "equivariant Verlinde formula" on S^2 takes the curious values

$$1 - t^3 \quad \text{and} \quad 1 - t^3 - t^5 + t^8$$

for $G = SU(2)$ and $G = SU(3)$ respectively. Notice that negative coefficients are manifestation of the fact that the stack $\mathfrak{M}_H$ is now derived.

As the general index formula still applies for $g = 0$, we obtain the following identity concerning θ_t evaluated on $F_{\rho,t}$.

Proposition 4. *The structure constants $\theta_t(f)$ satisfies the following equation,*

$$\sum_{f \in F^{reg}_{\rho,t}} \theta_t(f) = P_{-t^{1/2}}(G). \tag{33}$$

When $\mathbf{P}^1$ has two marked point with parabolic weights λ_1 and λ_2, over both of which the structure group reduces to T, the index associated with it is another fundamental building block for the 2D TQFT.

When λ_1 and λ_2 are not dual representations, the substack of semi-stable Higgs bundles is empty, and the quantization is trivial. We claim the following.

Proposition 5. *We have that*

$$d_{0,2}(\lambda_1, \lambda_2) = \delta_{\lambda_1, \lambda_2^*} d_\lambda^{-1}, \tag{34}$$

where

$$d_\lambda = P_{t^{1/2}}(BG_\lambda). \tag{35}$$

Proof. The moduli stack $\mathfrak{M}^{\mathrm{ss}}$ is given by G/T quotient by an adjoint G-action, which acts only on the base of the fibration

$$G_\lambda/T \to G/T \to G/G_\lambda.$$

So

$$\mathfrak{M}^{\mathrm{ss}} = G_\lambda/T \times BT.$$

The tangent complex is $\mathfrak{t}[1] \oplus (\mathfrak{g}/\mathfrak{t})^+$ which only has cohomology at degree zero. Then

$$d_\lambda^{-1} = \dim S_t(\mathfrak{t}[1] \oplus (\mathfrak{g}/\mathfrak{t})^+) = (1-t)^{\mathrm{rk}G} \cdot \dim S_t \mathfrak{n}_\lambda = \prod_{m_i} (1 - t^{m_i}),$$

where we used the fact that for nilpotent Lie algebra $\mathfrak{n}_\lambda = (\mathfrak{g}/\mathfrak{t})^+$, one has

$$\dim S_t \mathfrak{n}_\lambda = \prod_{m_i} \frac{1-t^{m_i}}{1-t}.$$

□

Just as in the case of $\mathbb{P}^1$, by comparing the above expression with the general index formula, one obtains interesting identities, involve the deformed characters Θ in the parabolic cases.

Theorem 19. *In the case where we use maximal parabolic structure, the deformed characters satisfy the following two orthogonality relations.*

- *The first orthogonality*

$$\sum_{f \in F_{\rho,t}^{reg}/W} \theta_t(f) \Theta_{\lambda_1}(f) \Theta_{\lambda_2}(f) = \delta_{\lambda_1, \lambda_2^*} d_\lambda^{-1}. \tag{36}$$

- *The second orthogonality*

$$\sum_\lambda \Theta_\lambda(f_i) \Theta_{\lambda^*}(f_j) \theta_t(f_i) = \delta_{ij} d_\lambda^{-1}, \tag{37}$$

where $f_i, f_j \in F_{\rho,t}^{reg}$.

Proof. The first orthogonality relation can be obtained by computing the index for the moduli stack associated to $\mathbb{P}^1$ with two (maximal) parabolic points in two different ways,

$$d_{0,2}(\lambda_1, \lambda_2) = \sum_{f \in F_{\rho,t}^{\mathrm{reg}}/W} \theta_t(f) \Theta_{\lambda_1}(f) \Theta_{\lambda_2}(f) = \delta_{\lambda_1, \lambda_2^*} d_\lambda^{-1}. \tag{38}$$

The middle expression comes from the general index formula, while the last is obtained by directly evaluating the index over this specific moduli stack.

The second orthogonality can be deduced from the first by considering the following expression,

$$D = \sum_{\lambda_2} \sum_{f_i} \Theta_{\lambda_1}(f_i) \Theta_{\lambda_2}(f_i) \theta_t(f_i) \Theta_{\lambda_2^*}(f_j). \tag{39}$$

Using the first orthogonality, we know that

$$D = \Theta_{\lambda_1}(f_j) d_\lambda^{-1}. \tag{40}$$

At $t = 0$, the Θ's form a complete basis of functions on $F^{\text{reg}}_{\rho,t}/W$, and this property extends to a neighborhood of $t = 0$. Then the completeness of Θ_t's necessarily implies that

$$\sum_\lambda \Theta_\lambda(f_i)\Theta_{\lambda^*}(f_j)\theta_t(f_i) = \delta_{ij} d_\lambda^{-1}. \tag{41}$$

□

6. 2D TQFT viewpoint

In this section, we will study the 2D TQFT underlying the index formula. The existence of a TQFT structure from the index theory of $\mathfrak{M}$ is foreseen in [T5] and highlighted in [GP] and [GPYY]. We call the TQFT functor $Z(G, k, t)$ and we will often suppress G, k and t dependence when no confusion will be caused. The functor Z associates to a circle S^1 a vector space

$$Z(S^1) = V.$$

The dimension of V does not depend on t and equals the size of the index set

$$Z(T^2) = |F^{\text{reg}}_{\rho,t}/W|.$$

The index associated with a closed Riemann surface Σ with genus g gives the partition function of Z on Σ

$$Z(\Sigma) = d_{g,0}. \tag{42}$$

The index for parabolic bundles over a genus-g Riemann surface with parabolic weights $\vec{\lambda}$

$$Z(\Sigma, \vec{\lambda}) = d_{g,n}(\vec{\lambda}) \tag{43}$$

defines a $\mathbb{C}[[t]]$-valued linear functional on $V^{\otimes n}$. They can be nicely assembled into a 2D TQFT provided the that *factorization formulae*

$$d_{g+1,n-2}(\vec{\lambda}) = \sum_{\lambda\in\Lambda_k} d_{g,n}(\vec{\lambda}, \lambda, \lambda^*) d_\lambda$$

and

$$d_{g_1+g_2,n_1+n_2}(\vec{\lambda}_1, \vec{\lambda}_2) = \sum_{\lambda\in\Lambda_k} d_{g_1,n_1}(\vec{\lambda}_1, \lambda) d_{g_2,n_2}(\vec{\lambda}_2, \lambda^*) d_\lambda$$

from Theorem 4, which we now prove.

Proof. Theorem 4 immediately follows from the second orthogonality relation of the deformed characters,

$$\sum_\lambda \Theta_\lambda(f_i)\Theta_{\lambda^*}(f_j)\theta_t(f_i) = \delta_{ij} d_\lambda^{-1}. \tag{44}$$

□

In order to discuss Z on two dimensional manifolds with boundary in an explicit way, it is convenient to choose a basis in V. This also corresponds to choosing a basis of the Fröbenius algebra associated with Z. There are three special choices that are particularly interesting:

a. The "idempotents basis" $\{e_i\}$. In this basis, the basis vectors are in one-to-one correspondence with $f_i \in F^{\text{reg}}_{\rho,t}/W$ and the multiplication is diagonalized as

$$e_i \cdot e_j = \delta_{ij} e_i$$

and the trace map on them is given by

$$(e_i, e_j) = \delta_{ij}\theta(f_i),$$

which is giving the link between e_i and f_i.

b. The "representation basis" $\{v_\lambda\}$, which are labeled by weights in Λ_k. In this basis, elements in V corresponds to virtual representations of the loop group of G at level k. We use v_U to denote the element associated with a virtual representation U. Associate to a marked point x on the Riemann surface a vector v_U corresponds to tensoring $E_x^* U$ to the K-theory class whose index we are computing if there were no punctures. A simple application of Theorem 5 gives us the decomposition of v_U in the idempotent basis as

$$v_U = \sum_i \mathrm{Tr}_U(f_i) e_i. \tag{45}$$

It is easy to check that

$$v_{U_1} \cdot v_{U_2} = v_{U_1 \otimes U_2}. \tag{46}$$

and

$$v_U + v_{U'} = v_{U \oplus U'}. \tag{47}$$

The inner product is however different from the undeformed Verlinde algebra

$$(v_{U_1}, v_{U_2}) = \sum_{f_i \in F^{\text{reg}}_{\rho,t}} \mathrm{Tr}_{U_1 \otimes U_2}(f_i)\theta(f_i). \tag{48}$$

c. The "parabolic basis" $\{w_\lambda\}$, whose base vectors are labeled by parabolic weights, and adding a marked point with w_λ create a maximal parabolic structure at the point of insertion. In general, one can consider any parabolic types between T and G_λ, but for uniformity, we have chosen $\{w_\lambda\}$ such that they correspond to the maximal reduction of the structure group $G \to T$. As $\{w_\lambda\}$ form a complete basis of V in a neighborhood of $t = 0$, all other parabolic types give vectors that can be written as linear combinations of w_λ's. Mathematically, we define w_λ by the equation

$$w_\lambda = \sum_i \Theta_\lambda(f_i) e_i. \tag{49}$$

In the rest of the paper, we will describe the TQFT for $G = SU(2)$ in the parabolic basis and give many explicit examples for the index formula associated to various (parabolic) moduli spaces of Higgs bundles.

6.1. **The TQFT for** $SU(2)$**.** For $G = SU(2)$ at level k, V is now spanned by w_λ labeled by integrable weights $\lambda = 1, 2, \ldots, k$. For later convenience we assume that $k > 2$. A 2D TQFT is determined by its value on the three-punctured sphere and the cylinder, or equivalently "fusion coefficients" $f \in V^{*\otimes 3}$ and the "metric"

$\eta \in V^{*\otimes 2}$. These are worked out using intuitive geometric argument in [GP] and are given by the following expressions

$$\eta^{\lambda_1\lambda_2} = \mathrm{diag}\{1-t^2, 1-t, \ldots, 1-t, 1-t^2\}, \tag{50}$$

and

$$f^{\lambda_1\lambda_2\lambda_3} = \begin{cases} 1 & \text{if } \lambda_1+\lambda_2+\lambda_3 \text{ is even and } \Delta\lambda \le 0, \\ t^{\Delta\lambda/2} & \text{if } \lambda_1+\lambda_2+\lambda_3 \text{ is even and } \Delta\lambda > 0, \\ 0 & \text{if } \lambda_1+\lambda_2+\lambda_3 \text{ is odd.} \end{cases} \tag{51}$$

Here, $\Delta\lambda = \max(d_0, d_1, d_2, d_3)$ with

$$\begin{aligned} d_0 &= \lambda_1+\lambda_2+\lambda_3-2k = 0 \\ d_1 &= \lambda_1-\lambda_2-\lambda_3 = 0 \\ d_2 &= \lambda_2-\lambda_3-\lambda_1 = 0 \\ d_3 &= \lambda_3-\lambda_1-\lambda_2 = 0. \end{aligned}$$

f and η related by the "cap state"

$$w_\emptyset = w_0 - tw_2$$

by

$$\eta^{\lambda_1\lambda_2} = f^{\lambda_1\lambda_2\emptyset} = f^{\lambda_1\lambda_2 0} - tf^{\lambda_1\lambda_2 2}.$$

f and η are obviously symmetric. η defines a set of basis dual to w_λ in V^*, which we will denote by w^λ. It also gives a trace map on V^* which we will denote by the same symbol and write $\eta_{\lambda_1\lambda_2}$ the $\{w^\lambda\}$ basis. With the identification of V and V^* using η, f can now define a product on V, which we write in component language as ${f^{\lambda_1\lambda_2}}_{\lambda_3}$.

6.2. **Sphere with four punctures.** The claim in [GP] is that the f and η given above gives a well-defined TQFT. In order to check this, one just needs to verify that $d_{0,4} \in V^{*\otimes 4}$ associated with a fourth-punctured $\mathbf{P}^1$ is well-defined,

$$d_{0,4}^{\lambda_1\lambda_2\lambda_3\lambda_4} = f^{\lambda_1\lambda_2\mu}\eta_{\mu\nu}f^{\nu\lambda_3\lambda_4} = f^{\lambda_1\lambda_3\mu}\eta_{\mu\nu}f^{\nu\lambda_2\lambda_4}.$$

This is explicitly checked in the appendix A, and it is verified there that

$$d_{0,4}^{\lambda_1\lambda_2\lambda_3\lambda_4} = t^{kh_0}\left(\frac{\widetilde{d}_{0,4}}{1-t} + \frac{2t}{1-t^2}\right) + \frac{\sum_i t^{kh_i}}{(1-t^{-1})(1-t^2)}. \tag{52}$$

Here all $h_0, \ldots, h_4$ are functions of λ's whose explicit form will be given in appendix A and

$$\widetilde{d}_{0,4} = \lim_{t\to 0}\left(t^{-kh_0}d_{0,4}\right).$$

It gives the (undeformed) Verlinde formula when $h_0 = 0$. The existence of a consistent answer for $d_{0,4}$ establishes the following theorem.

Theorem 20. *The data (V, f, η) given above defines a 2D TQFT.*

The formula (52) can also be interpreted as the $\mathbb{C}^*$-equivariant Atiyah-Bott localization formula applied to the moduli space of Higgs bundles.

Recall that the Hitchin moduli space in this case is a elliptic surface. The nilpotent cone $\mathcal{N}$ of the Hitchin fibration is the only singular fiber with affine D_4 singularity (or, equivalently, of Kodaira type I_0^*). It has five components

$$\mathcal{N} = M \cup_{i=1}^4 D_i. \tag{53}$$

And the first term in (52) comes from M where the moment map of the $\mathbb{C}^*$-action is h_0, and the last comes from the higher fixed points where the moment map evaluates to be h_i.[1] Indeed, M is $\mathbb{P}^1$ and its contribution in the localization formula is

$$\mathrm{Ind}(\mathbb{P}^1, \mathcal{L}\otimes S_t O(-2)) = \sum_{i=0}^{\infty} t^i \left(\mathrm{Ind}(\mathbb{P}^1,\mathcal{L}) - 2n\right),$$

where $\mathcal{L}$ is the restriction of L^k to M. After summation, this indeed agrees with the first term in (52), given that the degree of $\mathcal{L}$ is $\widetilde{d}_{0,4}-1$ when $h_0=0$ and $-\widetilde{d}_{0,4}+1$ when $h_0\neq 0$.

6.3. **Diagonalizing the fusion rule.** Now we have given an explicit description of the 2D TQFT, one can proceed to check that it agrees with the index formula given in Theorem 3 and 16 after a basis change from $\{w_\lambda\}$ to $\{e_i\}$. This is already done in section 7 of [GP]. It is even easier to go from the diagonalized basis $\{e_i\}$ and deduce the TQFT in the parabolic basis $\{w_\lambda\}$. As η already agrees with the index over the moduli stack associated to $\Sigma_{0,2}$, one only needs to verify that the algebra of the deformed characters gives the fusion coefficients f,

$$\Theta_{\lambda_1}\Theta_{\lambda_2} = \sum_{\lambda_3} f^{\lambda_1\lambda_2}{}_{\lambda_3}\Theta_{\lambda_3}, \tag{54}$$

when evaluated on $F^{\mathrm{reg}}_{\rho,t}$. The verification is straightforward.

6.4. **Genus two with odd determinant.** The moduli space of Higgs bundles often has singularities which can be avoided by considering the co-prime cases, where the underlying principal G-bundle is twisted by a line bundle. As was found in [GP] and [GPYY], twisting by a line bundle also has a nice interpretation in the TQFT language. For $SU(2)$, twisting by an odd line bundle is equivalent to first adding a puncture and then attaching a "twisted-cap" that is associated with the following vector in V,

$$w_\psi = w_k - t w_{k-2}. \tag{55}$$

In this section, we will test this conjecture by computing the index for $g=2$ and compare with the localization formula. The index is given by[2]

$$d_2^{\mathrm{odd}} = \frac{1}{12(1-t)^6(1+t)^3}(a_3k^3+a_2k^2+a_1k+a_0) \tag{56}$$

where

$$a_3 = \left(1-t^2\right)^3$$

$$a_2 = 6(1-t)^2(1+t)^4$$

$$a_1 = 2\left(1-t^2\right)\left(144t^{\frac{k}{2}+2}+7t^4+12t^3+10t^2+12t+7\right)$$

$$a_0 = 12\left(48t^{\frac{k}{2}+2}+32t^{\frac{k}{2}+3}+48t^{\frac{k}{2}+4}+t^6-6t^5-33t^4-52t^3-33t^2-6t+1\right).$$

[1]The normalization we used for the moment map is related to Hitchin's in [H1] by a factor of π. As a consequence, our moment map is always rational at the critical points.

[2]Computation using functoriality of TQFT is done in Mathematica, and we are very grateful for Ke Ye for his extensive help.

When k is odd, the index is zero.[3]

Now we show that it agrees with direct computation using localization formula. The fixed points under the $\mathbb{C}^*$-action are studied in [H1] and there are two components M_0 and M_1. The first is the moduli space of flat connections M sitting at $\mu = 0$. The cohomology of M is generated by $\alpha \in H^2(M,\mathbb{Z})$, $\psi_1, \psi_2, \psi_3, \psi_4 \in H^3(M, Z)$ and $\beta \in H^4(M,\mathbb{Z})$. The non-zero intersection numbers are [N1]

$$\int_M \alpha\wedge\beta = -4, \quad \int_M \alpha^3 = 4, \quad \int_M \gamma = \int_M \left(\sum_{i=1}^4 \psi_i\right)^2 = 4.$$

And the total Chern character of TM is given by [N2]

$$\mathrm{ch}(TM) = 3 + 2\alpha + \beta + \frac{1}{3}\alpha\beta - \frac{4}{3}\gamma.$$

Then one can explicitly compute the index for $S^nTM \otimes \mathcal{L}^k$, and a perfect match is found with (56). For example, for $n = 1$,

$$\chi(TM\otimes\mathcal{L}^k) = \int_M e^{(k/2+1)\alpha}(3+2\alpha+\beta+\frac{1}{3}\alpha\beta-\frac{4}{3}\gamma)\left(\frac{\sqrt{\beta}/2}{\sinh\sqrt{\beta}/2}\right)^2 = \frac{1}{4}\left(k^3 + 10k^2 + 22k - 12\right).$$

For $n = 2$, with the help of splitting principle, one finds

$$\mathrm{ch}(S^2TM) = 6 + 8\alpha + 5\beta + \frac{13}{3}\alpha\beta - \frac{28}{3}\gamma + 2\alpha^2.$$

And one can easily compute the index to be

$$\chi(S^2TM \otimes \mathcal{L}^k) = \frac{1}{2}\left(k^3 + 14k^2 + 34k - 84\right).$$

One can compute the index to all orders of t by expressing the $\mathrm{ch}(S_tTM)$ in terms of α, β and γ. For the purpose of computing the index, one can make the substitution $\beta \to -\alpha^2$ and $\gamma \to \alpha^3$. Then after a short computation, one finds

$$\mathrm{ch}(S_tTM) \sim \left[(1-t)^3 - 2\alpha t(1-t)^2 + \alpha^2 t(1-t)(1+2t) - \alpha^3 t(1 + 3t + \frac{4}{3}t^2)\right]^{-1}.$$

And this enables us to obtain a formula for the index of $S_tTM \otimes \mathcal{L}^k$ that is valid to all order of t:

$$\chi(S_tTM \otimes \mathcal{L}^k) = \int_M e^{(k/2)\alpha}\wedge\mathrm{ch}(S_tTM)\wedge\mathrm{Td}(TM) \tag{57}$$

$$= \frac{1}{(1-t^3)}(a_3'k^3 + a_2'k^2 + a_1'k + a_0'), \tag{58}$$

[3]In fact, in the convention that we are using, $\mathcal{L}^k$ is only well-defined over the moduli space for even k. The convention commonly used in the literature is related to ours by a factor of 2.

where

$$a'_3 = \frac{1}{12}$$
$$a'_2 = \frac{1+t}{2(1-t)}$$
$$a'_1 = \frac{7-2t+7t^2}{6(1-t)^2} \tag{59}$$
$$a'_0 = \frac{(1+t)(3+2t+3t^2)}{3(1-t)^3}.$$

Notice that the expression above is simply (56) without all terms proportional to $t^{k/2}$. These extra terms

$$t^{k/2+2}\left[\frac{24k}{(1-t)^5(1+t)^2} + \frac{16(3+2t+3t^2)}{(1-t)^6(1+t)^3}\right] \tag{60}$$

on one hand comes from the higher sheaf cohomology groups of $S_t TM \otimes \mathcal{L}^k$, and, on the other, is the contribution from the higher critical manifold M_1.

6.5. **The case of genus three.** As noted before, the $g = 2$ case is special for $G = SU(2)$, and we will postpone the discussion of it after $g = 3$. From the TQFT rule, one obtains

$$d_3 = \frac{1}{(1-t)^6}\left(b_6k^6 + b_5k^5 + b_4k^4 + b_3k^3 + b_2k^2 + b_1k + b_0\right) \tag{61}$$

where

$$b_6 = \frac{1}{180}$$
$$b_5 = \frac{(1+t)}{15(1-t)}$$
$$b_4 = \frac{\left(7t^2-2t+7\right)}{18(1-t)^2}$$
$$b_3 = \frac{4\left(t^6-t^5-4t^4-10t^3-4t^2-t+1\right)}{3(1-t^2)^3}$$
$$b_2 = \frac{1}{180(1-t^2)^4}\big(31680t^{4+k} + 469t^8 - 2280t^7 + 44t^6 - 6360t^5 + 7614t^4 - 6360t^3 + 44t^2 - 2280t + 469\big)$$
$$b_1 = \frac{1}{5(1-t^2)^5}\big(4160t^{4+k} + 3200t^{5+k} + 4160t^{6+k} + 13t^{10} - 114t^9 + 361t^8 + 296t^7 + 2986t^6 + 1556t^5 + 2986t^4 + 296t^3 + 361t^2 - 114t + 13\big)$$
$$b_0 = \frac{1}{(1-t^2)^6}\big(960t^{4+k} + 1536t^{5+k} + 2944t^{6+k} + 1536t^{7+k} + 960t^{8+k} + 64t^{6+2k} + t^{12} - 12t^{11} + 66t^{10} - 220t^9 - 465t^8 - 2328t^7 - 2084t^6 - 2328t^5 - 465t^4 - 220t^3 + 66t^2 - 12t + 1\big)$$

The term involving t^{2k} can be written as

$$\frac{64t^{2k}}{(1-t^{-1})^6(1-t^2)^6}$$

and agrees with the existence of 64 critical points at value of moment map $h = 2$ whose 12-dimensional normal bundle splits as $\mathbb{C}^6[-1] \oplus \mathbb{C}^6[2]$. And the terms proportional to t^k are related to the existence of a two-dimensional critical manifold at $h = 1$.

6.6. **The $SU(2)$ genus-two case.** The Verlinde formula for Higgs bundles in the $G = SU(2)$ and $g = 2$ case is given by

$$d_2 = \frac{1}{(1-t)^3}(c_3k^3 + c_2k^2 + c_1k + c_0) \tag{62}$$

where

$$\begin{aligned} c_3 &= \frac{1}{6}, \\ c_2 &= \frac{1+t^2}{1-t^2}, \\ c_1 &= \frac{11 - 36t - 9t^2 + 9t^4 + 36t^5 - 11t^6}{6(1-t^2)^3}, \\ c_0 &= \frac{1 - 6t + 15t^2 - 4t^3 + 15t^4 - 6t^5 + t^6 - 16t^{k+3}}{(1-t^2)^3}. \end{aligned}$$

The term proportional to t^k can be written as

$$\frac{16t^k}{(1-t^{-1})^3(1-t^2)^3}$$

and could be identified with the contribution from the 16 higher fixed points. The rest in (62) should be coming from the bottom component $M_0 = \mathbb{P}^3$. The normal bundle of M_0 in the Higgs bundle moduli space is not $T^*\mathbb{P}^3$ any more. The correction from the additional divisor is given by

$$\begin{aligned} &d_2 - \mathrm{Ind}(\mathbb{P}^3, S_tT \otimes \mathcal{L}^k) - \frac{16t^k}{(1-t^{-1})^3(1-t^2)^3} = \\ &\frac{1}{(1-t)^3}\left[\frac{2t}{1-t^2}k^2 + \frac{18t}{(1-t)(1-t^2)}k + \frac{18t(1+t^2)}{(1-t)(1-t^2)^2} - \frac{12t^2}{(1-t)(1-t^2)}\right]. \end{aligned} \tag{63}$$

Appendix A. Check of associativity

The associativity can be checked by computing

$$d_{0,4}^{\lambda_1\lambda_2\lambda_3\lambda_4} = f^{\lambda_1\lambda_2\nu} f^{\lambda_3\lambda_4\rho}\eta_{\nu\rho},$$

and verifying that it is symmetric in all indices.

From this expression, it is clear that the sum of λ's has to be an even number in order for d to be non-zero. Define

$$\begin{aligned}\delta_1 =&|\lambda_1-\lambda_2|,\\ \delta_2 =&\min(\lambda_1+\lambda_2, 2k-\lambda_1-\lambda_2),\\ \delta_3 =&|\lambda_3-\lambda_4|,\\ \delta_4 =&\min(\lambda_3+\lambda_4, 2k-\lambda_3-\lambda_4),\end{aligned}$$

then for ν such that $\nu+\lambda_1+\lambda_2$ is even,

$$f^{\lambda_1\lambda_2\nu} = \begin{cases} t^{(\delta_1-\nu)/2} & \text{if } \nu\leq\delta_1,\\ 1 & \text{if } \delta_1\leq\nu\leq\delta_2,\\ t^{(\nu-\delta_2)/2} & \text{if } \nu\geq\delta_2,\end{cases}$$

and the form of $f^{\lambda_3\lambda_4\rho}$ is similar. Now it is convenient to define

$$\begin{aligned}L_1 &= \min(\delta_1,\delta_3),\\ L_2 &= \max(\delta_1,\delta_3),\\ L_3 &= \min(\delta_2,\delta_4),\\ L_4 &= \max(\delta_2,\delta_4).\end{aligned}$$

There are two different possibilities about values of L_i's. We may have

$$L_1\leq L_2\leq L_3\leq L_4 \tag{64}$$

or

$$L_1\leq L_3\leq L_2\leq L_4, \tag{65}$$

which respectively correspond to either the moduli space of parabolic bundles being non-empty or empty. We start with the first situation and assume that $\lambda_1+\lambda_2$ is even, then the L's are all even. We also assume that k is even. Now we break the summation into different parts

$$f^{\lambda_1\lambda_2\nu}f^{\lambda_3\lambda_4\rho}\eta_{\nu\rho} = \sum_{\mu=0,2,4\ldots k} f^{\lambda_1\lambda_2\nu}f^{\lambda_3\lambda_4\mu}\eta_{\nu\nu} = A+B+C+D+E+F+G,$$

where

$$A=\frac{t^{(L_1+L_2)/2}}{1-t^2}$$

is the contribution of $\nu=0$,

$$B=\frac{1}{1-t}\sum_{\nu=2}^{L_1} t^{(L_1+L_2)/2-\nu},$$

$$C=\frac{1}{1-t}\sum_{\nu=L_1+2}^{L_2} t^{(L_2-\nu)/2},$$

$$D=\frac{(L_3-L_2)/2-1}{1-t}$$

comes from $\nu = L_2 + 2, L_2 + 4, \dots, L_3 - 2$,

$$E = \frac{1}{1-t} \sum_{\nu=L_3}^{L_4-2} t^{(\nu-L_3)/2},$$
$$F = \frac{1}{1-t} \sum_{\nu=L_4}^{k-2} t^{\nu-(L_3+L_4)/2},$$

and finally

$$G = \frac{t^{k-(L_3+L_4)/2}}{1-t^2}$$

comes from $\nu = k$. After performing summation of various geometric series, we have:

$$B = \frac{1}{1-t} \cdot \frac{t^{(L_2-L_1)/2} - t^{(L_2+L_1)/2}}{1-t^2},$$
$$C = \frac{1-t^{L_2-L_1}}{(1-t)^2},$$
$$E = \frac{1-t^{L_4-L_3}}{(1-t)^2},$$
$$F = \frac{1}{1-t} \cdot \frac{t^{(L_4-L_3)/2} - t^{k-(L_3+L_4)/2}}{1-t^2}.$$

Adding all the seven pieces up gives

$$d_{0,4} = \frac{(L_3-L_2)/2+1}{1-t} + \frac{2t}{(1-t)^2} + \frac{t^{(L_1+L_2)/2} + t^{(L_2-L_1)/2} + t^{k-(L_3+L_4)/2} + t^{(L_4-L_3)/2}}{(1-t^{-1})(1-t^2)}. \tag{66}$$

Although we have assume that k is even and $\lambda_1 + \lambda_2$ is even, one can verify that equation (66) is completely general. For example, if k is odd and $\lambda_1 + \lambda_2$ is even, then A, B, C, D, E will be unchanged, while $G = 0$ and

$$F = \frac{1}{1-t} \sum_{\nu=L_4}^{k-1} t^{\nu-(L_3+L_4)/2} = \frac{1}{1-t} \cdot \frac{t^{(L_4-L_3)/2} - t^{k-(L_3+L_4)/2+1}}{1-t^2}.$$

As a consequence, if we add all terms up, the coefficient of $t^{k-(L_3+L_4)/2}$ is still $\frac{1}{(1-t^2)(1-t^{-1})}$. Similarly, if $\lambda_1 + \lambda_2$ is odd and k is odd, then A becomes zero but B is changed to compensate for the vanishing of A so that the final expression stays the same. If $\lambda_1 + \lambda_2$ is odd while k is even, A and G are both zero while B and F are changed and final expression still remains to be (66).

When $t = 0$, and $L_2 \le L_3$

$$\widetilde{d}_{0,4} = (L_3 - L_2)/2 + 1.$$

So in this case (66) is exactly the same as (52) with $h = 0$ and

$$h_1 = \frac{1}{2k}(2k - L_3 - L_4)$$
$$h_2 = \frac{1}{2k}(L_4 - L_3)$$
$$h_3 = \frac{1}{2k}(L_1 + L_2)$$
$$h_4 = \frac{1}{2k}(L_2 - L_1).$$

The case with $L_2 > L_3$ is very analogous, where various geometric series will start with t^{h_0} instead of 1, with

$$h_0 = \frac{1}{2k}(L_2 - L_3).$$

Compared to the $L_2 \leq L_3$ situation, only C, D and E in the seven-term decomposition are changed. They are now

$$C = \frac{1}{1-t} \sum_{\nu=L_1+2}^{L_3} t^{(L_2-\nu)/2},$$
$$D = t^{(L_2-L_3)} \frac{(L_2 - L_3)/2 - 1}{1-t},$$
$$E = \frac{1}{1-t} \sum_{\nu=L_2}^{L_4-2} t^{(\nu-L_3)/2}.$$

So in this case we also have:

$$Z = t^{kh_0} \left[\frac{(L_2 - L_3)/2 + 1}{1-t} + \frac{2t}{(1-t)^2} \right] + \frac{\sum_{i=1,2,3,4} t^{kh_i}}{(1-t^{-1})(1-t^2)}. \tag{67}$$

Appendix B. Index Formula Made Explicit

In [GP], the "equivariant Verlinde formula" was written down in physics language for $G = SU(N)$ using a quantity called "twisted effective superpotential" $\widetilde{W}$.[4] It is a function on $\mathfrak{t}$ and after choosing $T \subset SU(N)$ to be parametrized by diagonal matrices with the i-th diagonal entry being $e^{2\pi i \sigma}$, $\widetilde{W}$ can be expressed as

$$\widetilde{W}(\sigma) = \pi i(k+N) \sum_{a=1}^{N} \sigma_a^2 + \frac{1}{2\pi i} \sum_{a \neq b} \mathrm{Li}_2 \left[t e^{2\pi i(\sigma_a - \sigma_b)} \right] + \pi i \sum_{a>b} (\sigma_a - \sigma_b).$$

Up to an additive constant, $-2\pi i \widetilde{W}$ is exactly the function D_t on T in (1) with $\xi = 2\pi i$. The "Bethe ansatz equations" are given by

$$\exp \left[\frac{\partial \widetilde{W}}{\partial(\sigma_a - \sigma_b)} \right] = 1, \quad \text{for all } a, b = 1, 2, \ldots, N. \tag{68}$$

Using

$$\frac{\partial \mathrm{Li}_2(e^x)}{\partial x} = -\ln(1 - e^x),$$

[4] It was in fact for $U(N)$ along with a method to convert it into $SU(N)$. All formulae quoted from [GP] in this section is after the conversion.

it is straightforward to check that (68) is exactly the equation $\chi'_t(f) = e^{2\pi i\rho}$. Then the [GP] gives the following expression of the equivariant Verlinde formula

$$d_g = \sum_{\sigma\in F^{\mathrm{reg}}_{\rho,t}} \left(N(1-t)^{(N-1)(1-R)} \det \left| \frac{1}{2\pi i} \frac{\partial^2 \widetilde{W}}{\partial\sigma_a \partial\sigma_b} \right| \frac{\prod_\alpha (1-te^{\alpha(\sigma)})^{1-R}}{\prod_\alpha (1-e^{\alpha(\sigma)})} \right)^{g-1}. \tag{69}$$

As $N \det \left| \frac{1}{2\pi i} \frac{\partial^2 \widetilde{W}}{\partial\sigma_a \partial\sigma_b} \right|$ is exactly the same as $|F| \det(H_t^\dagger)$, (69) agrees with the index formula in Theorem 6 that we have derived in this paper.

For $G = SU(2)$, we parametrize the Cartan subgroup $U(1)$ by $e^{i\vartheta}$. The index formula is then given by

$$Z = \sum_\vartheta \theta_t(e^{i\vartheta})^{1-g},$$

where we are summing over the solutions to the equation

$$e^{2i(k+2)\vartheta} \left(\frac{1-te^{2i\vartheta}}{1-te^{-2i\vartheta}} \right)^2 = 1$$

in $(0,\pi)$ and

$$\theta_t(e^{i\vartheta}) = (1-t)^{R-1} \frac{4(\sin\vartheta)^2 \cdot |1-te^{2i\vartheta}|^{2R-2}}{|F| \det H^\dagger},$$

with

$$|F| \det H^\dagger = 4 \cdot \left[\frac{k+2}{2} + \frac{2t\cos 2\vartheta - 2t^2}{|1-te^{2i\vartheta}|^2} \right].$$

For $R = 2$, the index indeed agrees with (7.51) in [GP], and for $R = 0$, it gives the index for the total lambda class of the cotangent bundle to $\mathfrak{M}$ as in [TW] and [T6] (see also [OY] where a quantum field theory approach was used).

The indices for the parabolic moduli spaces are given by

$$d_{g,n}(\{\lambda_i\}) = \sum_{f\in F^{\mathrm{reg}}_{\rho,t}/W} \theta_t(f)^{1-g} \prod_i \Theta_{t,\lambda_i}(f),$$

and for $SU(2)$ we have

$$\Theta_\lambda(e^{i\vartheta}) = \frac{\sin[(\lambda+1)\vartheta] - t\sin[(\lambda-1)\vartheta]}{\sin\vartheta |1-te^{2i\vartheta}|^2}.$$

In particular,

$$\Theta_0 = \frac{(1+t)}{|1-te^{2i\vartheta}|^2},$$

and the constant function on T can be decomposed as

$$1 = \Theta_0 - t\Theta_2,$$

in agreement with the decomposition of the "cap state" in the 2d TQFT,

$$w_\emptyset = w_0 - tw_2.$$

A version of the deformed characters with different normalization factors appeared in (7.47) of [GP], where it has the meaning of the transfer matrix between the diagonal basis e_i and the parabolic basis w_j. The normalization constant there is chosen such that all the basis vectors has norm 1.

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Center for Quantum Geometry of Moduli Spaces, Department of Mathematics, University of Aarhus, DK-8000, Denmark

Walter Burke Institute for Theoretical Physics, California Institute of Technology, Pasadena, CA 91125, USA,
Max-Planck-Institut für Mathematik,, Vivatsgasse 7, D-53111 Bonn, Germany
E-mail address: gukov@theory.caltech.edu

Walter Burke Institute for Theoretical Physics, California Institute of Technology, Pasadena, CA 91125, USA,
Center for Quantum Geometry of Moduli Spaces, Deparment of Mathematics, University of Aarhus, DK-8000, Denmark
E-mail address: pei@caltech.edu