Gain and Dispersion Focusing in a High Gain Laser

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The transverse modes of a laser resonator containing a medium with a strong radial gain profile differ greatly from the modes of a similar resonator containing a low gain medium. Focusing and defocusing effects result from the gain profile and from the associated dispersion profile. The dispersion focusing causes an asymmetry in the power output as the laser is tuned across the gain line. The theory has been verified using a high gain 3.51-μ xenon laser.

I. Introduction

In lenslike media having an approximately quadratic radial variation of either the gain or the index of refraction there is a focusing or defocusing of propagating Gaussian beams.[1,2] A positive index profile results in a spot size of an unmatched beam which oscillates periodically with distance, while a positive gain profile causes the beam to approach a steady-state spot size. In a high gain gas discharge there is often a strong profile of the gain, so that gain focusing is likely to be important. However, whenever a medium has a gain spectrum, it must also have an associated dispersion spectrum. Therefore, dispersion focusing effects must always accompany gain focusing. It is the purpose of this paper to show that the two types of focusing are comparable in importance and that, moreover, the dispersion focusing may lead to asymmetry in Lamb dip measurements. Ordinarily asymmetries in the output of simple gas lasers are attributed to collisions between the atoms.[3,4] The lasers considered are assumed to be operated very near threshold, so that self-focusing is unimportant.[5] This treatment is based on one given elsewhere.[6]

II. Theory

A lenslike medium is one in which the complex propagation constant k varies quadratically with radius according to

$$k = k_0 - \tfrac{1}{2}k_2 r^2. \qquad (1)$$

The real and imaginary parts of k are given by

$$k = \beta + i\alpha. \qquad (2)$$

Here α is the exponential gain constant for the electric field and β is related to the index of refraction by

$$\beta = 2\pi n/\lambda. \qquad (3)$$

The easiest way to study the propagation of Gaussian beams through such lenslike media is in terms of the complex beam parameter q given by

$$1/q = (1/R) - i(\lambda_m/\pi w^2), \qquad (4)$$

where R is the radius of curvature of the phase fronts, w is the spot size of the beam, and λ_m is the wavelength in the medium. The same beam parameter may also be used to characterize higher order Laguerre-Gaussian and Hermite-Gaussian beams.

From the wave equation one finds that the propagation of q is governed by the Ricatti equation[1]

$$(1/q)^2 + (d/dz)(1/q) + (k_2/k_0) = 0. \qquad (5)$$

For simplicity we assume here that saturation is unimportant and the coefficients k_0 and k_2 are independent of position along the axis of the medium. The solution of Eq. (5) is

$$1/q_2 = \{[(1/q_1)\cos(k_2/k_0)^{\frac{1}{2}}z] - [(k_2/k_0)^{\frac{1}{2}}\sin(k_2/k_0)^{\frac{1}{2}}z]\}$$
$$\div \{[(1/q_1)(k_0/k_2)^{\frac{1}{2}}\sin(k_2/k_0)^{\frac{1}{2}}z] + [\cos(k_2/k_0)^{\frac{1}{2}}z]\}. \qquad (6)$$

Except when $(k_2/k_0)^{\frac{1}{2}}$ is real, the beam parameter at large distances approaches the limiting value

$$\div \{1/q_\infty = \mp i(k_2/k_0)^{\frac{1}{2}}\}, \qquad (7)$$

where the upper sign is used if $\mathrm{Im}(k_2/k_0)^{\frac{1}{2}} > 0$ and the lower sign is used if $\mathrm{Im}(k_2/k_0)^{\frac{1}{2}} < 0$.

In the special case $\mathrm{Im}(k_2/k_0)^{\frac{1}{2}} = 0$, $\mathrm{Re}(k_2/k_0)^{\frac{1}{2}} \neq 0$ (no gain profile), the beam parameter oscillates periodically without approaching a steady-state value. If $(k_2/k_0)^{\frac{1}{2}} = 0$ so that there is no profile at all, Eq. (6) reduces to the free space result

$$q_2 = q_1 + z. \qquad (8)$$

Otherwise, after some oscillation the beam goes to the limit given in Eq. (7). From Eqs. (4) and (7) the limit-

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ing spot size is

$$w_\infty = (\lambda_m/\pi)^{\frac{1}{2}}[\pm\mathrm{Re}(k_2/k_0)^{\frac{1}{2}}]^{-\frac{1}{2}} \tag{9}$$

and the limiting radius of curvature of the phase fronts is

$$R_\infty = [\pm\mathrm{Im}(k_2/k_0)^{\frac{1}{2}}]^{-1}. \tag{10}$$

The spot size given by Eq. (9) must be real if the beam is to be confined by the medium. With our sign convention this means that the real and imaginary parts of $(k_2/k_0)^{\frac{1}{2}}$ must have the same sign, or that

$$\mathrm{Im}(k_2/k_0) > 0. \tag{11}$$

With Eq. (2) this stability condition is simply

$$\alpha_2 > 0, \tag{12}$$

provided that the gain per wavelength is small ($\beta_0 \gg \alpha_0$). In other words, the gain profile causes a *damped focusing* such that the beam approaches a stable steady-state value if and only if the gain is highest on the axis of the medium. It is only this stable situation which is of interest in the remainder of this paper. From Eqs. (9) and (10) the spot size is

$$w_\infty = (\lambda_m/\pi)^{\frac{1}{2}}\{\pm\mathrm{Re}[(\beta_2 + i\alpha_2)/\beta_0]^{\frac{1}{2}}\}^{-\frac{1}{2}} \tag{13}$$

and the radius of curvature of the phase fronts is

$$R_\infty = \{\pm\mathrm{Im}[(\beta_2 + i\alpha_2)/\beta_0]^{\frac{1}{2}}\}^{-1}. \tag{14}$$

We have described so far the propagation of Gaussian beams in long high gain laser amplifiers. The transverse modes of a laser resonator containing a high gain medium are most easily found using complex beam matrices.[2] In the remainder of this paper we consider only the simplest type of high gain laser, which consists of a plane parallel resonator filled with a lenslike medium. Such a configuration can be well approximated in practice. The spot size everywhere in this laser is equal to the limiting value given by Eq. (13), which can be written as

$$w = \left((\pi\alpha_2/4\lambda_m)\{[1 + (\beta_2/\alpha_2)^2]^{\frac{1}{2}} + (\beta_2/\alpha_2)\}\right)^{-\frac{1}{2}}. \tag{15}$$

This result can be applied to lasers with various types of index profiles. In the following, only profiles due to the gain and dispersion will be considered.

To find how the spot size in Eq. (15) depends on frequency, it is necessary to know the frequency dependence of α_2 and β_2. In a Doppler-broadened medium the unsaturated intensity gain g is a Gaussian function of frequency and $\alpha_2(x)$ may be written

$$\alpha_2(x) = (g_2'/2)e^{-x^2}, \tag{16}$$

where $x = 2(\nu - \nu_0)(\ln 2)^{\frac{1}{2}}/\Delta\nu_D$ is a normalized frequency and g_2' is the line center value of the quadratic term in the intensity gain constant. The quadratic term in the index of refraction of a Doppler-broadened medium is[7]

$$n_2(x) = (\lambda g_2'/2\pi^{3/2})F(x), \tag{17}$$

where $F(x)$ is Dawson's integral given by

$$F(x) = e^{-x^2}\int_0^x e^{t^2}dt. \tag{18}$$

Therefore $\beta_2(x)$ is simply

$$\beta_2(x) = (g_2'/\pi^{\frac{1}{2}})F(x). \tag{19}$$

Using Eqs. (16) and (19) in Eq. (15), one obtains the following expression for the frequency-dependent spot size of a plane parallel gain-focused laser:

$$w(x) = \left[\frac{\pi g_2'}{8\lambda_m}e^{-x^2}\left(\left\{1 + \left[\frac{2F(x)e^{x^2}}{\pi^{\frac{1}{2}}}\right]^2\right\}^{\frac{1}{2}} + \frac{2F(x)e^{x^2}}{\pi^{\frac{1}{2}}}\right)\right]^{-\frac{1}{2}} \tag{20}$$

It is convenient to define the normalized spot size $w^*(x)$ given by

$$w^*(x) = e^{x^2/4}\left(\left\{1 + \left[\frac{2F(x)e^{x^2}}{\pi^{\frac{1}{2}}}\right]^2\right\}^{\frac{1}{2}} + \frac{2F(x)e^{x^2}}{\pi^{\frac{1}{2}}}\right)^{-\frac{1}{2}}, \tag{21}$$

which is equal to the actual spot size divided by its line center value. A plot of $w^*(x)$ is given in Fig. 1.

Fig. 1

Also plotted in the figure is the function $e^{x^2/4}$, which would represent the frequency dependence of the normalized spot size if dispersion focusing were neglected. The asymmetry of the spot size results from the positive and negative dispersion focusing, which occur, respectively, for positive and negative values of the frequency x. The minimum value of the spot size is at a frequency of about $x \simeq 0.6$ rather than at line center.

We are primarily interested here in inhomogeneously broadened media, but the results for homogeneous broadening are similar. In an unsaturated homogeneous medium the gain is the Lorentzian

$$\alpha_2(y) = (g_2'/2)[1/(1 + y^2)], \tag{22}$$

where $y = 2(\nu - \nu_0)/\Delta\nu_n$ is a normalized frequency. The index of refraction is

$$n_2(y) = (\lambda g_2'/4\pi)[y/(1 + y^2)]. \tag{23}$$

Therefore, from Eq. (15) the spot size is given by

$$w = \{(\pi g_2'/8\lambda_m)[1/(1 + y^2)][(1 + y^2)^{\frac{1}{2}} + y]\}^{-\frac{1}{2}}. \tag{24}$$

These homogeneous results are not considered further.

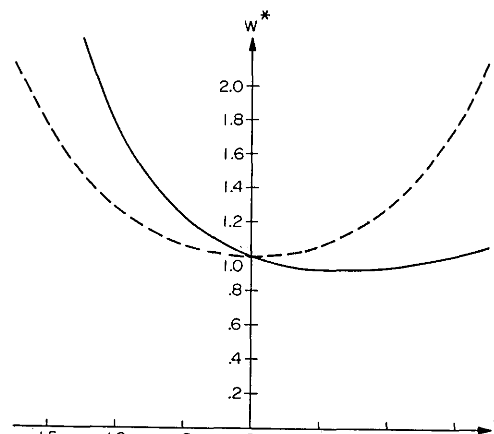

Fig. 1. Solid line is the normalized spot size as a function of frequency. Dashed line is the spot size neglecting dispersion focusing.

The difference between the spot sizes indicated by the two curves in Fig. 1 is at most of the order of 10%. Nevertheless, the dispersion focusing could, in principle, be detected directly by scanning the output beam profile of a laser if the oscillation frequency were known. A more important practical consequence of the dispersion focusing is that the total power output of the laser becomes asymmetric about line center. One expects that for equal values of the gain, the larger the beam spot size, the greater the output power. In particular, the maximum power output of the laser does not occur when the oscillation frequency is at gain center (neglecting the Lamb dip, of course). The asymmetry in the power output is easily measured and provides a fairly direct indication of dispersion focusing.

The growth of intensity in a saturating one-dimensional inhomogeneously broadened laser is governed by the well-known expression

$$dI/dz = gI/(1 + sI)^{\frac{1}{2}}, \tag{25}$$

where s is a saturation parameter. For simplicity distributed losses are neglected. For very weak saturation ($sI \ll 1$) Eq. (25) reduces to the homogeneous approximation

$$dI/dz = gI/(1 + \tfrac{1}{2}sI). \tag{26}$$

If the laser beam were uniform over an area of radius w and there were no gain profile, then the corresponding result for the growth of the total power would be

$$dP/dz = g_0P/[1 + \tfrac{1}{2}(sP/\pi w^2)]. \tag{27}$$

Equation (27) can be shown to be also valid for a Gaussian beam with nearly plane phase fronts in a medium with a quadratic gain profile, provided that the square of the spot size w is much smaller than the square of the discharge diameter r_0.[6] This is not always a good approximation but it is valid for our experiments. g_0 is the unsaturated gain at the axis of the medium. The result for a homogeneously broadened medium would be the same as Eq. (27) except that the factor $\frac{1}{2}$ would be missing.

If one mirror is highly reflecting, Eq. (27) can be integrated for one loop through the laser medium, and the result is

$$\tfrac{1}{2}(s/\pi w^2)(P_2 - P_1) = 2g_0l - \ln(P_2/P_1), \tag{28}$$

where l is the length of the medium. This integration is possible as long as the right and left traveling beams interact with different velocity classes of atoms. If the reflectivity of the output mirror is R and the transmission is T, then Eq. (28) may be solved for the output power P_0 as

$$P_0 = (2\pi w^2/s)[T/(1 - R)](2g_0l + \ln R). \tag{29}$$

A similar result can be obtained for homogeneously broadened lasers.

Using Eq. (20) for the frequency-dependent spot size with $g_0 = g_0'e^{-x^2}$, one obtains finally

$$P_0(x) = (4r_0/s)(\pi\lambda_m/1.44g_0')^{\frac{1}{2}}e^{x^2/2}[T/(1 - R)](2g_0'le^{-x^2} + \ln R) \times (\{1 + [2F(x)e^{x^2}/\pi^{\frac{1}{2}}]^2\}^{\frac{1}{2}} + [2F(x)e^{x^2}/\pi^{\frac{1}{2}}])^{-\frac{1}{2}}. \tag{30}$$

We have assumed here that the medium has a Bessel function radial gain profile, so that the quadratic term is[2]

$$g_2 = g_0(2.88/r_0^2). \tag{31}$$

Equation (30) is the general expression for the frequency dependence of the output power of a Doppler-broadened laser oscillator (neglecting the Lamb dip). The exact calculation of the Lamb dip is difficult in high gain lasers. However, as long as the homogeneous line width is small compared to the Doppler line width, the Lamb dip provides a useful indication of line center ($x = 0$) without significantly affecting the over-all line shape.

It is convenient to define the normalized power spectrum

$$P_0^*(x) = (e^{-x^2} - b)e^{x^2/2}(\{1 + [2F(x)e^{x^2}/\pi^{\frac{1}{2}}]^2\}^{\frac{1}{2}} + [2F(x)e^{x^2}/\pi^{\frac{1}{2}}])^{-\frac{1}{2}}, \tag{32}$$

where

$$b = -(\ln R/2g_0'l) \tag{33}$$

is a threshold parameter which is less than unity if the laser is saturated. Equation (32) is plotted in Fig. 2 for various values of the parameter b. Evidently when the laser is above threshold, the greatest output occurs at a slightly negative frequency rather than at line center. Near threshold ($b \rightarrow 1$) this effect diminishes and the greatest output occurs near $x = 0$. Also plotted in Fig. 2 is the gain spectrum e^{-x^2}.

The asymmetry of the power spectrum is most conveniently characterized by the location in frequency of the power maximum. In Fig. 5 is a plot of this frequency as a function of the threshold parameter b. The plots in Figs. 2 and 5 are not quantitatively correct for all values of b. From Eqs. (27) and (29) this homogeneous approximation is only valid in an inhomogeneously broadened medium as long as the product $(2g_0l + \ln R)(1 - R)^{-1}$ is small compared to unity. In a

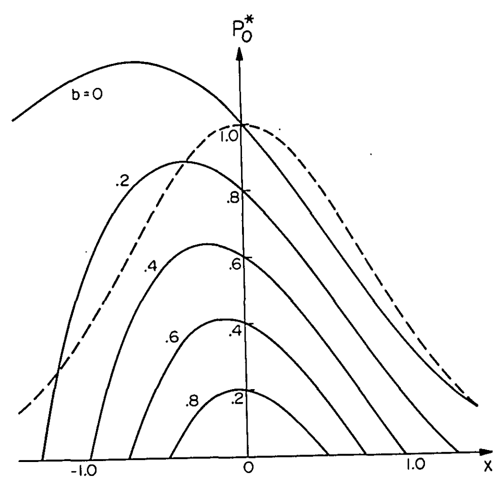

Fig. 2. Normalized power output as a function of frequency for various values of the threshold parameter. Dashed line is the gain spectrum.

high loss laser ($R \ll 1$) this is the same as requiring that b be nearly equal to unity. Nevertheless, these results are expected to be qualitatively correct for most values of b.

It is possible to obtain a simpler approximate expression for the output power as a function of frequency which is valid for small values of x. One finds after some algebra that Eq. (32) may be expanded to second order in x as

$$P_0^*(x) \simeq (1 - b) - [(1 - b)/\pi^{\frac{1}{2}}]x + \{[(1 - b)/2\pi] - [(1 + b)/2]\}x^2. \quad (34)$$

The maximum of this spectrum occurs at the frequency

$$x_{\max} = \{\pi^{-\frac{1}{2}} - [(1 + b)/(1 - b)]\pi^{\frac{1}{2}}\}^{-1}. \quad (35)$$

If b is nearly equal to unity, then the maximum is at

$$x_{\max} = -[(1 - b)/2\pi^{\frac{1}{2}}]. \quad (36)$$

This equation is plotted as a dashed line in Fig. 5. The two lines in the figure are in good agreement in the limit of weak saturation.

In this section we have discussed focusing effects which are due to the gain and dispersion profiles which may be associated with a high gain laser transition. It was shown that the spot size of a waveguided beam is greater for frequencies below gain center than for frequencies greater than gain center. As a consequence of this focusing asymmetry, the power output maximum occurs at a frequency slightly below gain center. Experimental verification of the theory is described in the next section.

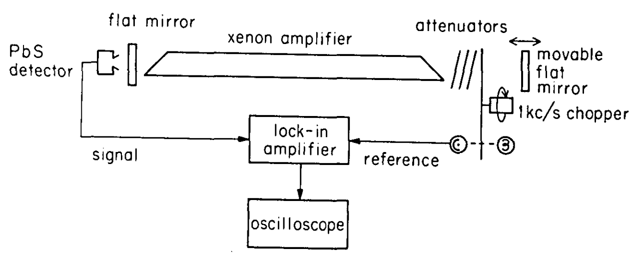

Fig. 3. Experimental setup.

III. Experiment

The gain focusing and dispersion focusing have been observed experimentally using a high gain 3.51-μ xenon laser. In a discharge laser of this type the gain maximum must be at the axis of the discharge with the gain falling to zero at the tube walls. This gain profile makes the xenon laser appropriate for studying focusing effects.

The spot size in a simple plane parallel resonator filled with an unsaturated high gain medium is given by Eq. (20). For frequencies near line center Dawson's integral $F(x)$ goes to zero and the spot size is simply

$$w(x) = (8\lambda_m/\pi g_2)^{\frac{1}{2}}. \quad (37)$$

Experimental investigations of this result have been reported previously.[2] The purpose of this section is to consider experimentally the more subtle frequency asymmetry resulting from dispersion focusing.

The apparatus is shown in Fig. 3. The dc discharge was about 5.5 mm in diameter and 1.1 m in length. The right mirror was highly reflecting and could be translated longitudinally by means of a motor drive. The cavity length was 1.29 m, so the empty cavity mode spacing would be about $c/2L = 116$ MHz. Synchronous detection was used to improve the signal-to-noise ratio, and monoisotopic xenon was used to prevent unnecessary asymmetries in the output. The xenon pressure was maintained at about 5 μ by means of a liquid nitrogen trap.[8]

A typical plot of the power output for decreasing cavity length (increasing frequency) is shown in Fig. 4. The laser was operated very near threshold and the peaks represent successive longitudinal modes. These peaks are to be compared to the theoretical curves shown in Fig. 2. In the experimental plot there is a dip in the output power on the high frequency side of the peak. This is the Lamb dip and it results from the interaction of the left and right traveling beams with atoms which have zero z-component of velocity. Thus the Lamb dip provides a convenient indication of the frequency $x = 0$.

Comparison of the experimental and theoretical plots shows that the power maximum is shifted down in frequency by roughly the amount predicted by the dispersion focusing theory. Some data are shown in Fig. 5. The value of the gain as a function of discharge current was determined by introducing known losses into the cavity and reducing the current until threshold was reached. The reflectivity of the output mirror was 4% and the value of b is given by Eq. (33). For the analysis of the data it was necessary to take into account mode pulling, because with a dispersive medium of this sort the rate of change of the frequency with mirror position may be greatly reduced near line center.[9] For a rigorous comparison of the experimental and theoretical results over the entire spectrum it would have been necessary to include the nonlinear mode pulling which occurs near the wings of the gain line. The pressure was low enough in these experiments that collision effects are believed to be completely unimportant. The experiments indicate that the threshold Lamb dip in xenon has a width of about 6 ± 1 MHz and a depth somewhat greater than the 10% *enhanced* Lamb dip reported previously.[10]

A possible cause of asymmetry in the power measurements is the mass motion of the emitting atoms.[11] Particularly in a low pressure dc discharge one might expect a drift of the ions toward the cathode compensated by a drift of neutral atoms toward the anode. In a high gain laser the Doppler shifts resulting from this mass motion would result in an asymmetry of the output power spectrum. However, the asymmetry for light emerging from one end of the laser would be expected to be in the opposite direction to the asymmetry for light emerging from the other end. To check this

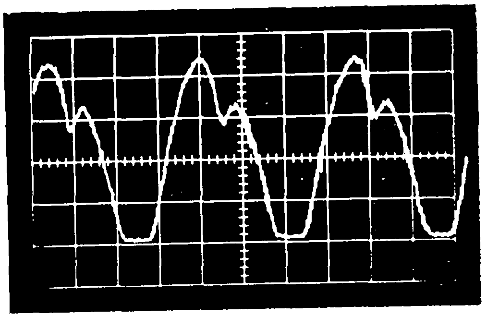

Fig. 4. Power output for decreasing cavity length with a discharge current of 18 mA.

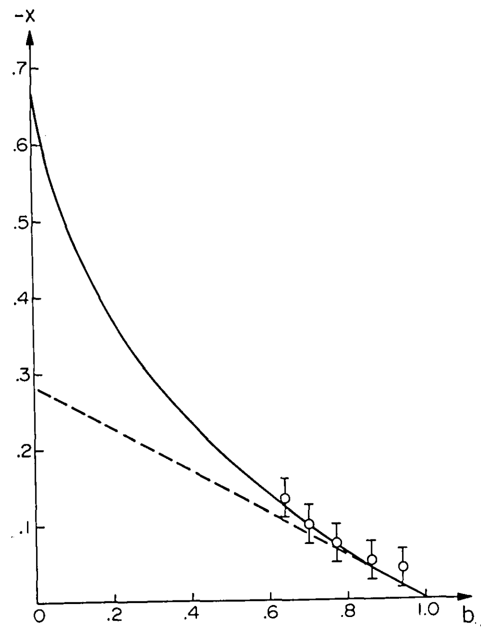

Fig. 5. Frequency of the power maximum vs the threshold parameter b. The solid line is a theoretical result obtained from Eq. (32) and the dashed line is a plot of Eq. (36). The circles are experimental values.

possibility a resonator was constructed having equally transmitting mirrors at the two ends. It was found that the power spectrum was identical at the two ends of the laser, indicating that for the conditions of our experiments drift of the atoms is unimportant. The shift resulting from the known abundances of impurity isotopes is estimated to be small compared to the observed asymmetry.

IV. Conclusion

The transverse modes of a laser containing a high gain medium may differ significantly from the modes of a similar low gain laser because of gain and dispersion focusing. The dispersion effects result in an asymmetry of the output power spectrum which could be important in any Lamb dip measurements in lasers with moderately high gain. Experiments with a high gain 3.51-μ xenon laser have yielded results in agreement with the theory. While only plane parallel resonators have been considered here, the results may readily may be extended to more complicated laser configurations.

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This photograph of participants at the OSA Tucson meeting last year includes Marlan Scully, Hideya Gamo, and G. Toraldo di Francia.