ABOUT THE PRODUCTION OF THE CONTINUOUS X-RAY SPECTRUM

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Twenty years ago, I gave an account[1] of the angular intensity distribution of the general x-radiation, "Brems-Strahlung," of an anticathode, on classical basis, with the result that the maximum intensity should be shifted from the position $\theta = \pi/2$ to smaller angles, θ being counted from the direction of propagation of the cathode ray to that of the x-ray observed. The assumption was made that the stopping process of the incident cathode particles can be considered as rectilinear and coincident with the direction of the impinging beam. During the stopping process, all velocities had to be considered between the initial velocity v_1, and the final velocity $v_2 = 0$, and the radiations corresponding to these different velocities had to be summed up. Because, according to the classical theory, or what is the same, according to the special relativity theory, electromagnetic field and radiation intensity in any state $(v,\dot{v})$ of the stopping process are given by:

$$E = H = \frac{e\dot{v}}{c^2 r} \frac{\sin\theta}{(1 - \beta\cos\theta)^3}, \quad J = \frac{e^2\dot{v}^2}{4\pi c^3 r^2} \frac{\sin\theta}{(1 - \beta\cos\theta)^6}, \tag{1}$$

the resultant radiation during the whole process of stopping ($\dot{v}$ = const., v decreasing from v to 0) would be[2]

$$J = \frac{e^2\dot{v}}{16\pi c^2 r^2} \frac{\sin^2\theta}{\cos\theta} \left\{ \frac{1}{(1 - \beta\cos\theta)^4} - 1 \right\}. \tag{2}$$

The shift of the maximum according to (2) of course is smaller than the shift according to (1), if $\beta = v/c$ corresponds in both cases to the initial velocity v. Also, the polarization first observed by Barkla of the general x-radiation is accounted for by the hypothesis of the rectilinear stopping.

At that time, only observations with solid anticathodes were available, the results of which seemed to agree in a general way with our formula (2). The observed polarization could be, of course, not complete, because of the multiple scattering and bending in the anticathode material. But, nowadays, by the work of Duane[3] on mercury vapor, by Webster[4] and Kulenkampff[5] on very thin foils (thickness, e.g., 0.5μ), we are informed about the stopping process in the single atom. We will show in this note that especially with the observations of Kulenkampff formulas (1) hold if the velocity ratio β occurring in (1) is interpreted in a convenient way; we

will point, also, to the method, by which the total problem of production of general x-radiation is to be treated in wave mechanics.

1. *General Assumptions.*—In the older quantum-theory, one would start with an hyperbolic orbit of the incident cathode electron and would consider the transition to another hyperbolic orbit of smaller energy, the difference of energies being radiated as an x-ray wave. But this view does not correspond to the spirit of wave mechanics. One must treat the incident cathode beam as an electronic wave, modified by scattering on the nucleus of the atom, *and one must treat as well, the emergent electron as an electronic wave of smaller energy,* modified also by scattering on the nucleus. Both waves are *present in the whole space,* the incident wave as well behind the nucleus as before, the emergent as well before the nucleus as behind. Multiplying both one has to compute the combined electric density $\rho = \psi_1\psi_2^*$ of the initial and the final state (ψ^* meaning the conjugate complex to ψ) and one has to compute the combined electric moment M of the transition, which is responsible for the energy emitted. This treatment is exactly equivalent to the usual treatment of the line spectra where one computes, e.g., M_x by the "matrix-element"

$$M_x = \int x\psi_1\psi_2^* d\tau, \tag{3}$$

the initial eigenfunction ψ_1, and the final ψ_2, being defined for all space and time and working together in the determination of M according to (3).

In our case, we deal with asymptotically plane waves and may put, following a paper by Temple[6] on the scattering of α particles,

$$\psi_1 = e^{ik_1x}L(u_1),\ \psi_2 = e^{ik_2x}L(u_2), \tag{4}$$

$$\begin{cases} k_1 = \dfrac{2\pi}{\lambda_1} = \dfrac{2\pi}{h}mv_1, \quad k_2 = \dfrac{2\pi}{\lambda_2} = \dfrac{2\pi}{h}mv_2, \\ u_1 = ik_1\eta, \quad u_2 = ik_2\eta, \end{cases} \tag{5}$$

η meaning the parabolic coördinate, which is symmetrical around the axis of incidence (x-axis), v_1 and v_2 meaning the initial and final velocity of the cathode particle; $L(u)$ is a certain solution of the differential equation of Laguerre (not a Laguerre's Polynomial, but a transcendental function, so to speak, a polynomial of an imaginary degree). The frequency ν of the radiation emitted is determined by the time factor of $\psi_1\psi_2^*$, which gives according to de Broglie's fundamental assumption:

$$\nu = \frac{\epsilon_1 - \epsilon_2}{h} = \frac{m}{2}\frac{v_1^2 - v_2^2}{h} = \frac{h}{8\pi^2 m}(k_1^2 - k_2^2). \tag{6}$$

As to the origin of the formula (4), we may mention that "scattering," or, what is the same, "diffraction" of an electron by a nucleus is a one-

electron problem. So, we understand that we need for this problem the eigenfunctions of the hydrogen atom, the above-used function $L(u)$ being, indeed, the eigenfunction of hydrogen in parabolic coördinates characteristic for the continuous hydrogen spectrum. If we had to treat, on the other hand, the diffraction of an electronic wave by an hydrogen atom, we would need the eigenfunctions of the helium atom, because we deal in this case, with a two-electron problem, though of course the "helium" atom would have the nuclear charge one instead of two; and so forth. While those problems could be handled until now only in an approximate way, by the method of perturbations, it can be shown that an exact and complete solution can be written down valid for any atom, the difficulty consisting only in the imperfect knowledge of those eigenfunctions. For the production of x-rays, it is evident that we need not consider complicated atomic structures, but may restrict ourselves to the consideration of the bare nucleus.

We are not concerned here with the detailed computation of the moment M; but will draw only some general conclusions from the form of the expression (3). In the first place, M has only the *one component* M_x, M_y and M_z being equal to zero because of the symmetry[7] of the incident and the emergent beam with respect to the axis of x. This statement is *equivalent to our old assumption as to the rectilinear way of stopping* and provides *complete polarization*, the plane of polarization being given by the incident cathode beam and the emitted x-ray. We mention that Ross[8] observed complete polarization with a thick anticathode *for the short wave-length limit*, and that we have to expect from our present (or former) theory the same result with a sufficiently thin anticathode for *any* part of the continuous spectrum.[7]

In the second place, we remark that our density distribution $\psi_1\psi_2^*$ is not, as in the optical case, restrained to a portion of the space small compared with the wave-length of radiation, but this distribution is in our present case extending over the whole space. So we have to refine the expression (3) by a "retardation factor" and have, also, to compute it not for a single frequency ν (cf. Eq. 6), but for an element $(\nu,\Delta\nu)$ of the continuous spectrum. But for what follows the general form of the expression (3) is sufficient without any detailed knowledge of its numerical value.

We now come to the main point, viz., the *angular distribution* of the radiation emitted by the oscillating electric moment (3). In the optical case, we deal with the radiation from an *atom at rest*, so that we are justified in putting $\beta = 0$ in (1). We have to replace in (1), of course, meantime, $e\ddot{v}$ by the general expression $\ddot{M}$. In this way, we get from (1) the well-known radiation law of a dipole.

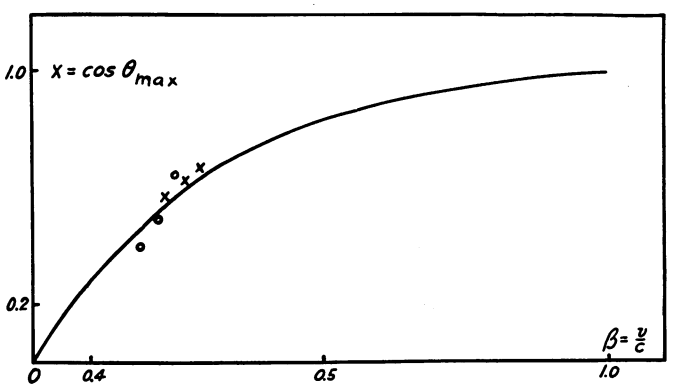
$$J = C \sin^2 \theta \qquad (7)$$

C not depending on θ. But this would be wrong in our case of x-ray production. The radiation is emitted really by a *moving charge distribution*, the initial velocity being v_1, the final v_2. It is obvious to take as the mean velocity of motion during the emission process

$$v = \frac{v_1 + v_2}{2}, \quad \beta = \frac{v}{c} = \sqrt{\frac{2e}{mc^2}} \frac{\sqrt{V_1} + \sqrt{V_2}}{2}, \tag{8}$$

V_1 meaning the voltage of the incident cathode beam, V_2 the voltage of that particular range of the continuous spectrum to be considered. Doing so, we get from (1) instead of (7)

$$J = C \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^6}. \tag{9}$$



We see, at once, from this expression, as mentioned already in connection with equation (2), that the shift of the maximum intensity toward small angles θ will now be *larger* than in our former theory. We see, moreover, that the *shift will be larger for the longer wave-lengths of the continuous spectrum than for the short wave-length limit.* This conclusion seems at the first view somewhat paradoxical, but it is checked, as we will see, by Kulenkampff's experiments.

If we consider that the denominator in (1) comes from the relativity transformation, say the motion of the radiating source relative to the observer, the interpretation of β given in (8) seems to be quite sensible. From this point of view, it would be meaningless to take, e.g., $v_1 - v_2$ instead of the mean velocity $((v_1 + v_2)/2)$; indeed, this assumption would be in contradiction with the experiments. We saw in (6), that $v_1^2 - v_2^2$ determines the wave-length of the radiation, but $v_1 - v_2$ does not at all determine the geometrical relation between the source and the observer, which is responsible for the intensity distribution.

Working with the corpuscular notion of v_1 and v_2 in our treatment of electronic waves, it is perhaps not out of place to point to the well-known fact that v_1 and v_2 mean, in wave mechanics, the group velocities for the incident and for the emergent wave.

It is true that our treatment of the emitted intensity is not a genuine wave mechanical one. But the same happens at the present state of the theory with any intensity calculation. I do not think that it is in the present case less approximate than in the optical case.

2. *Comparison with Experiments.*—From (9), we get immediately for the position of J_{max}:

$$0 = \frac{2 \sin \theta}{(1 - \beta \cos \theta)^7} (\cos \theta - \beta \cos^2 \theta - 3\beta \sin^2 \theta), \qquad (10)$$

so we have the quadric:

$$2\beta x^2 + x = 3\beta, \qquad x = \cos \theta_{max},$$

the solution of which is

$$x = \frac{1}{4\beta} \left\{ \sqrt{1 + \frac{3}{2} (4\beta)^2} - 1 \right\}. \qquad (11)$$

For $\beta = 0$, we get $x = 0$, $\theta_{max} = \pi/2$, as is known from the classical oscillator at rest, equation (7); for $\beta = 1$, we get $x = 1$, $\theta_{max} = 0$, which means "needle radiation." Figure 1 shows x as function of β. The points $\times$ and $\circ$ are observed values of θ_{max} under conditions which will be explained immediately.

Kulenkampff gives three distribution curves of intensity corresponding to the same initial voltage, $V_1 = 31.0$ kv., but to different wave-lengths of the continuous spectrum, viz.:

$$\lambda = 0.43_5, \qquad \lambda = 0.53, \qquad \lambda = 0.73 \text{ Å},$$

the short wave-length limit being

$$\lambda_m = 0.39$$

We may rewrite equation (6), dividing it by $\nu_{max} = c/\lambda_m$ and noticing that $V_2 = 0$ for $\lambda = \lambda_m$:

$$\frac{\lambda_m}{\lambda} = \frac{V_1 - V_2}{V_1} = 1 - \frac{V_2}{V_1}.$$

This gives

$$V_2 = V_1 \left(1 - \frac{\lambda_m}{\lambda}\right).$$

So we have for the three values of λ mentioned above:

$$V_2 = 3.2, \quad 8.2, \quad 14.4 \text{ kv.} \tag{12}$$

From (8) now, we get

$$\beta = 0.230, \quad 0.263, \quad 0.292,$$

and from (11)

x	0.550	0.600	0.640	calc.
$\theta_{max.}$	56.6°	53.1°	50.2°	
x	0.57	0.62	0.67	obs.[9]
$\theta_{max.}$	55°	52°	48°	

The agreement between the calculated and observed values is very satisfactory. The observed values of x are marked by $\times$ in figure 1.

Kulenkampff gives another set of observations corresponding to different initial voltages, viz.:

$$V_1 = 37.8, \quad 31.0, \quad 24.0 \quad 16.4 \text{ kv.} \tag{13}$$

The distribution curves for all these voltages are "taken for the short wave-length limit." But we may remark that in the first of the former cases (12), the observation has been made "as closely as possible to the short wave-length limit." Therefore, we may be justified in admitting that also in the present cases, the limit could not exactly be reached and we may take the same voltage

$$V_2 = 3.2 \tag{14}$$

as in the former case, for the final energy of the emergent electron. Then we get from (8) with the values (13) and (14),

$$\beta = 0.248, \quad 0.230, \quad 0.209, \quad 0.182,$$

and from (11)

x	0.578	0.550	0.515	0.467	calc.
$\theta_{max.}$	54.7°	56.6°	59.0°	62.2°	
x	0.643	0.574	0.500	0.422	obs.[10]
$\theta_{max.}$	50°	55°	60°	65°	

Also, here the agreement with the observed values is satisfactory, the shift to smaller θ no increasing with *decreasing* wave-lengths (increasing initial voltages) in opposition to the former case where the shift was increasing with *increasing* wave-lengths (increasing final voltages). The observed values of x are marked in figure 1 by O.

We shall not enter here in the discussion of the shape of the distribution curves as indicated by the experiments of Kulenkampff on the one side, and by the theory on the other side, because we would need for that

purpose a more detailed computation of our moment M. I must leave that for a fuller paper to be published later in the *Annalen der Physik*.

In conclusion, I think that the old problem of the stoppage of cathode particles, attacked by Stokes, J. J. Thomson and Wiechert immediately after Roentgen's discovery, can now be solved by the mathematical methods of wave mechanics in a satisfactory way, as is shown by the foregoing discussion of the few experimental data available for the shift of the intensity maximum in the continuous x-ray spectrum.

*Addendum.**—In the above, I have treated from the standpoint of wave mechanics, the continuous spectrum of x-rays produced in a thin target. The incident cathode ray was regarded as a plane de Broglie wave and the emergent beam was treated also as a plane wave of de Broglie type, *having the same direction as the primary beam.* Meanwhile I learned from the experimental results of Professor W. Duane, which he presented at the present meeting of The American Physical Society, that my previous assumption as to the direction of the emergent beam is too specialized. While this assumption would provide complete polarization for any part of the continuous spectrum, Duane observes, in general, incomplete polarization. While, furthermore, my assumption excludes radiation in the direction of the incident beam, there is evidence, according to Duane, of some radiation even in this direction.

I propose, therefore, to generalize my previous assumption so as to admit secondary beams to emerge from the scattering atom in all directions. This amounts to an averaging of the matrix elements defined in my former note over all possible directions. According to this new assumption, only beams emerging with zero velocity should give rise to completely polarized radiation, i.e., only radiation corresponding to the short wave-length limit will be completely polarized. This is in agreement with experiments of P. A. Ross. Furthermore, only radiation near this limit would give zero intensity in the direction of the incident beam; other parts of the continuous spectrum will, of course, furnish intensity in this direction because of the emergent beams forming an angle with the incident wave. This difference in the radiation from the limit and from other parts of the spectrum was already mentioned by Kulenkampff.

The main point of my previous note, viz., the account for the angular distribution of x-rays observed by Kulenkampff, is certainly not essentially influenced by this modified viewpoint.

I am indebted to Professor E. C. Kemble for having suggested to me this generalization, and also to Professor Kennard for having emphasized the analogy of this procedure with the usual treatment in the case of line spectra. Indeed, when calculating intensities of line spectra, we have to sum over all the different orientations of the atom characterized by the magnetic quantum number m, giving one weight to each of the magnetic levels.

Corresponding to this is in our case the averaging over all possible directions of the emerging beam, assuring equal weight for all of them.

* Read before the Academy, April 22, 1929.

[1] Sommerfeld, A., *Physikal. Zeitschr.* **10**, 969, 1909.

[2] Illustrated in *Atombau and Spektrallinien*, 4th Edition, Fig. II, p. 37 and p. 755.

[3] Duane, W., these PROCEEDINGS, **13**, 662, 1927; **14**, 450, 1928.

[4] Webster, D. L., Clark, A., Yeatman, R. M., and Hansen, W. W., these PROCEEDINGS, **14**, 679, 1928.

[5] Kulenkampff, H., *Ann. Phys.*, **87**, 597, 1928.

[6] Temple, G., *Proc. Roy. Soc.*, **121**, 673, 1928.

[7] However, the symmetry is disturbed by the retardation to be introduced for the emitted radiation as remarked below. As a consequence of this retardation the polarization would be complete only for an angle of observation perpendicular to the incident cathode beam, but in this case for every part of the continuous spectrum, so far as our assumption about the direction of the emergent cathode beam is justified. (*Note added in proof.*)

[8] Ross, G. A., *J. Opt. Soc. America*, **16**, 375, June, 1928.

[9] Cf. Fig. 9 in Kulenkampff's paper, loc. cit.

[10] Cf. Fig. 11, *Ibid.*

THE RELATION BETWEEN THE ELECTRIC MOMENT AND THE POTENTIAL DIFFERENCE AT AN INTERFACE

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The relation between the electrical properties of interfaces and the orientation of molecules was first recognized by Hardy[1] and discussed by Harkins, Davies and Clark.[2] It was stated in a definite way for the case of the water-mercury interface by Gouy[3] in 1917. One of us[4] has studied these potential differences at interfaces in their relation to the polarity and orientation of different linkages in the molecule. Quantitative calculations of the electric moment of the molecules from the potential difference data have been made by Guyot[5] and later by Rideal.[6] At an earlier date quantitative comparisons were made by Frumkin.[7] Rideal calculates the electric moment of the butyric acid molecule to be

$$\mu = 0.305 \times 10^{-18},$$

assuming the orientation to be complete and the dielectric constant of the surface layer to be unity, and compares it with the value calculated by Smyth[8] using the Gans theory

$$\mu = \text{ca } 2.0 \times 10^{-18}.$$