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ON THE ORDER OF DIRICHLET L-FUNCTIONS

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1. Introduction. Let $L(s, \chi)$ be a Dirichlet L-function, where χ is a nonprincipal character $(\bmod q)$ and $s = \sigma + it$. A standard estimate for $L(s, \chi)$ based on bounds for $\zeta(s, w)$, is

$$(1) \qquad |L(s, \chi)| \leqq C_1(\varepsilon)\cdot\tau^{c(1-\sigma)+\varepsilon}q^{1-\sigma}, \quad \frac{1}{2} \leqq \sigma \leqq 1 ,$$

where $\tau = |t| + 2$, $c = 1/6$ (see, for example, Prachar [5, (4.12)]), and in fact, c can be replaced by a constant $< 1/6$. An immediate application of Richert's work [6] gives

$$(2) \qquad |L(s, \chi)| \leqq C_1\tau^{100(1-\sigma)3/2}q^{1-\sigma}\log^{2/3}\tau, \quad \frac{1}{2} \leqq \sigma \leqq 1 ,$$

which is better than (1) if σ is near 1.

Another estimate can easily be obtained from $|L(1 + it, \chi)| \leqq C_2 \log \tau q$ and the functional equation of $L(s, \chi)$ as follows. First,

$$\begin{aligned}|L(it, \chi)| &= 2\cdot|(2\pi)^{it-1}q^{1/2-it} \\ &\times \cos\frac{1}{2}\pi\left(1 - it + \frac{1}{2} - \frac{1}{2}\,\bar{\chi}(-1)\right)\Gamma(1 - it)L(1 - it, \bar{\chi})| \\ &\leqq C_3\sqrt{\tau q}\log \tau q .\end{aligned}$$

Now the convexity principle yields for

$$(3) \qquad \begin{aligned}|L(s, \chi)| &\leqq (C_3\sqrt{\tau q}\log \tau q)^{1-\sigma}\cdot(C_2\log \tau q)^{\sigma} \leqq C_4(\tau q)^{1/2(1-\sigma)} \\ &\times \log \tau q, \ 0 \leqq \sigma \leqq 1 .\end{aligned}$$

Neglecting dependence on τ, Davenport [2], improved (3):

$$(4) \qquad |L(s, \chi)| \leqq C_2(\tau)q^{1/2(1-\sigma)}, \quad 0 \leqq \sigma \leqq 1 .$$

Also, Burgess [1] improved (4) by establishing

$$|L(s, \chi)| \leqq C_1(\varepsilon, \tau)q^{3/8(1-\sigma)+\varepsilon}, \quad \frac{1}{2} \leqq \sigma \leqq 1 .$$

By examining Burgess' proof, it can be seen that the constant $C(\varepsilon, \tau)$ can be taken to be $C_2(\varepsilon)\pi^{2(1-\sigma)}$ and his result can be further sharpened to yield

$$(5) \qquad |L(s, \chi)| \leqq C_6\tau^{2(1-\sigma)}q^{3/8(1-\sigma)}C^{\omega}\log \tau, \quad \frac{1}{2} \leqq \sigma \leqq 1 ,$$

where $\omega = \log q/\log\log q$. The estimates (3), (4), and (5) are better than (1) if q is large compared to τ.

For $\sigma = 1/2$, the previous estimates were improved by Fujii, Gallagher and Montgomery, [3], who showed that if P is a fixed set of primes and q is composed only of primes in P, then

$$\left| L\left(\frac{1}{2} + it,\ \chi\right)\right| \leqq C(\varepsilon, P)(\tau q)^{1/6+\varepsilon} . \tag{6}$$

In this paper we prove two more estimates which imply (1), (4), and (5) and which are better than (2), (3), and (6) in some range of σ, τ, and q. We prove:

THEOREM 1. *Let χ be a nonprincipal character* (mod q). *Let $1/2 \leqq \sigma \leqq 1, \tau = |t| + 2$ and $\omega = \log q/\log\log q$. Then*

$$|L(s, \chi)| \ll \tau^{-\sigma} q^{3/8(1-\sigma)} C^{\omega} \log \tau , \tag{7}$$

where C is some absolute constant.

THEOREM 2. *Let χ be a character* (mod q). *Let $1/2 \leqq \sigma \leqq 1$ and $\tau = |t| + 2$. Then*

$$|L(s, \chi)| \ll \tau^{35/108(1-\sigma)} q^{1-\sigma} \log^3 \tau q . \tag{8}$$

In particular, (7) *and* (8) *imply*

$$\left| L\left(\frac{1}{2} + it,\ \chi\ \right|\right) \ll \sqrt{\tau}\, q^{3/16} C^{\omega} \log \tau$$

and

$$\left| L\left(\frac{1}{2} + it,\ \chi\right)\right| \ll \tau^{35/216} \sqrt{q}\, \log^3 \tau q .$$

The estimates of $L(s, \chi)$ for $\sigma \in [0, 1/2]$ can be obtained by using (7) or (8) and the functional equation of $L(s, \chi)$.

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2. Notation.

$$e(f(x)) = \exp(2\pi i f(x)) .$$
$$\omega = \log q/\log\log q .$$
$$s = \sigma + it,\ \frac{1}{2} \leqq \sigma \leqq 1 .$$
$$\tau = |t| + 2 .$$

C denotes some appropriate absolute constant, not always the same.

3. Application of the estimate of Burgess. In this section we will show that

$$|L(s, \chi)| \ll \pi^{1-\sigma} q^{3/8(1-\sigma)} C^{\omega} \log^3 \tau .$$

We need the following result of E. Bombieri:

LEMMA. *Let N and m be nonnegative integers. Let α_j, β_j be numbers such that $|\alpha_j - \beta_j| \leqq (2\pi m N^j)^{-1}$ for $1 \leqq j \leqq m$, and let $f(x) = \alpha_1 x + \cdots + \alpha_m x^m$, $g(x) = \beta_1 x + \cdots + \beta_m x^m$. Let $c_1, c_2, \cdots$ be complex, and let*

$$S(\bar{\alpha}, N) = \max_{1 \leqq N_1 < N} \Big| \sum_{1 \leqq n \leqq N_1} c_n e(f(n)) \Big| ,$$

where $\bar{\alpha} = (\alpha_1, \cdots, \alpha_m)$. Then $S(\bar{\beta}, N) \leqq 6S(\bar{\alpha}, N)$.

Proof. For every $N_1 \in [1, N]$ we have:

$$\sum_{1 \leqq n \leqq N_1} c_n e(g(n)) = \sum_{1 \leqq n \leqq N_1} c_n e(f(n)) \prod_{j=1}^{m} e((\beta_j - \alpha_j) n^j)$$
$$= \sum_{k_1, \cdots, k_m = 0}^{\infty} \left(\prod_{j=1}^{m} \frac{\{2\pi i(\beta_j - \alpha_j)\}^{k_j}}{k_j!} \right) \sum_{1 \leqq n \leqq N_1} c_n n^{mk_m + \cdots + k_1} e(f(n)) .$$

Using Abel's summation formula, we obtain:

$$S(\bar{\beta}, N) \leqq \sum_{k_1, \ldots, k_m = 0}^{\infty} \prod_{j=1}^{m} \frac{|2\pi(\beta_j - \alpha_j)|^{k_j}}{k_j!} \cdot N^{mk_m + \cdots + k_1} \cdot 2S(\bar{\alpha}, N)$$
$$\leqq 2S(\bar{\alpha}, N) \cdot \sum_{k_1, \ldots, k_m = 0}^{\infty} \prod_{j=1}^{m} \frac{|(2\pi(\beta_j - \alpha_j) N^j|^{k_j}}{k_j!}$$
$$\leqq 2S(\bar{\alpha}, N) \left(\sum_{k=0}^{\infty} m^{-k}/k! \right)^m \leqq 6S(\bar{\alpha}, N) .$$

LEMMA 2. *Let $q \geqq 2$ and let M, N be integers. Let χ be a primitive character* (mod q). *Then*

$$\Big| \sum_{1 \leqq n \leqq N} \chi(n + M) \Big| \leqq \sqrt{N} q^{3/16} C^{\omega} .$$

This lemma can be proven similarly to Theorem 2, [1].

LEMMA 3. *Let q and N be integers such that $q \geqq 2$ and $N \leqq \tau q$. Let χ be a primitive character* (mod q). *Then*

$$|S = \max_{N \leqq N_1 \leqq 2N} \Big| \sum_{N+1 \leqq n \leqq N_1} \chi(n) n^{-it} \Big| \ll \sqrt{N\tau \log \tau} \cdot q^{3/16} C^{\omega} .$$

Proof. We can obviously suppose that $\tau \log \tau q \leqq N$ since otherwise the estimate is trivial. Taking $H = [N(\tau \log \tau q)^{-1}]$ and $m = [\log \tau q]$, and dividing the sum in S into $\leqq 2NH^{-1}$ subsums, we obtain:

$$|S| \leqq 2NH^{-1} \max_{N \leqq M \leqq 2N} \max_{1 \leqq H_1 \leqq H} \Big| \sum_{M+1 \leqq n \leqq M+H_1} \chi(n) n^{-it} \Big| .$$

For every M and H_1 in the above range , we get

$$(6) \qquad \sum_{M+1 \leqq n \leqq M+H_1} \chi(n)n^{-it}| = \Big| \sum_{1 \leqq n \leqq H_1} X(n+M) \left(\frac{n+M}{M}\right)^{-it} \Big|$$
$$\leqq \Big| \sum_{1 \leqq n \leqq H_1} \chi(n+M) e\left(-\frac{t}{2\pi}\left\{\frac{n}{M} - \frac{n^2}{2M^2} + \cdots + \frac{(-1)^m \cdot n^m}{mM^m}\right\}\right)\Big|$$
$$+ \frac{|t| H^{m+2}}{M^{m+1}} .$$

Let $\beta_j = 0$ and $\alpha_j = (-1)^j t/2\pi j M^j$. Then for $1 \leqq j \leqq m\, |\alpha_j - \beta_j| = |t| \cdot (2\pi j M^j)^{-1} \leqq (2\pi m H^j)^{-1}$. Applying Lemmas 1 and 2, we obtain:

$$|S| \leqq |2NH^{-1} \max_{N \leqq M \leqq 2N} \max_{1 \leqq H_1 \leqq H} \Big| \sum_{1 \leqq n \leqq H} \chi(n+M) \Big| + 2 \frac{\tau H^{m+1}}{N^m}$$
$$\ll NH^{-1} \sqrt{H} q^{3/16} C^{\omega} + N\tau(\tau \log \tau q)^{-\log \tau q}$$
$$\ll \sqrt{N \cdot \tau \log \tau q}\, q^{3/16} C^{\omega} .$$

From this, the result is easily obtained.

Now we can prove Theorem 1. First, we suppose that χ is primitive. Let $N = [\tau q]$, $M = [\tau q^{3/8}]$, $L = \log(N/M)/\log 2$, $N_l = M 2^l (l = 0, \cdots, L)$. Using Abel's formula, the Polya-Vinogradov estimate for character sums and Lemma 3, we get:

$$|L(s, \chi)| \leqq \sum_{n<M} n^{-\sigma} + \Big| \sum_{M \leqq n \leqq N} \chi(n) n^{-\sigma-it} \Big| + \Big| \sum_{n>N} \chi(n) n^{-s} \Big|$$
$$\ll M^{1-\sigma} \log M + \sum_{l=0}^{L} \max_{N_l \leqq N_l^1 \leqq 2N_l} \Big| \sum_{N_l \leqq n \leqq N_l^1} \chi(n) n^{-\sigma-it} \Big|$$
$$+ \sum_{n>N} \tau n^{-\sigma-1} \Big| \sum_{N \leqq x \leqq n} \chi(n) \Big|$$
$$\ll M^{1-\sigma} \log M + \sum_{l=0}^{L} N_l^{-\sigma} \max_{N_l \leqq N_l^1 \leqq 2N_l} \Big| \sum_{N_l \leqq n \leqq N_l^1} \chi(n)^{-it} \Big|$$
$$+ \tau \sqrt{q} N^{-\sigma} \log q$$
$$\ll M^{1-\sigma} \log M + \sum_{l=0}^{L} N_l^{1/2-\sigma} \sqrt{\tau}\, q^{3/16} C^{\omega} \sqrt{\log \tau} + \tau \sqrt{q} N^{-\sigma} \log q$$
$$\ll M^{1-\sigma} \log M + L M^{1/2-\sigma} \sqrt{\tau}\, q^{3/16} C^{\omega} \sqrt{\log \tau} + \tau \sqrt{q} N^{-\sigma} \log q$$
$$\ll \tau^{1-\sigma} q^{3/8(1-\sigma)} C^{\omega} \log \tau .$$

If X is not primitive, then there is a $q_1 | q$ and a primitive character $\chi_1 \pmod{q_1}$, associated with χ, such that we can write (see, for example, [5, (6.12)]):

$$|L(s, \chi)| = |L(s, \chi_1)| \prod_{p|q} \left|1 - \frac{\chi_1(p)}{p^s}\right| \leqq |L(s, \chi_1)| \cdot \prod_{p|q} 2 \leqq |L(s, \chi_1)| \cdot 2^{\omega} ,$$

and the theorem follows.

4. **The proof of Theorem 2.** To prove Theorem 2, we need two lemmas.

LEMMA 4. *Let $t \geqq 0$, $0 \leqq a \leqq 1$, and let X and X_1 be integers such that $0 < X \leqq X_1 \leqq 2X \leqq \tau^{143/108}$. Then*

$$S_1 \equiv \sum_{X \leqq x \leqq X_1} e(t \log (x + a)) \ll \sqrt{X} \tau^{35/216} \log^2 \tau .$$

Proof. If $X \leqq \sqrt{\tau}$, then the result can be proven similarly to Corollary 2, [4]. The same method yields

$$(9) \qquad \sum_{X \leqq x \leqq X_1} e(t \log x - ax) \ll \sqrt{X} \tau^{35/216} \log^2 \tau ,$$

for $X \leqq \sqrt{\pi}$. If $\sqrt{\tau} \leqq X \leqq \tau^{143/108}$, then, by Lemma 3 of [4]

$$|S_1| \leqq \sum_{t/(X_1+a) \leqq n \leqq t/(X+a)} \frac{\sqrt{t}}{n} e(t \log n - an)| + 0(X\tau^{-1/2}) .$$

Here $t/(X + a) \leqq \sqrt{\tau}$. With the use of Abel's inequality, (9) yields the result for $\sqrt{\tau} \leqq X \leqq \tau^{143/108}$.

LEMMA 5. *Let $1/2 \leqq \sigma \leqq 1$, $t \geqq 1$ and $0 \leqq a \leqq 1$. Then*

$$\zeta(s, a) \equiv \sum_{n=0}^{\infty} (n + a)^{-s} \ll a^{-\sigma} + \tau^{35(1-\sigma)/108} \log^3 \tau .$$

Proof. Let $N = \tau^{143/108}$. Using the Euler-Maclaurin formula [see, for example, [5], (1.7), p. 372]), we obtain similarly to [5], (5.8), p. 114:

$$\begin{aligned}\zeta(s, a) - \sum_{n=0}^{N-1} (n + a)^{-s} &= \frac{(N + a)^{1-s}}{1 - s} - s\int_N^{\infty} \frac{x - [x]}{(x + a)^{s+1}} dx \\ &= \frac{(N + a)^{1-s}}{1 - s} - \frac{1}{2} s \frac{(x - [x])^2}{(x + a)^{s+1}} \Big|_N^{\infty} + \frac{1}{2} s(s + 1) \int_N^{\infty} \frac{(x - [x])^2}{(x + a)^{s+2}} dx \\ &\ll 1 + \tau^2 \int_N^{\infty} u^{-\sigma-2} du \leqq 1 + \tau^2 \cdot N^{-\sigma-1} \ll \tau^{35(1-\sigma)/108} .\end{aligned}$$

If we denote $M = [\tau^{35/108}]$, $L = [\log (N/M)/\log 2]$, $N_l = M \cdot 2^l$ for $l = 0, \cdots, L$ and $N_{L+1} = N$, then we have

$$S \equiv \sum_{n=0}^{N-1} (n+a)^{-s} \ll \sum_{0<n<M} (n+a)^{-\sigma} + \sum_{0\leqq l \leqq L} \Big| \sum_{N_l \leqq n < N_{l+1}} (n+a)^{-s} \Big| .$$

Using Abel's formula and Lemma 4, we obtain:

$$S \ll a^{-\sigma} + M^{1-\sigma} \log M + \sum_{0\leqq l\leqq L} N_l^{-\sigma} \max_{N_l \leqq N_l' \leqq N_{l+1}} \Big| \sum_{N_l \leqq n \leqq N_l'} (n+a)^{-it} \Big|$$

$$\ll a^{-\sigma} + M^{1-\sigma} \log M + \sum_{0\leqq l\leqq L} N_l^{1/2-\sigma} \cdot \tau^{35/216} \log^2 \tau \ll a^{-\sigma}$$

$$+ \tau^{35(1-\sigma)/108} \log^3 \tau .$$

This proves the lemma.

To prove Theorem 2, we can obviously suppose $t \geqq 1$, otherwise the result follows from (1). Using Lemma 5, we obtain:

$$|L(s, \chi)| = \Big| q^{-s} \sum_{m=1}^{q} \chi(m)\zeta(s, m/q) \Big|$$

$$< q^{-\sigma} \sum_{m=1}^{q} ((q/m)^{\sigma} + \tau^{35(1-\sigma)/108} \log^3 \tau) \ll \tau^{35(1-\sigma)/108} q^{1-\sigma} \log^3 \tau q .$$

Note Added in Proof. We would like to draw attention to a recent paper by D. R. Heath-Brown, "Hybrid bounds for Dirichlet L-function," Inventiones Mathematicae, 44 (1978), 149–170, which contains a better result than our Theorem 7.

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