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Partially Conserved Axial-Vector Current and the Nonleptonic *K*-Meson Decays*

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Current commutation relations and a partially conserved axial-vector current hypothesis (PCAC) are applied to the nonleptonic K decays. On the assumption that the effective weak interactions should have the transformation property suggested by Gell-Mann in the quark model of the chiral symmetry, and that the axial-vector isospin charges to which the pions are related through PCAC should be generators of $U(2)\times U(2)$, the $K_{2\pi}$ and the $K_{3\pi}$ decay amplitudes, in which the four-momenta of all the final pions are continued to zero, are proved to obey the $\Delta I=\frac{1}{2}$ rule. In the approximation of neglecting the continuation, the decay rate of the $K_{3\pi}$ is related to that of the $K_{2\pi}$. The ratio of the decay rates is estimated and a reasonable agreement with experiment is obtained. Only the charge independence is assumed for the strong interactions.

I. INTRODUCTION

CURRENT commutation relations[1] and a partially conserved axial-vector current hypothesis[2] (PCAC) have been successfully applied to the parity-violating amplitudes of the nonleptonic hyperon decays.[3] What is remarkable is that the $\Delta I=\frac{1}{2}$ rules can be obtained for the Λ and Ξ decays without octet enhancement.

We shall give here further applications to the nonleptonic K decays. In the symmetric limit of $SU(3)$, all the $K_{2\pi}$ decays are forbidden. But the $K_{2\pi}$ decays really occur because of the large violation of $SU(3)$, and are actually the dominant modes. We shall therefore not work in the limit of $SU(3)$ symmetry, but assume only the charge independence for the strong interactions. Accordingly, we shall specify the weak Hamiltonian by the transformation property under the chiral $U(2)\times U(2)$,[4] which is generated by

$$F_i=\int d^3x\, \mathfrak{F}_{i0}(x)\,,\quad F_i^5=\int d^3x\, \mathfrak{F}_{i0}^5(x)\,,\qquad (i=0,\,1,\,2,\,3)\,,\qquad (1.1)$$

where $\mathfrak{F}_{i\mu}$ and $\mathfrak{F}_{i\mu}^5$ are the vector and the axial-vector isospin current densities, respectively. It should be understood at the outset, however, that we will not assume an $SU(2)\times SU(2)$ symmetry of the strong interactions. Our results come from the assumption that the weak Hamiltonian belongs to a sum of irreducible representations $\sum_n (n,1)$ $(n=2,\,4,\,\cdots)$ of $U(2)\times U(2)$ and from PCAC which relates $\partial_\mu\mathfrak{F}_\mu^5$ to the pion field.

With these assumptions, we can show that the $K_{2\pi}$ and the $K_{3\pi}$ decay matrix elements with all the pion four-momenta successively going to zero obey the $\Delta I=\frac{1}{2}$ rule. If these four-momenta are continued to the physical values, the $\Delta I=\frac{3}{2}$ amplitudes might arise. However, the final pions are in an s wave in the physical process of $K_{2\pi}$, and dominantly in an s wave in the physical process of $K_{3\pi}$. Therefore, the nonphysical amplitudes (zero four-momenta) should be a good approximation to the physical ones. Further, we can relate the rate of $K_{3\pi}$ to that of $K_{2\pi}$.

II. ASSUMPTIONS

We enumerate the assumptions here as follows:

(1) The effective weak Hamiltonian should have the following transformation properties under the chiral $U(2)\times U(2)$;

$$H^{(\omega)}=H^{(\omega)}(2,1)+H^{(\omega)}(4,1)\,,\qquad (2.1)$$

where the entries denote the dimensions of the irreducible representations to which $H^{(\omega)}$ belongs. The weak Hamiltonian is of the current×current type, and the strangeness-conserving and changing currents belong to the (3,1) and (2,1) representations, respectively, under the chiral $U(2)\times U(2)$ as in the quark model.[4] $H^{(\omega)}(2,1)$ and $H^{(\omega)}(4,1)$ contribute as the $\Delta I=\frac{1}{2}$ and the $\Delta I=\frac{3}{2}$ spurions, respectively. By definition, the commutator of $H^{(\omega)}(2,1)$ with the generator of $U(2)\times U(2)$ always gives back the (2,1) representation, and similarly for $H^{(\omega)}(4,1)$.

(2) Partially conserved axial vector current hypothesis (PCAC) or the generalized Goldberger-Treiman relation.[5]

(3) The axial vector currents to the divergences of which the pion fields are related generate the chiral $U(2)\times U(2)$.

III. DECAY MATRIX ELEMENTS

Let us begin with the $K_{2\pi}$ decay matrix element. It is defined as

$$K_{2\pi}=\left\langle K\left|\int d^3x\,\mathcal{H}^{(\omega)}(x,0)\right|\pi^i\pi^i\right\rangle.\qquad (3.1)$$

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For the convenience of the following discussions, we shall normalize state vectors as

$$\langle\alpha|\beta\rangle=2\omega_\alpha\delta_{\alpha\beta}\delta(\mathbf{p}_\alpha-\mathbf{p}_\beta)\,. \tag{3.2}$$

The reduction formula leads us to

$$K_{2\pi}=-\int d^4x d^4y\,\frac{e^{-ipx-iqy}+e^{-ipy-iqx}}{\sqrt{2}}\times(\Box_x-\mu^2)(\Box_y-\mu^2)\times\langle K|T\{H^{(\omega)}(0),\phi^i(x),\phi^j(y)\}|0\rangle, \tag{3.3}$$

or

$$K_{2\pi}=-i\int d^4x\,\frac{e^{-ipx}}{\sqrt{2}}(\Box-\mu^2)\times\langle K|T\{H^{(\omega)}(0),\phi^i(x)\}|\pi^j\rangle-i\int d^4x\,\frac{e^{-ipx}}{\sqrt{2}}(\Box-\mu^2)\times\langle K|T\{H^{(\omega)}(0),\phi^j(x)\}|\pi^i\rangle, \tag{3.4}$$

where ϕ^i is the renormalized pion field and $H^{(\omega)}(0)$ is the space integral of $\mathcal{H}^{(\omega)}(x,0)$. We have explicitly symmetrized the spatial wave function to maintain the symmetry property of the two-pion system at every step of the calculation. As previously stated,[3] we replace the pion field ϕ by the divergence of the axial-vector current by means of PCAC,

$$\partial_\mu\mathfrak{F}_{i\mu}{}^5(x)=(1/c)\phi^i(x) \tag{3.5a}$$

or

$$\frac{d}{dt}\int d^3x\,\mathfrak{F}_{i0}{}^5(x,t)=\frac{1}{c}\int d^3x\,\phi^i(x,t)\,. \tag{3.5b}$$

Partial integration of Eq. (3.4) over the time gives us

$$\begin{aligned}K_{2\pi}={}&i\frac{c}{\sqrt{2}}\int d^4x\,e^{-ipx}(\Box-\mu^2)\times\langle K|[H^{(\omega)}(0),\mathfrak{F}_{i0}{}^5(x)]_-\delta(x_0)|\pi^j\rangle\\&+i\frac{c}{\sqrt{2}}\int d^4y\,e^{-ipy}(\Box-\mu^2)\times\langle K|[H^{(\omega)}(0),\mathfrak{F}_{j0}{}^5(y)]_-\delta(y_0)|\pi^i\rangle\\&+\frac{c}{\sqrt{2}}\int d^4y\,e^{-ipx}(\Box-\mu^2)\times\langle K|T\{H^{(\omega)}(0),p_\mu\mathfrak{F}_{i\mu}{}^5(x)\}|\pi^j\rangle\\&+\frac{c}{\sqrt{2}}\int d^4x\,e^{-ipy}(\Box-\mu^2)\times\langle K|T\{H^{(\omega)}(0),p_\mu\mathfrak{F}_{j\mu}{}^5(y)\}|\pi^i\rangle.\end{aligned} \tag{3.6}$$

We should remark that the surface terms appearing on the way of partial integration have been neglected in the above formula.

Now let p go to zero, keeping the four-momentum of the remaining pion on the mass shell. Then we can eliminate $\Box$ by partial integration. We must be careful in taking the limit of $p\to 0$ in the third and the fourth terms. However, we can see that these terms apart from p_μ have no singularity like $1/p$, if we keep the four-momenta q finite. In this way, we get

$$\lim_{p\to0}K_{2\pi}=-i(c\mu^2/\sqrt{2})\langle K|[H^{(\omega)}(0),F_i{}^5(0)]_-|\pi^j\rangle-i(c\mu^2/\sqrt{2})\langle K|[H^{(\omega)}(0),F_j{}^5(0)]_-|\pi^i\rangle, \tag{3.7}$$

with the definition

$$F_i{}^5(t)=\int d^3x\,\mathfrak{F}_{i0}{}^5(x,t)\,. \tag{3.8}$$

We proceed to make contraction of the remaining pion and repeat the procedure described above. Then we can rewrite Eq. (3.7) into

$$\lim_{q\to0}\lim_{p\to0}K_{2\pi}=-(c^2\mu^4/\sqrt{2})\times\langle K|[[H^{(\omega)}(0),F_i{}^5(0)]_-F_j{}^5(0)]_-|0\rangle-(c^2\mu^4/\sqrt{2})\langle K|[[H^{(\omega)}(0),F_j{}^5(0)]_-F_i{}^5(0)]_-|0\rangle. \tag{3.9}$$

Let us decompose $H^{(\omega)}$ into the irreducible representations of the chiral $U(2)\times U(2)$, namely the $\Delta I=\frac{1}{2}$ spurion and the $\Delta I=\frac{3}{2}$ spurion, and label them as $H^{(\omega)}(2I+1,\,1)$. By definition, a commutator of $H^{(\omega)}(2I+1,\,1)$ with a generator $F_i{}^5$ always gives back the $(2I+1,\,1)$ representation and never goes out of it. Since the kaon has $I=\frac{1}{2}$, only the (2,1) piece of $H^{(\omega)}$, or the $\Delta I=\frac{1}{2}$ spurion, contributes to the matrix elements in the right-hand side of Eq. (3.9).

$$[[H^{(\omega)}(4,1),F_i{}^5(0)]_-F_j{}^5(0)]_-$$

transforms like (4,1) under $U(2)\times U(2)$, or like $I=\frac{3}{2}$ under the isospin group. Thus, we conclude that the $K_{2\pi}$ decay matrix element, in which the pion four-momenta go to zero as described in Eq. (3.9), obeys the $\Delta I=\frac{1}{2}$ rule.

It is easy to extend our argument to the $K_{3\pi}$ decays. We repeat the manipulation described above to reduce the $K_{3\pi}$ decay matrix element into

$$\lim_{p\to0}K_{3\pi}=-i(c\mu^2/\sqrt{3})\langle K|[H^{(\omega)}(0),F_i{}^5(0)]_-|\pi^j\pi^k\rangle-i(c\mu^2/\sqrt{3})\langle K|[H^{(\omega)}(0),F_j{}^5(0)]_-|\pi^i\pi^k\rangle-i(c\mu^2/\sqrt{3})\langle K|[H^{(\omega)}(0),F_k{}^5(0)]_-|\pi^i\pi^j\rangle. \tag{3.10}$$

Since the commutator of $H^{(\omega)}$ with F^5 belongs to the same representation of $U(2)\times U(2)$ to which $H^{(\omega)}$ belongs, the three terms in the right-hand side are essentially the same as the $K_{2\pi}$ decays. We have shown above that the $K_{2\pi}$ decays with the pion four-momenta equal to zero obey the $\Delta I=\frac{1}{2}$ rule. Therefore, the $K_{3\pi}$

decays also obey the $\Delta I=\frac{1}{2}$ rule, if the four-momenta of all the pions are properly continued to zero.

IV. RATES OF $K_{2\pi}$ AND $K_{3\pi}$

The foregoing discussions enable us to relate the rate of the $K_{3\pi}$ decay to that of the $K_{2\pi}$ decay. We must be prepared for the error amounting to several percent in the predicted rates, if we consider the accuracy of the $\Delta I=\frac{1}{2}$ rule in the experimental data of $K_{2\pi}$ and $K_{3\pi}$. Since we have implicitly assumed that the final pions in the $K_{3\pi}$ decays are in a relative s wave, the ratios among the $K_{2\pi}$ decays and among the $K_{3\pi}$ decays are uniquely determined. We have, therefore, only to calculate one of the ratios between $K_{2\pi}$ and $K_{3\pi}$.

The experimental accuracy is highest in the $K_1^0\to\pi^+\pi^-$ decay for $K_{2\pi}$ and in the $K^+\to\pi^+\pi^+\pi^-$ decay for $K_{3\pi}$. We shall estimate the ratio of these two decay rates. By use of Eq. (3.10), we get

$$K^+(++-)=-i(\sqrt{2}c\mu^2/\sqrt{3})K_1^0(+-)\,, \qquad (4.1)$$

with the present definition of the matrix elements. The constant c is given by PCAC to be

$$c=ig_rK(0)/M\mu^2g_A\,, \qquad (4.2)$$

where g_r is the renormalized πN coupling constant $(g_r^2/4\pi=14)$, g_A is the renormalization of the strangeness-conserving axial-vector weak current $(g_A=1.18)$, $K(0)$ is the πN vertex renormalized as $K(-\mu^2)=1$, and M is the nucleon mass. Straightforward calculations lead us to the ratio

$$\frac{\Gamma(K^+(++-))}{\Gamma(K_1^0(+-))}=\frac{2}{9}\frac{m(g_r^2/4\pi)K^2(0)\rho}{M^2(m^2-4\mu^2)^{1/2}g_A^2}\,, \qquad (4.3)$$

where the phase volume ρ is given by

$$\rho=\frac{1}{\pi^3}\int\frac{\delta^4(p_i+p_j+p_k-P)}{16\omega_1\omega_2\omega_3}d^3p_id^3p_jd^3p_k\,, \qquad (4.4)$$

and P and m denote the four-momentum and the mass of the kaon.

Equation (4.4) is rewritten to be[6]

$$\rho=\frac{1}{2}\int d\omega_1 d\omega_2\,, \qquad (4.5)$$

which is given by

$$\rho_{\mathrm{N.R.}}=(\sqrt{3}/72)Q^2 \qquad (4.6)$$

in the nonrelativistic limit, where Q is the Q value of $K_{3\pi}(++-)$ given by $m-3\mu^\pm$. We have taken account of the identity of the two π^+ and the *symmetrization of the final state vectors* which was made in the previous section. After making a small correction to $\rho_{\mathrm{N.R.}}$ due to relativistic kinematics, we substitute the numerical values to get finally

$$\Gamma(K^+(++-))/\Gamma(K_1^0(+-))=4.8\times10^{-4}\,, \quad (4.7)$$

where $K^2(0)$ has been put to unity. The experimental data available at present[7] gives

$$\Gamma(K^+(++-))/\Gamma(K_1^0(+-))=(6.3\pm0.5)\times10^{-4}. \quad (4.8)$$

If we rewrite Eqs. (4.7) and (4.8) in terms of the amplitudes instead of the decay rates,

$$[K^+(++-)/K_1^0(+-)]_{\mathrm{theor}} =(1.15\pm0.09)\times[K^+(++-)/K_1^0(+-)]_{\mathrm{exp}}. \quad (4.9)$$

From the beginning, we have anticipated that a small amount of the $\Delta I=\frac{3}{2}$ amplitudes may be contained in the physical amplitudes. If we consider the present approximation that the continuation of the pion mass from zero to μ is entirely neglected, the numerical agreement with the accuracy of about 15% in amplitudes should be regarded to be a support for our discussions.

V. DISCUSSION

It should be emphasized again that the chiral $U(2)\times U(2)$ symmetry has not been assumed for the strong interactions. What is crucial to draw our conclusions is the assumption that the effective weak interactions have the transformation property of $\sum_n (n,1)$ under the chiral $U(2)\times U(2)$, as well as the assumption that the space-integrals of the fourth components of the axial vector currents to which the pions are related should be generators of $U(2)\times U(2)$.

If the weak Hamiltonian has a term transforming like (i,j) with $i,j\neq1$, this term may be transformed from $I=\frac{3}{2}$ to $I=\frac{1}{2}$ or conversely under the chiral $U(2)\times U(2)$ transformations. Therefore, not only ambiguity due to continuation in the pion momenta, but also the deviation of $H^{(\omega)}$ from the form suggested in the quark model of $U(2)\times U(2)$, if any, may be responsible for the deviations from the $\Delta I=\frac{1}{2}$ rules and for the discrepancy between our estimates and experiment on the $K_{3\pi}/K_{2\pi}$ ratio.

An important question which is left unclarified here is how accurately the matrix elements defined by Eq. (3.9) approximate the physical amplitudes. The operator $(\Box-\mu^2)$ in decay matrix elements [see, for example, Eq. (3.4)] picks up the singularity like $(\Box-\mu^2)^{-1}$ in the Green's function. A physical matrix element is given by the singular part, after the integration over the time of $H^{(\omega)}(t)$ is carried out. When we put $\Box$ equal to zero by partial integration and by the energy-momentum conservation in the limit of $p\to0$, we are left with the term of the zeroth order of μ^2 which is the true continuation of the physical amplitude and the continuum term of the order of μ^2. Since the lowest continuum state with the same quantum number as

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the pion is the 3π state, we could roughly estimate that the continuum term is at most of the order of $\mu^2/(3\mu)^2=\frac{1}{9}$ as compared with the term representing the physical process. Although this reasoning is not rigorous, it seems that contributions of the continuum states are not very large. This question should be subject to further investigations.

Finally, we should like to add a comment on the $\Delta I=\frac{1}{2}$ rules of the Λ and the Ξ hyperon decays. The method stated here proves that they hold good even within the charge independence instead of $SU(3)$. It is because $\bar{\Lambda}N$ and $\bar{\Xi}\Lambda$ systems do not contain an $I=\frac{1}{2}$ component. We cannot get the $\Delta I=\frac{1}{2}$ rule for the Σ decays in general.

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PHYSICAL REVIEW VOLUME 144, NUMBER 4 29 APRIL 1966

Threshold Factors in Partial-Wave Dispersion Relations*†

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If the solution of an approximate partial-wave dispersion relation is required to satisfy unitarity and possess the correct threshold behavior, additional poles (representing effective forces) are introduced into the amplitude. These poles are explicitly exhibited. To enforce l-wave threshold conditions, approximately l *arbitrary* parameters appear (the precise number depending upon the properties of the amplitude considered). This difficulty is in principle *independent* of the method of solution. Using the nucleon exchange force as input, the $J=\frac{3}{2}$ amplitudes for π-N scattering are obtained by numerical solution of the N/D equations. The solutions are extremely sensitive to the arbitrary parameters. It appears that, in the present formulation, partial-wave dispersion relations do not provide a reliable means of calculating detailed properties of the amplitudes.

I. INTRODUCTION

THE literature of strong-interaction physics contains many discussions and solutions of partial-wave dispersion relations. We quote a representative sample[1–11] to which the reader may refer for details. Although practically all solutions have employed a "threshold factor" to guarantee that approximations do not spoil the desired threshold behavior of the amplitude, there have been rather few discussions[4,12–16] of the related questions of uniqueness and consistency.

We require that the partial-wave amplitude satisfy unitarity and possess the correct threshold behavior even when approximations are made. These constraints require the introduction of additional poles in the approximate amplitude, whose positions are essentially arbitrary. Roughly speaking, there are l arbitrary parameters in the lth partial-wave amplitude, and we demonstrate by explicit calculation that the solutions are quite sensitive to these parameters. For definiteness, we have concentrated mainly on π-N scattering, although most of these considerations apply to any partial-wave dispersion relation.

Section II contains a brief summary of the kinematics, definitions of the amplitudes, and other details. In Sec. III, the need for a threshold factor is briefly discussed and the extra poles introduced thereby are explicitly

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