

# Topological Field Theory and Matrix Product States

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## Abstract

It is believed that most (perhaps all) gapped phases of matter can be described at long distances by Topological Quantum Field Theory (TQFT). On the other hand, it has been rigorously established that in 1+1d ground states of gapped Hamiltonians can be approximated by Matrix Product States (MPS). We show that the state-sum construction of 2d TQFT naturally leads to MPS in their standard form. In the case of systems with a global symmetry  $G$ , this leads to a classification of gapped phases in 1+1d in terms of Morita-equivalence classes of  $G$ -equivariant algebras. Non-uniqueness of the MPS representation is traced to the freedom of choosing an algebra in a particular Morita class. In the case of Short-Range Entangled phases, we recover the group cohomology classification of SPT phases.

## 1 Introduction and Overview

It is a widely held belief that the universal long-distance behavior of a quantum phase of matter at zero temperature can be encoded into an effective field theory.<sup>1</sup> In the case of gapped phases of matter, the extreme infrared should be described by a Topological Quantum Field Theory.

It has been rigorously shown that the ground state of any gapped 1+1d Hamiltonian with a short-range interaction can be approximated by a Matrix Product State [1]. This representation is very efficient, especially in the translationally-invariant case, and is well-suited to the Renormalization Group analysis. In particular, it leads to a classification of Short-Range Entangled Phases of 1+1d matter in terms of group cohomology [2, 3, 4].

It is natural to ask about the connection between these two approaches to gapped phases of matter. In this note we answer this question in the case of bosonic systems with a finite symmetry  $G$ . For simplicity, we assume that all elements of  $G$  act unitarily (i.e. we do not allow for time-reversing symmetries). The case of time-reversing symmetries and fermionic phases will be addressed in a separate publication [5].

In brief, the results are as follows. We show that a standard-form MPS is naturally associated with a module  $M$  over a finite-dimensional semi-simple algebra  $A$ . The universality class of the MPS depends only on the center  $Z(A)$ . On the other hand, every unitary 2d TQFT has a state-sum construction which uses a semi-simple algebra as an input. Further, given a module  $M$  over this algebra, one naturally gets a particular state in the TQFT space of states. We show that this state is precisely the MPS associated to the pair  $(A, M)$ . Since the TQFT depends only on  $Z(A)$ , we reproduce the fact that the universality class of the MPS depends only on  $Z(A)$ .

In the case of an MPS with a symmetry  $G$ , a similar story holds. A  $G$ -equivariant MPS is encoded in a  $G$ -equivariant module  $M$  over a  $G$ -equivariant semi-simple algebra  $A$ . Such an algebra can be used to give a state-sum construction of a  $G$ -equivariant TQFT, while every  $G$ -equivariant module  $M$  gives rise to a particular state. This state is an equivariant MPS state. Again, different  $A$  can give rise to the same TQFT. This leads to an equivalence relation on  $G$ -equivariant algebras which is a special case of Morita equivalence.

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<sup>1</sup>It is hard to make this rigorous since neither the notion of a phase of matter nor that of an effective field theory has been formalized.

An indecomposable phase with symmetry  $G$  is therefore associated with a Morita-equivalence class of indecomposable  $G$ -equivariant algebras. The classification of such algebras is well known [6] and leads to an (also well-known [2, 3, 4]) classification of bosonic 1+1d gapped phases of matter with symmetry  $G$ . In the special case of Short-Range Entangled gapped phases, we recover the group cohomology classification of SPT phases.

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## 2 Matrix Product States at RG Fixed Points

### 2.1 Matrix Product States

In this section, we review Matrix Product States (MPS) and extract the algebraic data that characterizes them at fixed points of the Renormalization Group (RG). We find that a fixed point MPS is described by a module over a finite-dimensional semisimple algebra. We discuss the notion of a gapped phase and argue that they are classified by finite-dimensional semisimple commutative algebras. Given a fixed point MPS and the corresponding semisimple algebra  $A$ , the commutative algebra characterizing the gapped phase is the center of  $A$ , denoted  $\mathcal{A} = Z(A)$ .

The models we consider are defined on Hilbert spaces that are tensor products of finite-dimensional state spaces  $A$  on the sites of a 1D chain. We are interested in Hamiltonians with an energy gap that persists in the thermodynamic limit of an infinite chain. A large class of examples of gapped systems come from local commuting projector (LCP) Hamiltonians; that is,  $H = \sum h_{s,s+1}$ , where the  $h_{s,s+1}$  are projectors that act on sites  $s$ ,  $s + 1$  and commute with each other. Since the local projectors commute, an eigenstate of  $H$  is an eigenstate of each projector. It follows that the gap of  $H$  is at least 1. Thus LCP Hamiltonians are gapped in the thermodynamic limit. In one spatial dimension, ground states of gapped Hamiltonians are efficiently approximated by an ansatz called a matrix product state (MPS) [1], which we recall below.<sup>2</sup> From each MPS, one can construct a LCP parent Hamiltonian, which has the MPS as a ground state. To discuss and classify 1D gapped Hamiltonians, it suffices to consider the parent Hamiltonians of the MPS that approximate their ground states.

Consider a closed chain of  $N$  sites, each with a copy of a *physical* Hilbert space  $A \simeq \mathbb{C}^d$  and two copies  $V^L$ ,  $V^R$  of a *virtual* space  $\mathbb{C}^D$ . We identify  $V^L = V$  and  $V^R = V^*$  and choose a Hilbert space structure on  $V$ . Between each adjacent pair  $(s, s + 1)$  of sites, place the maximally entangled state

$$|\omega\rangle_{s,s+1} = \sum_{i=1}^D |i\rangle \otimes |i\rangle \in V_s^R \otimes V_{s+1}^L \quad (1)$$

An MPS tensor<sup>3</sup> is a linear map  $\mathcal{P} : V^L \otimes V^R \rightarrow A$ . The MPS associated to  $\mathcal{P}$  is the state

$$|\psi_{\mathcal{P}}\rangle = (\mathcal{P}_1 \otimes \mathcal{P}_2 \otimes \cdots \otimes \mathcal{P}_N) (|\omega\rangle_{12} \otimes |\omega\rangle_{23} \otimes \cdots \otimes |\omega\rangle_{N1}) \in A \otimes A \otimes \cdots \otimes A. \quad (2)$$

Since  $|\psi_{\mathcal{P}}\rangle$  lies in the image of  $\mathcal{P}^{\otimes N}$ , we do not lose generality by truncating  $A$  to  $\text{im } P$ . We will assume we have done so in the following. Equivalently, we assume that the adjoint MPS tensor  $T = \mathcal{P}^\dagger$  is injective<sup>4</sup>. The MPS wavefunction can be expressed as a trace of a product of matrices, hence its name. In the basis

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<sup>2</sup>We only consider translationally-invariant MPS.

<sup>3</sup>More generally, the tensors  $\mathcal{P}_s$  may depend on the site index  $s$ . But any translationally-invariant state has an MPS representation with a site-independent tensor [8].

<sup>4</sup>To avoid confusion, we stress that injectivity of  $T$  is unrelated to the notion of an injective MPS state in the sense of [9]. In particular, while we will always assume that  $T$  is injective, we will not assume that the the ground state of the parent Hamiltonian is unique.

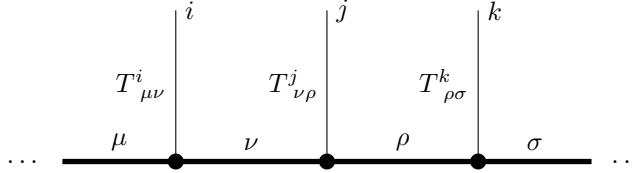


Figure 1: An MPS represented as a tensor network

$\{e_i\}_{i=1,\dots,d}$  of  $A$ , the conjugate state takes the form

$$\langle \psi_T | = \sum_{i_1 \dots i_N=1}^d \text{Tr}[T(e_{i_1}) \cdots T(e_{i_N})] \langle i_1 \cdots i_N | \quad (3)$$

For the tensor  $T$ , one can construct a LCP Hamiltonian  $H_T$ , called the *parent Hamiltonian*<sup>5</sup> of  $|\psi_T\rangle$ , which has  $|\psi_T\rangle$  as a ground state. It is given as a sum of 2-site terms  $h_{s,s+1}$  that project onto the orthogonal complement of  $\ker h = (\mathcal{P} \otimes \mathcal{P})(V \otimes |\omega\rangle \otimes V^*)$ . Explicitly,

$$H_T = \sum_s h_{s,s+1} \quad \text{where} \quad h_{s,s+1} = \mathbb{1} - (\mathcal{P}_s \otimes \mathcal{P}_{s+1}) \delta(\mathcal{P}_s^+ \oplus \mathcal{P}_{s+1}^+) \quad (4)$$

where  $\delta$  is the projector onto  $(V_s \otimes |\omega\rangle \otimes V_{s+1}^*)$  and  $\mathcal{P}_s^+ := (T_s \mathcal{P}_s)^{-1} T_s$  is a left inverse of  $\mathcal{P}_s$ . The local projectors  $h_{s,s+1}$  commute, so  $H_T$  is gapped.  $|\psi_T\rangle$  is annihilated by  $h_{s,s+1}$  for all  $s$  and therefore also by  $H_T$ .

In general,  $H_T$  has other ground states. They are all of the form

$$|\psi_T^X\rangle = (\mathcal{P}_1 \otimes \mathcal{P}_2 \otimes \cdots \otimes \mathcal{P}_N) (|\omega\rangle_{12} \otimes |\omega\rangle_{23} \otimes \cdots \otimes |\omega^X\rangle_{N1}) \quad (5)$$

for some virtual state

$$|\omega^X\rangle = \sum_{i=1}^D X_{ij} |i\rangle \otimes |j\rangle \in V^* \otimes V \quad (6)$$

where  $X$  is a matrix that commutes with  $T(a)$  for all  $a \in A$ . Note that  $|\omega^1\rangle = |\omega\rangle$  and so  $|\psi_T^1\rangle = |\psi_T\rangle$ . The states (5) are clearly annihilated by  $h_{s,s+1}$  for  $s \neq 1$ . To see that they are annihilated by  $h_{12}$ , note that tensor  $T(e_i)XT(e_j)$  is expressible as a linear combination of tensors  $T(e_i)T(e_j)$  if and only if  $X$  commutes with every  $T(e_i)$ . The conjugate states have wavefunctions

$$\langle \psi_T^X | = \sum \text{Tr}[X^\dagger T(e_{i_1}) \cdots T(e_{i_n})] \langle i_1 \cdots i_N | \quad (7)$$

We will call them generalized MPS.

There may be many different ways to represent a given state in  $A^{\otimes N}$  in an MPS form. Even the dimension of the virtual space  $V$  is not uniquely defined. In general, it is not immediate to read off the properties of the state  $\psi_T$  from the tensor  $T$ .

## 2.2 RG-fixed MPS and gapped phases

Under real-space renormalization group (RG) flow [11], adjacent pairs of sites are combined into blocks with physical space  $A \otimes A$ . The MPS form of the state is preserved, with the new MPS tensor being

$$T'(a \otimes b) = T(a)T(b), \quad (8)$$

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<sup>5</sup>There is a more general notion of a parent Hamiltonian where  $h$  is any operator with this kernel; however, we will always take  $h$  to be the projector.

where on the r.h.s. the multiplication is matrix multiplication. We also define  $\mathcal{P}' = T'^\dagger$ . Though an RG step squares the dimension of the codomain of the MPS tensor, the rank is bounded above by  $D^2$ , and so the truncated physical space  $\text{im}(\mathcal{P}')$  never grows beyond dimension  $D^2$ .

An *RG fixed MPS tensor* is an MPS tensor such that  $\mathcal{P}$  and  $\mathcal{P}'$  have isomorphic images and are identical (up to this isomorphism) as maps. That is, there exists an injective map  $\mu : A \rightarrow A \otimes A$  such that

$$\mu \circ \mathcal{P} = \mathcal{P}'. \quad (9)$$

If we denote  $m = \mu^\dagger$ , this is equivalent to

$$T(m(a \otimes b)) = T(a)T(b). \quad (10)$$

Since  $T$  was assumed to be injective, this equation completely determines  $m$ . Similarly, the fact that matrix multiplication is associative implies that  $m : A \otimes A \rightarrow A$  is an associative multiplication on  $A$ . The map  $T : A \rightarrow \text{End}(V)$  then gives  $V$  the structure of a module over  $A$ . Since  $T$  is injective, this module is faithful (all nonzero elements of  $A$  act nontrivially).

As previously stated, a state in  $A^{\otimes N}$  may have multiple distinct MPS descriptions. Given an RG fixed MPS, one can always choose  $T$  to have a certain *standard form* [9]. When this is done, the matrices  $T(a)$  are simultaneously block-diagonalized, for all  $a \in A$ . Moreover, if we denote by  $T^{(\alpha)}$  the  $\alpha^{\text{th}}$  block, say of size  $L_\alpha \times L_\alpha$ , then the matrices  $T^{(\alpha)}(e_i)$  span the space of  $L_\alpha \times L_\alpha$  matrices. That is,  $T^{(\alpha)}$  defines a surjective map from  $A$  to the space of  $L_\alpha \times L_\alpha$  matrices.

For an RG-fixed MPS tensor in its standard form, one can easily see that  $A$  is a direct sum of matrix algebras. Indeed, each block  $A^\alpha$  defines a surjective homomorphism  $T^\alpha$  from  $A$  to the algebra of  $L_\alpha \times L_\alpha$  matrices, and if an element of  $A$  is annihilated by all these homomorphisms, then it must vanish. Thus we get a decomposition

$$A = \bigoplus_\alpha A^\alpha, \quad (11)$$

where each  $A^\alpha = (\ker T^\alpha)^\perp$  is isomorphic to a matrix algebra. We stress that some of these homomorphisms might be linearly dependent, so the number of summands may be smaller than the number of blocks in the standard form of  $T$ . An algebra of such a form is semi-simple, that is, any module is a direct sum of irreducible modules. More specifically, any module over a matrix algebra of  $L \times L$  matrices is a direct sum of several copies of the obvious  $L$ -dimensional module. This basic module is irreducible. If, for a particular  $A^\alpha$ ,  $T$  contains more than one copy of the irreducible module, the corresponding blocks in the standard form of  $T$  are not independent.

The ground-state degeneracy is simply related to the properties of the algebra  $A$ . Namely, the number of ground states is equal to the number of independent blocks in a standard-form MPS, or equivalently the number of summands in the decomposition (11). Since the center of a matrix algebra consists of scalar matrices and thus is isomorphic to  $\mathbb{C}$ , one can also say that the number of ground states is equal to the dimension of  $\mathcal{A} = Z(A)$ .

Two gapped systems are said to be in the same phase if their Hamiltonians can be connected by a Local Unitary (LU) evolution, i.e. if they are related by a conjugation with a finite-dime evolution operator for a local time-dependent Hamiltonian [10]. Clearly, the ground-state degeneracy is the same for all systems belonging to a particular phase. In fact, for 1+1d gapped bosonic systems, it completely determines the phase [9, 2].

It is convenient to introduce an addition operation  $\oplus$  on systems and phases. Given two 1+1d systems with local Hilbert spaces  $A_1$  and  $A_2$ , we can form a new 1+1d system with the local Hilbert space  $A_1 \oplus A_2$ . The Hamiltonian is taken to be the sum of the Hamiltonians of the two systems plus projectors which enforce the condition that neighboring “spins” are either both in the  $A_1$  subspace or in the  $A_2$  subspace. The ground state degeneracy is additive under this operation. A phase is called decomposable if it is a sum of two phases, otherwise it is called indecomposable. Clearly, it is sufficient to classify indecomposable phases.

It is easy to see that if  $A$  decomposes as a sum of two subalgebras, then the corresponding phase is decomposable. Further, an indecomposable semi-simple algebra  $A$  is isomorphic to a matrix algebra. The corresponding ground state is unique. Moreover, while the parent Hamiltonians for different matrix algebras

are the different, they all correspond to the same phase [2], i.e. are related by a Local Unitary evolution. Hence the phase is determined by the number of components in the decomposition (11), or in other words, by  $Z(A)$ .

### 3 Topological Quantum Field Theory

We have seen above that an RG-fixed MPS state is associated with a finite-dimensional semi-simple algebra  $A$ , and that the universality class of the corresponding phase depends only on the center of  $A$ . On the other hand, it is known since the work of Fukuma, Hosono, and Kawai [12] that for any finite-dimensional semi-simple algebra  $A$  with an invariant scalar product one can construct a unitary 2D TQFT, and that the isomorphism class of the resulting TQFT depends only on the center of  $A$ . In this section we show that this is not a mere coincidence, and that the ground states of this TQFT can be naturally written in an MPS form, with an RG-fixed MPS tensor.

#### 3.1 State-sum construction of 2d TQFTs

A (closed) 2D TQFT associates a space of states  $\mathcal{A}$  to an oriented circle, and a vector space  $\mathcal{A}^{\otimes n}$  to  $n$  disjoint oriented circles. Further, suppose we are given an oriented bordism from  $n$  circles to  $l$  circles, i.e. a compact oriented 2d manifold  $\Sigma$  whose boundary consists of  $l$  circles oriented in the same way as  $\Sigma$  and  $n$  circles oriented in the opposite way. A 2d TQFT associates to  $\Sigma$  a linear map from  $\mathcal{A}^{\otimes n}$  to  $\mathcal{A}^{\otimes l}$ . This map does not change if we replace  $\Sigma$  by a diffeomorphic oriented manifold. Also, gluing bordisms taking care that orientations agree corresponds to composing linear maps.

Let us mention some special cases. If  $\Sigma$  is closed (i.e. has an empty boundary), then the 2D TQFT associates to it a linear map  $\mathbb{C} \rightarrow \mathbb{C}$ , i.e. a complex number  $Z_\Sigma$ , called the partition function. If  $\Sigma$  is a pair-of-pants bordism from two circles to one circle, the corresponding map  $m : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$  defines an associative, commutative product on  $\mathcal{A}$ . The cap bordism defines a symmetric trace function  $\text{Tr} : \mathcal{A} \rightarrow \mathbb{C}$  such that the scalar product  $\eta(a, b) = \text{Tr}(ab)$  is symmetric and non-degenerate. These data make  $\mathcal{A}$  into a commutative Frobenius algebra. It is known that a two-dimensional TQFT is completely determined by the commutative Frobenius algebra structure on  $\mathcal{A}$  [13, 14, 15]. The state-operator correspondence identifies  $\mathcal{A}$  with the algebra of local operators. This Frobenius algebra encodes the 2- and 3-point functions on the sphere, from which all other correlators, including the partition function, can be reconstructed.

There is alternative construction of a 2d TQFT called the state-sum construction [12] which is combinatorial and manifestly local. The input for this construction is a finite-dimensional Frobenius algebra  $A$ , which is not necessarily commutative. To compute the linear maps associated to a particular bordism  $\Sigma$ , one needs to choose a triangulation of  $\Sigma$ . Nevertheless, the result is independent of the choice of the triangulation [12]. The connection between the not-necessarily commutative algebra  $A$  and the commutative algebra  $\mathcal{A}$  is that  $\mathcal{A}$  is  $Z(A)$ , the center of  $A$ . From the perspective of open-closed TQFTs,  $A$  is the algebra of states on the interval for a particular boundary condition.

Let us describe the state-sum construction for the partition function  $Z_\Sigma$  of a closed oriented 2D manifold  $\Sigma$  [12]. Fix a basis  $e_i$ ,  $i \in S$ , of  $A$ . We define the following tensors:

$$\eta_{ij} = \eta(e_i, e_j) = \text{Tr } e_i e_j \quad C_{ijk} = \text{Tr } e_i e_j e_k \quad (12)$$

The tensor  $\eta_{ij}$  is symmetric and non-degenerate, the tensor  $C_{ijk}$  is cyclically-symmetric. We also denote by  $\eta^{ij}$  the inverse to the tensor  $\eta_{ij}$ . Note that  $C_{ijk}$  is related to the structure constants  $C^i_{jk}$  in this basis by

$$C^i_{jk} = \sum_l \eta^{il} C_{ljk}. \quad (13)$$

Let  $T(\Sigma)$  be a triangulation of  $\Sigma$ . A coloring of a 2-simplex  $F$  of  $T(\Sigma)$  is a choice of a basis vector  $e_i$  for each 1-simplex  $E \in \partial F$ . A coloring of  $T(\Sigma)$  is a coloring of all 2-simplices of  $T(\Sigma)$ . Note that each 1-simplex of  $T(\Sigma)$  has two basis vectors attached to it, one from each 2-simplex that it bounds. The weight

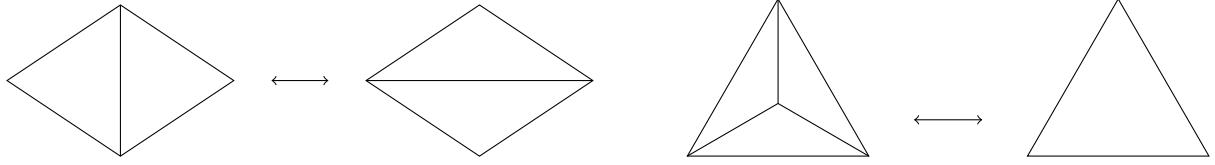


Figure 2: The 2-2 (left) and the 3-1 (right) Pachner moves

of a coloring is the product of  $C_{ijk}$  over 2-simplexes and  $\eta^{ij}$  over 1-simplices, where the cyclic ordering of indices for each 2-simplex is determined by the orientation of  $\Sigma$ . The partition function is the sum of these weights over all colorings.

Topological invariance of  $Z_\Sigma$  can be shown as follows. It is known that any two triangulations of a smooth manifold are related by a finite sequence of local moves [16]. In two dimensions, there are two moves - the 2-2 move and the 3-1 move, depicted in Figure 2 - which swap two or three faces of a tetrahedron with their complement. Invariance of the state-sum under the 2-2 “fusion” move reads

$$C_{ij}{}^p C_{pk}{}^l = C_{jk}{}^p C_{ip}{}^l \quad (14)$$

Similarly the 3-1 move reads

$$C_i{}^{mn} C_{nl}{}^k C_{mj}^l = C_{ij}{}^k \quad (15)$$

These axioms are satisfied by any symmetric Frobenius algebra  $A$  [12]; therefore, the partition sum is a topological invariant<sup>6</sup>.

### 3.2 Open-closed 2d TQFT

So far we have discussed what is known as closed 2D TQFTs. That is, the boundary circles were interpreted as spacelike hypersurfaces, and thus each spatial slice had an empty boundary. The notion of a TQFT can be extended to incorporate spatial boundaries; such theories are called open-closed TQFTs. In such a theory a spatial slice is a compact oriented manifold, possibly with a nonempty boundary. That is, it is a finite collection of oriented intervals and circles. A bordism between such spatial slices is a smooth oriented *surface with corners*: paracompact Hausdorff spaces for which each point has a neighborhood homeomorphic to an open subset of a half-plane. Surfaces with corners are homeomorphic, but typically not diffeomorphic, to smooth surfaces with a boundary.

The corner points subdivide the boundary of the bordism into two parts: the initial and final spatial slices, and the rest. We will refer to the initial and final spatial slices as the cut boundary, while the rest will be referred to as the brane boundary. The cut boundary can be thought of as spacelike, while the brane boundary is timelike. Bordisms are composed along their cut boundary (hence the name), while on the brane boundary one needs to impose boundary conditions (known as D-branes in the string theory context, hence the name). More precisely, if  $\mathcal{C}$  is the set of boundary conditions, one needs to label each connected component of the brane boundary with an element of  $\mathcal{C}$ .

An open-closed 2d TQFT associates a vector space  $V_{MM'}$  to every oriented interval with the endpoints labeled by  $M, M' \in \mathcal{C}$ , and a vector space  $\mathcal{A}$  to every oriented circle. To a collection of thus labeled compact oriented 1D manifolds it attaches the tensor product of spaces  $V_{MM'}$  and  $\mathcal{A}$ . To every bordism with corners labeled in the way explained above, it attaches a linear map from a vector space of the ‘incoming’ cut boundary to the vector space of the ‘outgoing’ cut boundary. Gluing bordisms along their cut boundaries corresponds to composing the linear maps.

Just like in the case of a closed 2d TQFT, one can describe algebraically the data which are needed to construct a 2d open-closed TQFT. We refer to [14] for details. Suffice it to say that each space  $V_{MM}$  is a (possibly noncommutative) Frobenius algebra, and each space  $V_{MM'}$  is a left module over  $V_{MM}$  and a right module over  $V_{M'M'}$ . That is, to every element  $x \in V_{MM}$  one associates a linear operator  $T^M(x)$ :

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<sup>6</sup>In two dimensions, there is no difference between topological and smooth manifolds.

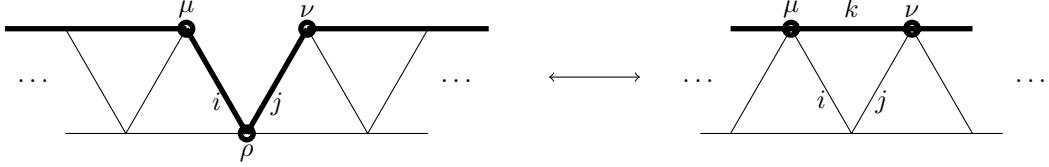


Figure 3: An elementary shelling, representing  $T_{\rho i}^\mu T_{\nu j}^\rho = C_{ij}^k T_{\nu k}^\mu$  (Eq. 16). The thick line represents a physical boundary.

$V_{MM'} \rightarrow V_{MM'}$  so that composition of elements of  $V_{MM}$  corresponds to the composition of linear operators:  $T^M(x)T^M(x') = T^M(xx')$  (and similarly for  $V_{M'M'}$ ). Also, for every  $M \in \mathcal{C}$  there is a map  $\iota^M : \mathcal{A} \rightarrow V_{MM}$  which is a homomorphism of Frobenius algebras. The dual map  $\iota_M : V_{MM} \rightarrow \mathcal{A}$  is known as the generalized boundary-bulk map. In particular, if we act with  $\iota_M$  on the identity element of the algebra  $V_{MM}$ , we get a distinguished element  $\psi_M \in \mathcal{A}$  called the boundary state corresponding to the boundary condition  $M$ . Geometrically,  $\psi_M$  is the element of  $\mathcal{A}$  which the open-closed TQFT associates to an annulus whose interior circle is a brane boundary labeled by  $M$ , while the exterior circle is an outgoing cut boundary.

One may wonder if it is possible to reconstruct the open-closed TQFT from the closed TQFT. The answer turns out to be yes if  $\mathcal{A}$  is a semisimple, i.e. if every module over  $\mathcal{A}$  is a sum of irreducible modules [14].<sup>7</sup> Then  $\mathcal{C}$  is the set of finite-dimensional modules over  $\mathcal{A}$ , and  $V_{MM'}$  is the space of linear maps from the module  $M$  to the module  $M'$  commuting with the action of  $\mathcal{A}$  (i.e.  $V_{MM'}$  is the space of module homomorphisms). Conversely, one can reconstruct the algebra  $\mathcal{A}$  from any “sufficiently large” brane  $M \in \mathcal{C}$ : if we assume that the module  $M$  is faithful (i.e. all nonzero elements of  $\mathcal{A}$  act nontrivially), then  $\mathcal{A} = Z(V_{MM})$ .

The state-sum construction generalizes to the open-closed case [19]. Let us describe it for a semisimple  $\mathcal{A}$ , and assuming that the bordism  $\Sigma$  only has a brane boundary. Each connected component of  $\partial\Sigma$  is then labeled by a brane  $M \in \mathcal{C}$ . We pick a sufficiently large brane  $M_0$  such that  $\mathcal{A} = Z(V_{M_0 M_0})$ . Let  $A = V_{M_0 M_0}$ . We also choose a basis  $f_\mu^M$ ,  $\mu \in S_M$  in each module  $M$ . Denote the matrix elements of the action of  $A$  on  $M$  by  $T_{M\nu i}^\mu$ . We choose a triangulation of  $\Sigma$ , which also gives us a triangulation of each connected component of the boundary. 2-simplices of  $\Sigma$  are labeled as before. Label boundary 0-simplices on any  $M$ -labeled boundary component by the basis vectors  $f_\mu^M$ . Thus each boundary 1-simplex is labeled by a basis vector of  $A$  and a pair of basis vectors of a module. We assign a weight to each 2-simplex and each interior 1-simplex before. We also assign a weight to each boundary 1-simplex as follows. Suppose the boundary 1-simplex is labeled by  $e_i \in A$  and  $f_\mu^M, f_\nu^M \in M$ . Then the weight of the boundary 1-simplex is  $T_{M\nu i}^\mu$ . The total weight is the product of weights of all 2-simplices and all 1-simplices (both interior and exterior).

Due to the introduction of brane boundaries, there are two more moves, called the 2-2 and 3-1 *elementary shellings* and depicted in Figures 3 and 4, that must be considered when demonstrating topological invariance [19]. They yield conditions

$$T_{M\rho i}^\mu T_{M\nu j}^\rho = C_{ij}^k T_{M\nu k}^\mu \quad (16)$$

and

$$T_{M\rho j}^\mu T_{M\nu k}^\rho C_i^{jk} = T_{M\nu i}^\mu \quad (17)$$

respectively. The first one is the definition of a module, the second one follows from the definition of a module and the semisimplicity of  $A$ . Therefore the state-sum is a well-defined open-closed TQFT. Moreover, such structures are precisely those required to define a topologically invariant state-sum.

### 3.3 Unitary TQFTs and semisimplicity

The state-sum construction defines a perfectly good topological invariant for any finite-dimensional Frobenius algebra  $A$ ; however, if it is to model an actual physical system, its space of states must carry a Hilbert space

<sup>7</sup>This might seem like a rather uninteresting case, since by the Wedderburn theorem every commutative semisimple algebra is isomorphic to a sum of several copies of  $\mathbb{C}$ . But as explained below unitarity forces  $\mathcal{A}$  to be semisimple. Also, in the case of TQFTs with symmetries and fermionic TQFTs the classification of semisimple algebras is more interesting.

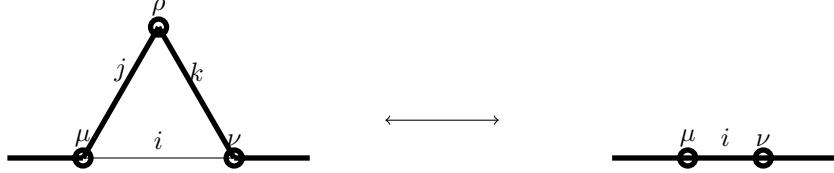


Figure 4: An elementary shelling representing  $T_{\rho j}^\mu T_{\nu k}^\rho C_i^{jk} = T_{\nu i}^\mu$  (Eq. 17).

structure, and linear maps corresponding to bordisms must be compatible in some sense with this structure. To be precise, for any oriented bordism  $\Sigma$  whose source is a disjoint union of  $n$  circles and whose target is a disjoint union of  $l$  circles, let  $-\Sigma$  denote its orientation-reversal.  $-\Sigma$  has  $l$  circles in its source and  $n$  circles in its target. A 2d TQFT attaches to  $\Sigma$  a linear map  $\mathcal{A}^{\otimes n} \rightarrow \mathcal{A}^{\otimes l}$ , and to  $-\Sigma$  a linear map  $\mathcal{A}^{\otimes l} \rightarrow \mathcal{A}^{\otimes n}$ . A unitary structure on a 2d TQFT is a Hilbert space structure on  $\mathcal{A}$  such that the maps corresponding to  $\Sigma$  and  $-\Sigma$  are adjoint to each other. For an open-closed 2D TQFT, we require that the state-space assigned to each boundary-colored interval has a non-degenerate Hermitian metric, and that cobordisms with nonempty brane boundary also satisfy the Hermiticity condition.

Let  $\langle a, b \rangle$  denote the Hilbert space inner product of  $a, b \in \mathcal{A}$ . Since  $\mathcal{A}$  also has a bilinear scalar product  $\eta$ , we can define an antilinear map

$$*: \mathcal{A} \rightarrow \mathcal{A}, \quad a \mapsto a^*, \quad (18)$$

such that  $\langle a, b \rangle = \eta(a^*, b)$ . It can be shown that this map is an involution (i.e.  $a^{**} = a$ ) and an anti-automorphism (i.e.  $(ab)^* = b^*a^*$ ) [17]. This can also be expressed by saying that  $\mathcal{A}$  is a  $*$ -algebra. Conversely, one can show that any commutative Frobenius  $*$ -algebra such that the sesquilinear product  $\eta(a^*, b)$  is positive-definite gives rise to a unitary 2d TQFT [17].

A corollary of this result is that for a unitary 2d TQFT the algebra  $\mathcal{A}$  is semisimple. To see this, note first that any nonzero self-adjoint element  $a$ ,  $a = a^*$ , cannot be nilpotent. Indeed, if  $n$  is the smallest  $n$  such that  $x^n = 0$ , then  $x^{2m} = 0$ , where  $m = [(n+1)/2]$ . Then  $\langle x^m | x^m \rangle = \langle 1 | x^{2m} | 1 \rangle = 0$ , and therefore  $x^m = 0$ . Since  $n \leq m$ , we must have  $n = 1$ , i.e.  $x = 0$ . Now we can use the result of [18] which says that a  $*$ -algebra with no nilpotent self-adjoint elements (apart from zero) is semisimple.

By the Artin-Wedderburn theorem, a finite-dimensional semisimple algebra over complex numbers is isomorphic to a sum of matrix algebras. Since  $\mathcal{A}$  is also commutative, this means that it is isomorphic to a sum of several copies of  $\mathbb{C}$ . The Frobenius and  $*$ -algebra structures are unique up to isomorphism. This means that the only invariant of the 2d TQFT is the dimension of  $\mathcal{A}$ , i.e. the ground-state degeneracy of the corresponding phase.

As discussed above, for a semisimple algebra  $\mathcal{A}$  boundary conditions correspond to finite-dimensional modules over  $\mathcal{A}$ . It is easy to see that for the open-closed TQFT to be unitary, the algebra  $V_{MM}$  must also have a Hilbert space structure such that

$$T(a)^\dagger = T(a^*) \quad (19)$$

Such a structure always exists and is unique. Thus a boundary condition for a unitary 2d TQFT can be simply identified with a module over  $\mathcal{A}$ . One can use any faithful module over  $\mathcal{A}$  as an input for the state-sum construction.

### 3.4 State-sum construction of the space of states

We have discussed above the state-sum construction of the partition function  $Z(\Sigma)$  for an oriented 2d manifold  $\Sigma$  without boundary (or more generally, with only brane boundary). More generally, one also needs to describe in similar terms the state space  $\mathcal{A}$  and a linear map  $\mathcal{A}^{\otimes n} \rightarrow \mathcal{A}^{\otimes l}$  for every bordism  $\Sigma$  whose source is a disjoint union of  $n$  circles and target is a disjoint union of  $l$  circles. That is, one needs to describe  $Z(\Sigma)$  for the case when  $\Sigma$  has nonempty cut boundary.

Consider a bordism  $\Sigma$  with a nonempty cut boundary. For simplicity let us assume that there is no brane boundary; the general case is a trivial generalization, but requires a more cumbersome notation. We

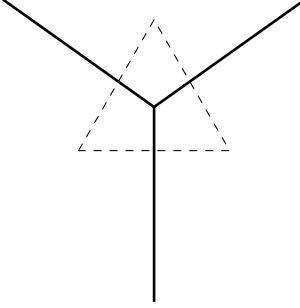


Figure 5: Taking the Poincaré dual of a triangle

choose a triangulation  $\mathcal{T}$  of  $\Sigma$ . It induces a triangulation of each boundary circle. We label the edges of 2-simplices with basis elements of  $A$ , as before. The only difference is that boundary 1-simplices have only one label rather than two. If we assign the weights to every 2-simplex and every internal 1-simplex as before and sum over the labelings of internal 1-simplices, we get a number  $Z_{\mathcal{T}}(\Sigma)$  which depends on the labelings of the boundary 1-simplices. Suppose some boundary circle is divided into  $N$  intervals. Then a labeling by  $e_{i_1}, \dots, e_{i_N}$  corresponds to a vector

$$e_{i_1} \otimes \dots \otimes e_{i_N} \in A^{\otimes N}. \quad (20)$$

We can think of the number  $Z_{\mathcal{T}}(\Sigma)$  computed by the state-sum as a matrix element of linear map from

$$A^{\otimes N_1} \otimes \dots \otimes A^{\otimes N_n} \quad (21)$$

to

$$A^{\otimes M_1} \otimes \dots \otimes A^{\otimes M_l}, \quad (22)$$

where  $N_1, \dots, N_n$  denote the number of 1-simplices in the source circles, and  $M_1, \dots, M_l$  denote the number of 1-simplices in the target circles of  $\Sigma$ . It can be shown [12] that the map  $Z_{\mathcal{T}}(\Sigma)$  does not depend on the triangulation of  $\Sigma$ , provided we fix the triangulation of the boundary circles.

$Z_{\mathcal{T}}(\Sigma)$  is not yet the desired  $Z(\Sigma)$  because it depends on the way the boundary circles are triangulated. To get rid of this dependence, we need to restrict this map to a certain subspace in each source factor  $A^{\otimes N_i}$  and project to a certain subspace in each target factor  $A^{\otimes M_j}$ . Both tasks are accomplished by means of projectors  $C_N : A^{\otimes N} \rightarrow A^{\otimes N}$ . The projector  $C_N$  is simply  $Z_{\mathcal{T}_N}(C)$ , where  $C$  is a cylinder and  $\mathcal{T}_N$  is any triangulation of  $C$  such that both boundary circles are subdivided into  $N$  intervals. The image of each  $C_N$  is a certain subspace of  $A^{\otimes N}$  isomorphic to  $Z(A)$  [12]. Restricting  $Z_{\mathcal{T}}(\Sigma)$  to these subspaces and then projecting to the image of each  $C_{M_j}$  gives us the desired map

$$Z(\Sigma) : \mathcal{A}^{\otimes n} \rightarrow \mathcal{A}^{\otimes l}, \quad (23)$$

where  $\mathcal{A} = Z(A)$ .

### 3.5 MPS from TQFT

Let us consider the special case when  $\Sigma$  is an annulus such that one of the circles is a cut boundary, while the other one is a brane boundary corresponding to an  $A$ -module  $M$ . Let  $T(a) \in \text{Hom}(M, M)$  represent an action of  $a \in A$  in this module. For definiteness, we choose the cut boundary to be the source of  $\Sigma$ , while the target is empty. Thus  $Z(\Sigma)$  is a linear map  $\mathcal{A} \rightarrow \mathbb{C}$ . It is the dual of the boundary state corresponding to the module  $M$ .

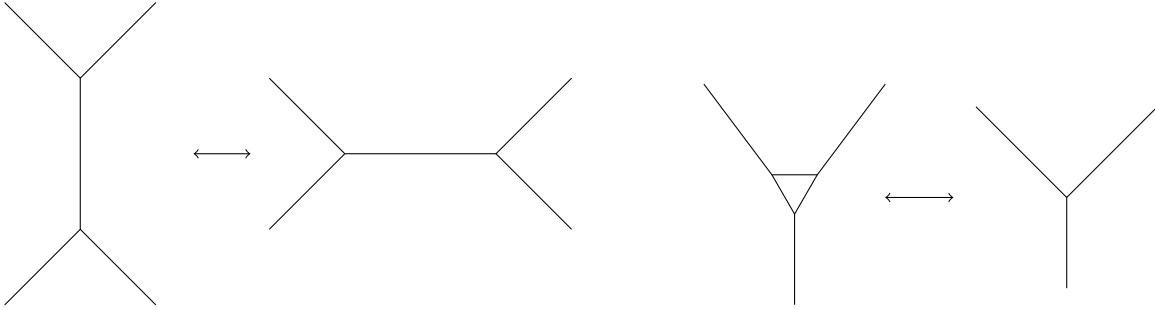


Figure 6: The dual 2-2 (left) and 3-1 (right) moves

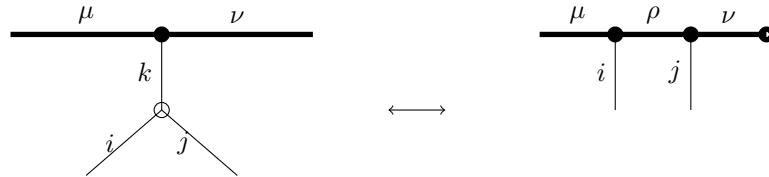


Figure 7: The dual shelling of Eq. 16. A filled dot represents  $T$ , while an empty dot represents  $C$ .

Let us now pick a triangulation of the annulus such that the cut boundary is divided into  $N$  intervals. Then  $Z_{\mathcal{T}}(\Sigma)$  is a linear map  $A^{\otimes N} \rightarrow \mathbb{C}$  which depends only on  $\mathcal{T}$  and  $N$ . We claim that this map is the dual of the MPS state with the dual MPS tensor given by  $T : A \rightarrow \text{Hom}(M, M)$ .

To see this, it is convenient to reformulate the state-sum on the Poincaré dual complex. This complex is built from the triangulation  $\mathcal{T}(\Sigma)$  by replacing  $k$ -cells with  $(2 - k)$ -cells, as in Figure 5. The dual of a triangulation is not a simplicial complex but a more general cell complex; since we will only be interested in the edges and vertices of this dual complex, we will refer to it as a *skeleton* for  $\Sigma$ . The Pachner moves are the same for skeleton as for triangulations, see Figure 4. Recall that for a unitary TQFT, one can choose  $\eta_{ij} = \delta_{ij}$ , so that indices may be freely raised and lowered; nonetheless, keeping track of index positions now will pay off later when we generalize to equivariant theories. Choose a direction for each edge; the state-sum does not depend on this choice. Choose these directions so that all edges on incoming boundaries are incoming and all edges on outgoing boundaries are outgoing. To define a state-sum on a skeleton, label its non-boundary edges with elements  $e_i$  and assign structure coefficients  $C$  to each non-boundary vertex according to orientation and using lower indices for incoming arrows and upper for outgoing. With these conventions, the Pachner moves algebraize to (14) and (15) as before. To incorporate brane boundaries, color brane boundary edges by elements  $v_\mu$  attach the module action  $T$  to each boundary vertex. The boundary moves recover (16) and (17). The dual state-sum is naturally a tensor network: it defines a circuit between the incoming and outgoing legs. Note that the “virtual” module indices are all contracted, so these legs are physical.

Consider the triangulation, shown in Figure 9 (a), of the annulus with boundary condition  $T$  on one of its boundary components. Its state-sum defines a state in the physical space  $\mathcal{A}^N$ . We claim that this state is the fixed point MPS  $|\psi_T\rangle$ . The proof of this fact is straightforward: by Pachner invariance, the annulus

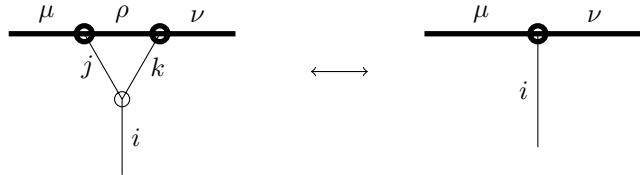


Figure 8: The dual shelling of Eq. 17, representing  $T_{\mu\rho}^j T_{\rho\nu}^k C_{ijk} = T_{\mu\nu}^i$

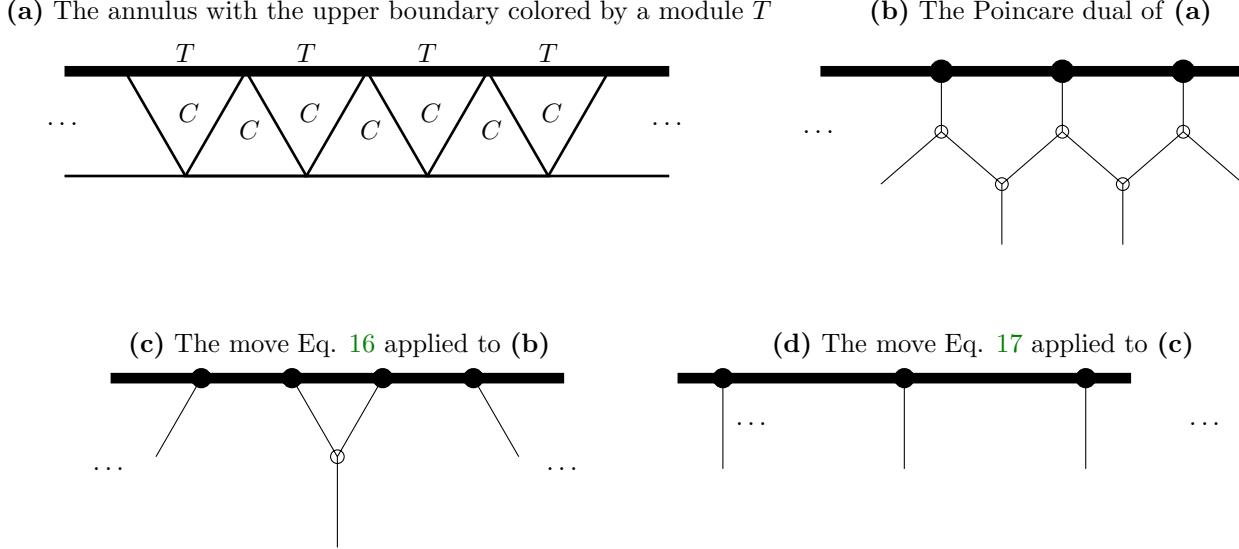


Figure 9: The equivalence of the annulus to the tensor network representation of an MPS

and MPS tensor networks are equivalent, see Figure 9.

More generally, one can insert a local observable on the brane boundary of the annulus. Such a local observable is parameterized by  $X \in \text{Hom}(M, M)$  which commutes with  $T(a)$  for all  $a \in A$ . The corresponding dual state is  $\text{tr}[X^\dagger TT \cdots T]$ , i.e. it is a generalized MPS state, with  $A$  being the physical space.

Since the linear operators  $T(a)$  satisfy  $T(a)T(b) = T(ab)$ , all these MPS states are RG-fixed MPS states. The RG-step is described by the algebra structure on  $A$ ,  $m : A \otimes A \rightarrow A$ . Moreover, the MPS is automatically in a standard form. The module  $T : A \rightarrow \text{End}(V)$  is semisimple, so it has a decomposition into simple modules  $T^{(\alpha)} : A \rightarrow \text{End}(V^{(\alpha)})$ . The collection of spaces  $\text{End}(V^{(\alpha)})$  form a block-diagonal subspace of  $\text{End}(V)$ . Since  $V^{(\alpha)}$  is simple,  $T^{(\alpha)}$  surjects onto the block  $\text{End}(V^{(\alpha)})$ .

The parent Hamiltonian of the MPS on an  $N$ -site closed chain has a TQFT interpretation as well: it is the linear map  $C_N = Z_{\mathcal{T}_N}(C) : A^{\otimes N} \rightarrow A^{\otimes N}$  assigned to a triangulated cylinder  $C$  whose boundary consists of two circles triangulated into  $N$  intervals. As previously stated,  $C_N$  projects onto a subspace  $\mathcal{A} = Z(A) \subset A^{\otimes N}$ , precisely the space of ground states of the parent Hamiltonian. In the continuum TQFT, topological invariance requires that the cylinder is the identity; this is consistent with our already having projected to  $\mathcal{A}$  in defining the continuum state spaces.

We have seen that a unitary TQFT is completely determined by its space of states  $\mathcal{A}$  on a circle and that each finite-dimensional commutative algebra  $\mathcal{A}$  defines a unitary TQFT. Therefore, the classification of unitary TQFTs is quite simple: there is one for every positive integer  $n$ , in agreement with the MPS-based classification of gapped phases [2, 3, 4].

## 4 Equivariant TQFT and Equivariant MPS

In this section, we generalize the relation between 2D TQFT and MPS states to systems with a global symmetry  $G$ . We show that both  $G$ -equivariant TQFTs and  $G$ -equivariant RG-fixed MPS states are described by semisimple  $G$ -equivariant algebras. In particular, we show that invertible  $G$ -equivariant TQFTs correspond to short-range entangled phases with symmetry  $G$ , and that both are classified by  $H^2(G, U(1))$ .

## 4.1 $G$ -equivariant Matrix Product States

Let  $G$  be a finite symmetry group acting on the physical space  $A$  via a unitary representation  $R$ ,  $g \mapsto R(g) \in \text{End}(A)$ . A  $G$ -invariant MPS tensor is a map  $\mathcal{P} : U \otimes U^* \rightarrow A$  equivariant in the following sense:

$$R(g)\mathcal{P}(X) = \mathcal{P}(Q(g)XQ(g^{-1})), \quad (24)$$

where the linear maps  $Q(g) \in \text{End}(U)$  form a projective representation of  $G$ . Let  $T = \mathcal{P}^\dagger$ . In terms of  $T$ , the equivariance condition looks as follows:

$$T(R(g)a) = Q(g)T(a)Q(g)^{-1}, \quad (25)$$

for any  $a \in A$  and any  $g \in G$ . The dual MPS state corresponding to  $T$  is

$$\langle \psi_T | = \sum_{i_1, \dots, i_N} \text{Tr}_U T(e_{i_1}) \dots T(e_{i_N}) \langle i_1 \dots i_N | \quad (26)$$

It is easy to see that the state  $\psi_T$  is  $G$ -invariant, thanks to the equivariance condition on  $P$ . More generally, let  $X \in \text{End}(U)$ . Note that  $\text{End}(U)$  is a genuine (not projective) representation of  $G$ . Then the generalized MPS state  $\text{Tr}(XTT \dots T)$  transforms in the same way as  $X$ .

## 4.2 $G$ -equivariant TQFT

Roughly speaking, a definition of a  $G$ -equivariant TQFT is obtained from the definition of an ordinary TQFT by replacing oriented manifolds with oriented manifolds with principal  $G$ -bundles. This reflects the intuition that a model with a global non-anomalous symmetry  $G$  can be coupled to a background  $G$  gauge field. (For a finite group  $G$ , there is no difference between a  $G$  gauge field and a principal  $G$ -bundle.)

Some care is required regarding marked points and trivializations. Namely, each source and each target circle must be equipped with a marked point and a trivialization of the  $G$ -bundle at this point. This means that the holonomy of the gauge field around the circle is a well-defined element  $g \in G$ , rather than a conjugacy class. A  $G$ -equivariant TQFT associates a vector space  $\mathcal{A}_g$  to a circle with holonomy  $g$ . A generic  $G$ -equivariant bordism has more than one marked point, and the holonomies between marked points along chosen paths are well-defined elements of  $G$  as well. Of course, these holonomies depend only on the homotopy classes of paths. For example, a  $G$ -equivariant cylinder bordism has two marked points (one for each boundary circle) and depends on two arbitrary elements of  $G$ . On the other hand, a  $G$ -equivariant torus, regarded as bordism with an empty source and empty target, has no marked points and depends on two commuting elements of  $G$  defined up to an overall conjugation.

One can describe  $G$ -equivariant TQFTs purely algebraically in terms of  $G$ -crossed Frobenius algebras [17, 14]. This notion generalizes the notion of a commutative Frobenius algebra. But this notion is rather complicated, and encodes the linear maps  $Z(\Sigma, \mathcal{P})$  in a fairly complex way.

We will use instead a state-sum construction of 2D equivariant TQFTs which is manifestly local. Its starting point is a  $G$ -equivariant Frobenius algebra  $A$ . This is a Frobenius algebra with an action of  $G$  which preserves both the scalar product  $\eta$  and the multiplication map  $m : A \otimes A \rightarrow A$ . That is,  $G$  acts on  $A$  via a linear representation  $R(g)$ ,  $g \in G$ , such that

$$m(R(g)a \otimes R(g)b) = R(g)m(a \otimes b) \quad (27)$$

and

$$\eta(R(g)a, R(g)b) = \eta(a, b). \quad (28)$$

A large class of examples of  $G$ -equivariant Frobenius algebras is obtained by taking  $A = \text{End}(U)$ , where  $U$  is a vector space, and  $G$  acts on  $U$  via a projective representation  $Q(g)$ . It is clear that this gives rise to a genuine action of  $G$  on  $\text{End}(U)$  which preserves the usual matrix multiplication on  $\text{End}(U)$ . Moreover, the standard Frobenius structure

$$\eta(a, b) = \text{Tr}(ab) \quad (29)$$

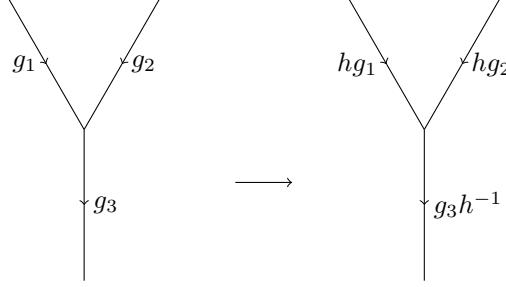


Figure 10: A lattice gauge transformation at the vertex by  $h$

is clearly  $G$ -invariant.

A  $G$ -equivariant module over a  $G$ -equivariant algebra  $A$  is a vector space  $V$  with compatible actions of both  $A$  and  $G$ . That is, for every  $a \in A$  we have a linear map  $T(a) : V \rightarrow V$  such that  $T(a)T(a') = T(aa')$ , and for every  $g \in G$  we have an invertible linear map  $\mathcal{Q}(g) : V \rightarrow V$  such that  $\mathcal{Q}(g)\mathcal{Q}(g') = \mathcal{Q}(gg')$ . The compatibility condition that they satisfy reads

$$T(R(g)a) = \mathcal{Q}(g)T(a)\mathcal{Q}(g)^{-1} \quad (30)$$

If we take  $A = \text{End}(U)$ , where  $U$  is a projective representation of  $G$  with a 2-cocycle  $\omega \in H^2(G, U(1))$ , then  $U$  is not a  $G$ -equivariant module over  $A$  unless  $\omega$  vanishes. However, if  $W$  is a projective representation of  $G$  with a 2-cocycle  $-\omega$ , then  $U \otimes W$  is a  $G$ -equivariant module.<sup>8</sup>

Equivariant TQFTs admit a lattice description as well. It is simplest to describe a Poincare dual formulation in the sense of Section 3.5; spaces in this formulation also have direct interpretations as tensor networks. A trivialized background gauge field is represented on a skeleton as a decoration of each oriented edge with an element  $g \in G$ . Flipping the orientation of the edge replaces  $g$  with  $g^{-1}$ . We require that the field is flat: that the product of the group elements around the boundary of each face is the identity element.<sup>9</sup> In a basis  $e_i$ ,  $i \in S$  of  $A$ , the weight of a coloring of the skeleton is the product of the structure constants  $C_{ijk}$  over vertices (with the cyclic order given by the orientation) and the matrix elements  $R(g)^i{}_k\eta^{jk}$  over edges. The partition sum is the sum of these weights over all colorings; we emphasize that the group labels represent a background gauge field and are not summed. To incorporate brane boundaries, choose a  $G$ -equivariant module  $V$  over  $A$ . Fix a basis  $f_\mu$  of  $V$ . For each brane boundary vertex, label its adjacent boundary edges each with a basis element, so that each boundary edge has a total of two labels. The weight of a skeleton with a brane boundary is a product of  $C$ 's and  $R$ 's as well as the module tensor  $T$  for each brane boundary vertex and the matrix elements  $\mathcal{Q}(g)^\mu{}_\nu$  for each brane boundary edge.

As before, topological invariance of the state-sum amounts to checking the conditions (14), (15), (16), and (17). These are satisfied by  $A$ . In order for the equivariant state-sum to constitute a well-defined equivariant TQFT, it must also be independent of the choice of triangulation of the background gauge field; in other words, it must be gauge invariant. A gauge transformation by  $h \in G$  on a vertex acts by changing the decorations of the three edges whose boundary contains the vertex: incoming edges with  $g$  become  $hg$ , outgoing  $gh^{-1}$ , as in Figure 10. Depending on whether the gauge transformation is on a vertex in the interior of the skeleton or in the brane boundary, invariance is ensured by the definitions of  $G$ -equivariant algebras and modules.<sup>10</sup>

<sup>8</sup>In fact, the category of projective representations of  $G$  with a 2-cocycle  $-\omega$  is equivalent to the category of  $G$ -equivariant modules over  $\text{End}(U)$ , and the equivalence sends a projective representation  $W$  to  $U \otimes W$ .

<sup>9</sup>In the triangulation picture, we require the product of all group elements corresponding to edges entering a particular vertex to be the identity element.

<sup>10</sup>Here it is crucial that linear transformations  $\mathcal{Q}(g)$  form an ordinary (i.e. not projective) representation of  $G$ .

### 4.3 $G$ -equivariant semi-simple algebras

The classic Wedderburn theorem implies that every finite-dimensional semi-simple algebra is a sum of matrix algebras. Let us discuss a generalization of this result to the  $G$ -equivariant case following [6, 7].

First, we can write every  $G$ -equivariant semi-simple algebra as a sum of indecomposable ones, so it is sufficient to classify indecomposable  $G$ -equivariant semi-simple algebras. A large class of examples is given by algebras of the form  $\text{End}(U)$ , where  $U$  is a projective representation of  $G$ . Another set of examples is obtained as follows: let  $H \subset G$  be a subgroup. Consider the space of complex-valued functions on  $G$  invariant with respect to left translations by  $H$ , i.e.  $f(h^{-1}g) = f(g)$  for all  $g \in G$  and all  $h \in H$ . The group  $G$  acts on this space by right translations:

$$(R(g)f)(g') = f(g'g) \quad (31)$$

Pointwise multiplication makes this space of functions into an associative algebra, and it is clear that the  $G$ -action commutes with the multiplication. This  $G$ -equivariant algebra is indecomposable for any  $H$ .

The most general indecomposable  $G$ -equivariant semi-simple algebra is a combination of these two constructions [6, 7]. One picks a subgroup  $H \subset G$  and a projective representation  $(U, Q)$  of  $H$ . Here  $U$  is a vector space and  $Q$  is a map  $H \rightarrow \text{End}(U)$  defining a projective action with a 2-cocycle  $\omega \in H^2(H, U(1))$ . Then one considers the space of functions on  $G$  with values in  $\text{End}(U)$  which have the following transformation property under the left  $H$  action:

$$f(h^{-1}g) = Q(h)f(g)Q(h)^{-1} \quad (32)$$

It is easy to check that the right  $G$  translations act on this space of functions. Pointwise multiplication makes this space into a  $G$ -equivariant algebra, and one can show that it is indecomposable. To summarize, indecomposable  $G$ -equivariant semi-simple algebras are labeled by triples  $(H, U, Q)$ , where  $H \subset G$  is a subgroup, and  $(U, Q)$  is a projective representation of  $H$ . All these algebras are actually Frobenius algebras: the trace function  $A \rightarrow \mathbb{C}$  is given by

$$\sum_{g \in G} \text{Tr}_U f(g) \quad (33)$$

A  $G$ -equivariant module over such an algebra  $A$  is obtained as follows. Start with an  $H$ -equivariant module  $(M, Q)$  over  $\text{End}(U)$ . Here  $M$  is a module over  $\text{End}(U)$  and  $Q : H \rightarrow \text{End}(M)$  is a compatible action of  $H$  on  $M$ . As explained above,  $M$  must have the form  $U \otimes W$ , where  $W$  carries a projective action  $S(h)$  of  $H$  with a 2-cocycle  $-\omega$ . Then consider functions on  $G$  with values in  $M$  which transform as follows under the left  $H$ -translations:

$$m(h^{-1}g) = (Q(h) \otimes S(h))m(g), \quad m : G \rightarrow U \otimes W. \quad (34)$$

The group  $G$  acts on this space by right translations, and it is easy to see that the pointwise action of  $A = (H, U, Q)$  makes it into a  $G$ -equivariant module over  $A$ . One can show that any  $G$ -equivariant module over such an  $A$  is a direct sum of modules of this sort.

### 4.4 $G$ -equivariant MPS from $G$ -equivariant TQFT

It is sufficient to consider indecomposable TQFTs and  $G$ -equivariant algebras. Let us begin with the case  $H = G$ . Then the algebra  $A = (G, U, Q)$  is isomorphic to the algebra  $\text{End}(U)$ , and a  $G$ -equivariant module over it is simply a vector space  $M$  with a  $G$ -equivariant action of  $\text{End}(U)$ . In other words,  $M = U \otimes W$ , where  $U$  carries a projective representation of  $G$  with the 2-cocycle  $\omega$ , and  $V$  carries a projective representation of  $G$  with a 2-cocycle  $-\omega$ .

Consider an annulus whose outer boundary is a brane boundary labeled by  $M$ , and inner boundary is a cut boundary. Let us triangulate both boundary circles into  $N$  intervals. Let  $g_{i,i+1}$  be the element of  $G$  labeling the interval between the  $i^{\text{th}}$  and  $(i+1)^{\text{th}}$  points on the boundary. We also assume that the holonomy of the gauge field between the points labeled by 1 on the two boundary circles is trivial. We get the the following dual state:

$$\langle \psi_T | = \sum \text{Tr}_{U \otimes W} [T(e_{i_1})Q(g_{12}) \cdots T(e_{i_N})Q(g_{N1})] \langle i_1 \cdots i_N | \quad (35)$$

Note that although  $T(e_i)$  is an operator on  $U \otimes W$ , it has the form  $T(e_i) \otimes \mathbf{1}_W$ . Therefore, if  $g_{i,i+1} = 1$  for all  $i$ , the trace over  $W$  gives an overall factor  $\dim W$ , and up to this factor we get the equivariant MPS (26). Inserting an observable  $X \in \text{End}(U)$  on the brane boundary, we get a generalized equivariant MPS. The case when  $X \in \text{End}(U \otimes W)$  does not give anything new, since the trace over  $V$  factors out.

The generalized equivariant MPS (cf. eq 7)

$$\langle \psi_T^X | = \sum \text{Tr}[X^\dagger T(e_{i_1}) \cdots T(e_{i_n})] \langle i_1 \cdots i_n | \quad (36)$$

may be charged under the action of  $h \in G$ :

$$\begin{aligned} R(h)^{\otimes N} \langle \psi_T^X | &= \sum \text{Tr}[X^\dagger T(e_{i_1}) \cdots T(e_{i_n})] \langle (h^{-1} \cdot i_1) \cdots (h^{-1} \cdot i_n) | \\ &= \sum \text{Tr}[X^\dagger T(h \cdot e_{i_1}) \cdots T(h \cdot e_{i_n})] \langle i_1 \cdots i_n | \\ &= \sum \text{Tr}[Q(h^{-1}) X^\dagger Q(h) T(e_{i_1}) \cdots T(e_{i_n})] \langle i_1 \cdots i_n | \end{aligned} \quad (37)$$

Let us now consider the case when  $H$  is a proper subgroup of  $G$  and  $A = \text{End}(U)$  [???  $A$  can't be  $\text{End}(U)$ ], for some projective representation  $U$  of  $H$ . If we choose right  $H$ -coset representatives  $g_a$ ,  $a \in H \backslash G$ , and a basis  $e_i$  in  $\text{End}(U)$ , then a basis in  $A$  is given by  $e_i^a$ . Similarly, if  $f_\mu$  is a basis in an  $H$ -equivariant module  $U \otimes W$ , then a basis in the corresponding  $G$ -equivariant module  $M$  is  $f_\mu^a$ .

The action of  $A$  on  $M$  is diagonal as far as the  $a$  index is concerned. Therefore the dual state corresponding to a triangulated annulus with  $g_{i,i+1} = 1$  for all  $i$  vanishes unless all  $a$  indices are the same. Then

$$\langle \psi_T | = \dim(W) \sum_{a,i_1,\dots,i_N} \text{Tr}_U[T(e_{i_1}) \cdots T(e_{i_N})] \langle i_1 a i_2 a \cdots i_N a |. \quad (38)$$

This state has equal components along all  $|H \backslash G|$  directions. We can get a state concentrated at a particular value of  $a$  by inserting a suitable observable  $X \in \text{End}(M)$  on the brane boundary. Such an observable must commute with the action of  $A$ , which means that it must have the form  $X_{\nu}^{\mu a} = f(a) \delta_{\nu}^{\mu} \delta_b^a$ . Choosing the function  $f(a)$  to be supported at a particular value of  $a$  gives a generalized MPS state supported at this value of  $a$ .

The symmetry group  $G$  acts transitively on  $H \backslash G$ . This suggests that we are dealing with a phase where the symmetry  $G$  is spontaneously broken down to  $H$ , so that we get  $|H \backslash G|$  sectors labeled by the index  $a$ . To confirm this, consider the partition function of this TQFT on a closed oriented 2-manifold  $\Sigma$  with a trivial  $G$ -bundle. After we choose a skeleton of  $\Sigma$ , we can represent this  $G$ -bundle by labeling every 1-simplex with the identity element of  $G$ . In addition, every 1-simplex is labeled by a pair of basis vectors of  $A$ . Since both the multiplication in the algebra  $A$  and the scalar product are pointwise in  $H \backslash G$ , the partition function receives contributions only from those labelings where all  $a$  labels are the same. Furthermore, turning on a gauge field which takes values in  $H$  does not destroy this property. We conclude that the theory has superselection sectors labeled by elements of  $H \backslash G$ , and each sector has unbroken symmetry  $H$ .

## 4.5 Twisted-sector states

Now let us not assume that  $g_{i,i+1} = 1$ , but instead allow the gauge field around the circle to have a nontrivial holonomy. Let us take  $H = G$  first, i.e. the case of unbroken symmetry. Consider the MPS (35). Applying a gauge transformations (by  $g_{k-1,k} \cdots g_{2,1}$  at vertex  $k$ ) to the boundary vertices it can be written as

$$\langle \psi_{T,g} | = \sum \text{Tr}_{U \otimes W}[\mathcal{Q}(g) T(e_{i_1}) \cdots T(e_{i_N})] \bigotimes_{k=1}^N R(g_{(k-1),k} \cdots g_{2,1})^{i_k} {}_{j_k} \langle j_k | \quad (39)$$

where  $g = g_{N,1} g_{(N-1),N} \cdots g_{1,2}$  is the holonomy of the gauge field. This is LU equivalent to the state

$$\langle \psi_{T,g} | = \sum \text{Tr}_{U \otimes W}[\mathcal{Q}(g) T(e_{i_1}) \cdots T(e_{i_N})] \langle i_1 \cdots i_N | \quad (40)$$

so we have effectively set  $g_{i,i+1} = 1$  for all  $i \neq N$  and  $g_{N,1} = g$ . Note that  $\mathcal{Q} = Q \otimes S$ , so the trace factors into a product of a trace over  $U$  and a trace over  $W$ . The latter gives us an overall factor, and we have

$$\langle \psi_{T,g} | = \text{Tr}_W[S(g)] \sum \text{Tr}_U[Q(g_{N1})T(e_{i_1}) \cdots T(e_{i_N})] \langle i_1 \cdots i_N |. \quad (41)$$

This state transforms under  $h \in G$  into

$$\begin{aligned} R(h)^{\otimes N} \langle \psi_{T,g} | &= (\text{Tr}_W[S(g)]) \sum \text{Tr}[Q(g)T(e_{i_1}) \cdots T(e_{i_n})] \langle (h^{-1} \cdot i_1) \cdots (h^{-1} \cdot i_n) | \\ &= (\text{Tr}_W[S(g)]) \sum \text{Tr}[Q(h)^{-1}Q(g)Q(h)T(e_{i_1}) \cdots T(e_{i_n})] \langle i_1 \cdots i_n | \\ &= (\text{Tr}_W[S(g)])\omega(g,h)\omega(h^{-1},gh) \sum \text{Tr}[Q(h^{-1}gh)T(e_{i_1}) \cdots T(e_{i_n})] \langle i_1 \cdots i_n | \end{aligned} \quad (42)$$

Note that the  $g$ -twisted sector becomes the  $hgh^{-1}$ -twisted sector.

Now suppose  $H$  is a proper subgroup of  $G$ . Since  $T$  acts pointwise in the  $a$  label, while  $G$  acts on  $a \in H \backslash G$  by right translations, the annulus state vanishes unless the holonomy around the circle is in  $H$ . This confirms once again that  $H$  is the unbroken subgroup. Indeed, when the holonomy does not belong to the unbroken subgroup, there must be a domain wall somewhere on the circle. Its energy is nonzero in the thermodynamic limit, which means that the TQFT space of states must be zero-dimensional for holonomies not in  $H$ .

## 4.6 Morita equivalence

We have seen that to any semi-simple  $G$ -equivariant algebra one can associate a  $G$ -equivariant 2d TQFT. But different algebras may give rise to the same TQFT. In particular, we would like to argue that the TQFT corresponding to an indecomposable algebra  $A = (H, U, Q)$ , where  $(U, Q)$  is a projective representation of  $H$ , depends only on the subgroup  $H$  and the 2-cocycle  $\omega$ , but not on the specific choice of  $(U, Q)$ .

To show this, note first of all that the partition function vanishes if the holonomy does not lie in  $H$  (this again follows from the fact that multiplication in the algebra  $A$  is pointwise with respect to the  $a$  index). Thus it is sufficient to consider oriented 2-manifolds with  $H$ -bundles. Further, if  $U$  and  $U'$  are projective representations of  $H$  with the same 2-cocycle, then  $U' = U \otimes W$ , where  $W$  is an ordinary representation of  $H$ . Thus we only need to show that the partition functions corresponding to algebras  $(H, U, Q)$  and  $(H, U \otimes W, Q \otimes S)$  are the same, where  $S : H \rightarrow \text{End}(W)$  is a representation of  $H$ . But it is clear from the state sum construction that the two partition functions differ by a factor which is the partition function of 2d  $H$ -equivariant TQFT corresponding to the algebra  $(H, W, S)$ .

We reduced the problem to showing that the  $H$ -equivariant TQFT for the algebra  $(H, W, S)$  is trivial when  $(W, S)$  is an ordinary (not projective) representation of  $H$ . This is straightforward: the equation  $S(h_1) \dots S(h_n) = S(h_1 \dots h_n)$  and the flatness condition for the  $H$  gauge field imply that the partition function is independent of the  $H$ -bundle, and for the trivial  $H$ -bundle the partition function is the same as for the trivial TQFT with  $A = \mathbb{C}$ .

From the mathematical viewpoint,  $G$ -equivariant algebras with the same  $H$  and  $\omega$  are *Morita-equivalent* [6]. Thus we have shown that Morita-equivalent algebras lead to identical  $G$ -equivariant TQFTs.<sup>11</sup>

## 4.7 Symmetry Protected Topological Phases

Finally, let us discuss the case of Short-Range Entangled phases with symmetry  $G$ . These are the same as Symmetry Protected Topological phases. Such phases have a one-dimensional space of ground states for every  $G$ -bundle on a circle. Since the space of states of a decomposable TQFT on a circle with a trivial bundle has dimension greater than one, a TQFT corresponding to an SPT phase must be indecomposable. We also showed that when  $H$  is a proper subgroup of  $G$ , the TQFT space of states is zero-dimensional

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<sup>11</sup>Strictly speaking, we only showed this for closed 2d TQFTs, but the argument easily extends to the open-closed case.

whenever the holonomy does not lie in  $H$ . Hence an equivariant TQFT corresponding to an indecomposable  $G$ -equivariant algebra  $(H, U, Q)$  cannot correspond to an SPT unless  $H = G$ .

Finally, a  $G$ -equivariant algebra of the form  $\text{End}(U)$ , where  $U$  is a projective representation of  $G$ , is simply a matrix algebra, if we ignore the  $G$  action. Hence the corresponding non-equivariant TQFT is trivial, or equivalently, the corresponding Hamiltonian is connected to the trivial one by a Local Unitary transformation. Hence SPT phases with symmetry  $G$  are labeled by 2-cocycles  $\omega \in H^2(G, U(1))$ . This is a well-known result [2, 3, 4].

## References

- [1] M.B. Hastings, “Entropy and Entanglement in Quantum Ground States,” Phys. Rev. B **76**, 035114 (2007)
- [2] X. Chen, Z. Gu, X. Wen, “Classification of Gapped Symmetric Phases in 1D Spin Systems”, Phys. Rev. B **83**, 035107 (2011).
- [3] X. Chen, Z. Gu, X. Wen, “Complete Classification of 1D Gapped Quantum Phases in Interacting Spin Systems”, Phys. Rev. B **84**, 235128 (2011).
- [4] L. Fidkowski, A. Kitaev, “Topological Phases of Fermions in One Dimension,” Phys. Rev. B **83**, 075103 (2011).
- [5] A. Kapustin, A. Turzillo, M. You, to appear.
- [6] V. Ostrik “Module Categories, Weak Hopf Algebras and Modular Invariants,” Transformation Groups Vol.2, no. 2, 177-206 (2003)
- [7] P. Etingof, “Semisimple and  $G$ -equivariant semisimple algebras over operads,” arXiv:1512.07658[math.RA]
- [8] D. Perez-García, F. Verstraete, M.M. Wolf and J.I. Cirac, “Matrix Product State Representations,” arXiv:0608197
- [9] N. Schuch, D. Perez-García, I. Cirac, “Classifying quantum phases using MPS and PEPs,” arXiv:1010.3732v3
- [10] X. Chen, Z. Gu, X. Wen, “Local unitary transformation, long-range quantum entanglement, wave function renormalization, and topological order”, Phys. Rev. B **82**, 155138 (2010)
- [11] F. Verstraete, J.I. Cirac, J.I. Latorre, E. Rico, M.M. Wolf, “Renormalization Group Transformations on Quantum States,” Phys. Rev. Lett. **94**, 140601 (2005) [arXiv:quant-ph/0410227].
- [12] S. Fukuma, A. Hosono, H. Kawai, “Lattice Topological Field Theory in two dimensions”, Comm. Math. Phys. **161**, no. 1, 157–175 (1994)
- [13] M. Atiyah, “Topological quantum field theories,” Inst. Hautes Etudes Sci. Publ. Math. **68**, 175 (1989)
- [14] G. Moore, G. Segal, “D-branes and K-theory in 2D topological field theory,” arXiv: hep-th/0609042.
- [15] L. Abrams, “Two-dimensional topological quantum field theories and Frobenius algebras,” J. Knot Theory Ramifications **5**, 569-587 (1996).
- [16] U. Pachner, “P.L. homeomorphic manifolds are equivalent by elementary shellings”, European Journal of Combinatorics **12**, no. 2, 129-145 (1991).
- [17] V. Turaev, “Homotopy Quantum Field Theory,” EMS Tracts in Mathematics Vol. 10 (2010).
- [18] A. Kapustin, “Is There Life Beyond Quantum Mechanics?” arXiv:1303.6917
- [19] A. Lauda and H. Pfeiffer, “State sum construction of two-dimensional open-closed Topological Quantum Field Theories,” arXiv:0602047

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